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To cite this version:

Sophie Laruelle, Gilles Pagès. Randomized Urn Models revisited using Stochastic Approximation. 2011. hal-00555752v3

HAL Id: hal-00555752

https://hal.science/hal-00555752v3

Preprint submitted on 13 Dec 2011 (v3), last revised 17 Jan 2017 (v6)

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Randomized Urn Models revisited using Stochastic Approximation

Sophie Laruelle [*] Gilles Pagès [†]

First Version : January 14, 2011
This Version : December 13, 2011

Abstract

This paper presents the link between stochastic approximation and clinical trials based on randomized urn models investigated in [4, 5, 6]. We reformulate the dynamics of both the urn composition and the assigned treatments as standard stochastic approximation (SA) algorithms with remainder. Then, we derive the $a.s.$ convergence and the asymptotic normality (Central Limit Theorem CLT) of the normalized procedure under less stringent assumptions by calling upon the ODE and SDE methods. As a second step, we investigate a more involved family of models, known as multi-arm clinical trials, where the urn updating depends on the past performances of the treatments. By increasing the dimension of the state vector, our SA approach provides this time a new asymptotic normality result.

Keywords *Stochastic approximation, extended Pòlya urn models, nonhomogeneous generating matrix, strong consistency, asymptotic normality, multi-arm clinical trials.*

2010 AMS classification: 62L20, 62E20, 62L05 secondary: 62F12, 62P10.

1 Introduction

The aim of this paper is to illustrate the efficiency of Stochastic Approximation (SA) Theory by revisiting several recent results on randomized urn models applied to clinical trials (especially [4, 5, 6]). We will first retrieve the $a.s.$ convergence (strong consistency) and asymptotic normality results obtained in these papers under less stringent assumptions. Then we will take advantage of this more synthetic approach to establish a new Central Limit Theorem (CLT) in the more sophisticate randomized urn model known as "multi-arm clinical test". In this model, the urn updating which produces the adaptive design is based on statistical estimators of the past efficiency of the assigned treatments.

In these adaptive models, the starting point is the equation which governs the urn composition updated after each new treated patient. Basically, we will show that a normalized version of this urn composition can be formulated as a classical recursive stochastic algorithm with step $\gamma_n = \frac{1}{n}$

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which classical Stochastic Approximation Theory deals with. Doing so we will be in position to establish the $a.s.$ convergence of the procedure by calling upon the so-called Ordinary Differential Equation Method (ODE method) and to derive the asymptotic normality - a CLT, to be precise - from the standard CLT for stochastic algorithms (sometimes called the Stochastic Differential Equation Method (SDE method), see $e.g.$ [12, 8]). These two main theoretical results are recalled in a self-contained form in the Appendix. They can be found in all classical textbooks on SA ([8], [11], [12], [19]) and go back to [18] and [10].

Clinical trials essentially deal with the asymptotic behaviour of the patient allocation to several treatments during the procedure. Adaptive designs in clinical trials aim at detecting "on line" which treatment should be assigned to more patients, while keeping randomness enough to preserve the basis of treatments. This adaptive approach relies on the cumulative information provided by the responses to treatments of previous patients in order to adjust treatment allocation to the new patients. To this end, many urn models have been suggested in the literature (see [17], [23], [22], [13] and [20]). The most widespread random adaptive model is the Generalized Friedman Urn (GFU) (see [2]), also called Generalized Pólya Urn (GPU). The idea of this modeling is that the urn contains balls of d different types representative of the treatments. All random variables involved in the model are supposed to be defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Denote $Y_0 = (Y_0^i)_{i=1,\dots,d} \in \mathbb{R}_+^d \setminus \{0\}$ the initial composition of the urn, where Y_0^i denotes the number of balls of type i, $i = 1, \dots, d$. The allocation of the treatments is sequential and the urn composition at draw n is denoted by $Y_n = (Y_n^i)_{i=1,\dots,d}$. When the nth patient presents, one draws randomly ($i.e.$ uniformly) a ball from the urn with instant replacement. If the ball is of type j, then the treatment j is assigned to the nth patient, $j = 1, \dots, d$, $n \geq 1$. The urn composition is updated by taking into account the response of the nth patient to the treatment j, or the responses of all patients up to the nth one ($i.e.$ the efficiency of the assigned treatment), namely by adding D_n^{ij} balls of type i, $i = 1, \dots, d$. The procedure is iterated as long as patients present. Consequently the larger the number of balls of a given type is, the more efficient the treatment is. The urn composition at n, modeled by an $\mathbb{R}^d$-valued vector Y_n satisfies the following recursive procedure:

$$Y_n = Y_{n-1} + D_n X_n, \quad Y_0 \in \mathbb{R}_+^d \setminus \{0\}, \tag{1.1}$$

with $D_n = (D_n^{ij})_{1 \leq i,j \leq d}$ is the addition rule matrix and X_n is the result of the nth draw and $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \to \{e^1, \cdots, e^d\}$ models the selected treatment ($\{e^1, \cdots, e^d\}$ denotes the canonical basis of $\mathbb{R}^d$ and e^i stands for treatment i). We model the drawing in the urn by setting

$$X_n = \sum_{i=1}^{d} \mathbb{1}_{\left\{ \frac{\sum_{\ell=1}^{i-1} Y_{n-1}^\ell}{\sum_{\ell=1}^{d} Y_{n-1}^\ell} < U_n \leq \frac{\sum_{\ell=1}^{i} Y_{n-1}^\ell}{\sum_{\ell=1}^{d} Y_{n-1}^\ell} \right\}} e^i, \quad n \geq 1, \tag{1.2}$$

where $(U_n)_{n \geq 1}$ is i.i.d. with distribution $U_1 \overset{\mathcal{L}}{\sim} \mathcal{U}_{[0,1]}$.

Let $\mathcal{F}_n = \sigma(Y_0, U_k, D_k, 1 \leq k \leq n)$ be the filtration of the procedure. The *generating matrices* are defined as the $\mathcal{F}_n$-compensator of the additions rule sequence $i; e$.

$$H_n = \left(\mathbb{E}\left[D_n^{ij} \,|\, \mathcal{F}_{n-1} \right] \right)_{1 \leq i,j \leq d}, \; n \geq 1.$$

The first designs under consideration were the homogeneous GFU models where the addition rules D_n are i.i.d. and the so-called generating matrices $H_n = H = \mathbb{E} D_n$ are identical, non-random,

nonnegative and irreducible. Hence by the Perron-Frobenius theorem H has a unique and positive maximal eigenvalue and an eigenvector with positive components (see [2, 3, 15, 16]). But the homogeneity of the generating matrix is often not satisfied in practice and inhomogeneous GFU models have been introduced (see [4]) in which H_n are not random but converge to a deterministic limit H, under the assumption that the total number of balls added at each stage is constant. As a third step, the homogeneous Extended Pòlya Urn (EPU) models have been introduced in [21] in which only the mean total number of balls added at each stage is constant.

Finally, in [5] the authors proposed a nonhomogeneous EPU model because in applications, the addition rule D_n depends on the past history of previous trials (see [1]), so that the general generating matrix H_n is usually random. Thus the entries of H may not be all nonnegative (*e.g.*, when there is no replacement after the draw), and they assume that the matrix H has a unique maximal eigenvalue λ with associated (right) eigenvector $v^* = (v^{*,i})_{i=1,\dots,d}$ with $\sum_{i=1}^d v^{*,i} = 1$. Furthermore the conditional expectation of the total number of balls added at each stage were constant.

The first theoretical investigations on these models focused on the asymptotic properties of the urn composition (consistency and asymptotic normality). However, for practical matter, it is clear that the asymptotic behaviour of the vector $N_n := \sum_{k=1}^n X_k$ which stores the treatment allocation among the first n patients is of high interest, especially its variance structure in order to compare several adaptive designs. Thus, in [5] is proved the strong consistency of both (normalized) quantities Y_n/n and N_n/n (under a summability assumption on the generating matrices).

By considering an appropriate recursive procedure for the normalized urn composition derived from (1.1) we prove by the ODE method its $a.s.$ convergence toward v^* under a significantly less stringent assumption, namely the minimal requirement that $H_n \xrightarrow[n\to\infty]{a.s.} H$. The $a.s.$ convergence of the treatment allocation frequency $\frac{N_n}{n}$ toward the same v^*.

As concerns asymptotic normality, separate results on these two quantities are obtained in [5] under an additional assumption on the rate of convergence of the generating matrices H_n toward H. On our side we propose to consider a stochastic approximation procedure with remainder satisfied by the higher dimensional vector $(\frac{Y_n}{n}, \frac{N_n}{n})$. Then, the standard CLT for SA procedures with remainder directly provides the expected asymptotic normality result for the whole vector under an assumption on the L^2-rate of convergence of the generating matrices towards their limit (namely *i.e.* $|||H_n - H||| = o\left(n^{-\frac{1}{2}}\right)$) which is again slightly less stringent than the original one. As a result, we obtain the asymptotic joint distribution with an explicit global covariance structure matrix.

In the end of [5], an application to multi-arm clinical trials randomized urn models is proposed. This adaptive design has already been introduced in [6] with first consistency results. This kind of models is clearly the most interesting for practitioners since it takes into account the past results of the assigned treatments in the addition rule matrices, denoted S_n at time n (S_n^i denotes the number of cured patients by treatment i among the N_n^i treated ones). The above strong consistency results apply but none of the asymptotic normality works as stated since the generating matrices H_n do not – in fact *cannot* as we will emphasize – converge at the requested rate since they themselves satisfy a CLT. However by increasing again the structural dimension of the problem by considering the triplet $(\frac{Y_n}{n}, \frac{N_n}{n}, \frac{S_n}{n})$ which can be shown again to satisfy a SA algorithm with remainder for which $a.s.$ convergence and the CLT hold (provided the limiting generating matrix is still irreducible, etc). Thus we illustrate on this example that SA Theory is a powerful tool to investigate this kind

of adaptive design problem. The main difficulty is to exhibit the appropriate form for the recursion by making *a priori* the balance between significant asymptotic terms and remainder terms.

The paper is organized as follows. We rewrite the dynamics (1.1) of the urn composition as a stochastic approximation procedure with state variable for $\widetilde{Y}_n := \frac{Y_n}{n}$ in Section 2.1. In Section 2.2 the $a.s.$ convergence of $\frac{1}{n}\sum_{i=1}^d Y_n^i$ is established which implies that of $\widetilde{Y}_n$ and $\widetilde{N}_n := \frac{N_n}{n}$ by using the ODE method of SA under slightly lighten assumption than in [5]. The rate of convergence is investigated in Section 2.3: we obtain a CLT, once again under slightly less stringent assumptions on the limit generating matrix H than in [5]. Section 3 is devoted to multi-arm clinical tests. In Section 3.1 we briefly recall the Wei GFU model introduced [22, 6] where the generating matrices H_n are not random. In this case, the strong consistency and the asymptotic normality follow from the results of Section 2 (like in [5]). In Section 3.2 we study the adaptive design proposed in [6] where the addition rule matrices depend on the responses of all the past patients. We use result in Section 2.2 to prove the strong consistency. We prove in Section 3.3 a new CLT for this model, when the generating matrix H_n satisfies itself a CLT, which relies again on Stochastic Approximation techniques.

NOTATIONS $\forall u = (u^i)_{i=1,\dots,d} \in \mathbb{R}^d$, $\|u\|$ denotes the canonical Euclidean norm of the column vector u on $\mathbb{R}^d$, $\mathrm{Tr}(u) = \sum_{k=1}^d u^d$ denotes its "trace", u^t denotes its transpose; $|||A|||$ denotes the operator norm of the matrix $A \in \mathcal{M}_{d,q}(\mathbb{R})$ with d rows and q columns with respect to canonical Euclidean norms. When $d = q$, $\mathrm{Sp}(A)$ denotes the set of eigenvalues of A. $\mathbf{1} = (1 \cdots 1)^t$ denotes the unit column vector in $\mathbb{R}^d$ and I_d denotes the $d \times d$ identity matrix.

2 Convergence and first rate result

With the notations and definitions described in the introduction, we then formulate the main assumptions to establish the $a.s.$ convergence of the urn composition.

(A1) The generating matrices $H_n = (H_n^{ij})_{1\le i,j\le d}$, $n \ge 1$, satisfies $a.s.$

$$\forall\, i,j \in \{1,\dots,d\}, \quad H_n^{ij} \ge 0 \quad \text{and} \quad \forall\, j \in \{1,\dots,d\}, \quad \sum_{i=1}^d H_n^{ij} = c > 0. \tag{2.3}$$

We may assume up to a renormalization of Y_n without loss of generality that $c = 1$.

(A2) The addition rule D_n is conditionally independent of the drawing procedure X_n given $\mathcal{F}_{n-1}$ and satisfies

$$\forall 1 \le j \le d, \quad \sup_{n\ge1} \mathbb{E}\left[\left\|D_n^{\cdot j}\right\|^2 \mid \mathcal{F}_{n-1}\right] < +\infty. \tag{2.4}$$

(A3) Assume that there exists an irreducible $d \times d$ matrix H (with non-negative entries) such that

$$H_n \xrightarrow[n\to\infty]{a.s.} H. \tag{2.5}$$

H is called the limit generating matrix.

This assumption guarantees by the Perron-Frobenius Theorem (see [9]) that 1 is the maximal eigenvalue of H and that the components of its right eigenvector v can be chosen all positive. Therefore, we may normalize this vector v^* such that $\mathrm{Tr}(v^*) = 1$.

2.1 The dynamics as a stochastic approximation procedure

Our aim in this section is to reformulate the dynamics (1.1)-(1.2) into a recursive stochastic algorithm. Then we aim at applying the most powerful tools of SA, namely the "ODE" and the "SDE" methods to elucidate the asymptotic properties ($a.s.$ convergence and weak rate) of both the urn composition and the treatment allocation. We start from (1.1) with $Y_0 \in \mathbb{R}_+^d \setminus \{0\}$.

$$Y_{n+1} = Y_n + D_{n+1}X_{n+1} = Y_n + \mathbb{E}\left[D_{n+1}X_{n+1} \,|\, \mathcal{F}_n\right] + \Delta M_{n+1}, \tag{2.6}$$

where

$$\Delta M_{n+1} := D_{n+1}X_{n+1} - \mathbb{E}\left[D_{n+1}X_{n+1} \,|\, \mathcal{F}_n\right]$$

is an $\mathcal{F}_n$-martingale increment. By the definition of the generating matrix H_n, we have

$$\begin{aligned}
\mathbb{E}\left[D_{n+1}X_{n+1} \,|\, \mathcal{F}_n\right] &= \sum_{i=1}^{d} \mathbb{E}\left[D_{n+1}\mathbb{1}_{\{X_{n+1}=e^i\}}e^i \,|\, \mathcal{F}_n\right] = \sum_{i=1}^{d} \mathbb{E}\left[D_{n+1} \,|\, \mathcal{F}_n\right] \mathbb{P}\left(X_{n+1} = e^i \,|\, \mathcal{F}_n\right) e^i \\
&= H_{n+1} \sum_{i=1}^{d} \frac{Y_n^i}{\mathrm{Tr}(Y_n)} e^i = H_{n+1} \frac{Y_n}{\mathrm{Tr}(Y_n)}
\end{aligned}$$

so that
$$Y_{n+1} = Y_n + H_{n+1}\frac{Y_n}{\mathrm{Tr}(Y_n)} + \Delta M_{n+1}.$$

Now we can derive a stochastic approximation for the normalized urn composition Y_n. First we have for every $n \geq 1$,

$$\frac{Y_{n+1}}{n+1} = \frac{Y_n}{n} + \frac{1}{n+1}\left(H_{n+1}\frac{Y_n}{\mathrm{Tr}(Y_n)} - \frac{Y_n}{n}\right) + \frac{\Delta M_{n+1}}{n+1}.$$

Consequently, $\widetilde{Y}_n = \dfrac{Y_n}{n}$, $n \geq 1$, satisfies a canonical recursive stochastic approximation procedure

$$\begin{aligned}
\widetilde{Y}_{n+1} &= \widetilde{Y}_n + \frac{1}{n+1}\left(H_{n+1} - I_d\right)\widetilde{Y}_n + \frac{1}{n+1}\left(\Delta M_{n+1} + \left(\frac{n}{\mathrm{Tr}(Y_n)} - 1\right) H_{n+1}\widetilde{Y}_n\right) \\
&= \widetilde{Y}_n - \frac{1}{n+1}\left(I_d - H\right)\widetilde{Y}_n + \frac{1}{n+1}\left(\Delta M_{n+1} + r_{n+1}\right)
\end{aligned} \tag{2.7}$$

with step $\gamma_n = \frac{1}{n}$ and a remainder term given by

$$r_{n+1} := \left(\frac{n}{\mathrm{Tr}(Y_n)} - 1\right) H_{n+1}\widetilde{Y}_n + (H_{n+1} - H)\widetilde{Y}_n. \tag{2.8}$$

Furthermore, in order to establish the $a.s.$ boundedness of $(\widetilde{Y}_n)_{n\geq 1}$ we will rely on the following recursive equation satisfied by $\mathrm{Tr}(Y_n)$:

$$\mathrm{Tr}(Y_{n+1}) = \mathrm{Tr}(Y_n) + \frac{\mathrm{Tr}(H_{n+1}Y_n)}{\mathrm{Tr}(Y_n)} + \mathrm{Tr}(\Delta M_{n+1}).$$

By the properties of the generating matrix H_{n+1}, we obtain

$$\mathrm{Tr}(H_{n+1}Y_n) = \sum_{i=1}^{d}(H_{n+1}Y_n)_i = \sum_{i=1}^{d}\sum_{j=1}^{d} H_{n+1}^{ij}Y_n^j = \sum_{j=1}^{d}\left(\sum_{i=1}^{d} H_{n+1}^{ij}\right) Y_n^j = \mathrm{Tr}(Y_n).$$

Consequently

$$\mathrm{Tr}(Y_{n+1}) = \mathrm{Tr}(Y_n) + 1 + \mathrm{Tr}(\Delta M_{n+1}). \tag{2.9}$$

2.2 Convergence results

Theorem 2.1. *Under the assumptions* **(A1)**, **(A2)** *and* **(A3)**,

(a) $\frac{Tr(Y_n)}{n} \xrightarrow[n\to\infty]{a.s.} 1$ *and* $\frac{Y_n}{Tr(Y_n)} \xrightarrow[n\to\infty]{a.s.} v^*$.

(b) $\widetilde{N}_n := \frac{N_n}{n} = \frac{1}{n}\sum_{k=1}^{n} X_k \xrightarrow[n\to\infty]{a.s.} v^*$.

Remarks. • We simply need that $H_n \xrightarrow[n\to\infty]{a.s.} H$ whereas the assumption in [5], namely

$$\sum_{n\geq 1} \frac{\|H_n - H\|_\infty}{n} < +\infty \quad a.s.$$

• Assumption **(A3)** is not necessary to prove that $\frac{\mathrm{Tr}(Y_n)}{n} \xrightarrow[n\to\infty]{a.s.} 1$.

Proof. We will first prove that $(a) \Rightarrow (b)$, then we will prove (a).
$(a) \Rightarrow (b)$. We have

$$\mathbb{E}\left[X_n \,|\, \mathcal{F}_{n-1}\right] = \sum_{i=1}^{d} \frac{Y_{n-1}^i}{\mathrm{Tr}(Y_{n-1})} e^i = \frac{Y_{n-1}}{\mathrm{Tr}(Y_{n-1})}$$

and, by construction $\|X_n\|^2 = 1$ so that $\mathbb{E}\left[\|X_n\|^2 \,|\, \mathcal{F}_{n-1}\right] = 1$. Hence the martingale

$$\widetilde{M}_n = \sum_{k=1}^{n} \frac{X_k - \mathbb{E}\left[X_k \,|\, \mathcal{F}_{k-1}\right]}{k} \xrightarrow[n\to\infty]{a.s.\&L^2} \widetilde{M}_\infty \in L^2,$$

and by the Kronecker Lemma we obtain

$$\frac{1}{n}\sum_{k=1}^{n} X_k - \frac{1}{n}\sum_{k=1}^{n} \frac{Y_{k-1}}{\mathrm{Tr}(Y_{k-1})} \xrightarrow[n\to\infty]{a.s.} 0.$$

This yields the announced implication owing to the Cesaro Lemma.
(a) First Step: We have

$$D_{n+1}X_{n+1} = \sum_{j=1}^{d} D_{n+1}^{\cdot j} \mathbb{1}_{\{X_{n+1}=e^j\}}$$

where $D_{n+1}^{\cdot j} = (D_{n+1}^{ij})_{i=1,\ldots,d}$. Therefore

$$\begin{aligned}
\|D_{n+1}X_{n+1}\|^2 &= \sum_{j=1}^{d} \left\|D_{n+1}^{\cdot j}\right\|^2 \mathbb{1}_{\{X_{n+1}=e^j\}}, \\
\text{so that} \qquad \mathbb{E}\left[\|D_{n+1}X_{n+1}\|^2 \,|\, \mathcal{F}_n\right] &= \sum_{j=1}^{d} \mathbb{E}\left[\left\|D_{n+1}^{\cdot j}\right\|^2 \,|\, \mathcal{F}_n\right] \mathbb{P}\left(X_{n+1} = e^j \,|\, \mathcal{F}_n\right) \\
&\leq d \sup_{n\geq 0} \sup_{1\leq j\leq d} \mathbb{E}\left[\left\|D_{n+1}^{\cdot j}\right\|^2 \,|\, \mathcal{F}_n\right] < +\infty \quad a.s.
\end{aligned}$$

Consequently $\sup_{n\geq 1}\mathbb{E}\left[\|\Delta M_{n+1}\|^2 \mid \mathcal{F}_n\right] < +\infty$. Therefore thanks to the strong law of large numbers for conditionally L^2-bounded martingale increments, we have $\frac{M_n}{n} \xrightarrow[n\to\infty]{a.s.} 0$. Consequently it follows from (2.9) that

$$\frac{\mathrm{Tr}(Y_n)}{n} = 1 + \frac{\mathrm{Tr}(Y_0)-1}{n} + \frac{\mathrm{Tr}(M_n)}{n} \xrightarrow[n\to\infty]{a.s.} 1. \tag{2.10}$$

Second Step: Since the components of $\widetilde{Y}_n$ are non-negative and $\mathrm{Tr}(\widetilde{Y}_n) = \frac{\mathrm{Tr}(Y_n)}{n} \xrightarrow[n\to\infty]{a.s.} 1$, it is clear that $(\widetilde{Y}_n)_{n\geq 1}$ is $a.s.$ bounded and that $a.s.$ the set $\mathcal{Y}_\infty$ of all its limiting value is contained in

$$\mathcal{V} = \mathrm{Tr}^{-1}\{1\} = \left\{u \in \mathbb{R}_+^d \mid \mathrm{Tr}(u) = 1\right\}.$$

So we may try applying the ODE method (see Appendix Theorem A.1). Since $\widetilde{Y}_n$ and $H_{n+1}\widetilde{Y}_n$ are $a.s.$ bounded, (2.10) and **(A3)** imply that $r_n \xrightarrow[n\to\infty]{a.s.} 0$.

The ODE associated to the recursive procedure reads

$$ODE_{I_d-H} \equiv \dot{y} = -(I_d - H)y.$$

Owing to Assumption **(A3)**, $I_d - H$ admits v^* as unique zero in $\mathcal{V}$. The restriction of ODE_{I_d-H} to the affine hyperplane $\mathcal{V}$ is the linear system $\dot{z} = -(I_d - H)z$, where $z = y - v^*$ takes values in $\mathcal{V}_0 = \left\{u \in \mathbb{R}^d \mid \mathrm{Tr}(u) = 0\right\}$. Since $\mathrm{Sp}\left((I_d - H)_{|\mathcal{V}_0}\right) \subset \{\lambda \in \mathbb{C}, \mathcal{R}e(\lambda) > 0\}$, owing to Assumption **(A3)**. As a consequence v^* is an uniformly stable equilibrium for the restriction of ODE_{I_d-H} to $\mathcal{V}$, the whole hyperplane, as an attracting area. The fundamental result derived from the ODE method (see Theorem A.1 in Appendix) yields the expected result

$$\widetilde{Y}_n \xrightarrow[n\to\infty]{a.s.} v^*. \qquad \square$$

Remark: If we assume that the addition rule matrices $(D_n)_{n\geq 1}$ satisfy **(A1)**, then we can directly write a stochastic approximation for $\frac{Y_n}{\mathrm{Tr}(Y_n)}$ in which the remainder simply reads $(H_{n+1} - H)\frac{Y_n}{\mathrm{Tr}(Y_n)}$ and prove the $a.s.$ convergence under the same assumptions.

Comments. We could apply directly the ODE method because we first proved that $(\widetilde{Y}_n)_{n\geq 1}$ is $a.s.$ bounded without using the standard Lyapunov machinery developed in SA Theory. That is why the assumption on the remainder sequence $(r_n)_{n\geq 1}$ simply reads

$$r_n \xrightarrow[n\to\infty]{a.s.} 0.$$

Another approach is the martingale one. It relies on the existence of a Lyapunov function $V : \mathbb{R}^d \to \mathbb{R}_+$ associated to the algorithm satisfying

$$\exists\, a > 0, \quad \forall y \in \mathbb{R}^d,, \quad y \neq v^*, \quad \langle \nabla V \mid I_d - H\rangle(y) > 0 \quad \text{and} \quad \langle \nabla V \mid I_d - H\rangle > a\,|\nabla V|^2. \tag{2.11}$$

In this framework the existence of a Lyapunov function can be established. Hence, the natural condition on the remainder sequence $(r_n)_{n\geq 1}$ reads (see [11])

$$\sum_{n\geq 1} \frac{\|r_n\|^2}{n} < +\infty \; a.s.$$

In that perspective, the assumption on the generating matrices would read $\sum_{n\geq 1} \frac{|||H_n - H|||^2}{n} < +\infty$ $a.s.$ which is still slightly less stringent than assumption on the generating matrices made in [5].

2.3 Rate of convergence

In the previous section we proved the $a.s.$ convergence of both quantities of interest, namely $\widetilde{Y}_n$ and $\widetilde{N}_n$, toward v^*. In this section we establish a "joint CLT" for the couple $\theta_n := (\widetilde{Y}_n, \widetilde{N}_n)^t$ with an explicit asymptotic joint normal distribution (including covariances). To this end we will show that θ_n satisfies a SA recursive procedure which ($a.s.$ converges toward $\theta^* = (v^*, v^*)^t$ and) fulfills the assumptions of the CLT Theorem A.2 for SA algorithms (see Appendix), with a special attention paid to Condition (A.23) about the remainder term. As concerns $\widetilde{Y}_n$, we derive from (2.7) that

$$\forall\, n \geq 1, \qquad \widetilde{Y}_{n+1} = \widetilde{Y}_n - \frac{1}{n+1}\Big(I_d - (2 - \mathrm{Tr}(\widetilde{Y}_n))H\Big)\widetilde{Y}_n + \frac{1}{n+1}\left(\Delta M_{n+1} + \bar{r}_{n+1}\right),$$

where $$\bar{r}_{n+1} := \left(\frac{H_{n+1} - H}{\mathrm{Tr}(\widetilde{Y}_n)} + \frac{(\mathrm{Tr}(\widetilde{Y}_n) - 1)^2}{\mathrm{Tr}(\widetilde{Y}_n)} H\right)\widetilde{Y}_n.$$

For $\widetilde{N}_n$ we have, still for every $n \geq 1$,

$$\widetilde{N}_{n+1} = \widetilde{N}_n - \frac{1}{n+1}\Big(\widetilde{N}_n - (2 - \mathrm{Tr}(\widetilde{Y}_n))\widetilde{Y}_n\Big) + \frac{1}{n+1}\Big(\Delta\widetilde{M}_{n+1} + \widetilde{r}_{n+1}\Big)$$

with $\Delta\widetilde{M}_{n+1} := X_{n+1} - \mathbb{E}\left[X_{n+1} \,|\, \mathcal{F}_n\right] = X_{n+1} - \frac{Y_n}{\mathrm{Tr}(Y_n)}$ and $\widetilde{r}_{n+1} := \frac{(\mathrm{Tr}(\widetilde{Y}_n) - 1)^2}{\mathrm{Tr}(\widetilde{Y}_n)}\widetilde{Y}_n$.

Thus, we obtain a new recursive SA procedure, still with step $\gamma_n = \frac{1}{n}$, namely

$$\theta_{n+1} = \theta_n - \frac{1}{n+1}h(\theta_n) + \frac{1}{n+1}\left(\Delta\mathbf{M}_{n+1} + R_{n+1}\right), \quad n \geq 1,$$

with $\Delta\mathbf{M}_{n+1} := \begin{pmatrix}\Delta M_{n+1} \\ \Delta\widetilde{M}_{n+1}\end{pmatrix}$, $R_{n+1} := \begin{pmatrix}\bar{r}_{n+1} \\ \widetilde{r}_{n+1}\end{pmatrix}$ and

$$\forall\, \theta = \begin{pmatrix} y \\ \nu \end{pmatrix},\ y \in \mathbb{R}^d,\ \nu \in \mathbb{R}^d, \quad h(\theta) := \begin{pmatrix} (I_d - (2 - \mathrm{Tr}(y))H)y \\ \nu - (2 - \mathrm{Tr}(y))y \end{pmatrix} \quad \text{with} \quad h(\theta^*) = 0.$$

The function h is differentiable on $\mathbb{R}^{2d}$ and its differential at point θ^* is given by

$$Dh(\theta^*) = \begin{pmatrix} I_d - H + v^*\mathbf{1}^t & 0_{\mathcal{M}_d(\mathbb{R})} \\ v^*\mathbf{1}^t - I_d & I_d \end{pmatrix}.$$

To establish a CLT for the sequence $(\theta_n)_{n\geq 1}$ we need to make the following additional assumptions:

(A4) The addition rules D_n $a.s.$ satisfy

$$\forall 1 \leq j \leq d, \quad \begin{cases} \sup_{n\geq 1} \mathbb{E}\left[\|D_n^{\cdot j}\|^{2+\delta} \,|\, \mathcal{F}_{n-1}\right] \leq C < \infty \quad \text{for a } \delta > 0, \\ \mathrm{Cov}\left[D_n^{\cdot j}(D_n^{\cdot j})^t \,|\, \mathcal{F}_{n-1}\right] \underset{n\to\infty}{\longrightarrow} C^j, \end{cases}$$

where $C^j = (C^j_{il})_{1\le i,l\le d}$, $j = 1,\dots,d$, are $d\times d$ positive definite matrices.

(A5) The matrix H satisfies

$$n\,\mathbb{E}\left[|||H_n - H|||^2\right] \underset{n\to\infty}{\longrightarrow} 0. \tag{2.12}$$

Theorem 2.2. *Assume* **(A1)**, **(A3)**, **(A4)** *and* **(A5)** *and*

$$\mathcal{R}e\left(\mathrm{Sp(H)}\setminus\{1\}\right) < 1/2 \tag{2.13}$$

Then, $\theta_n \to \theta^*$ *a.s. as* $n\to\infty$ *and*

$$\sqrt{n}\left(\theta_n - \theta^*\right) \underset{n\to\infty}{\overset{\mathcal{L}}{\longrightarrow}} \mathcal{N}\left(0,\Sigma\right) \qquad \textit{with} \qquad \Sigma = \int_0^{+\infty} e^{u(Dh(\theta^*)-\frac{I}{2})}\Gamma e^{u(Dh(\theta^*)-\frac{I}{2})^t}du$$

$$\textit{and} \quad \Gamma = \begin{pmatrix} \displaystyle\sum_{k=1}^d v^{*k}C^k & H\left(\mathrm{diag}(v^*) - v^*(v^*)^t\right) \\ \left(\mathrm{diag}(v^*) - v^*(v^*)^t\right)^t H^t & \mathrm{diag}(v^*) - v^*(v^*)^t \end{pmatrix} = a.s.\text{-}\lim_{n\to\infty}\mathbb{E}\left[\Delta\mathbf{M}_n\Delta\mathbf{M}_n^t \,|\, \mathcal{F}_{n-1}\right]. \tag{2.14}$$

Proof. We will check the three assumptions of the CLT for SA algorithms recalled in the Appendix (Theorem A.2). Firstly, the condition (A.24) on the spectrum of $Dh(\theta^*)$ requested for algorithms with step $\frac{1}{n}$ in Theorem A.2 reads $\mathcal{R}e\left(\mathrm{Sp}(Dh(\theta^*))\right) > \frac{1}{2}$. This follows from our Assumption (2.13) since by decomposing $\mathbb{R}^d = \mathbb{R}v^* \oplus \mathrm{Ker(Tr)}$, one checks that

$$\mathrm{Sp}(Dh(\theta^*)) = \{1\}\cup\{1-\lambda,\ \lambda\in \mathrm{Sp}(H)\setminus\{1\}\}.$$

Secondly Assumption **(A4)** ensures that Condition (A.22) is satisfied since

$$\sup_{n\ge1}\mathbb{E}\left[\|\Delta\mathbf{M}_n\|^{2+\delta}\,|\,\mathcal{F}_{n-1}\right] < +\infty \quad a.s. \quad \text{and} \quad \mathbb{E}\left[\Delta\mathbf{M}_n\Delta\mathbf{M}_n^t\,|\,\mathcal{F}_{n-1}\right] \underset{n\to\infty}{\overset{a.s.}{\longrightarrow}} \Gamma \quad \text{as} \quad n\to\infty,$$

where Γ is the symmetric nonnegative matrix given by (2.14) as established below. To this end we have to determine three blocks since Γ reads

$$\Gamma = \begin{pmatrix} \Gamma_1 & \Gamma_{12} \\ \Gamma_{12}^t & \Gamma_2 \end{pmatrix} \quad \text{where} \quad \Gamma_1,\Gamma_2,\Gamma_{12}\in\mathcal{M}_d(\mathbb{R}).$$

Computation of Γ_1.

$$\begin{aligned}\mathbb{E}\left[\Delta M_{n+1}\Delta M_{n+1}^t\,|\,\mathcal{F}_n\right] &= \sum_{q=1}^d \mathbb{P}(X_{n+1}=e^q\,|\,\mathcal{F}_n)\left(\mathbb{E}\left[D_{n+1}^{\cdot q}(D_{n+1}^{\cdot q})^t\,|\,\mathcal{F}_n\right] - \mathbb{E}\left[D_{n+1}X_{n+1}\,|\,\mathcal{F}_n\right]\mathbb{E}\left[D_{n+1}X_{n+1}\,|\,\mathcal{F}_n\right]^t\right) \\ &= \sum_{q=1}^d \frac{Y_n^q}{\mathrm{Tr}(Y_n)}\mathrm{Cov}\left(D_{n+1}^{\cdot q}(D_{n+1}^{\cdot q})^t\,|\,\mathcal{F}_n\right) \underset{n\to\infty}{\overset{a.s.}{\longrightarrow}} \Gamma_1 = \sum_{q=1}^d v^{*q}C^q.\end{aligned}$$

Computation of Γ_2.

$$\begin{aligned}\mathbb{E}\left[\Delta\widetilde{M}_{n+1}\Delta\widetilde{M}_{n+1}^t\,|\,\mathcal{F}_n\right] &= \mathbb{E}\left[X_{n+1}X_{n+1}^t\,|\,\mathcal{F}_n\right] - \frac{Y_n}{\mathrm{Tr}(Y_n)}\left(\frac{Y_n^q}{\mathrm{Tr}(Y_n)}\right)^t \\ &= \mathrm{diag}\left(\frac{Y_n}{\mathrm{Tr}(Y_n)}\right) - \frac{Y_n}{\mathrm{Tr}(Y_n)}\left(\frac{Y_n^q}{\mathrm{Tr}(Y_n)}\right)^t \underset{n\to\infty}{\overset{a.s.}{\longrightarrow}} \Gamma_2 = \mathrm{diag}(v^*) - v^*(v^*)^t.\end{aligned}$$

Computation of Γ_{12}.

$$\begin{aligned}
\mathbb{E}\left[\Delta M_{n+1}\Delta\widetilde{M}_{n+1}^t \,|\, \mathcal{F}_n\right] &= \mathbb{E}\left[D_{n+1}X_{n+1}X_{n+1}^t \,|\, \mathcal{F}_n\right] - \mathbb{E}\left[D_{n+1}X_{n+1} \,|\, \mathcal{F}_n\right]\mathbb{E}\left[X_{n+1} \,|\, \mathcal{F}_n\right]^t \\
&= \mathbb{E}\left[D_{n+1} \,|\, \mathcal{F}_n\right]\mathbb{E}\left[X_{n+1}X_{n+1}^t \,|\, \mathcal{F}_n\right] - \mathbb{E}\left[D_{n+1} \,|\, \mathcal{F}_n\right]\mathbb{E}\left[X_{n+1} \,|\, \mathcal{F}_n\right]\mathbb{E}\left[X_{n+1} \,|\, \mathcal{F}_n\right]^t \\
&= H_{n+1}\mathrm{diag}\left(\frac{Y_n}{\mathrm{Tr}(Y_n)}\right) - H_{n+1}\frac{Y_n}{\mathrm{Tr}(Y_n)}\left(\frac{Y_n}{\mathrm{Tr}(Y_n)}\right)^t \\
&\xrightarrow[n\to\infty]{a.s.} \Gamma_{12} = H\left(\mathrm{diag}(v^*) - v^*(v^*)^t\right).
\end{aligned}$$

Finally, it remains to check that the remainder sequence $(R_n)_{n\geq 1}$ satisfies (A.23) for an $\epsilon > 0$:

$$\mathbb{E}\left[(n+1)\left\|R_{n+1}\right\|^2 \mathbb{1}_{\{\|\theta_n-\theta^*\|\leq\epsilon\}}\right] \underset{n\to\infty}{\longrightarrow} 0. \tag{2.15}$$

We note that $\|R_{n+1}\|^2 = \|\bar{r}_{n+1}\|^2 + \|\widetilde{r}_{n+1}\|^2$. It follows from the definition of $\bar{r}_{n+1}$ and the elementary facts $\|\widetilde{Y}_n - v^*\| \leq \|\theta_n - \theta^*\|$ and $\mathrm{Tr}(\widetilde{Y}_n) \geq \|\widetilde{Y}_n\|$ that

$$\begin{aligned}
\|\bar{r}_{n+1}\|^2 \mathbb{1}_{\left\{\|\theta_n-\theta^*\|\leq\frac{\|v^*\|}{2}\right\}} &\leq 2\left(\frac{(\mathrm{Tr}(\widetilde{Y}_n)-1)^4}{\frac{\|v^*\|}{2}} + \frac{|||H_{n+1}-H|||^2}{\frac{\|v^*\|}{2}}\right)\frac{3}{2}\|v^*\|\mathbb{1}_{\left\{\|\theta_n-\theta^*\|\leq\frac{\|v^*\|}{2}\right\}} \\
&\leq 6\left((\mathrm{Tr}(\widetilde{Y}_n)-1)^4 + |||H_{n+1}-H|||^2\right)\mathbb{1}_{\left\{\|\theta_n-\theta^*\|\leq\frac{\|v^*\|}{2}\right\}}.
\end{aligned}$$

But $\mathrm{Tr}(\widetilde{Y}_n) - 1 = \frac{\mathrm{Tr}(\Delta M_n)}{n}$ where $\sup_{n\geq 0}\mathbb{E}\left[|\mathrm{Tr}(\Delta M_{n+1})|^{2+\delta} \,|\, \mathcal{F}_n\right] \leq C'$, $\delta > 0$, owing to (A4). Now using that $|\mathrm{Tr}(y)| \leq C_d\|y\|$,

$$\mathbb{E}\left[n\left|\mathrm{Tr}(\widetilde{Y}_n)-1\right|^4 \mathbb{1}_{\left\{\|\theta_n-\theta^*\|\leq\frac{\|v^*\|}{2}\right\}}\right] \leq C_\delta^* n\mathbb{E}\left[\left|\mathrm{Tr}(\widetilde{Y}_n)-1\right|^{2+\delta}\right] = \frac{C_d}{n^{1+\delta}}\mathbb{E}\left[|\mathrm{Tr}(\Delta M_n)|^{2+\delta}\right] \leq \frac{C'_d}{n^{1+\delta}},$$

where $C_\delta^* > 0$ is a real constant. Consequently

$$\mathbb{E}\left[\left|\mathrm{Tr}(\widetilde{Y}_n)-1\right|^4 \mathbb{1}_{\left\{\|\theta_n-\theta^*\|\leq\frac{\|v^*\|}{2}\right\}}\right] = o\left(\frac{1}{n}\right).$$

Thus, by (A5) we obtain

$$\mathbb{E}\left[\|\bar{r}_{n+1}\|^2 \mathbb{1}_{\left\{\|\theta_n-\theta^*\|\leq\frac{\|v^*\|}{2}\right\}}\right] = o\left(\frac{1}{n}\right).$$

The same argument yields $\mathbb{E}\left[\|\widetilde{r}_{n+1}\|^2 \mathbb{1}_{\left\{\|\theta_n-\theta^*\|\leq\frac{\|v^*\|}{2}\right\}}\right] = o\left(\frac{1}{n}\right)$, therefore (2.15) is satisfied. □

3 Application to urn models for multi-arm clinical trials

3.1 The Wei *GFU* Model

We consider here the model presented in [22] and in [6], where balls are added depending on the success probabilities of each treatment. Define an *efficiency indicator* as follows: let $(T_n^i)_{n\geq 1}$, $1 \leq i \leq d$, be d independent sequences of $[0,1]$-valued i.i.d. random variables, independent of the i.i.d.*sampling* sequence $(U_n)_{n\geq 1}$ so that

$$\mathbb{E}\left[T_n^i\right] = p^i, \quad 0 < p^i < 1, \quad 1 \leq i \leq d. \tag{3.16}$$

Remark. If $(T_n^i)_{n\geq 1}$, $1 \leq i \leq d$, is simply a *success indicator*, namely d independent sequences of i.i.d. $\{0,1\}$-valued Bernoulli trials with respective parameter p^i, then the convention is to set $T_n^i = 1$ to indicate that the response of the ith treatment in the nth trial is a success and $T_n^i = 0$ otherwise.

In this framework one considers the filtration $\mathcal{F}_n = \sigma\left(Y_0, U_k, T_k, 1 \leq k \leq n\right)$, $n \geq 0$. Consider the following addition rules: a success on the treatment i adds a ball of type i to the urn and a failure on the treatment i adds $\frac{1}{d-1}$ balls for each of the other $d-1$ types. Thus the addition rule proposed in [22] is as follows

$$D_{n+1} = \begin{pmatrix} T_{n+1}^1 & \frac{1-T_{n+1}^2}{d-1} & \cdots & \frac{1-T_{n+1}^d}{d-1} \\ \frac{1-T_{n+1}^1}{d-1} & T_{n+1}^2 & \cdots & \frac{1-T_{n+1}^d}{d-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1-T_{n+1}^1}{d-1} & \frac{1-T_{n+1}^2}{d-1} & \cdots & T_{n+1}^d \end{pmatrix}$$

so that

$$H_{n+1} = \mathbb{E}\left[D_{n+1} \,|\, \mathcal{F}_n\right] = \mathbb{E}\, D_{n+1} = H = \begin{pmatrix} p^1 & \frac{q^2}{d-1} & \cdots & \frac{q^d}{d-1} \\ \frac{q^1}{d-1} & p^2 & \cdots & \frac{q^d}{d-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{q^1}{d-1} & \frac{q^2}{d-1} & \cdots & p^d \end{pmatrix},$$

where $q^i = 1 - p^i$, $1 \leq i \leq d$. The strong consistency has been first established in [3], then redone in [5] and in this paper with Theorem 2.1, and the asymptotic normality

$$\frac{Y_n - nv^*}{\sqrt{n}} \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Sigma)$$

results from Theorem 3.2 in [5] and from Theorem 2.2 of this paper where we obtain a joint CLT for $(\widetilde{Y}_n, \widetilde{N}_n)$. Furthermore we know that

$$v^{*i} = \frac{1/q^i}{\sum_{j=1}^d 1/q^j}, \quad 1 \leq i \leq d.$$

Note that if $p^i > p^j$, then $v^{*i} > v^{*j}$. Hence the components v^{*i} are ordered according to the increasing efficiency p^i of the treatments. Furthermore, it is clear that, if $p^i \uparrow 1$ and all other probabilities p^j stand still, then

$$\lim_{p^i \to 1} v^{*j} = \delta_{ij}$$

where δ_{ij} denotes the Kronecker symbol. Consequently, since v^{*i} is the asymptotic probability of assigning treatment i to a patient, the procedure asymptotically allocates more patients to the most efficient treatment(s). Following the practitioners, the fact that a marginal allocation of less efficient treatments is preserved is justified by some comparison matter.

However this model only takes into account in the addition rule matrix D_n the response of the nth patient without considering the ones of past patients. This led the author of [6] to introduce a new model based on statistical observations of the efficiency of the assigned treatments to past patients.

3.2 The Bai-Hu-Shen *GFU* Model

We consider now the model introduced in [6] (and considered again in [5]) where $(T_n^i)_{n\geq 1}, 1 \leq i \leq d$, are d independent sequences of i.i.d. $\{0,1\}$-valued Bernoulli trials satisfying (3.16) and the filtration $(\mathcal{F}_n)_{n\geq 0}$ is defined as in the previous section. Let $N_n = (N_n^1, \ldots, N_n^d)^t$ and $S_n = (S_n^1, \ldots, S_n^d)^t$, where $N_n^i = N_{n-1}^i + X_n^i$, $n \geq 1$, still denotes the number of times the ith treatment is selected among the first n stages and

$$S_n^i = S_{n-1}^i + T_n^i X_n^i, \quad n \geq 1,$$

denotes the *number of successes* of the ith treatment among these N_n^i trials, $i = 1, \ldots, d$. However, to avoid degeneracy of the procedure, we will make the following initialization assumption

$$N_0^i = 1, \; S_0^i = 1, \quad i = 1, \ldots, d$$

(which makes the above interpretation of these quantities correct "up to one unit").

Remark. Like with the Wei model, we can simply assume that T_n^i is a $[0,1]$-valued efficiency indicator.

Define $Q_n = (Q_n^1, \ldots, Q_n^d)^t$, where $Q_n^i = \frac{S_n^i}{N_n^i}$, $i = 1, \ldots, d$ and $E_n = \sum_{i=1}^d Q_n^i$. In [6] the authors consider the following addition rule matrices,

$$D_{n+1} = \begin{pmatrix} T_{n+1}^1 & \frac{Q_n^1(1-T_{n+1}^2)}{E_n - Q_n^2} & \cdots & \frac{Q_n^1(1-T_{n+1}^d)}{E_n - Q_n^d} \\ \frac{Q_n^2(1-T_{n+1}^1)}{E_n - Q_n^1} & T_{n+1}^2 & \cdots & \frac{Q_n^2(1-T_{n+1}^d)}{E_n - Q_n^d} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{Q_n^d(1-T_{n+1}^1)}{E_n - Q_n^1} & \frac{Q_n^d(1-T_{n+1}^2)}{E_n - Q_n^2} & \cdots & T_{n+1}^d \end{pmatrix},$$

i.e. at stage $n+1$, if the response of the ith treatment is a success, then one ball of type i is added in the urn. Otherwise, $\frac{Q_n^j}{E_n - Q_n^i}$ (virtual) balls of type j, $j \neq i$, are added. Then, one easily checks that the generating matrices are given by

$$H_{n+1} = \mathbb{E}\left[D_{n+1} \,|\, \mathcal{F}_n\right] = \begin{pmatrix} p^1 & \frac{Q_n^1}{E_n - Q_n^2} q^2 & \cdots & \frac{Q_n^1}{E_n - Q_n^d} q^d \\ \frac{Q_n^2}{E_n - Q_n^1} q^1 & p^2 & \cdots & \frac{Q_n^2}{E_n - Q_n^d} q^d \\ \vdots & \vdots & \ddots & \vdots \\ \frac{Q_n^d}{E_n - Q_n^1} q^1 & \frac{Q_n^d}{E_n - Q_n^2} q^2 & \cdots & p^d \end{pmatrix}.$$

As soon as $Y_0 \in \mathbb{R}_+^d \setminus \{0\}$, $H_n \xrightarrow{a.s.} H$ (see Lemma 3.1 below or [6] when $Y_0 \in (0,\infty)^d$) where

$$H = \begin{pmatrix} p^1 & \frac{p^1}{E-p^2}q^2 & \cdots & \frac{p^1}{E-p^d}q^d \\ \frac{p^2}{E-p^1}q^1 & p^2 & \cdots & \frac{p^2}{E-p^d}q^d \\ \vdots & \vdots & \ddots & \vdots \\ \frac{p^d}{E-p^1}q^1 & \frac{p^d}{E-p^2}q^2 & \cdots & p^d \end{pmatrix} \quad \text{where } E = p^1 + \cdots + p^d.$$

The matrix H is clearly irreducible since $0 < p^i < 1$, $1 \le i \le d$. Then Theorem 2.1 (or following the direct proof from [6]) we obtain

$$\widetilde{Y}_n = \frac{Y_n}{n} \underset{n\to\infty}{\overset{a.s.}{\longrightarrow}} v^* \quad \text{and} \quad \widetilde{N}_n = \frac{N_n}{n} \underset{n\to\infty}{\overset{a.s.}{\longrightarrow}} v^*. \tag{3.17}$$

Note that the normalizes maximal eigenvector v^* (associated to the eigenvalue 1) is given by

$$v^{*i} = \frac{p^i \frac{E-p^i}{1-p^i}}{\sum_{1\le j\le d} p^j \frac{E-p^j}{1-p^j}}, \quad i = 1, \ldots, d.$$

Note that if $p^i > p^j$, $\frac{p^i}{p^j}\frac{E-p^i}{E-p^j} > 1$ and $\frac{1-p^j}{1-p^i} > 1$ so that $v^{*i} > v^{*j}$. Hence the entries v^{*i} are ordered according to the increasing efficiency p^i of the treatments. This model can be considered as more ethical than the Wei model since a better treatment will be administrated to more patients. Indeed, when $d > 2$, for any $i \neq j$, $1 \le i, j \le d$, if $p^i > p^j$,

$$\frac{v^{*i}_{BHS}}{v^{*j}_{BHS}} > \frac{v^{*i}_{W}}{v^{*j}_{W}} > 1$$

(when $d = 2$ both matrices H coincide).

Remark. Note that in that model the "balls" in the urn become virtual since there exists no $N \in \mathbb{N}$ such that, for every $n \ge 1$, $ND_n \in \mathcal{M}_d(\mathbb{N})$.

3.3 Asymptotic normality for multi-arm clinical trials for the BHS GFU model

In order to derive a CLT, not with the bias $\mathbb{E}Y_n$ but with nv^*, from their own general asymptotic normality result (which statement is similar to Theorem 2.2) they need to fulfill the following convergence rate assumption for H_n

$$\sum_{n\ge 1} \frac{|||H_n - H|||_\infty}{\sqrt{n}} < \infty \quad a.s. \tag{3.18}$$

In [6], the $a.s.$ rate of decay $|||H_n - H|||_\infty = o(n^{-\frac{1}{4}})$ is shown which is clearly not fast enough to fulfill (3.18).

However, by enlarging the dimension of the structure process of the procedure by considering the $3d$-dimensional random sequence

$$\widetilde{\theta}_n = \begin{pmatrix} \widetilde{Y}_n \\ \widetilde{N}_n \\ \widetilde{S}_n \end{pmatrix} \quad \text{where} \quad \widetilde{S}_n = \frac{S_n}{n}, \quad n \geq 1,$$

we will establish that a CLT does hold for the BHS GFU model.

The first step is to notice that the generating matrix H_{n+1} can may be written as a function depending on $\widetilde{S}_n$ and $\widetilde{N}_n$, *i.e.* $H_{n+1} = \Phi(\widetilde{S}_n, \widetilde{N}_n)$, where $\Phi : \mathbb{R}_+^d \times (0,\infty)^d \to \mathcal{M}_d(\mathbb{R})$ is a differentiable function defined by

$$\Phi(s,\nu) = \left(\Phi^{ij}(s,\nu)\right)_{1\leq i,j\leq d} \quad \text{where} \quad \begin{cases} \Phi^{ii}(s,\nu) = p^i & 1 \leq i \leq d \\ \Phi^{ij}(s,\nu) = \frac{s^i/\nu^i}{\sum_{k\neq j} s^k/\nu^k}\, q^j & 1 \leq i,j \leq d,\ i \neq j. \end{cases}$$

Then the following strong consistency and CLT hold for $(\widetilde{\theta}_n)_{n\geq 1}$.

Theorem 3.1. *Assume that* **(A1)**, **(A4)**, **(A5)** *and (2.13) hold. Then, as soon as* $Y_0 \in \mathbb{R}_+^d \setminus \{0\}$,

$$\widetilde{\theta}_n \xrightarrow[n\to\infty]{a.s.} \widetilde{\theta}^* \quad \text{and} \quad \sqrt{n}\left(\widetilde{\theta}_n - \widetilde{\theta}^*\right) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \widetilde{\Sigma}\right),$$

where

$$\widetilde{\theta}^* := \left(v^*, v^*, \mathrm{diag}(p)v^*\right)^t, \quad \widetilde{\Sigma} = \int_0^{+\infty} e^{u(D\widetilde{h}(\widetilde{\theta}^*)-\frac{I}{2})}\widetilde{\Gamma} e^{u(D\widetilde{h}(\widetilde{\theta}^*)-\frac{I}{2})^t} du$$

with $\widetilde{\Gamma} = a.s.\ \lim_{n\to+\infty} \mathbb{E}\left[\Delta\widetilde{\mathbf{M}}_{n+1}\Delta\widetilde{\mathbf{M}}_{n+1}^t \,|\, \mathcal{F}_n\right]$ *(with* $(\Delta\widetilde{\mathbf{M}}_n)_{n\geq 1}$ *a sequence of martingale increments defined in the proof below) reads*

$$\widetilde{\Gamma} = \begin{pmatrix} \sum_{k=1}^d v^{*k} C^k & H\left(\mathrm{diag}(v^*) - v^*(v^*)^t\right) & \left(\mathrm{diag}(v^*) - v^*(v^*)^t\right)\mathrm{diag}(p) \\ \left(\mathrm{diag}(v^*) - v^*(v^*)^t\right)^t H^t & \mathrm{diag}(v^*) - v^*(v^*)^t & \left(\mathrm{diag}(v^*) - v^*(v^*)^t\right)\mathrm{diag}(p) \\ \mathrm{diag}(p)\left(\mathrm{diag}(v^*) - v^*(v^*)^t\right)^t & \mathrm{diag}(p)\left(\mathrm{diag}(v^*) - v^*(v^*)^t\right)^t & \mathrm{diag}(p)\left(v^* - v^* v^{*t}\mathrm{diag}(p)\right) \end{pmatrix}$$

and

$$D\widetilde{h}(\widetilde{\theta}^*) = \begin{pmatrix} I_d - H + v^*\mathbf{1}^t & -\frac{\partial}{\partial\nu}\left(\Phi(s,\nu)y\right)_{|\widetilde{\theta}=\widetilde{\theta}^*} & -\frac{\partial}{\partial s}\left(\Phi(s,\nu)y\right)_{|\widetilde{\theta}=\widetilde{\theta}^*} \\ v^*\mathbf{1}^t - I_d & I_d & 0_{\mathcal{M}_d(\mathbb{R})} \\ diag(p)\left(v^*\mathbf{1}^t - I_d\right) & 0_{\mathcal{M}_d(\mathbb{R})} & I_d \end{pmatrix}$$

which is invertible.

Proof. Step 1(*Strong consistency*). We will show with Lemma 3.1 that $\widetilde{S}_n \xrightarrow[n\to\infty]{a.s.} \mathrm{diag}(p)v^*$ and that $H_n \xrightarrow[n\to\infty]{a.s.} H$ so that, using what precedes $\widetilde{\theta}_n \xrightarrow[n\to\infty]{a.s.} \widetilde{\theta}^*$.

Lemma 3.1. *If the assumptions (1.1), (2.3), (2.4) hold and $Y_0 \in \mathbb{R}_+^d$ such that $Tr(Y_0) > 0$, then,*

$$Q_n \xrightarrow{a.s.} p = (p^1, \ldots, p^d) \qquad \text{as } n \to \infty$$

so that Assumption (2.5) holds i.e. $H_n \xrightarrow[n\to\infty]{a.s.} H$.

Remark. If we assume that $Y_0^i > 0$, $1 \le i \le d$, then we can prove that $\lim_n N_n^i = +\infty$ *a.s.*, $1 \le i \le d$, faster than below by using that $Y_n^i \ge Y_0^i$, $1 \le i \le d$, $n \ge 1$. The following proof consider the more general case where $Y_0 \in \mathbb{R}_+^d \setminus \{0\}$.

Proof of the lemma. *Step 1.* It follows from the dynamics (1.1) and the definition of D_{n+1} and H_{n+1} that, for every $n \ge 0$, $\mathrm{Tr}(Y_n) = \mathrm{Tr}(Y_0) + n$ and that, for every $i \in \{1, \ldots, d\}$,

$$Y_{n+1}^i = Y_n^i + \sum_{j=1}^d H_{n+1}^{ij} \frac{Y_n^i}{\mathrm{Tr} Y_n} + \Delta M_{n+1}^i$$

where $(\Delta M_n^i)_{n\ge1}$ is a sequence of martingale increments satisfying $\sup_n \mathbb{E}\left[|\Delta M_n^i|^2 \,|\, \mathcal{F}_{n-1}\right] < +\infty$ owing to (2.4). Now using that $S_0^i = N_0^i = 1$ by convention, one derives that

$$\forall\, i \ne j,\ H_{n+1}^{ij} \ge \frac{\kappa_0}{n}, \quad \text{with } \kappa_0 = \frac{1}{2d} \min_{1\le i\le d} \left(p^i, q^i\right) > 0$$

so that, using that $H_{n+1}^{ii} = p^i$, there exists a deterministic integer n_0 such that for every $n \ge n_0$,

$$\begin{aligned} Y_{n+1}^i &\ge \Big(1 + \frac{p_i}{n} - \frac{\kappa_0}{n \mathrm{Tr} Y_n}\Big) Y_n^i + \frac{\kappa_0}{n} + \Delta M_{n+1}^i \\ &\ge \Big(1 + \frac{p_i}{2\mathrm{Tr}(Y_n)}\Big) Y_n^i + \frac{\kappa_0}{n} + \Delta M_{n+1}^i. \end{aligned}$$

Standard computations show that, setting $a_n^i = \prod_{k=n_0}^{n-1} (1 + \frac{p_i}{2\mathrm{Tr}(Y_n)})$, $i = 1, \ldots, d$,

$$\forall\, n \ge n_0, \quad \frac{Y_n^i}{a_n^i} \ge \frac{Y_{n_0}^i}{a_{n_0}^i} + \sum_{k=n_0+1}^n \frac{\kappa_0}{a_k^i} + \sum_{k=n_0+1}^n \frac{\Delta M_k^i}{a_k^i}$$

Since there exists κ_1, $\kappa_2 > 0$ such that $\kappa_1 n^{\frac{p^i}{2}} \le a_n^i \le \kappa_2 n^{\frac{p^i}{2}}$, one has

$$\forall\, \eta > 0, \quad \sum_{k=n_0+1}^n \frac{\Delta M_k^i}{a_k^i} = o\big(n^{\frac{1-p^i+\eta}{2}}\big).$$

Finally, there exists a positive real constant c' such that, for every $i = 1, \ldots, d$,

$$Y_n^i \ge c' n^{\frac{p^i}{2}} \sum_{k=n_0+1}^n k^{-\frac{p^i}{2}} + o\big(n^{\frac{1+\eta}{2}}\big)$$

so that

$$\forall\, i \in \{1, \ldots, d\}, \quad \liminf_n \widetilde{Y}_n^i \ge c' \int_0^1 u^{-\frac{p^i}{2}} du > 0$$

and, as a consequence, $\sum_{n\geq 1}\widetilde{Y}_n^i = +\infty$-$a.s.$ Now using that for every $i = 1,\dots,d$,

$$N_n^i = \sum_{k=1}^n \mathbf{1}_{\{X_k = e^i\}} \quad \text{and} \quad \mathbb{P}(X_n = e^i \,|\, \mathcal{F}_{n-1}) = \widetilde{Y}_{n-1}^i\left(1 - \frac{\mathrm{Tr}(Y_0)}{\mathrm{Tr}(Y_{n-1})}\right), \quad n \geq 1,$$

we get by the conditional Borel-Cantelli Lemma that $N_\infty^i = \lim_n N_n^i = +\infty$ $a.s.$

Step 2. First we note that

$$Q_n^i = \frac{\sum_{k=1}^n T_k^i \Delta N_k^i}{N_n^i}$$

and we introduce the sequence $(\widetilde{Q}_n)_{n\geq 1}$ defined by

$$\widetilde{Q}_n^i = \sum_{k=1}^n (T_k^i - p^i)\frac{\Delta N_k^i}{N_{k-1}^i + 1}, \quad n \geq 1.$$

It is an $\mathcal{F}_n$-martingale since, T_k^i being independent of $\mathcal{F}_{k-1}$ and X_k,

$$\mathbb{E}\Big((T_k^i - p^i)\Delta N_k^i \,|\, \mathcal{F}_{k-1}\Big) = \mathbb{E}(T_k^i - p^i)\mathbb{P}(X_k = e^i \,|\, \mathcal{F}_{k-1}) = 0.$$

It has bounded increments since $|T_k^i - p^i| \leq 1$ and

$$\langle \widetilde{Q}^i\rangle_n \leq \sum_{k=1}^n \frac{\mathbb{E}((\Delta N_k^i)^2 \,|\, \mathcal{F}_{k-1})}{(N_{k-1}^i + 1)^2}.$$

It follows, using $(\Delta N_k^i)^2 = \Delta N_k^i$, that, for every $n \geq 1$,

$$\mathbb{E}\langle \widetilde{Q}^i\rangle_n \quad \leq \quad \mathbb{E}\Big(\sum_{k=1}^n \frac{\Delta N_k^i}{(N_{k-1}^i + 1)^2}\Big) \leq \mathbb{E}\Big(\sum_{k=1}^n \frac{\Delta N_k^i}{N_{k-1}^i N_k^i}\Big) \leq \frac{1}{N_0^i} = 1.$$

Consequently $\widetilde{Q}_n^i \to \widetilde{Q}_\infty^i \in L^1(\mathbb{P})$ $a.s.$ as $n \to \infty$. This in turn implies by Kronecker's Lemma that

$$Q_n^i \xrightarrow{a.s.} p^i \quad \text{as} \quad n \to \infty$$

since $N_n^i \to \infty$ by the first step. $\square$

It follows from the lemma and Theorem 2.1 that $(\widetilde{Y}_n, \widetilde{N}_n) \to (v^*, v^*)$. Furthermore $\mathrm{diag}(\widetilde{S}_n) = \mathrm{diag}(Q_n)\widetilde{N}_n \to \mathrm{diag}(p)v^* = u^*$ so that $\widetilde{\theta}_n \to \widetilde{\theta}^*$ as $n \to \infty$.

Step 2 (*Asymptotic normality*). We will show now that $(\widetilde{\theta}_n)_{n\geq 1}$ satisfies an appropriate recursion to apply Theorem A.2 (CLT). First, we write a recursive procedure for $\widetilde{S}_n$. Having in mind that $S_n = 1 + \sum_{1\leq k\leq n}\mathrm{diag}(T_k)X_k$, we get

$$\begin{aligned}
\widetilde{S}_{n+1} &= \widetilde{S}_n - \frac{1}{n+1}\Big(\widetilde{S}_n - \mathrm{diag}(T_{n+1})X_{n+1}\Big) \\
&= \widetilde{S}_n - \frac{1}{n+1}\left(\widetilde{S}_n - \mathrm{diag}(p)\frac{\widetilde{Y}_n}{\mathrm{Tr}(\widetilde{Y}_n)}\right) + \frac{1}{n+1}\Delta\widehat{M}_{n+1} \\
&= \widetilde{S}_n - \frac{1}{n+1}\Big(\widetilde{S}_n - \mathrm{diag}(p)(2 - \mathrm{Tr}(\widetilde{Y}_n))\widetilde{Y}_n\Big) + \frac{1}{n+1}\Big(\Delta\widehat{M}_{n+1} + \widehat{r}_{n+1}\Big) \qquad (3.19)
\end{aligned}$$

where $\quad \Delta\widehat{M}_{n+1} := \mathrm{diag}(T_{n+1})X_{n+1} - \mathbb{E}\left[\mathrm{diag}(T_{n+1})X_{n+1} \,|\, \mathcal{F}_n\right] = \mathrm{diag}(T_{n+1})X_{n+1} - \mathrm{diag}(p)\dfrac{\widetilde{Y}_n}{\mathrm{Tr}(Y_n)}$

is an $\mathcal{F}_n$-martingale increment and $\widehat{r}_{n+1} = \mathrm{diag}(p)\frac{\left(\mathrm{Tr}(\widetilde{Y}_n)-1\right)^2}{\mathrm{Tr}(\widetilde{Y}_n)}\widetilde{Y}_n$. Then we rewrite the dynamics satisfied by $\widetilde{Y}_n$ as follows

$$\widetilde{Y}_{n+1} = \widetilde{Y}_n - \frac{1}{n+1}\left(I_d - (2-\mathrm{Tr}(\widetilde{Y}_n))H_{n+1}\right)\widetilde{Y}_n + \frac{1}{n+1}\left(\Delta M_{n+1} + \check{r}_{n+1}\right), \tag{3.20}$$

where $\check{r}_{n+1} := \dfrac{\left(\mathrm{Tr}(\widetilde{Y}_n)-1\right)^2}{\mathrm{Tr}(\widetilde{Y}_n)}H_{n+1}\widetilde{Y}_n$. Finally, we get the following recursive procedure for $\widetilde{\theta}_n$

$$\widetilde{\theta}_{n+1} = \widetilde{\theta}_n - \frac{1}{n+1}\widetilde{h}(\widetilde{\theta}_n) + \frac{1}{n+1}\left(\Delta\widetilde{\mathbf{M}}_{n+1} + \widetilde{R}_{n+1}\right), \quad n \geq 1,$$

where, for every $\widetilde{\theta} = (y,\nu,s)^t \in \mathbb{R}_+^{3d}$,

$$\widetilde{h}(\widetilde{\theta}) := \begin{pmatrix} (I_d - (2-\mathrm{Tr}(y))\Phi(s,\nu))y \\ \nu - (2-\mathrm{Tr}(y))y \\ s - (2-\mathrm{Tr}(y))\mathrm{diag}(p)y \end{pmatrix}, \ \Delta\widetilde{\mathbf{M}}_{n+1} := \begin{pmatrix} \Delta M_{n+1} \\ \Delta\widetilde{M}_{n+1} \\ \Delta\widehat{M}_{n+1} \end{pmatrix} \ \text{ and } \ \widetilde{R}_{n+1} := \begin{pmatrix} \check{r}_{n+1} \\ \widetilde{r}_{n+1} \\ \widehat{r}_{n+1} \end{pmatrix}.$$

The function Φ being differentiable at the equilibrium point $\widetilde{\theta}^*$, we have

$$D\widetilde{h}(\widetilde{\theta}^*) = \begin{pmatrix} I_d - H + v^*\mathbf{1}^t & -\frac{\partial}{\partial\nu}\left(\Phi(s,\nu)y\right)_{|\widetilde{\theta}=\widetilde{\theta}^*} & -\frac{\partial}{\partial s}\left(\Phi(s,\nu)y\right)_{|\widetilde{\theta}=\widetilde{\theta}^*} \\ v^*\mathbf{1}^t - I_d & I_d & 0_{\mathcal{M}_d(\mathbb{R})} \\ \mathrm{diag}(p)\left(v^*\mathbf{1}^t - I_d\right) & 0_{\mathcal{M}_d(\mathbb{R})} & I_d \end{pmatrix}$$

which is invertible since by Schur complement we have $\det(D\widetilde{h}(\widetilde{\theta}^*)) = \det(I_d - H + v^*\mathbf{1}^t)$ thanks to $\frac{\partial}{\partial\nu}\left(\Phi(s,\nu)y\right)_{|\widetilde{\theta}=\widetilde{\theta}^*} = -\mathrm{diag}(p)\frac{\partial}{\partial s}\left(\Phi(s,\nu)y\right)_{|\widetilde{\theta}=\widetilde{\theta}^*}$.

At this stage, the proof follows the lines of that of Theorem 2.2: the computation of the covariance matrix $\widetilde{\Gamma}$ and the treatment of the remainder term uses the same tools as before. The details are left to the reader. $\square$

Remarks. • The asymptotic variances of $\widetilde{Y}_n$ and $\widetilde{N}_n$ in Theorem 3.1 are different from those in Theorem 2.2 because the differential matrices $Dh(\theta^*)$ and $D\widetilde{h}(\widetilde{\theta}^*)$ are not the same.
• In the SA Theory, there also exists some results on the rate of convergence without CLT. For the sake of simplicity, in this paper we only treat the case where $L > \frac{1}{2}$, with $L = \min\left\{\mathcal{R}e(\lambda);\ \lambda \in \mathrm{Sp}\left(Dh(\theta^*)\right)\right\}$. When $L = \frac{1}{2}$, $\sqrt{n}$ can be replaced by $\sqrt{\frac{n}{(\ln n)^\alpha}}$, $\alpha > 0$, and when $L < \frac{1}{2}$, we have an $a.s.$ convergence toward a finite random variable at a rate n^{-L} (see [12]).

Corollary 3.1. *Under the assumptions of Theorem 3.1,*

$$\sqrt{n}\big(H_n - H\big) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0;\Gamma_H)$$

where Γ_H is a $d^2 \times d^2$ matrix given by $\Gamma_H = D\Phi(u^*,v^*)[\widetilde{\Sigma}_{i+d,j+d}]_{1\leq i,j\leq 2d}D\Phi(u^*,v^*)^t$.

Proof. This is an easy consequence of the so-called Δ-method since

$$H_n = \Phi(\widetilde{S}_n, \widetilde{N}_n) = \Phi(u^*, v^*) + D\Phi(u^*, v^*).(\widetilde{S}_n - u^*, \widetilde{N}_n - v^*) + \|(\widetilde{S}_n - u^*, \widetilde{N}_n - v^*)\|\varepsilon(\widetilde{S}_n, \widetilde{N}_n)$$

with $\lim_{y \to (u^*, v^*)} \varepsilon(y) = 0$. Consequently

$$\sqrt{n}\big(H_n - H\big) = D\Phi(u^*, v^*).(\sqrt{n}(\widetilde{S}_n - u^*), \sqrt{n}(\widetilde{N}_n - v^*)) + \varepsilon_{\mathbb{P}}(n)$$

where $\varepsilon_{\mathbb{P}}(n)$ goes to 0 in probability (as the product of a tight sequence and an $a.s.$ convergent sequence). This concludes the proof. $\square$

Remark. This corollary shows *a posteriori* that it was hopeless to try applying Theorem 2.2 in its standard form to establish asymptotic normality for multi-arm clinical trials since the assumption $\mathbb{E}|||H_n - H|||^2 = o(n^{-1})$ cannot be satisfied. Our global SA approach breaks the vicious circle.

Numerical Example: BHS model. We consider the case $d = 2$, so v^* as the same form as in the example in Subsection 2.3.

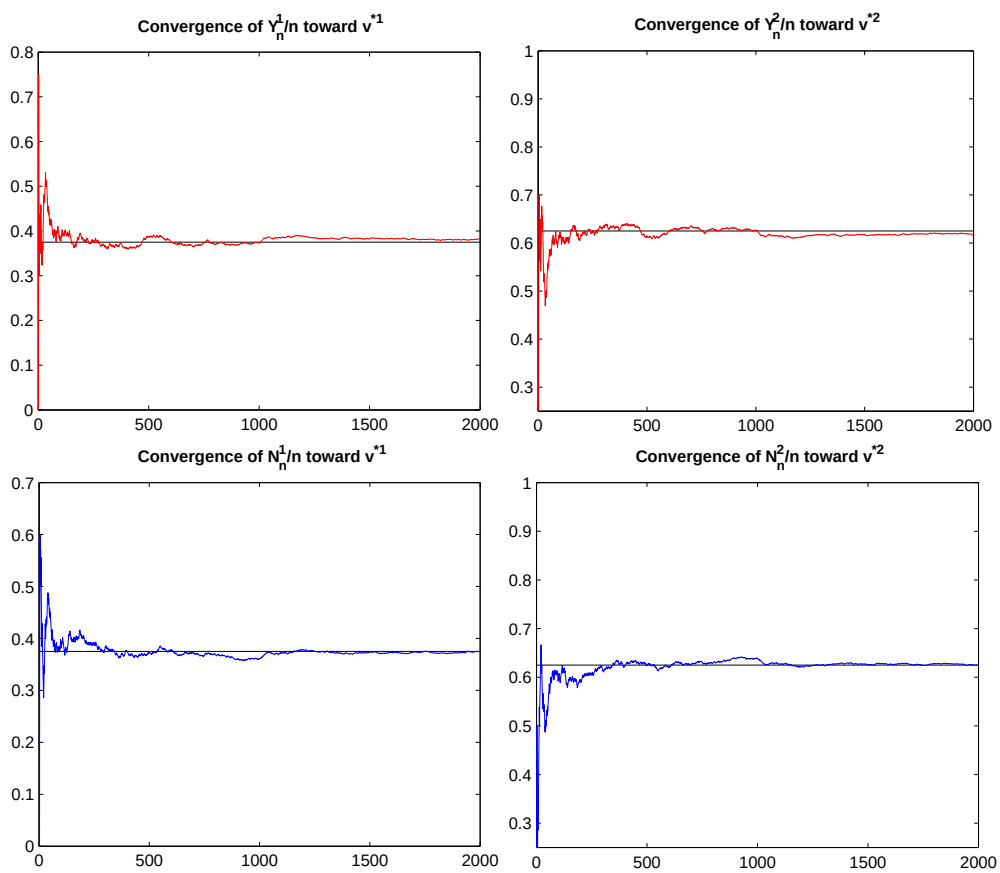

Figure 1: Convergence of $\frac{Y_n}{n}$ toward v^* (up-windows) and of $\frac{N_n}{n}$ toward v^* (down-windows): $d = 2$, $n = 2.10^3$, $p^1 = 0.5$, $p^2 = 0.7$, $Y_0 = (0.5, 0.5)^t$ and $N_0 = (1, 1)^t$.

Appendix

A Basic tools of Stochastic Approximation

Consider the following recursive procedure defined on a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_n)_{n\geq 0}, \mathbb{P})$

$$\forall\, n \geq n_0, \quad \theta_{n+1} = \theta_n - \gamma_{n+1} h(\theta_n) + \gamma_{n+1} \left(\Delta M_{n+1} + r_{n+1}\right), \tag{A.21}$$

where $h : \mathbb{R}^d \to \mathbb{R}^d$ is a locally Lipschitz continuous function, θ_{n_0} an $\mathcal{F}_{n_0}$-measurable finite random vector and, for every $n \geq n_0$, ΔM_{n+1} is an $\mathcal{F}_n$-martingale increment and r_n is an $\mathcal{F}_n$-adapted remainder term.

Theorem A.1. *(a.s. convergence with ODE method, see e.g. [8, 12, 19, 14, 7]). Assume that h is locally Lipschitz, that*

$$r_n \xrightarrow[n\to\infty]{a.s.} 0 \quad \textit{and} \quad \sup_{n\geq n_0} \mathbb{E}\left[\|\Delta M_{n+1}\|^2 \,|\, \mathcal{F}_n\right] < +\infty \quad a.s.,$$

and that $(\gamma_n)_{n\geq 1}$ is a positive sequence satisfying

$$\sum_{n\geq 1} \gamma_n = +\infty \quad \textit{and} \quad \sum_{n\geq 1} \gamma_n^2 < +\infty.$$

Then the set Θ^∞ of its limiting values as $n \to +\infty$ is a.s. a compact connected set, stable by the flow of

$$ODE_h \equiv \dot{\theta} = -h(\theta).$$

Furthermore if $\theta^* \in \Theta^\infty$ is a uniformly stable equilibrium on Θ^∞ of ODE_h, then

$$\theta_n \xrightarrow[n\to\infty]{a.s.} \theta^*.$$

Comments. By uniformly stable we mean that

$$\sup_{\theta\in\Theta^\infty} |\theta(\theta_0, t) - \theta^*| \longrightarrow 0 \quad \text{as} \quad t \to +\infty,$$

where $\theta(\theta_0, t)_{\theta_0\in\Theta^\infty, t\in\mathbb{R}_+}$ is the flow of ODE_h on Θ^∞.

Theorem A.2. *(CLT see e.g. [8, 12, 19]). Let θ^* be an equilibrium point of $\{h = 0\}$. Assume that the function h is differentiable at θ^* and all the eigenvalues of $Dh(\theta^*)$ have positive real parts. Assume that for some $\delta > 0$,*

$$\sup_{n\geq n_0} \mathbb{E}\left[\|\Delta M_{n+1}\|^{2+\delta} \,|\, \mathcal{F}_n\right] < +\infty \, a.s., \quad \mathbb{E}\left[\Delta M_{n+1} \Delta M_{n+1}^t \,|\, \mathcal{F}_n\right] \xrightarrow[n\to\infty]{a.s.} \Gamma, \tag{A.22}$$

where Γ is a nonrandom symmetric definite positive matrix and for an $\epsilon > 0$,

$$\mathbb{E}\left[(n+1)\|r_{n+1}\|^2 \mathbb{1}_{\{\|\theta_n - \theta^*\| \leq \epsilon\}}\right] \xrightarrow[n\to\infty]{} 0. \tag{A.23}$$

Specify the gain parameter sequence as follows

$$\forall n \geq 1, \quad \gamma_n = \frac{\alpha}{n}, \quad \alpha > \frac{1}{2\mathcal{R}e(\lambda_{min})} \tag{A.24}$$

where λ_{min} denotes the eigenvalue of $Dh(\theta^*)$ with the lowest real part. Then, the above a.s. convergence is ruled on the convergence set $\{\theta_n \longrightarrow \theta^*\}$ by the following Central Limit Theorem

$$\sqrt{n}\,(\theta_n - \theta^*) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0, \alpha\Sigma) \quad \textit{with} \quad \Sigma := \int_0^{+\infty} \left(e^{-\left(Dh(\theta^*)-\frac{I_d}{2\alpha}\right)u}\right)^t \Gamma e^{-\left(Dh(\theta^*)-\frac{I_d}{2\alpha}\right)u} du.$$

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