The type III manufactory

Sébastien Palcoux

THE TYPE III MANUFACTORY

SÉBASTIEN PALCOUX

ABSTRACT. Using unusual objects in the theory of von Neumann algebra, as the chinese game Go or the Conway game of life (generalized on finitely presented groups), we are able to build, by hands, many type III factors.

CONTENTS

Warning 0.1. *This paper is just a first draft, it contains very few proofs. It is possible that some propositions are false, or that some proofs are incomplete or trivially false.*

1. INTRODUCTION

2. Recall on von Neumann algebras

Let H be an Hilbert space and $\mathcal{A}$ a unital $\star$-algebra of bounded operators.

Definition 2.1. *The commutant $\mathcal{A}'$ of $\mathcal{A}$ is the set of $b \in B(H)$ such that, $\forall a \in \mathcal{A}$, then $[a,b] := ab - ba = 0$*

Definition 2.2. *The weak operator topology closure $\bar{\mathcal{A}}$ of $\mathcal{A}$ is the set of $a \in B(H)$ such that $\exists a_n \in \mathcal{A}$ with $(a_n\eta, \xi) \to (a\eta, \xi)$, $\forall \eta, \xi \in H$.*

Reminder 2.3. *(Bicommutant theorem)   Let $\mathcal{M}$ be a unital $\star$-algebra, then:*

$$\mathcal{M}'' = \mathcal{M} \iff \bar{\mathcal{M}} = \mathcal{M}$$

Definition 2.4. *Such a $\mathcal{M}$ verifying one of these equivalents properties is called a von Neumann algebra.*

Definition 2.5. *The von Neumann algebra generated by a set S of bounded operators on H is $\{s, s^\star \mid s \in S\}''$.*

Definition 2.6. *A factor is a von Neumann algebra $\mathcal{M}$ with $\mathcal{M} \cap \mathcal{M}' = \mathbb{C}$.*

Reminder 2.7. *(Murray and von Neumann theorem) The set of all the factors on H is a standard borelian space X and every von Neumann algebra $\mathcal{M}$ decompose into a direct integral of factors: $\mathcal{M} = \int_X^\oplus \mathcal{M}_x d\mu_x$*

Reminder 2.8. *(Murray and von Neumann's classification of factors)*
Let $\mathcal{M} \subset B(H)$ be a factor. We shall consider H as a representation of $\mathcal{M}'$. Thus subrepresentations of H correspond to projections in $\mathcal{M}$. If $p, q \in \mathcal{M}$ are projections, then pH and qH are unitarily equivalent as representations of $\mathcal{M}'$ iff there is a partial isometry $u \in \mathcal{M}$ between pH and qH; thus $u^*u = p$ and $uu^* = q$. We can immediately distinguish three mutually exclusive cases:
I. H has an irreducible subrepresentation.
II. H has no irreducible subrepresentation, but has a subrepresentation not equivalent to any proper subrepresentation of itself.
III. H has no irreducible subrepresentation and every subrepresentation is equivalent to some proper subrepresentation of itself.
We shall call $\mathcal{M}$ a factor of type I , II or III acording to the above cases.

Reminder 2.9. *The type I and II corresponds to factors admitting non-trivial trace, with only integer values on the projectors for the type I ($M_n(\mathbb{C})$ or $B(H)$), and non-integer values for the type II (factors generated by ICC groups for example). On the type III, the values are only 0 or ∞.*

Reminder 2.10. *(Tomita-Takesaki theory) We suppose the existence of a vector Ω (called vacuum vector) such that $\mathcal{M}\Omega$ and $\mathcal{M}'\Omega$ are dense in H (ie Ω is cyclic and separating). Let $S : H \to H$ the closure of the antilinear map: $\star : x\Omega \to x^\star\Omega$. Then, S admits the polar decomposition*

$S = J\triangle^{\frac{1}{2}}$ with J antilinear unitary, and $\triangle^{\frac{1}{2}}$ positve; so that $JMJ = \mathcal{M}'$, $\triangle^{it}\mathcal{M}\triangle^{-it} = \mathcal{M}$ and $\sigma_t^{\Omega}(x) = \triangle^{it}x\triangle^{-it}$ gives the one parameter modular group action.

Reminder 2.11. *(Radon-Nikodym theorem) Let Ω' be another vacuum vector, then there exists a Radon-Nikodym map $u_t \in \mathcal{U}(\mathcal{M})$, define such that $u_{t+s} = u_t\sigma_t^{\Omega'}(u_s)$ and $\sigma_t^{\Omega'}(x) = u_t\sigma_t^{\Omega}(x)u_t^{\star}$. Then, modulo $Int(\mathcal{M})$, σ_t^{Ω} is independant of the choice of Ω, ie, there exist an intrinsic $\delta : \mathbb{R} \to Out(\mathcal{M}) = Aut(\mathcal{M})/Int(\mathcal{M})$. On type I or II the modular action is internal, and so δ trivial. It's non-trivial for type III.*

Definition 2.12. *We can then define two invariants of $\mathcal{M}$, $T(\mathcal{M}) = ker(\delta)$ and $\mathcal{S}(\mathcal{M}) = Sp(\delta) = \bigcap Sp(\triangle_\Omega) \setminus \{0\}$ called the Connes spectrum of $\mathcal{M}$.*

Reminder 2.13. *(see [3]) Let $\mathcal{M}$ be a type III factor, then $\mathcal{S}(M) = \{1\}$, $\lambda^{\mathbb{Z}}$ or $\mathbb{R}_+^{\star}$, and then, $\mathcal{M}$ is called a III_0, III_λ or III_1 factor (with $0 < \lambda < 1$).*

Reminder 2.14. *A von Neumann algebra $\mathcal{M} \neq \mathbb{C}$ is a type III_1 factor if the modular action σ is ergodic (i.e. fixing only the scalar operator), or iff the cross-product $\mathcal{M} \rtimes_\sigma \mathbb{R}$ is equal to $\mathcal{N} \otimes B(H)$, with $\mathcal{N}$ a II_1 factor, called the core of $\mathcal{M}$.*

Definition 2.15. *Let $\mathcal{N}$ be a II_1 factor, then its fundamental group $\mathcal{F}(\mathcal{N})$ is the set of real numbers λ such that there is an automorphism rescaling the trace of $\mathcal{N} \otimes B(H)$ by a factor λ.*

Reminder 2.16. *If $\mathcal{N}$ is the core of a III_1 factor then $\mathcal{F}(\mathcal{N}) = \mathbb{R}_+^{\star}$ (see [8]).*

Remark 2.17. *The converse is false, see [6].*

3. Play Go on a Cayley graph

Definition 3.1. *Let (Γ, S) be a marked group, $\mathcal{G}$ its Cayley graph, $\ell(.)$ the word lenght. Let E be the set $\{-1, 0, 1\}$. A **finite state** of $\mathcal{G}$ is a map $\Psi : \Gamma \to E$ with $\Psi^{-1}(\{-1, 1\})$ of finite cardinal. In analogy with the game of Go, $\Psi(g) = -1$ means there is a black stone on the vertex g, $\Psi(g) = 1$ means a white stone and $\Psi(g) = 0$ means no stone.*

Definition 3.2. *Let Ψ be a state, a stone on g (ie $\Psi(g) \neq 0$) is **adjacent** to another on h if $\ell(h^{-1}g) = 1$, ie there is an edge between g and h on the Cayley graph $\mathcal{G}$. A stone is **connected** to another if there is a sequence of adjacent stones between them.*

Definition 3.3. *A **cluster** c is a connected component of stones with a same color. A **liberty** of a cluster c is an empty vertex adjacent to c. Let $n(c)$ be the number of liberties. An **eye** of a cluster c is an empty vertex such that every vertices adjacent to it admit a stone of the cluster c. A cluster is **immortal** if it admits at least two eyes.*

Definition 3.4. *We define by induction the set $\mathcal{S}$ of **admissible states**. First of all the vacuum state Ω (i.e. $\Omega = \Psi$ with $\Psi(\Gamma) = \{0\}$) is admissible. Next if Ψ is an admissible state then $B_g(\Psi)$ and $W_g(\Psi)$ are also admissible, where the operation B_g (resp. W_g) grossly means 'put a black (resp. white) stone on the vertex g', more precisely :*

(1) *if there is a stone on the vertex g (i.e. $\Psi(g) \neq 0$) then $B_g(\Psi) = W_g(\Psi) = \Psi$,*
(2) *else if the vertex g is not a liberty of a cluster, then $B_g(\Psi)(g) = -1$, $W_g(\Psi)(g) = 1$ and $\forall h \neq g$, $B_g(\Psi)(h) = W_g(\Psi)(h) = \Psi(h)$,*
(3) *else if it's a liberty of a cluster but not the last one and not an eye, then we apply the rule (2).*
(4) *else if the vertex g is an eye of a cluster but not the only one, then we apply the rule (1).*
(5) *else the vertex g is the last liberty of the clusters $(c_i)_{i \in I}$, then $\forall i \in I$ with c_i white (resp. black), $\forall h \in c_i$, $W_g(\Psi)(h) = \Psi(h) = 1$, $W_g(\Psi)(g) = 0$ (i.e. no suicide), $B_g(\Psi)(h) = 0$, $B_g(\Psi)(g) = -1$ (resp. $B_g(\Psi)(h) = \Psi(h) = -1$, $B_g(\Psi)(g) = 0$, $W_g(\Psi)(h) = 0$, $W_g(\Psi)(g) = 1$), the others vertices are unchanged.*

Remark 3.5. *The set of states is countable, so is for the set of admissible states $\mathcal{S}$.*

Definition 3.6. *Let H be the Hilbert space $\ell^2(\mathcal{S})$, then the operations B_g and W_g on $\mathcal{S}$ extend into bounded operators on H.*

Definition 3.7. *We call $\mathcal{G}o(\Gamma, S) = \{B_g, W_g, B_g^\star, W_g^\star \mid g \in G\}''$ the Go von Neumman algebra of the marked group (Γ, S).*

Question 3.8. *Does $\mathcal{G}o(\Gamma, S)$ depends on the choice of the generating set S ?*

Question 3.9. *$\mathcal{G}o(\mathbb{Z})$? , $\mathcal{G}o(\mathbb{Z}^n)$?, $\mathcal{G}o(\mathbb{F}^n)$? (with canonical generating set)*

Conjecture 3.10. *Let (Γ, S) be a (non-amenable) finitely presented ICC marked group then $\mathcal{G}o(\Gamma, S)$ is a (non-hyperfinite) III_1 factor.*

4. CONWAY'S GAME OF LIFE ON A CAYLEY SPACE

Reminder 4.1. *The game of life is a cellular automaton on a tilling of $\mathbb{R}^2$ with square cells. Each cell admits two state: 'alive' or 'death', and admits 8 neighbours. From an initial configuration the game progress step by step using the following rules:*

- *A death cell with exactly 3 alive neighbours becomes alive (birth).*
- *An alive cell with 2 or 3 alive neighbours remains alive, otherwise dies.*

Remark 4.2. *In fact we can build many others rules using the 8 neighbours of a cell, changing the number of alive neighbours to remain alive etc... We count 2^{16} rules.*

We can see this game as a particular case on $\mathbb{Z}^2$ of a game on a Cayley space of any finitely presented group $\Gamma = \langle S \mid R \rangle$. We recall here the construction of such a Cayley space using the presentation of the group, we choose the notation n-block instead of n-cell:

Definition 4.3. *Let Γ_n be the set of irreducible n-blocks, defined by induction:*

- $\Gamma_0 = \Gamma$.
- $\Gamma_1 := \{\{g, gs\} \mid g \in \Gamma, s \in S\}$

An $(n+2)$-block is a finite set a of $(n+1)$-blocks such that:
$$\forall b \in a, \forall c \in b, \exists! b' \in a \text{ such that } b \cap b' = \{c\}.$$
Let a, a' be n-blocks then the commutative and associative composition:
$$a.a' := a \triangle a' = (a \cup a') \backslash a \cap a'$$
gives also an n-block if it's non empty (we take $n \neq 0$).
*Let $n > 1$, an n-block a'' is called **irreducible** if $\forall a, a'$ n-blocks:*

(1) $a'' = a.a' \Rightarrow card(a)$ or $card(a') \geq card(a'')$

(2) $\forall b \in a''$, b is an irreducible $(n-1)$-block.

- Γ_{n+2} is the set of irreducible $(n+2)$-blocks.

*Note that if $b \in \Gamma_n$, we call n the **dimension** of b.*

Definition 4.4. *An n-block is called **admissible** if it decomposes into irreducibles.*

Example 4.5. *Let $\mathbb{Z} = \langle s^{\pm 1} \mid \rangle$ then $a = \{e, s^{10}\}$ is an admissible 1-block because $a = \{e, s\}.\{s, s^2\}...\{s^9, s^{10}\}$; but, $b = \{\{e, s\}, \{e, s^{-1}\}, \{s^{-1}, s\}\}$ is a non-admissible 2-block, because there is no irreducible 2-block in this case.*

Remark 4.6. *The graph with vertices Γ_0 and edges Γ_1 is the Cayley graph $\mathcal{G}$.*

Remark 4.7. *Let a be an n-block then $a.a = \emptyset$ and if $a = \{b_1, .., b_r\}$ then $b_i = b_1.b_2...b_{i-1}.b_{i+1}...b_r$ and $b_1.b_2...b_r = \emptyset$.*

Remark 4.8. *$\Gamma_{n+1} \neq \emptyset$ iff $\exists r > 1; a_1, ..., a_r \in \Gamma_n$ all distincts with $a_1...a_r = \emptyset$.*

Remark 4.9. *Let $\Gamma = \langle S \mid R \rangle$ be a finitely presented group, then $\exists N$ such that $\Gamma_N \neq \emptyset$ and $\forall n > N$, $\Gamma_n = \emptyset$. In fact $N \leq card(S)$*

Examples 4.10. *For $\mathbb{F}_r = \langle s_1^{\pm 1}, ..., s_r^{\pm 1} \mid \rangle$, we have $N = 1$.*
For $\mathbb{Z}^r = \langle s_1^{\pm 1}, ..., s_r^{\pm 1} \mid s_i s_j s_i^{-1} s_j^{-1}, i, j = 1, ..., r \rangle$, we have $N = r$.
Here an n-block ($n \leq r$) is just an n-dimensional hypercube.

Definition 4.11. *Let a and b be blocks, then we say that $b \Subset a$ if $b = a$ or if $b \in a$ or if $\exists c \in a$ such that $b \Subset c$ (recursive definition).*

Definition 4.12. *Let $n > 1$ then an n-block c is **connected** if $\forall b \subset c$: 'b is an n-block' $\Rightarrow b = c$.*

Definition 4.13. *An n-block b is called **maximal** if there is no $(n+1)$-block c with $b \in c$. We note $\Gamma_{\mathbf{max}}$ the set of maximal irreducible blocks.*

Example 4.14. *Let $\Gamma = \mathbb{Z}^2 \star \mathbb{Z} = \langle s_1^{\pm 1}, s_2^{\pm 1}, s_3^{\pm 1} \mid s_1 s_2 s_1^{-1} s_2^{-1} \rangle$, then $\{e, s_3\}$ is a maximal 1-block, $\{\{e, s_1\}, \{s_1, s_1 s_2\}, \{s_1 s_2, s_2\}, \{s_2, e\}\}$ is a maximal 2-block.*

Definition 4.15. *We define the action of Γ on Γ_n recursively:*
  - *Γ acts on $\Gamma_0 = \Gamma$ as: $u_g : h \to g.h$ with $g, h \in \Gamma$.*
  - *Action on Γ_{n+1}: $u_g : a \to g.a = \{g.b \mid b \in a\}$ with $g \in \Gamma$, $a \in \Gamma_{n+1}$.*

Note that the action is well-defined: $g.\Gamma_n = \Gamma_n$, $\forall g \in \Gamma$.

Remark 4.16. *Note that Γ acts on Γ_{max} with finitely many orbits because the group is finitely presented, so there are finitely many types of maximal irreducible blocks : $\tilde{c}_1, ..., \tilde{c}_r$.*

Definition 4.17. *Let $\mathcal{G}$ be the Cayley space (or Cayley complex) i.e. the canonical geometric realization of the blocks. Then by construction $\mathcal{G}$ is tiled by the tiles $\tilde{c}_1, ..., \tilde{c}_r$. We can see $\mathcal{G}$ as the universal covering of a classifying space of Γ, as $\mathbb{R}^n$ is the universal covering of $\mathbb{T}^n$ the classifying space of $\mathbb{Z}^n$.*

We now come back to the **Conway's game of life** on the Cayley space $\mathcal{G}$:

Definition 4.18. *The cells of the game on $\mathcal{G}$ are the irreducible maximal block of type $\tilde{c}_1$ or $\tilde{c}_2$ or ... or $\tilde{c}_r$. A cell admits two states: 'alive' or 'death'. A neighbour of a cell c is a cell $c' \neq c$ such that $\exists g \in \Gamma$ with $g \Subset c$ and $g \Subset c'$. A cell admits finitely many neighbours, the number of neighbours depends only of the type. Let n_i be the number of neighbours of a cell of type $\tilde{c}_i$.*

Remark 4.19. *We count $2^{2n_1}...2^{2n_r}$ rules.*

Definition 4.20. *A rule is called **admissible** if it does not contain:*
          'A death cell with no alive neighbours becomes alive'
We count $[2^{n_1}(2^{n_1} - 1)]...[2^{n_r}(2^{n_r} - 1)]$ admissible rules.

Definition 4.21. *A **state** of the game is a map: $\Psi : \Gamma_{max} \to \{alive, death\}$. The state Ψ is called **finite** if $\Psi^{-1}\{alive\}$ has a finite cardinal.*
Let $\mathcal{S}$ be the countable set of all the finite states.

Definition 4.22. *Let H be the Hilbert space $\ell^2(\mathcal{S})$, then an admissible rule R extend, in general, into an unbounded operator on H.*

Definition 4.23. *Let $\mathcal{M}(\Gamma, S, R_1, ..., R_m)$ be the von Neumann algebra on H generated by the finitely presented marked group (Γ, S) and the rules (R_i).*
We need to complete the definition because the operators (R_i) are unbounded...

Conjecture 4.24. *Let Γ be a (non-amenable) finitely presented ICC group, then it exists a finite presentation $\langle S \mid R \rangle$ and rules $R_1, ..., R_m$, such that $\mathcal{M}(\Gamma, S, R_1, ..., R_m)$ is a (non-hyperfinite) III_1 factor.*

5. The truncated algebra of a marked group

Definition 5.1. *A **marked group** is a group Γ with a generating set S.*

Definition 5.2. *Let $\mathcal{G}_{\Gamma,S}$ be the Cayley of the marked group (Γ, S) and $\ell(.)$ the word lenght (see [5] for more details).*

Definition 5.3. *Let $(U_s)_{s \in S}$ be operators on $H = \ell^2(\Gamma)$ (of basis $(e_g)_{g \in \Gamma}$) defined by $U_s.e_g = e_{gs}$, then $\{U_s \mid s \in S\}''$ is the von Neumann algebra $\mathcal{R}(\Gamma)$ generated by Γ and the right action on H. It's well-known that $\mathcal{R}(\Gamma)$ is a II_1 factor iff Γ is an ICC group (ie $\forall g \neq e$, $\{hgh^{-1} \mid h \in \Gamma\}$ is infinite).*

Definition 5.4. *Let $(X_s)_{s \in S}$ be operators on $H = \ell^2(\Gamma)$ defined by:*
$$X_s.e_g = \begin{cases} e_{gs} & \text{if } \ell(gs) = \ell(g) + 1 \\ e_g & \text{if } \ell(gs) \neq \ell(g) + 1 \end{cases}$$
*We call $\mathcal{T}(\Gamma, S) = \{X_s, X_s^\star \mid s \in S\}''$ the **truncated** von Neumann algebra of the marked group (Γ, S).*

Question 5.5. *Does $\mathcal{T}(\Gamma, S)$ depends on the choice of the generating set S ?*

Remark 5.6. $X_s^\star.e_g = \begin{cases} e_{gs^{-1}} & \text{if } \ell(gs) = \ell(g) - 1 \\ e_g & \text{if } \ell(gs) \neq \ell(g) - 1 \end{cases}$

Then $X_s + X_{s^{-1}}^\star - Id = U_s$, and $\mathcal{R}(\Gamma) \subset \mathcal{T}(\Gamma, S)$.

Question 5.7. *Does $\mathcal{T}(\Gamma, S)$ is a factor ? type ? If type III, ok with the topic.*
If type II_1, does $\mathcal{R}(\Gamma) \subset \mathcal{T}(\Gamma, S)$ is a finite index subfactor ?

6. Action of free groups on the circle

Definition 6.1. *Let $s, r_\theta : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, defined by $s(x) = x^2$ (choosing representatives in $[0, 1[$) and $r_\theta(x) = x + \theta$. Now, identifying $\mathbb{R}/\mathbb{Z}$ and $\mathbb{S}^1$, we define the action α of $\mathbb{F}_2 = \langle a, b \mid \; \rangle$, generated by $\alpha(a) = s$ and $\alpha(b) = r_\theta$ in $Homeo(\mathbb{S}^1)$.*

Lemma 6.2. *If θ is transcendantal, the action α is faithful.*

Proof. A relation $s^{n_1} r_\theta^{m_1}...s^{n_k} r_\theta^{m_k} = e$ can be translated into an algebraic equation in x and θ, which θ has to be a root $\forall x$. Then, if θ is transcendantal, we are sure that there is no relation. $\qquad\square$

Remark 6.3. *For a fixed transcendantal θ, each non-trivial relations can be realized for at most finitely many $x \in \mathbb{R}/\mathbb{Z}$, ie roots of the related algebraic equation.*

Theorem 6.4. $\mathcal{M} = L^\infty(\mathbb{S}^1, Leb) \rtimes_\alpha \mathbb{F}_2$ *is a non-hyperfinite type III factor.*

Proof. The action α of $\mathbb{F}_2$ on $\mathbb{S}^1$ is:
   (a) Measure class preserving: the set of null measure subspaces is invariant.
   (b) Free: a fixed point set for $\gamma \neq e$ is at most finite, so with null measure.
   (c) Properly ergodic: ergodicity comes from irrationnal rotation, next, every $\mathbb{F}_2$-orbit have null measure.
   (d) Non-amenable: by Connes-Feldman-Weiss [2], if such an action is amenable, there exist a transformation T of $\mathbb{S}^1$, such that $\forall x \in \mathbb{S}^1$ up to a null set, $\mathbb{F}_2.x = T^\mathbb{Z}.x$. Then, it exists $n \in \mathbb{Z}$ and $\gamma \in \mathbb{F}_2$, such that $a.x = T^n.x$ and $T.x = \gamma.x$. So, $a.x = \gamma^n.x$ and x is in the null set of algebraics with θ.
   (e) Non equivalent measure preserving: by ergodicity, an equivalent invariant measure m is proportionnal to Leb. Then $m([1/4, 1/2]) = 2m([1/16, 1/4])$, and by $\alpha(a)$ invariance, $m([1/4, 1/2]) = m([1/16, 1/4])$. In fact, the only invariant measure are 0 or ∞.

(a), (b), (c) give a factor, (d) gives non-hyperfinite, (e) gives a type III. $\quad\square$

Remark 6.5. *See [1] for groups acting on $\mathbb{S}^1$ without preserving finite measure.*

Exercice 6.6. *Extend the faithful action α to a faithful action of $\mathbb{F}_n$ on the circle, satifying the condition $(a), (b), ...(e)$ of the previous theorem.*

Corollary 6.7. $\mathcal{M} = L^\infty(\mathbb{S}^1, Leb) \rtimes_\alpha \mathbb{F}_n$ *is a non-hyperfinite type III factor.*

Question 6.8. *Let Γ be non-amenable ICC group. Does they exist a faithful action of Γ on the circle, satifying the condition $(a), (b), ...(e)$?*

References

[1] L. A. Beklaryan, *On analogues of the Tits alternative for groups of homeomorphisms of the circle and the line.* (Russian) Mat. Zametki 71 (2002), no. 3, 334–347; translation in Math. Notes 71 (2002), no. 3-4, 305–315.
[2] A. Connes, J. Feldman, B. Weiss, *An amenable equivalence relation is generated by a single transformation.* Ergodic Theory Dynamical Systems 1 (1981), no. 4, 431–450 (1982).
[3] A. Connes, *Noncommutative geometry*, Academic Press, Inc., San Diego, CA, 1994.

[4] V. J. Golodec, *Conditional expectations and modular automorphisms of von Neumann algebras.* (Russian) Funkcional. Anal. i Prilozhen. 6 (1972), no. 3, 68–69. (English translation: Functional Anal. Appl. 6 (1972), no. 3, 231–232).

[5] P. de la Harpe, *Topics in geometric group theory.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2000.

[6] S. Popa, S. Vaes *On the fundamental group of II1 factors and equivalence relations arising from group actions*

[7] M. Takesaki, *Conditional expectations in von Neumann algebras.* J. Functional Analysis 9 (1972), 306–321.

[8] M. Takesaki, *Duality for crossed products and the structure of von Neumann algebras of type III.* Acta Math. 131 (1973), 249–310.

INSTITUT DE MATHÉMATIQUES DE LUMINY, MARSEILLE, FRANCE.
E-mail address: `palcoux@iml.univ-mrs.fr`, `http://iml.univ-mrs.fr/∼palcoux`