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► **To cite this version:**

Yves Achdou, Thibaut Deheuvels, Nicoletta Tchou. JLip versus Sobolev Spaces on a Class of Self-Similar Fractal Foliages. *Journal de Mathématiques Pures et Appliquées*, 2012, 97 (2), pp.142-172. 10.1016/j.matpur.2011.07.002 . hal-00555205

**HAL Id: hal-00555205**

**<https://hal.science/hal-00555205>**

Submitted on 12 Jan 2011

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# *JLip* versus Sobolev Spaces on a Class of Self-Similar Fractal Foliages

Yves Achdou <sup>\*</sup>; Thibaut Deheuvels <sup>†</sup>; Nicoletta Tchou <sup>‡</sup>

January 11, 2011

## Abstract

For a class of self-similar sets  $\Gamma^\infty$  in  $\mathbb{R}^2$ , supplied with a probability measure  $\mu$  called the self-similar measure, we investigate if the  $B_s^{q,q}(\Gamma^\infty)$  regularity of a function can be characterized using the coefficients of its expansion in the Haar wavelet basis. Using the Lipschitz spaces with jumps recently introduced by Jonsson, the question can be rephrased: when does  $B_s^{q,q}(\Gamma^\infty)$  coincide with  $JLip(s, q, q; 0; \Gamma^\infty)$ ? When  $\Gamma^\infty$  is totally disconnected, this question has been positively answered by Jonsson for all  $s, q$ ,  $0 < s < 1$  and  $1 \leq q < \infty$  (in fact, Jonsson has answered the broader question of characterizing  $B_s^{p,q}(\Gamma^\infty)$ ,  $s > 0$ ,  $1 \leq p, q < \infty$ , using possibly higher degree Haar wavelets coefficients). Here, we fully answer the question in the case when  $0 < s < 1$  and  $\Gamma^\infty$  is connected.

## 1 Introduction

There is a growing interest in analysis on self-similar fractal sets, see for instance Kigami [16], Strichartz [25, 26], Mosco [23, 22] and references therein. These works aim at intrinsically defining function spaces using Dirichlet forms and a different metric from the Euclidean one. The results in this direction are often subject to the important assumption that the set is post-critically finite (or p.c.f.), see [16], page 23 for the definition.

In a different direction, Jonsson has studied Lipschitz functions spaces on a self-similar fractal set  $\mathcal{S}$  under a technical condition which yields a Markov inequality at any order, see the pioneering works [12, 13]. More precisely, in [13], Haar wavelets of arbitrary order on  $\mathcal{S}$  were introduced and used for constructing a family of Lipschitz function spaces allowing jumps at some special points in  $\mathcal{S}$ . These function spaces are named  $JLip(t, p, q; m; \mathcal{S})$ , where  $t$  is a positive real number,  $p, q$  are two real numbers not smaller than 1 and  $m$  is an integer ( $m$  is the order of the Haar wavelets used for constructing the space). Here  $J$  stands for *jumps*, since the considered functions may jump at some points of  $\mathcal{S}$ . The theory in [13], which does not need the assumption that  $\mathcal{S}$  be p.c.f., plays an important role in the present paper. It will be partially reviewed in § 4.1 (we will focus on the case when  $m = 0$ ,  $p = q$  and  $0 < t < 1$ ).

In the present work, for a class of self-similar sets contained in  $\mathbb{R}^2$ , we aim at studying the relationships between some *JLip* spaces and the more classical Besov spaces introduced and studied by Jonsson and Wallin [14] for closed sets: consider a closed subset  $F$  of  $\mathbb{R}^n$  supplied

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with a Borel measure  $\mu$  such that there exists a positive real number  $d$  and two positive constants  $c_1$  and  $c_2$  with

$$c_1 r^d \leq \mu(B(x, r)) \leq c_2 r^d,$$

for all  $x \in F$  and  $r < 1$  (here  $B(x, r)$  is the ball in  $F$  with center  $x$  and radius  $r$ , with respect to the Euclidean distance in  $\mathbb{R}^n$ ); the set  $F$  is said to be a  $d$ -set. In [14], Sobolev and Besov spaces are defined on  $d$ -sets. For example, for  $0 < s < 1$ , the Sobolev space  $B_s^{p,p}(F)$  is defined as

$$B_s^{p,p}(F) = \left\{ f \in L_\mu^p(F); \int_{x,y \in F, |x-y| < 1} \frac{|f(x) - f(y)|^p}{|x - y|^{d+sp}} d\mu(x) d\mu(y) < \infty \right\}, \quad (1)$$

see the definition in [14] page 103. In the same book, results on the extension (in  $\mathbb{R}^n$ ) of functions belonging to Besov and Sobolev spaces on  $F$  and trace results are proved using as a main ingredient Whitney extension theory. In particular, there exists a continuous trace operator from  $W^{1,p}(\mathbb{R}^n)$  onto  $B_{1-\frac{n-d}{p}}^{p,p}(F)$ , if  $\max(1, n-d) < p < \infty$ . A more general trace theorem is available, see Theorem 1, page 141 in [14].

The approach of Triebel [27] is somewhat different. In [27] chapter IV, paragraph 18, it is proved that the space of the traces on  $F$  of functions in  $B_{\frac{n-d}{p}}^{p,q}(\mathbb{R}^n)$  is  $L_\mu^p(F)$  for  $0 < d < n$ ,  $d/n < p < \infty$  and  $0 < q \leq \min(1, p)$ ; Besov spaces on  $F$  are then defined as spaces of the traces of Besov spaces on  $\mathbb{R}^n$  and embeddings properties are studied.

In [13], Jonsson has proved that if the self-similar set  $\mathcal{S}$  is totally disconnected, then the  $JLip$  spaces coincide with Lipschitz or Besov spaces, more precisely that the spaces  $JLip(t, p, q; m; \mathcal{S})$  coincide with the Lipschitz spaces  $Lip(t, p, q; m; \mathcal{S})$  also introduced in [13]. The latter are a generalization of the more classical spaces  $Lip(t, p, q; \mathcal{S})$  introduced in [14] since  $Lip(t, p, q; [t]; \mathcal{S}) = Lip(t, p, q; \mathcal{S})$ . Note that  $Lip(t, p, q; [t]; \mathcal{S}) = B_t^{p,q}(\mathcal{S})$ , see [15]. When the fractal set is not totally disconnected, the  $JLip$  space may not coincide with  $Lip$  or Besov spaces.

In the present work, we focus on a class of self-similar sets noted  $\Gamma^\infty$  below, see for example Figure 1. The set  $\Gamma^\infty$  is the unique compact subset of  $\mathbb{R}^2$  such that

$$\Gamma^\infty = f_1(\Gamma^\infty) \cup f_2(\Gamma^\infty),$$

where  $f_1$  and  $f_2$  are two similitudes with rotation angles  $\pm\theta$  and contraction factor  $a$ ,  $0 < a \leq a^*(\theta)$ . As we shall see,  $\Gamma^\infty$  can be seen as a part of the boundary of a ramified domain  $\Omega$  in  $\mathbb{R}^2$ , see Figure 1, and the restriction  $a \leq a^*(\theta)$  allows for the construction of  $\Omega$  as a union of non-overlapping sub-domains, see (21). In § 2.2.3, we will recall the notion of self-similar measure  $\mu$  defined in the triplet  $(\Gamma^\infty, f_1, f_2)$ , see [16]. With the Borel regular probability measure  $\mu$ ,  $\Gamma^\infty$  is a  $d$ -set where  $d \equiv -\log 2 / \log a$  is the Hausdorff dimension of  $\Gamma^\infty$ .

The notion of traces on  $\Gamma^\infty$  for functions in  $W^{1,q}(\Omega)$  has been defined in the earlier work [2]. In [4], some of the authors of the present paper have characterized the space of the traces on  $\Gamma^\infty$  of functions in  $W^{1,q}(\Omega)$  as  $JLip(1 - \frac{2-d}{q}, q, q; 0; \Gamma^\infty)$ , for  $1 < q < \infty$  (with  $d = -\log 2 / \log a$ ). Note that  $JLip(1 - \frac{2-d}{q}, q, q; 0; \Gamma^\infty)$  always contains  $Lip(1 - \frac{2-d}{q}, q, q; 0; \Gamma^\infty) = B_{1-\frac{2-d}{q}}^{q,q}(\Gamma^\infty)$ , and that  $JLip(1 - \frac{2-d}{q}, q, q; 0; \Gamma^\infty) = Lip(1 - \frac{2-d}{q}, q, q; 0; \Gamma^\infty)$  if  $a < a^*(\theta)$ .

Therefore, the question considered here is to know for  $a = a^*(\theta)$ , in which case the identity

$$JLip(t, q, q; 0; \Gamma^\infty) = Lip(t, q, q; 0; \Gamma^\infty) = B_t^{q,q}(\Gamma^\infty) \quad (2)$$

holds, and if not, to find the parameters  $s$  such that  $JLip(t, q, q; 0; \Gamma^\infty) \subset B_s^{q,q}(\Gamma^\infty)$ . The first part of the question covers the following one: when do the spaces containing the traces on  $\Gamma^\infty$

of the functions in  $W^{1,q}(\Omega)$  and  $W^{1,q}(\mathbb{R}^2)$  coincide? This is also linked to the possibility of constructing an extension operator from  $W^{1,q}(\Omega)$  to  $W^{1,q}(\mathbb{R}^2)$ . Note that a partial answer was given in [3] (before the characterization of the trace space as a *JLip* space was found) in the special case when  $q = 2$  and for a special geometry ( $\theta = \pi/4$ ).

We will see that two different situations occur:

- if there does not exist an integer  $\mathfrak{p}$  such that  $\mathfrak{p}\theta = \pi/2$  then  $f_1(\Gamma^\infty) \cap f_2(\Gamma^\infty)$  is a singleton, and we will see that (2) holds if  $qt < d$ .
- Otherwise, the Hausdorff dimension of  $f_1(\Gamma^\infty) \cap f_2(\Gamma^\infty)$  is  $d/2$  and (2) holds only for  $qt < d/2$ .

Finally, note that the question of extensions or traces naturally arises in boundary value or transmission problems in domains with fractal boundaries. Results in this direction have been given in [24, 18, 17] for the Koch flake. There also, the assumption that the fractal set is p.c.f. is generally made. Boundary value problems posed in the domains  $\Omega$  displayed in Figure 1 were studied in [2].

The paper is organized as follows: the geometry is presented in Section 2. In Section 3, we recall some of the results of [2] on the space  $W^{1,q}(\Omega)$  and the construction of the trace operator. The theory proposed in [13] is reviewed in Section 4, where we also recall the characterization of the trace space proved in [4]. The main results of the paper are Theorems 6 and 7 which are stated in §5 and respectively proved in §5.2 and §5.3. For the ease of the reader, the geometrical lemmas, which are crucial but technical, are proved in the Appendix at the end of the paper.

## 2 The Geometry

### 2.1 The similitudes $f_1$ and $f_2$ and the self-similar set $\Gamma^\infty$

#### 2.1.1 Definitions

Consider four real numbers  $a, \alpha, \beta, \theta$  such that  $0 < a < 1/\sqrt{2}$ ,  $\alpha > 0$ ,  $\beta > 0$  and  $0 < \theta < \pi/2$ . Let  $f_i$ ,  $i = 1, 2$  be the two similitudes in  $\mathbb{R}^2$  given by

$$\begin{aligned} f_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} -\alpha \\ \beta \end{pmatrix} + a \begin{pmatrix} x_1 \cos \theta - x_2 \sin \theta \\ x_1 \sin \theta + x_2 \cos \theta \end{pmatrix}, \\ f_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + a \begin{pmatrix} x_1 \cos \theta + x_2 \sin \theta \\ -x_1 \sin \theta + x_2 \cos \theta \end{pmatrix}. \end{aligned} \tag{3}$$

The two similitudes have the same dilation ratio  $a$  and opposite angles  $\pm\theta$ . One can obtain  $f_2$  by composing  $f_1$  with the symmetry with respect to the axis  $\{x_1 = 0\}$ .

We denote by  $\Gamma^\infty$  the self-similar set associated to the similitudes  $f_1$  and  $f_2$ , i.e. the unique compact subset of  $\mathbb{R}^2$  such that

$$\Gamma^\infty = f_1(\Gamma^\infty) \cup f_2(\Gamma^\infty).$$

For  $n \geq 1$ , we denote by

- $\mathcal{A}_n$  the set containing all the  $2^n$  mappings from  $\{1, \dots, n\}$  to  $\{1, 2\}$
- $\mathcal{A}$  the set defined by  $\mathcal{A} = \cup_{n \geq 1} \mathcal{A}_n$

- $\mathcal{A}_\infty = \{1, 2\}^{\mathbb{N} \setminus \{0\}}$  the set of the sequences  $\sigma = (\sigma(i))_{i=1, \dots, \infty}$  with values  $\sigma(i) \in \{1, 2\}$ .

Consider  $1 \leq m < n \leq \infty$  and  $\sigma \in \mathcal{A}_n$ : We say that  $\sigma_m \in \mathcal{A}_m$  defined by  $\sigma_m(i) = \sigma(i)$ ,  $i = 1, \dots, m$  is a prefix of  $\sigma$ . We also define for  $\eta \in \mathcal{A}_n$  and  $\sigma \in \mathcal{A}_k$  the sequence  $\eta + \sigma \in \mathcal{A}_{n+k}$  by

$$\eta + \sigma = (\eta(1), \dots, \eta(n), \sigma(1), \dots, \sigma(k)). \quad (4)$$

For a positive integer  $n$  and  $\sigma \in \mathcal{A}_n$ , we define the similitude  $f_\sigma$  by

$$f_\sigma = f_{\sigma(1)} \circ \dots \circ f_{\sigma(n)}. \quad (5)$$

Similarly, if  $\sigma \in \mathcal{A}_\infty$ ,

$$f_\sigma = \lim_{n \rightarrow \infty} f_{\sigma(1)} \circ \dots \circ f_{\sigma(n)} = \lim_{n \rightarrow \infty} f_{\sigma_n} \quad (6)$$

Let the subset  $\Gamma^{\infty, \sigma}$  of  $\Gamma^\infty$  be defined by

$$\Gamma^{\infty, \sigma} = f_\sigma(\Gamma^\infty). \quad (7)$$

The definition of  $\Gamma^\infty$  implies that for all  $n > 0$ ,  $\Gamma^\infty = \bigcup_{\sigma \in \mathcal{A}_n} \Gamma^{\infty, \sigma}$ . We also define the set  $\Xi^\infty$ :

$$\Xi^\infty = f_1(\Gamma^\infty) \cap f_2(\Gamma^\infty). \quad (8)$$

The following theorem was stated by Mandelbrot et al, [19] (a complete proof is given in [7]):

**Theorem 1** *For any  $\theta$ ,  $0 < \theta < \pi/2$ , there exists a unique positive number  $a^*(\theta) < 1/\sqrt{2}$ , (which does not depend of  $(\alpha, \beta)$  see [4]) such that*

$$\begin{aligned} 0 < a < a^*(\theta) &\Rightarrow \Xi^\infty = \emptyset &\Rightarrow \Gamma^\infty \text{ is totally disconnected,} \\ a = a^*(\theta) &\Rightarrow \Xi^\infty \neq \emptyset &\Rightarrow \Gamma^\infty \text{ is connected, (from Th. 1.6.2 in [16]).} \end{aligned} \quad (9)$$

The critical parameter  $a^*(\theta)$  is the unique positive root of the polynomial equation:

$$\sum_{i=0}^{\mathfrak{p}-1} X^{i+2} \cos i\theta = \frac{1}{2}, \quad (10)$$

where

$$\mathfrak{p} \text{ is the smallest integer such that } \mathfrak{p}\theta \geq \pi/2. \quad (11)$$

**Remark 1** *From (10), it can be seen that  $\theta \mapsto a^*(\theta)$  is a continuous and increasing function from  $(0, \pi/2)$  onto  $(1/2, 1/\sqrt{2})$  and that  $\lim_{\theta \rightarrow 0} a^*(\theta) = 1/2$ .*

Hereafter, for a given  $\theta$ ,  $0 < \theta < \pi/2$ , we will write for brevity  $a^*$  instead of  $a^*(\theta)$  and we will only consider  $a$  such that  $0 < a \leq a^*$ .

### 2.1.2 Characterization of $\Xi^\infty$

We aim at characterizing  $\Xi^\infty$  defined in (8). We already know that  $\Xi^\infty \neq \emptyset$  if and only if  $a = a^*$ . Let us denote by  $\Lambda$  the vertical axis:  $\Lambda = \{x : x_1 = 0\}$  and by  $O$  the origin  $O = (0, 0)$ . Since  $f_1(\Gamma^\infty) = \Gamma^\infty \cap \{x_1 \leq 0\}$  and  $f_2(\Gamma^\infty) = \Gamma^\infty \cap \{x_1 \geq 0\}$ , we immediately see that  $\Xi^\infty = \Gamma^\infty \cap \Lambda$ . It can be observed (see [7] for the proof) that the sequences  $\sigma \in \mathcal{A}_\infty$  such that  $f_\sigma(O) \in \Lambda$  and that  $\sigma(1) = 1$  are characterized by the following property: for all  $n \leq 1$ , the truncated sequence  $\sigma_n$  achieves the maximum of the abscissa of  $f_\eta(O)$  over all  $\eta \in \mathcal{A}_n$  such that  $\eta(1) = 1$ .

Let us make out two cases, according to the value of  $\mathfrak{p}$  defined in (11):

**The case when  $\mathfrak{p}\theta > \pi/2$**

**Proposition 1** *If  $\mathfrak{p}\theta > \pi/2$  and  $a = a^*$ , then  $\Xi^\infty$  contains the single point*

$$\xi = \lim_{n \rightarrow \infty} f_1 \circ f_2^{\mathfrak{p}+1} \circ (f_1 \circ f_2)^n(O) = \lim_{n \rightarrow \infty} f_2 \circ f_1^{\mathfrak{p}+1} \circ (f_2 \circ f_1)^n(O). \quad (12)$$

**Proof.** For brevity, we skip the proof, which is available in [19, 7]. ■

**The case when  $\mathfrak{p}\theta = \pi/2$**  We need some specific notation:

- for  $i = 1, 2$ , we define  $\bar{i} = \begin{cases} 2 & \text{si } i = 1 \\ 1 & \text{si } i = 2 \end{cases}$ ,

- we introduce

$$F_1 = f_1 \circ f_2 \quad \text{and} \quad F_2 = f_2 \circ f_1, \quad (13)$$

- for  $\eta \in \mathcal{A}_k$ , we define  $\eta^{(1)}, \eta^{(2)} \in \mathcal{A}_{2(k+1)+\mathfrak{p}}$  by:

$$\left\{ \begin{array}{l} \eta^{(1)}(1) = 1 \text{ and } \eta^{(1)}(\ell) = 2 \text{ for all } \ell \in [2, \mathfrak{p} + 2], \\ \eta^{(2)}(1) = 2 \text{ and } \eta^{(2)}(\ell) = 1 \text{ for all } \ell \in [2, \mathfrak{p} + 2], \\ \forall j, 1 \leq j \leq k, \quad \left\{ \begin{array}{l} \eta^{(1)}(\mathfrak{p} + 2j + 1) = \eta(j) \\ \eta^{(1)}(\mathfrak{p} + 2(j + 1)) = \overline{\eta(j)} \end{array} \right. \text{ and } \left\{ \begin{array}{l} \eta^{(2)}(\mathfrak{p} + 2j + 1) = \overline{\eta(j)} \\ \eta^{(2)}(\mathfrak{p} + 2(j + 1)) = \eta(j) \end{array} \right. \end{array} \right. \quad (14)$$

In an equivalent manner,

$$\left\{ \begin{array}{l} \eta^{(1)} = (1, \underbrace{2, \dots, 2}_{\mathfrak{p}+1}, \eta(1), \overline{\eta(1)}, \dots, \eta(k), \overline{\eta(k)}), \\ \eta^{(2)} = (2, \underbrace{1, \dots, 1}_{\mathfrak{p}+1}, \overline{\eta(1)}, \eta(1), \dots, \overline{\eta(k)}, \eta(k)), \end{array} \right. \quad (15)$$

which yields

$$\left\{ \begin{array}{l} f_{\eta^{(1)}} = f_1 \circ \underbrace{f_2 \circ \dots \circ f_2}_{\mathfrak{p}+1} \circ F_{\eta(1)} \circ \dots \circ F_{\eta(k)}, \\ f_{\eta^{(2)}} = f_2 \circ \underbrace{f_1 \circ \dots \circ f_1}_{\mathfrak{p}+1} \circ F_{\overline{\eta(1)}} \circ \dots \circ F_{\overline{\eta(k)}}. \end{array} \right. \quad (16)$$

**Proposition 2** *If  $\mathfrak{p}\theta = \pi/2$  and  $a = a^*$ , then*

$$\Xi^\infty = \left\{ \lim_{n \rightarrow \infty} f_{\sigma_n^{(1)}}(O) = \lim_{n \rightarrow \infty} f_{\sigma_n^{(2)}}(O); \sigma \in \mathcal{A}_\infty \right\}. \quad (17)$$

Moreover, for  $x \in \Xi^\infty$ , there exists a unique  $\sigma \in \mathcal{A}_\infty$  such that

$$x = \lim_{n \rightarrow \infty} f_{\sigma_n^{(1)}}(O) = \lim_{n \rightarrow \infty} f_{\sigma_n^{(2)}}(O). \quad (18)$$

The set  $\Xi^\infty$  is not countable.

**Proof.** For brevity, we skip the proof, which is available in [7]. ■

## 2.2 Ramified domains

### 2.2.1 The construction

Call  $P_1 = (-1, 0)$  and  $P_2 = (1, 0)$  and  $\Gamma^0$  the line segment  $\Gamma^0 = [P_1, P_2]$ . We impose that  $f_2(P_1)$ , and  $f_2(P_2)$  have positive coordinates, i.e. that

$$a \cos \theta < \alpha \quad \text{and} \quad a \sin \theta < \beta. \quad (19)$$

We also impose that the open domain  $Y^0$  inside the closed polygonal line joining the points  $P_1, P_2, f_2(P_2), f_2(P_1), f_1(P_2), f_1(P_1), P_1$  in this order is convex. With (19), this is true if and only if

$$(\alpha - 1) \sin \theta + \beta \cos \theta \geq 0. \quad (20)$$

Under assumptions (19) and (20), the domain  $Y^0$  is either hexagonal or trapezoidal in degenerate cases, contained in the half-plane  $x_2 > 0$  and symmetric w.r.t. the vertical axis  $x_1 = 0$ .

We introduce  $K^0 = \overline{Y^0}$ . It is possible to glue together  $K^0, f_1(K^0)$  and  $f_2(K^0)$  and obtain a new polygonal domain, also symmetric with respect to the axis  $\{x_1 = 0\}$ . The assumptions (19) and (20) imply that  $Y^0 \cap f_1(Y^0) = \emptyset$  and  $Y^0 \cap f_2(Y^0) = \emptyset$ . We also define the ramified open domain  $\Omega$ , see Figure 1:

$$\Omega = \text{Interior} \left( K^0 \cup \left( \bigcup_{\sigma \in \mathcal{A}} f_\sigma(K^0) \right) \right). \quad (21)$$

Note that  $\Omega$  is symmetric with respect to the axis  $x_1 = 0$ , and that for  $a < 1/\sqrt{2}$ , the measure of  $\Omega$  is finite.

For a given  $\theta$ , with  $a^*$  defined as above, we shall make the following assumption on  $(\alpha, \beta)$ :

**Assumption 1** For  $0 < \theta < \pi/2$ , the parameters  $\alpha$  and  $\beta$  satisfy (20) and (19) for  $a = a^*$ , and are such that

$$\begin{cases} \text{i) for all } a, 0 < a \leq a^*, \text{ the sets } Y^0, f_\sigma(Y^0), \sigma \in \mathcal{A}_n, n > 0, \text{ are disjoint} \\ \text{ii) for all } a, 0 < a < a^*, f_1(\overline{\Omega}) \cap f_2(\overline{\Omega}) = \emptyset \\ \text{iii) for } a = a^*, f_1(\overline{\Omega}) \cap f_2(\overline{\Omega}) \neq \emptyset. \end{cases}$$

**Remark 2** Assumption 1 implies that if  $a = a^*$ , then  $f_1(\Omega) \cap f_2(\Omega) = \emptyset$ ; to prove this, we define the open set  $T = \text{Interior} (K^0 \cup f_1(K^0) \cup f_2(K^0))$ . It is easy to check that  $\Omega = T \cup \bigcup_{\sigma \in \mathcal{A}} f_\sigma(T)$ .

If  $f_1(\Omega) \cap f_2(\Omega) \neq \emptyset$ , there exist  $x \in \Omega$ , a positive number  $\rho$ , two positive integers  $n$  and  $n'$ , and  $\sigma \in \mathcal{A}_n$  and  $\sigma' \in \mathcal{A}_{n'}$  with  $\sigma(1) = 1$  and  $\sigma'(1) = 2$  such that  $B(x, \rho) \subset f_\sigma(T) \cap f_{\sigma'}(T)$ . It is then easy to prove that this contradicts point i) in Assumption 1.

The following theorem proved in [4] asserts that  $\forall \theta, 0 < \theta < \pi/2$ , there exists  $(\alpha, \beta)$  satisfying Assumption 1.

**Theorem 2** If  $\theta \in (0, \pi/2)$ , then for all  $\alpha > a^* \cos \theta$ , there exists  $\bar{\beta} > 0$  such that  $\bar{\beta} > a^* \sin \theta$  and  $(\alpha - 1) \sin \theta + \bar{\beta} \cos \theta \geq 0$  and for all  $\beta \geq \bar{\beta}$ ,  $(\alpha, \beta)$  satisfies Assumption 1.

It has been proved in [3] that if  $a < a^*$ , then there exists  $\epsilon > 0$  and  $\delta > 0$  such that  $\Omega$  is a  $\epsilon - \delta$  domain as defined by Jones [11], see also [14] or in an equivalent manner a quasi-disk, see [20]. On the contrary, if  $a = a^*$ , then  $\Omega$  is not a  $\epsilon - \delta$  domain because from Propositions 1 and 2, it is possible to construct two sequences  $(x_n^{(1)})_n$  and  $(x_n^{(2)})_n$ ,  $x_n^{(1)} \in f_1(\Omega)$  and  $x_n^{(2)} \in f_2(\Omega)$  such that  $\lim_{n \rightarrow \infty} |x_n^{(1)} - x_n^{(2)}| = 0$ ; then, any arc contained in  $\Omega$  and joining  $x_n^{(1)}$  to  $x_n^{(2)}$  has a length bounded from below by a positive constant.

### 2.2.2 The Moran condition

The Moran condition, (or open set condition), see [21, 16], is that there exists a nonempty bounded open subset  $\omega$  of  $\mathbb{R}^2$  such that  $f_1(\omega) \cap f_2(\omega) = \emptyset$  and  $f_1(\omega) \cup f_2(\omega) \subset \omega$ . For a given  $\theta \in (0, \pi/2)$ , let  $(\alpha, \beta)$  satisfy Assumption 1; for  $0 < a \leq a^*$ , the Moran condition is satisfied with  $\omega = \Omega$  because

- $f_1(\Omega) \cap f_2(\Omega) = \emptyset$ , which stems from point ii) in Assumption 1 if  $a < a^*$ , and from Remark 2 if  $a = a^*$ ;
- by construction of  $\Omega$ , we also have  $f_1(\Omega) \cup f_2(\Omega) \subset \Omega$ .

The Moran condition implies that the Hausdorff dimension of  $\Gamma^\infty$  is

$$\dim_H(\Gamma^\infty) = d \equiv -\log 2 / \log a,$$

see [21, 16]. If  $0 < \theta < \pi/2$ , we have  $0 < a \leq a^* < 1/\sqrt{2}$  and thus  $d < 2$ .

It can also be seen that if  $\mathfrak{p}\theta = \pi/2$  and  $a = a^*$ , then the Hausdorff dimension of  $\Xi^\infty$  is  $d/2$ .

### 2.2.3 The self-similar measure $\mu$

To define traces on  $\Gamma^\infty$ , we recall the classical result on self-similar measures, see [8, 10] and [16] page 26:

**Theorem 3** *There exists a unique Borel regular probability measure  $\mu$  on  $\Gamma^\infty$  such that for any Borel set  $A \subset \Gamma^\infty$ ,*

$$\mu(A) = \frac{1}{2}\mu(f_1^{-1}(A)) + \frac{1}{2}\mu(f_2^{-1}(A)). \quad (22)$$

The measure  $\mu$  is called the *self-similar measure defined in the self-similar triplet*  $(\Gamma^\infty, f_1, f_2)$ .

**Proposition 3** *The measure  $\mu$  is a  $d$ -measure on  $\Gamma^\infty$ , with  $d = -\log 2 / \log a$ , according to the definition in [14], page 28: there exists two positive constants  $c_1$  and  $c_2$  such that*

$$c_1 r^d \leq \mu(B(x, r)) \leq c_2 r^d,$$

for any  $r$   $0 < r < 1$  and  $x \in \Gamma^\infty$ , where  $B(x, r)$  is the Euclidean ball in  $\Gamma^\infty$  centered at  $x$  and with radius  $r$ . In other words the closed set  $\Gamma^\infty$  is a  $d$ -set, see [14], page 28.

**Proof.** The proof stems from the Moran condition. It is due to Moran [21] and has been extended by Kigami, see [16], §1.5, especially Proposition 1.5.8 and Theorem 1.5.7. ■

We define  $L_\mu^p$ ,  $p \in [1, +\infty)$  as the space of the measurable functions  $v$  on  $\Gamma^\infty$  such that  $\int_{\Gamma^\infty} |v|^p d\mu < \infty$ , endowed with the norm  $\|v\|_{L_\mu^p} = (\int_{\Gamma^\infty} |v|^p d\mu)^{1/p}$ . We also introduce  $L_\mu^\infty$ , the space of essentially bounded functions with respect to the measure  $\mu$ . A Hilbertian basis of  $L_\mu^2$  can be constructed with e.g. Haar wavelets.

### 2.2.4 Example

We make the choice  $\theta = \pi/4$ ,  $\alpha = 1 - a/\sqrt{2}$ ,  $\beta = 1 + a/\sqrt{2}$ . Hence  $\mathfrak{p} = 2$ . The critical parameter  $a^*(\pi/4)$  is the unique positive solution of  $X^3 + \sqrt{2}X^2 - \sqrt{2}/2 = 0$ , i.e.  $a \leq a^*(\pi/4) \simeq 0.593465$ . The construction described in § 2.2.1 with the critical value  $a = a^*(\pi/4)$  leads to the domain  $\Omega$  shown in the left part of Figure 1. If  $a > 1/2$ , the Hausdorff dimension of  $\Gamma^\infty$  is larger than one. For instance, if  $a = a^*(\pi/4)$ , then  $\dim_H(\Gamma^\infty) \simeq 1.3284371$ . In the right part of Figure 1, we show a similar construction with  $\theta = \pi/5$  (for which  $\mathfrak{p} = 3$ ) and  $a = a^*(\pi/5) \simeq 0.56658$ . Note the difference between the two cases: in the former case  $\mathfrak{p}(\theta) \cdot \theta = \pi/2$  and the set  $\Xi^\infty$  defined in (8) is not countable whereas in the latter case,  $\mathfrak{p}(\theta) \cdot \theta > \pi/2$  and the set  $\Xi^\infty$  is a singleton.



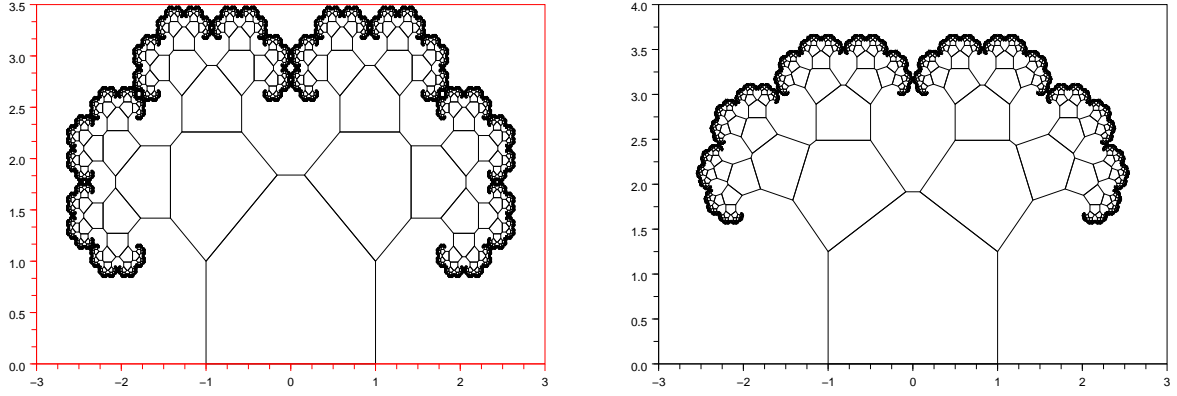


Figure 1: Left: the ramified domain  $\Omega$  for  $\theta = \pi/4$ ,  $a = a^*(\pi/4)$ ,  $\alpha = 1 - a^*/\sqrt{2}$ ,  $\beta = 1 + a^*/\sqrt{2}$ . Right: a similar construction for  $\theta = \pi/5$  and  $a = a^*(\pi/5)$ .

### 2.2.5 Additional notations

We define the sets  $\Gamma^\sigma = f_\sigma(\Gamma^0)$  and  $\Gamma^N = \cup_{\sigma \in \mathcal{A}_N} \Gamma^\sigma$ . The one-dimensional Lebesgue measure of  $\Gamma^\sigma$  for  $\sigma \in \mathcal{A}_N$  and of  $\Gamma^N$  are

$$|\Gamma^\sigma| = a^N |\Gamma^0| \quad \text{and} \quad |\Gamma^N| = (2a)^N |\Gamma^0|.$$

We will sometimes use the notation  $\lesssim$  or  $\gtrsim$  to indicate that there may arise constants in the estimates, which are independent of the index  $n$  in  $\Gamma^n$ , or of the index  $\sigma$  in  $\Gamma^\sigma$  or  $\Gamma^{\infty, \sigma}$ . We may also write  $A \simeq B$  if  $A \lesssim B$  and  $B \lesssim A$ .

## 3 The space $W^{1,q}(\Omega)$

Hereafter, we take  $\theta$  in  $(0, \pi/2)$  and suppose that the parameters  $(\alpha, \beta)$  satisfy Assumption 1.

**Basic facts** For a real number  $q \geq 1$ , let  $W^{1,q}(\Omega)$  be the space of functions in  $L^q(\Omega)$  with first order partial derivatives in  $L^q(\Omega)$ . The space  $W^{1,q}(\Omega)$  is a Banach space with the norm

$\left( \|u\|_{L^q(\Omega)}^q + \left\| \frac{\partial u}{\partial x_1} \right\|_{L^q(\Omega)}^q + \left\| \frac{\partial u}{\partial x_2} \right\|_{L^q(\Omega)}^q \right)^{\frac{1}{q}}$ , see for example [5], p 60. Elementary calculus shows that  $\|u\|_{W^{1,q}(\Omega)} \equiv \left( \|u\|_{L^q(\Omega)}^q + \|\nabla u\|_{L^q(\Omega)}^q \right)^{\frac{1}{q}}$  is an equivalent norm, with  $\|\nabla u\|_{L^q(\Omega)}^q \equiv \int_\Omega |\nabla u|^q$  and  $|\nabla u| = \sqrt{\left| \frac{\partial u}{\partial x_1} \right|^2 + \left| \frac{\partial u}{\partial x_2} \right|^2}$ .

The spaces  $W^{1,q}(\Omega)$  as well as elliptic boundary value problems in  $\Omega$  have been studied in [2], with, in particular Poincaré inequalities and a Rellich compactness theorem. The same results in a similar but different geometry were proved by Berger [6] with other methods.

**Traces** We first discuss very briefly the less interesting case when  $a < 1/2$ . If  $a < 1/2$ , then  $d < 1$  and  $\Gamma^\infty$  is totally disconnected, see [9], Lemma 4.1 page 54. This implies that  $f_1(\Gamma^\infty) \cap f_2(\Gamma^\infty) = \emptyset$ , see [16], theorem 1.6.2 page 33. The results of Jones[11] and of Jonsson

and Wallin [14] can be combined to prove that if  $q > \max(1, 2 - d)$ , then the space of the traces on  $\Gamma^\infty$  of the functions  $v \in W^{1,q}(\Omega)$  is  $B_{1-\frac{2-d}{q}}^{q,q}(\Gamma^\infty)$  (see the introduction for the definition). We will see in Theorem 4 below that in this case,  $B_{1-\frac{2-d}{q}}^{q,q}(\Gamma^\infty) = J\text{Lip}(1 - \frac{2-d}{q}, q, q; 0; \Gamma^\infty)$ .

Since the case  $a < 1/2$  is understood, in the remaining part of the paper, we will take  $a$  such that  $1/2 \leq a \leq a^*$ , so the Hausdorff dimension  $d$  of  $\Gamma^\infty$  is not smaller than 1. We recall the construction of the trace operator made in [2] by taking advantage of the self-similarity; this trace operator, called  $\ell^\infty$  below, is well defined even if  $a = a^*$ .

We first construct a sequence  $(\ell^n)_n$  of approximations of the trace operator: consider the sequence of linear operators  $\ell^n : W^{1,q}(\Omega) \rightarrow L_\mu^q$ ,

$$\ell^n(v) = \sum_{\sigma \in \mathcal{A}_n} \left( \frac{1}{|\Gamma^\sigma|} \int_{\Gamma^\sigma} v \, dx \right) \mathbf{1}_{f_\sigma(\Gamma^\infty)}, \quad (23)$$

where  $|\Gamma^\sigma|$  is the one-dimensional Lebesgue measure of  $\Gamma^\sigma$ .

**Proposition 4** *The sequence  $(\ell^n)_n$  converges in  $\mathcal{L}(W^{1,q}(\Omega), L_\mu^q)$  to an operator that we call  $\ell^\infty$ .*

**Proof.** See [2]. ■

**Remark 3** *For a given  $\theta$ ,  $0 < \theta < \pi/2$ , let  $(\alpha, \beta)$  satisfy Assumption 1 and  $\Omega$  be constructed as in § 2.2.1, with  $1/2 \leq a \leq a^*$ ; in a work in progress [1], we prove that  $\Omega$  is a 2-set as defined in e.g. [14] page 205, i.e. there exist three positive constants  $r_0$ ,  $c_1$  and  $c_2$  such that for any closed ball  $B(P, r)$ ,  $P \in \Omega$ ,  $0 < r \leq r_0$ ,  $c_1 r^2 \leq m_2(B(P, r) \cap \Omega) \leq c_2 r^2$ , where  $m_2$  is the Lebesgue measure in  $\mathbb{R}^2$ . Since  $\Omega$  is a 2-set, there is a classical definition of a trace operator on  $\partial\Omega$ , see for instance [14] page 206.*

*Although it has no bearing on the present paper, it is interesting to compare the operator  $\ell^\infty$ , whose construction is based on the self-similarity properties, with the latter classical trace operator. In [1], one of the goals is to prove that if  $q > 1$ , the two definitions of the trace of a function  $u \in W^{1,q}(\Omega)$  coincide  $\mu$ -almost everywhere. At the present stage of our investigations, we have proved the result for all  $\theta \in (0, \pi/2) \setminus \{\pi/2k, 0 < k \in \mathbb{N}\}$  and  $0 < a \leq a^*$ , (and also for all  $\theta = \pi/2k$ ,  $1 < k \in \mathbb{N}$  and  $0 < a < a^*$ ).*

## 4 The spaces $J\text{Lip}(t, q, q; 0; \Gamma^\infty)$ for $0 < t < 1$ and the trace theorem

In [13], A. Jonsson has introduced Haar wavelets of arbitrary order on self-similar fractal sets and has used these wavelets for constructing a family of Lipschitz spaces. These function spaces are named  $J\text{Lip}(t, p, q; m; \mathcal{S})$ , where  $\mathcal{S}$  is the fractal set,  $t$  is a nonnegative real number,  $p, q$  are two real numbers not smaller than 1 and  $m$  is an integer ( $m$  is the order of the Haar wavelets used for constructing the space). Here  $J$  stands for jumps, since the considered functions may jump at some points of  $\mathcal{S}$ . If the fractal set  $\mathcal{S}$  is totally disconnected, then these spaces coincide with the Lipschitz spaces  $\text{Lip}(t, p, q; m; \mathcal{S})$  also introduced in [13]. The latter are a generalization of the more classical spaces  $\text{Lip}(t, p, q; \mathcal{S})$  introduced in [14] since  $\text{Lip}(t, p, q; [t]; \mathcal{S}) = \text{Lip}(t, p, q; \mathcal{S})$ . Note that  $\text{Lip}(t, p, q; [t]; \mathcal{S}) = B_t^{p,q}(\mathcal{S})$ , see [15]. We will focus on the case when  $\mathcal{S} = \Gamma^\infty$ ,  $m = 0$  and  $p = q$ , since this is sufficient for what follows.

#### 4.1 Definition of $JLip(t, q, q; 0; \Gamma^\infty)$ for $0 < t < 1$ .

The definition of  $JLip(t, q, q; 0; \Gamma^\infty)$  presented below is adapted to the class of fractal sets  $\Gamma^\infty$  considered in the present paper. It was proved in [4] that this definition coincides with the original and more general one that was proposed in [13].

Consider a real number  $t$ ,  $0 < t < 1$ . Following [13], it is possible to characterize  $JLip(t, q, q; 0; \Gamma^\infty)$  by using expansions in the standard Haar wavelet basis on  $\Gamma^\infty$ . Consider the Haar mother wavelet  $g_0$  on  $\Gamma^\infty$ ,

$$g_0 = 1_{f_1(\Gamma^\infty)} - 1_{f_2(\Gamma^\infty)}, \quad (24)$$

and for  $n \in \mathbb{N}$ ,  $n > 0$ ,  $\sigma \in \mathcal{A}_n$ , let  $g_\sigma$  be given by

$$g_\sigma|_{\Gamma^\infty, \sigma} = 2^{n/2} g_0 \circ f_\sigma^{-1}, \quad \text{and} \quad g_\sigma|_{\Gamma^\infty \setminus \Gamma^\infty, \sigma} = 0. \quad (25)$$

It is proved in [12] §5 that a function  $f \in L_\mu^q$  can be expanded on the Haar basis as follows:

$$f = P_0 f + \beta_0 g_0 + \sum_{n \geq 1} \sum_{\sigma \in \mathcal{A}_n} \beta_\sigma g_\sigma,$$

where  $P_0 f = \int_{\Gamma^\infty} f d\mu$ . Let  $b_0$  be a real number and  $(b_\sigma)_{\sigma \in \mathcal{A}}$  be a sequence of real numbers; we define  $\|(b_0, (b_\sigma))\|_{b_t^{q,q}}$ :

$$\begin{aligned} \|(b_0, (b_\sigma))\|_{b_t^{q,q}} &= \left( |b_0|^q + \sum_{n=1}^{\infty} 2^{qtn/d} 2^{n(1/2-1/q)q} \sum_{\sigma \in \mathcal{A}_n} |b_\sigma|^q \right)^{\frac{1}{q}}. \\ &= \left( |b_0|^q + \sum_{n=1}^{\infty} a^{-qtn} 2^{n(1/2-1/q)q} \sum_{\sigma \in \mathcal{A}_n} |b_\sigma|^q \right)^{\frac{1}{q}}. \end{aligned} \quad (26)$$

**Definition 1** A function  $f \in L_\mu^q$  belongs to  $JLip(t, q, q; 0; \Gamma^\infty)$  if and only if the norm

$$\|f\|_{JLip(t, q, q; 0; \Gamma^\infty)} = |P_0 f| + |f|_{JLip(t, q, q; 0; \Gamma^\infty)} \quad (27)$$

is finite, where

$$|f|_{JLip(t, q, q; 0; \Gamma^\infty)} = \|(b_0, (\beta_\sigma))\|_{b_t^{q,q}}. \quad (28)$$

**Remark 4** An equivalent definition of  $JLip(t, q, q; 0; \Gamma^\infty)$  can be given using projection of  $f$  on constants on  $\Gamma^\infty, \sigma$ , see [13, 4].

If the fractal set  $\Gamma^\infty$  is totally disconnected, then  $JLip(t, q, q; 0; \Gamma^\infty)$  coincides with a more classical function space:

**Theorem 4** (Jonsson) If  $a < a^*$ , then  $f_1(\Gamma^\infty) \cap f_2(\Gamma^\infty)$  is empty and

$$JLip(t, q, q; 0; \Gamma^\infty) = Lip(t, q, q; 0; \Gamma^\infty) = B_t^{q,q}(\Gamma^\infty),$$

where the Lipschitz space  $Lip(t, q, q; 0; \Gamma^\infty)$  and the Sobolev space  $B_t^{q,q}(\Gamma^\infty)$  are defined in [14].

**Proof.** This is a particular case of Theorem 2 in [13], see also [12] for a partial proof.  $\blacksquare$

## 4.2 Characterization of the traces on $\Gamma^\infty$ of the function in $W^{1,q}(\Omega)$

The following theorem was proved in [4].

**Theorem 5** *For a given  $\theta$ ,  $0 < \theta < \pi/2$ , let  $(\alpha, \beta)$  satisfy Assumption 1 and  $\Omega$  be constructed as in § 2.2.1, with  $1/2 \leq a \leq a^*$ ; then for all  $q$ ,  $1 < q < \infty$ ,*

$$\ell^\infty(W^{1,q}(\Omega)) = J\text{Lip}(1 - \frac{2-d}{q}, q, q; 0; \Gamma^\infty). \quad (29)$$

A first consequence of Theorem 5 is that if  $1/2 \leq a < a^*$ , then  $d \geq 1$  and from Theorem 4,

$$\ell^\infty(W^{1,q}(\Omega)) = \text{Lip}(1 - \frac{2-d}{q}, q, q; 0; \Gamma^\infty) = B_{1-\frac{2-d}{q}}^{q,q}(\Gamma^\infty), \quad \forall q \in (1, +\infty). \quad (30)$$

**Remark 5** *Note that (30) has been proved in [3], without relying on the  $J\text{Lip}$  spaces: indeed  $\Omega$  is a  $\epsilon - \delta$  domain and  $\Gamma^\infty$  is a  $d$ -set; in this case, the extension result of Jones [11] (from  $W^{1,q}(\Omega)$  to  $W^{1,q}(\mathbb{R}^2)$ ) and the trace result of Jonsson and Wallin [14] (from  $W^{1,q}(\mathbb{R}^2)$  onto  $B_{1-\frac{2-d}{q}}^{q,q}(\Gamma^\infty)$ ) can be combined to obtain (30).*

In what follows, we will see that when  $a = a^*$ , then (30) does not hold for every  $q \in (1, +\infty)$ .

## 5 Embedding of the $J\text{Lip}$ spaces in Sobolev spaces for $a = a^*$

### 5.1 Main results

Since  $a = a^*$ , it is not possible to apply Theorem 4. Similarly,  $\Omega$  is not an  $\epsilon - \delta$  domain, so Jones extension result (from  $W^{1,q}(\Omega)$  to  $W^{1,q}(\mathbb{R}^2)$ ) does not hold for all  $q \in [1, +\infty)$ . Note that  $a = a^* > 1/2$  implies that  $d > 1$ . We are going to make out two cases: with  $\mathfrak{p}$  defined in (11), the simpler case is when  $\mathfrak{p}\theta > \pi/2$ , so  $\Xi^\infty$  is made of a single point; the case when  $\mathfrak{p}\theta = \pi/2$  will turn out to be more difficult because  $\Xi^\infty$  is not countable.

**Theorem 6** *Assume that  $a = a^*$  and  $\mathfrak{p}\theta > \pi/2$ .*

1. *For all  $t \in (0, 1)$  and  $s > \frac{d}{q}$ ,  $J\text{Lip}(t, q, q; 0; \Gamma^\infty) \not\subset B_s^{q,q}(\Gamma^\infty)$ .*
2. *If  $0 < t < \min(d/q, 1)$ , then  $J\text{Lip}(t, q, q; 0; \Gamma^\infty) = \text{Lip}(t, q, q; 0; \Gamma^\infty) = B_t^{q,q}(\Gamma^\infty)$ .*

The following corollary stems from Theorem 6:

**Corollary 1** *Assume that  $a = a^*$  and  $\mathfrak{p}\theta > \pi/2$ . For all  $q > d$  and  $t \in [d/q, 1)$ ,  $J\text{Lip}(t, q, q; 0; \Gamma^\infty) \subset B_s^{q,q}(\Gamma^\infty)$  with a continuous injection, for all  $s$ ,  $0 < s < \frac{d}{q}$ .*

As an easy consequence of Theorem 6, it is possible to find some relationships between  $J\text{Lip}(1 - \frac{2-d}{q}, q, q; 0; \Gamma^\infty)$ , the trace space of  $W^{1,q}(\Omega)$ , see (29), and some Sobolev spaces:

**Corollary 2** *Assume that  $a = a^*$  and  $\mathfrak{p}\theta > \pi/2$ .*

1. *If  $q \geq 2$ , then*
  - (a)  $J\text{Lip}(1 - \frac{2-d}{q}, q, q; 0; \Gamma^\infty) \subset B_s^{q,q}(\Gamma^\infty)$ , for all  $s$ ,  $0 < s < \frac{d}{q}$ .
  - (b)  $J\text{Lip}(1 - \frac{2-d}{q}, q, q; 0; \Gamma^\infty) \not\subset B_s^{q,q}(\Gamma^\infty)$ , for all  $s > \frac{d}{q}$ .

2. If  $1 \leq q < 2$ , then  $JLip(1 - \frac{2-d}{q}, q, q; 0; \Gamma^\infty) = Lip(1 - \frac{2-d}{q}, q, q; 0; \Gamma^\infty) = B_{1-\frac{2-d}{q}}^{q,q}(\Gamma^\infty)$ .

**Theorem 7** Assume that  $a = a^*$  and  $\mathfrak{p}\theta = \pi/2$ .

1. For all  $t \in (0, 1)$  and  $s > \frac{d}{2q}$ ,  $JLip(t, q, q; 0; \Gamma^\infty) \not\subset B_s^{q,q}(\Gamma^\infty)$ .

2. If  $0 < t < \min(d/(2q), 1)$ , then  $JLip(t, q, q; 0; \Gamma^\infty) = Lip(t, q, q; 0; \Gamma^\infty) = B_t^{q,q}(\Gamma^\infty)$ .

**Corollary 3** Assume that  $a = a^*$  and  $\mathfrak{p}\theta = \pi/2$ . For all  $q > d/2$  and  $t \in [d/(2q), 1)$ ,  $JLip(t, q, q; 0; \Gamma^\infty) \subset B_s^{q,q}(\Gamma^\infty)$  with a continuous injection, for all  $s$ ,  $0 < s < \frac{d}{2q}$ .

Here again, it is possible to find some relationships between the trace space  $JLip(1 - \frac{2-d}{q}, q, q; 0; \Gamma^\infty)$  and some Sobolev spaces:

**Corollary 4** Assume that  $a = a^*$  and  $\mathfrak{p}\theta = \pi/2$ .

1. If  $q \geq 2 - d/2$ , then

(a)  $JLip(1 - \frac{2-d}{q}, q, q; 0; \Gamma^\infty) \subset B_s^{q,q}(\Gamma^\infty)$ , for all  $s$ ,  $0 < s < \frac{d}{2q}$ .

(b)  $JLip(1 - \frac{2-d}{q}, q, q; 0; \Gamma^\infty) \not\subset B_s^{q,q}(\Gamma^\infty)$ , for all  $s > \frac{d}{2q}$ .

2. If  $1 \leq q < 2 - d/2$ , then  $JLip(1 - \frac{2-d}{q}, q, q; 0; \Gamma^\infty) = Lip(1 - \frac{2-d}{q}, q, q; 0; \Gamma^\infty) = B_{1-\frac{2-d}{q}}^{q,q}(\Gamma^\infty)$ .

Hereafter, when dealing with  $a = a^*$ , we will always write  $a$ .

## 5.2 Proof of Theorem 6

### 5.2.1 Geometrical lemmas

The proofs of the lemmas below are given in appendix.

For two subsets  $X$  and  $Y$  of  $\mathbb{R}^2$ , we define  $d(X, Y) = \inf_{x \in X, y \in Y} |y - x|$ . We will need to estimate  $d(\Gamma^{\infty, \sigma}, \Gamma^{\infty, \tau})$  for  $\sigma, \tau \in \mathcal{A}_n$ ,  $n \geq 1$ . We start by estimating the distance between  $\Gamma^\sigma$  and the horizontal line  $\mathcal{H}$  tangent to the upper part of  $\Gamma^\infty$ , i.e.  $\mathcal{H} = \{x : x_2 = h\}$ , where

$$h = \sup_{x \in \Omega} x_2 = \max_{x \in \Gamma^\infty} x_2 = \frac{\beta + a(\alpha \sin \theta + \beta \cos \theta)}{1 - a^2}. \quad (31)$$

**Lemma 1**

$$\sup_{x \in f_1 \circ f_1(\Omega)} x_2 = \sup_{x \in f_2 \circ f_2(\Omega)} x_2 < h.$$

**Lemma 2** Take  $n \geq 1$  and  $\sigma \in \mathcal{A}_n$ . Let  $k$  be the largest integer such that  $2k \leq n$  and for all  $j \in \{1, \dots, k\}$ ,  $\sigma(2j - 1) \neq \sigma(2j)$ . We have

$$d(\Gamma^{\infty, \sigma}, \mathcal{H}) + a^n \simeq a^{2k}. \quad (32)$$

**Remark 6** Note that Lemmas 1 and 2 hold if  $\mathfrak{p}\theta = \pi/2$ .

**Definition 2** Let us define the mapping  $\Pi : \mathcal{A} \rightarrow \mathbb{N}$  as follows : for  $\sigma \in \mathcal{A}_n$ ,  $n \geq 1$ ,

- if  $n < \mathfrak{p} + 4$  or  $\sigma_{\mathfrak{p}+2} \notin \{(1, 2, \dots, 2), (2, 1, \dots, 1)\}$ , then  $\Pi(\sigma) = 0$ ,

- *else,*

$$\begin{cases} \Pi(\sigma) = \max\{k \geq 0, \forall j \in \{1, \dots, k\}, \sigma(\mathbf{p} + 2j + 1) = 1 \text{ and } \sigma(\mathbf{p} + 2 + 2j) = 2\} & \text{if } \sigma(1) = 1, \\ \Pi(\sigma) = \max\{k \geq 0, \forall j \in \{1, \dots, k\}, \sigma(\mathbf{p} + 2j + 1) = 2 \text{ and } \sigma(\mathbf{p} + 2 + 2j) = 1\} & \text{if } \sigma(1) = 2. \end{cases}$$

Here, in other words,  $\Pi(\sigma)$  is the largest integer  $k \geq 0$  such that  $\mathbf{p} + 2 + 2k \leq n$  and  $f_{\sigma_{\mathbf{p}+2+2k}} = f_1 \circ f_2^{\mathbf{p}+1} \circ (f_1 \circ f_2)^k$  or  $f_{\sigma_{\mathbf{p}+2+2k}} = f_2 \circ f_1^{\mathbf{p}+1} \circ (f_2 \circ f_1)^k$ .

Therefore, if  $n < \mathbf{p} + 4$  then  $\Pi(\sigma) = 0$  and if  $n \geq \mathbf{p} + 4$ , then  $\Pi(\sigma)$  takes its values in  $\{0, \dots, [(n - \mathbf{p} - 2)/2]\}$ .

**Definition 3** For  $\sigma \in \mathcal{A}_n$ ,  $n \geq 1$ , we say that  $\Pi(\sigma)$  is maximal if  $n < \mathbf{p} + 4$  (in this case  $\Pi(\sigma) = 0$ ) or if  $n \geq \mathbf{p} + 4$  and  $\Pi(\sigma) = [(n - \mathbf{p} - 2)/2]$ .

The following lemma shows that the distance of  $\Gamma^{\infty, \sigma}$  to the vertical axis  $\Lambda = \{x : x_1 = 0\}$  can be estimated in terms of  $\Pi(\sigma)$ :

**Lemma 3** Take  $n \geq 1$  and  $\sigma \in \mathcal{A}_n$ ; for  $d_\sigma$  defined by

$$\begin{aligned} d_\sigma &= d(\Gamma^{\infty, \sigma}, \Lambda), & \text{if } \Pi(\sigma) \text{ is not maximal,} \\ d_\sigma &= d(\Gamma^{\infty, \sigma}, \Lambda) + a^n, & \text{if } \Pi(\sigma) \text{ is maximal,} \end{aligned} \quad (33)$$

we have

$$d_\sigma \simeq a^{2\Pi(\sigma)}. \quad (34)$$

**Lemma 4** Take  $n \geq 1$  and  $\sigma, \tau \in \mathcal{A}_n$  such that  $\sigma(1) \neq \tau(1)$ ; we have

$$d(\Gamma^{\infty, \sigma}, \Gamma^{\infty, \tau}) + a^n \simeq a^{2\min(\Pi(\sigma), \Pi(\tau))}. \quad (35)$$

**Remark 7** From Lemma 3, we also have that for all  $\sigma, \tau \in \mathcal{A}$  with  $\sigma(1) \neq \tau(1)$ ,

$$d(\Gamma^{\infty, \sigma}, \Gamma^{\infty, \tau}) \lesssim a^{2\min(\Pi(\sigma), \Pi(\tau))}.$$

**Definition 4** • Lemma 3 implies that there exists a positive constant  $c_1$ , such that, for all  $n \geq 1$ ,  $\sigma \in \mathcal{A}_n$  and  $x \in \Gamma^{\infty, \sigma}$ ,

$$c_1 a^{2\Pi(\sigma)} < d(x, \Lambda) \quad \text{if } \Pi(\sigma) \text{ is not maximal,} \quad (36)$$

and for all  $\eta \in \mathcal{A}$  such that  $\Pi(\eta) = 0$  and  $f_\eta$  is a similitude with rotation angle 0,

$$d(f_\eta(\Lambda), \Lambda) > c_1. \quad (37)$$

We must have  $c_1 < d(\Lambda, f_1 \circ f_2(\Lambda))$ , because  $\Pi((1, 2)) = 0$ .

- Let us define the positive number  $c_2 > 0$  by

$$c_2 = \frac{d(\Lambda, f_1 \circ f_2(\Lambda))}{a^2}. \quad (38)$$

Note that

$$\max_{x \in \Gamma^\infty} x_1 = \max_{x \in \Gamma^\infty} d(x, \Lambda) = \frac{d(\Lambda, f_1 \circ f_2(\Lambda))}{a^2} = c_2. \quad (39)$$

- Finally, from (34), we know that there exists a constant  $c_3$ , such that for all  $x \in \Gamma^{\infty, \sigma}$ ,  $\sigma \in \mathcal{A}_n$ ,

$$d(x, \Lambda) \leq c_3 a^{2\Pi(\sigma)}. \quad (40)$$

We must have  $c_3 \geq c_2$ .

From (36) and (40), we deduce that for all  $n \geq 1$ ,  $\sigma \in \mathcal{A}_n$  and  $x \in \Gamma^{\infty, \sigma}$ ,

$$\begin{cases} d(x, \Lambda) \leq c_3 a^{2\Pi(\sigma)}, & \text{if } \Pi(\sigma) \text{ is maximal,} \\ c_1 a^{2\Pi(\sigma)} < d(x, \Lambda) \leq c_3 a^{2\Pi(\sigma)}, & \text{if } \Pi(\sigma) \text{ is not maximal.} \end{cases} \quad (41)$$

**Lemma 5** For any  $\eta \in \mathcal{A}$  such that  $f_\eta$  is a similitude with rotation angle 0,

$$d(\Lambda, f_\eta(\Lambda)) > c_1 a^{2\Pi(\eta)}, \quad (42)$$

where  $c_1$  satisfies (36) and (37).

For what follows, we will need to partition  $f_1(\Gamma^\infty)$  into a sequence of subsets  $(X_i)_{i \in \mathbb{N}}$ . The measure of the set  $X_i \subset f_1(\Gamma^\infty)$  and its distance to the axis  $\Lambda$  will be decreasing as  $i$  grows. By similarity,  $(f_\eta(X_i))_{i \in \mathbb{N}}$  will be a partition of  $f_\eta \circ f_1(\Gamma^\infty)$ :

**Definition 5** • Let us define the subsets of  $\Gamma^\infty$ :

$$\begin{aligned} X_i &= \{x \in f_1(\Gamma^\infty), c_1 a^{2i} \leq d(x, \Lambda) < c_1 a^{2(i-1)}\}, \quad \forall i \geq 1, \\ X_0 &= \{x \in f_1(\Gamma^\infty), c_1 \leq d(x, \Lambda) \leq c_2\}, \end{aligned} \quad (43)$$

where  $c_1$  satisfies (36) and (37) and  $c_2$  is given by (38), see Figure 2. The

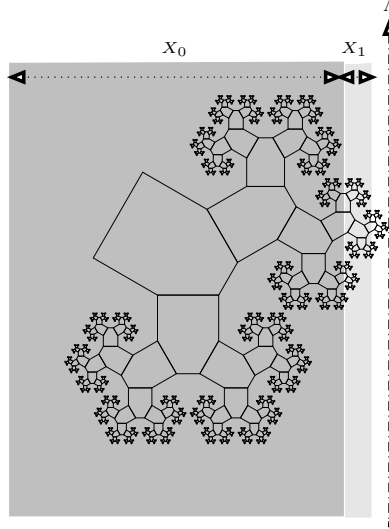


Figure 2:  $\theta = \pi/3$ : the domain  $f_1 \circ f_2^{p+1}(\Omega)$ , the fractal set  $f_1 \circ f_2^{p+1}(\Gamma^\infty)$  and parts of  $X_0$  and  $X_1$ :  $X_0$  (resp.  $X_1$ ) is the intersection of  $\Gamma^\infty$  with the dark grey half-plane (resp. light grey strip).

• For  $\ell \geq 0$ , we define the class  $\mathcal{Z}_\ell$  of subsets of  $\Gamma^\infty$ :

$$\mathcal{Z}_\ell = \{f_\eta(X_i), \eta \in \mathcal{A}_n, n + 2i = \ell\}. \quad (44)$$

• Let  $\phi \in [0, 2\pi)$  be such that there exists a similitude  $f_\sigma$ ,  $\sigma \in \mathcal{A}$ , with rotation angle  $\phi$ . For  $n \geq 0$ , we define  $\mathcal{A}_{n, \phi} = \{\eta \in \mathcal{A}_n, f_\eta \text{ is a similitude of angle } \phi\}$ , and the class  $\mathcal{Z}_{\ell, \phi}$  of subsets of  $\Gamma^\infty$ :

$$\mathcal{Z}_{\ell, \phi} = \{f_\eta(X_i), \eta \in \mathcal{A}_{n, \phi}, n + 2i = \ell\}. \quad (45)$$

**Lemma 6** For all  $i \geq 1$ ,

$$\mu(X_i) \lesssim 2^{-2i}. \quad (46)$$

**Remark 8** A direct consequence of Lemma 6 is that for all  $Y \in \mathcal{Z}_\ell$ ,  $\mu(Y) \lesssim 2^{-\ell}$ .

**Lemma 7** For all nonnegative integers  $n, m, i, j$  such that  $n + 2i = m + 2j$  and  $\eta \in \mathcal{A}_{n,\phi}$ ,  $\nu \in \mathcal{A}_{m,\phi}$ , the sets  $Y = f_\eta(X_i)$  and  $Z = f_\nu(X_j)$  are disjoint if  $(n, i, \eta) \neq (m, j, \nu)$ .

**Remark 9** We will see that when  $\mathfrak{p}\theta = \pi/2$ , the definition of  $\Pi$  differs, but once  $\Pi$  is defined, the definitions of  $c_1$  and  $c_2$  are the same. In that case, Lemma 5 and Lemma 7 are still true; by contrast, Lemma 6 does not hold, see Lemma 13.

**Lemma 8** If  $\mathfrak{p}\theta > \pi/2$ , then for any  $\ell \geq 0$ , any  $x \in \Gamma^\infty$ , there are at most a finite number of  $(i, \eta)$ ,  $0 \leq 2i \leq \ell$ ,  $\eta \in \mathcal{A}_{\ell-2i}$  such that  $x \in f_\eta(X_i)$ , and this number is independent of  $\ell$ .

**Remark 10** Although it seems clear that for a given  $Z \in \mathcal{Z}_\ell$ , there is a unique  $(i_Z, \eta_Z)$  such that  $0 \leq 2i_Z \leq \ell$ ,  $\eta_Z \in \mathcal{A}_{\ell-2i_Z}$  and  $Z = f_{\eta_Z}(X_{i_Z})$ , we have not found a short proof of this assertion. For what follows, it will be enough to use the following weaker result which stems from Lemma 8: there is at most a finite number of pairs  $(i, \eta)$  with  $0 \leq 2i \leq \ell$ ,  $\eta \in \mathcal{A}_{\ell-2i}$  and  $Z = f_\eta(X_i)$ , and this number is independent of  $Z$  and  $\ell$ .

### 5.2.2 Sobolev regularity of the Haar wavelet $g_0$

The following proposition will imply regularity results for the Haar wavelet  $g_0$ :

**Proposition 5** We have

$$\int_{f_1(\Gamma^\infty)} \int_{f_2(\Gamma^\infty)} \frac{1}{|x-y|^\gamma} d\mu(y) d\mu(x) < +\infty, \quad \text{if } 0 \leq \gamma < 2d, \quad (47)$$

$$\int_{f_1(\Gamma^\infty)} \int_{f_2(\Gamma^\infty)} \frac{1}{|x-y|^\gamma} d\mu(y) d\mu(x) = +\infty, \quad \text{if } \gamma > 2d. \quad (48)$$

**Proof.** Take  $n \geq \mathfrak{p} + 2$  and let  $\kappa$  be the largest integer such that  $n \geq \mathfrak{p} + 2 + 2\kappa$ . We have

$$\begin{aligned} \int_{f_1(\Gamma^\infty)} \int_{f_2(\Gamma^\infty)} \frac{d\mu(y) d\mu(x)}{(|x-y| + a^n)^\gamma} &= \sum_{\substack{\sigma \in \mathcal{A}_n \\ \sigma(1)=1}} \sum_{\substack{\tau \in \mathcal{A}_n \\ \tau(1)=2}} \int_{\Gamma^{\infty,\sigma}} \int_{\Gamma^{\infty,\tau}} \frac{d\mu(y) d\mu(x)}{(|x-y| + a^n)^\gamma} \\ &\simeq 2^{-2n} \sum_{\substack{\sigma \in \mathcal{A}_n \\ \sigma(1)=1}} \sum_{\substack{\tau \in \mathcal{A}_n \\ \tau(1)=2}} \frac{1}{(d(\Gamma^{\infty,\sigma}, \Gamma^{\infty,\tau}) + a^n)^\gamma}, \end{aligned}$$

because if  $x \in \Gamma^{\infty,\sigma}$  and  $y \in \Gamma^{\infty,\tau}$ , then  $|x-y| + a^n \simeq d(\Gamma^{\infty,\sigma}, \Gamma^{\infty,\tau}) + a^n$ . Thus, from Lemma 4, we have

$$\int_{f_1(\Gamma^\infty)} \int_{f_2(\Gamma^\infty)} \frac{d\mu(y) d\mu(x)}{(|x-y| + a^n)^\gamma} \simeq S_1 + S_2,$$

with

$$S_1 = 2^{-2n} \sum_{\substack{\sigma, \tau \in \mathcal{A}_n \\ \sigma(1)=1, \tau(1)=2 \\ \Pi(\sigma) \geq \Pi(\tau)}} \frac{1}{a^{2\gamma\Pi(\tau)}} \quad \text{and} \quad S_2 = 2^{-2n} \sum_{\substack{\sigma, \tau \in \mathcal{A}_n \\ \sigma(1)=1, \tau(1)=2 \\ \Pi(\sigma) < \Pi(\tau)}} \frac{1}{a^{2\gamma\Pi(\sigma)}}.$$



We can write  $S_1$  as follows:

$$S_1 = 2^{-2n} \sum_{0 \leq \ell \leq k \leq \kappa} \sum_{\substack{\sigma \in \mathcal{A}_n \\ \sigma(1)=1 \\ \Pi(\sigma)=k}} \sum_{\substack{\tau \in \mathcal{A}_n \\ \tau(1)=2 \\ \Pi(\tau)=\ell}} \frac{1}{a^{2\gamma\ell}}. \quad (49)$$

On the other hand, the number of  $\sigma \in \mathcal{A}_n$  such that  $\Pi(\sigma) = k$  is of the order of  $2^{n-2k}$ . Therefore, (49) leads to

$$S_1 \simeq 2^{-2n} \sum_{0 \leq \ell \leq k \leq \kappa} 2^{n-2k} 2^{n-2\ell} \frac{1}{a^{2\gamma\ell}} = \sum_{k=0}^{\kappa} 2^{-2k} \sum_{\ell=0}^k 2^{2\ell(-1+\frac{\gamma}{d})} \lesssim \sum_{k=0}^{\kappa} 2^{-2k} \max(k, 2^{2k(-1+\frac{\gamma}{d})}),$$

and  $S_1 \simeq \sum_{k=0}^{\kappa} 2^{2k(-2+\frac{\gamma}{d})}$  if  $\gamma > 2d$ . The same is true for  $S_2$ . Therefore, if  $\gamma > 2d$ , then

$$\int_{f_1(\Gamma^\infty)} \int_{f_2(\Gamma^\infty)} \frac{1}{(|x-y|+a^n)^\gamma} d\mu(y) d\mu(x) \simeq 2^{n(-2+\frac{\gamma}{d})}, \quad \xrightarrow{n \rightarrow \infty} +\infty,$$

which yields (48).

On the other hand, if  $\gamma < 2d$ , then

$$\int_{f_1(\Gamma^\infty)} \int_{f_2(\Gamma^\infty)} \frac{1}{(|x-y|+a^n)^\gamma} d\mu(y) d\mu(x) \lesssim \sum_{\ell=0}^{\infty} 2^{-2\ell} \max(\ell, 2^{2\ell(-1+\frac{\gamma}{d})}) < \infty,$$

which yields (47) from the monotone convergence theorem. ■

**Corollary 5** For any  $q, 1 \leq q < \infty$ ,  $g_0 \in B_s^{q,q}(\Gamma^\infty)$  if  $0 \leq s < \frac{d}{q}$  and  $g_0 \notin B_s^{q,q}(\Gamma^\infty)$  if  $s > \frac{d}{q}$ .

**Proof.** The result follows from the identity  $|g_0/2|_{B_s^{q,q}(\Gamma^\infty)}^q = \int_{f_1(\Gamma^\infty)} \int_{f_2(\Gamma^\infty)} \frac{1}{|x-y|^{d+qs}} d\mu(y) d\mu(x)$  and from Proposition 5. ■

### 5.2.3 Two lemmas

**Lemma 9 (discrete Hardy inequalities, [14], page 121, Lemma 3)** For any  $\gamma \in \mathbb{R}$ , any  $p \geq 1$  there exists a constant  $C$  such that, for any sequence of positive real numbers  $(c_k)_{k \in \mathbb{N}}$ ,

$$\sum_{n \in \mathbb{N}} 2^{\gamma n} \left( \sum_{k \leq n} c_k \right)^p \leq C \sum_{n \in \mathbb{N}} 2^{\gamma n} c_n^p \quad \text{if } \gamma < 0, \quad (50)$$

$$\sum_{n \in \mathbb{N}} 2^{\gamma n} \left( \sum_{k \geq n} c_k \right)^p \leq C \sum_{n \in \mathbb{N}} 2^{\gamma n} c_n^p \quad \text{if } \gamma > 0. \quad (51)$$

**Lemma 10** For any  $\gamma > d$ , we have

$$\int_{f_1(\Gamma^\infty)} \int_{f_2(\Gamma^\infty)} \frac{|v(x)|^q}{|x-y|^\gamma} d\mu(y) d\mu(x) \lesssim \int_{f_1(\Gamma^\infty)} \frac{|v(x)|^q}{d(x, \Lambda)^{\gamma-d}} d\mu(x), \quad \forall v \in L^q(\Gamma^\infty).$$

**Proof.** For any  $n \geq \mathfrak{p} + 2$ , let  $\kappa$  be the largest integer such that  $n \geq \mathfrak{p} + 2 + 2\kappa$ .

$$\begin{aligned}
& \int_{f_1(\Gamma^\infty)} \int_{f_2(\Gamma^\infty)} \frac{|v(x)|^q}{(|x-y|+a^n)^\gamma} d\mu(y) d\mu(x) \\
& \leq \sum_{\substack{\sigma, \tau \in \mathcal{A}_n \\ \sigma(1)=1, \tau(1)=2}} \int_{\Gamma^\infty, \sigma} \int_{\Gamma^\infty, \tau} \frac{|v(x)|^q}{(d(\Gamma^\infty, \sigma, \Gamma^\infty, \tau) + a^n)^\gamma} d\mu(y) d\mu(x) \\
& = 2^{-n} \sum_{\substack{\sigma, \tau \in \mathcal{A}_n \\ \sigma(1)=1, \tau(1)=2}} \frac{1}{(d(\Gamma^\infty, \sigma, \Gamma^\infty, \tau) + a^n)^\gamma} \int_{\Gamma^\infty, \sigma} |v(x)|^q d\mu(x) \lesssim S_1 + S_2
\end{aligned}$$

where, from Lemma 4,

$$\begin{aligned}
S_1 &= 2^{-n} \sum_{0 \leq \ell \leq k \leq \kappa} \sum_{\substack{\sigma \in \mathcal{A}_n \\ \sigma(1)=1 \\ \Pi(\sigma)=k}} \sum_{\substack{\tau \in \mathcal{A}_n \\ \tau(1)=2 \\ \Pi(\tau)=\ell}} \frac{1}{a^{2\gamma\ell}} \int_{\Gamma^\infty, \sigma} |v(x)|^q d\mu(x), \\
S_2 &= 2^{-n} \sum_{0 \leq k \leq \ell \leq \kappa} \sum_{\substack{\sigma \in \mathcal{A}_n \\ \sigma(1)=1 \\ \Pi(\sigma)=k}} \sum_{\substack{\tau \in \mathcal{A}_n \\ \tau(1)=2 \\ \Pi(\tau)=\ell}} \frac{1}{a^{2\gamma k}} \int_{\Gamma^\infty, \sigma} |v(x)|^q d\mu(x).
\end{aligned}$$

Since the number of  $\tau \in \mathcal{A}_n$  such that  $\tau(1) = 2$  et  $\Pi(\tau) = \ell$  is of the order of  $2^{n-2\ell}$ , we have

$$\begin{aligned}
S_1 &\simeq \sum_{0 \leq \ell \leq k \leq \kappa} \frac{2^{-2\ell}}{a^{2\gamma\ell}} \sum_{\substack{\sigma \in \mathcal{A}_n \\ \sigma(1)=1 \\ \Pi(\sigma)=k}} \int_{\Gamma^\infty, \sigma} |v(x)|^q d\mu(x) \\
&= \sum_{k=0}^{\kappa} \left( \sum_{\ell=0}^k \frac{1}{a^{2\ell(\gamma-d)}} \right) \sum_{\substack{\sigma \in \mathcal{A}_n \\ \sigma(1)=1 \\ \Pi(\sigma)=k}} \int_{\Gamma^\infty, \sigma} |v(x)|^q d\mu(x) \\
&\simeq \sum_{k=0}^{\kappa} \sum_{\substack{\sigma \in \mathcal{A}_n \\ \sigma(1)=1 \\ \Pi(\sigma)=k}} \int_{\Gamma^\infty, \sigma} \frac{|v(x)|^q}{a^{2k(\gamma-d)}} d\mu(x) \simeq \int_{f_1(\Gamma^\infty)} \frac{|v(x)|^q}{(d(x, \Lambda) + a^n)^{\gamma-d}} d\mu(x),
\end{aligned}$$

from Lemma 3. Similarly,  $S_2 \simeq \int_{f_1(\Gamma^\infty)} \frac{|v(x)|^q}{(d(x, \Lambda) + a^n)^{\gamma-d}} d\mu(x)$ .

Finally, we obtain the desired estimate by having  $n$  tend to  $\infty$  and using the monotone convergence theorem. ■

#### 5.2.4 Proof of Theorem 6

**Proof of 1)** Point 1) in Theorem 6 stems from Corollary 5 and from the fact that the wavelet  $g_0$  belongs to  $JLip(t, q, q; 0; \Gamma^\infty)$  for all  $q, t$ ,  $1 \leq q < \infty$ ,  $0 < t < 1$ .

**Proof of 2)** Consider  $t$ ,  $0 < t < \min(d/q, 1)$ .

$$\begin{aligned}
& \int_{\Gamma^\infty} \int_{\Gamma^\infty} \frac{|v(x) - v(y)|^q}{|x-y|^{d+qt}} d\mu(x) d\mu(y) - \sum_{i=1}^2 \int_{f_i(\Gamma^\infty) \times f_i(\Gamma^\infty)} \frac{|v(x) - v(y)|^q}{|x-y|^{d+qt}} d\mu(x) d\mu(y) \\
& \lesssim I_1 + I_2 + I_3
\end{aligned}$$

where

$$I_1 = 2 \int_{x \in f_1(\Gamma^\infty)} \int_{y \in f_2(\Gamma^\infty)} \frac{|v(x) - \langle v \rangle_{f_1(\Gamma^\infty)}|^q}{|x - y|^{d+qt}} d\mu(y) d\mu(x), \quad (52)$$

$$I_2 = 2 |\langle v \rangle_{f_2(\Gamma^\infty)} - \langle v \rangle_{f_1(\Gamma^\infty)}|^q \int_{f_1(\Gamma^\infty)} \int_{f_2(\Gamma^\infty)} \frac{1}{|x - y|^{d+qt}} d\mu(y) d\mu(x), \quad (53)$$

$$I_3 = 2 \int_{x \in f_1(\Gamma^\infty)} \int_{y \in f_2(\Gamma^\infty)} \frac{|\langle v \rangle_{f_2(\Gamma^\infty)} - v(y)|^q}{|x - y|^{d+qt}} d\mu(x) d\mu(y). \quad (54)$$

By iterating this argument and using Fatou lemma, we obtain that

$$\int_{\Gamma^\infty} \int_{\Gamma^\infty} \frac{|v(x) - v(y)|^q}{|x - y|^{d+qt}} d\mu(x) d\mu(y) \lesssim I_1 + I_2 + I_3 + \sum_{n \geq 1} \sum_{\eta \in \mathcal{A}_n} (I_{1,\eta} + I_{2,\eta} + I_{3,\eta})$$

where

$$I_{1,\eta} = 2 \int_{x \in f_\eta \circ f_1(\Gamma^\infty)} \int_{y \in f_\eta \circ f_2(\Gamma^\infty)} \frac{|v(x) - \langle v \rangle_{f_\eta \circ f_1(\Gamma^\infty)}|^q}{|x - y|^{d+qt}} d\mu(y) d\mu(x), \quad (55)$$

$$I_{2,\eta} = 2 |\langle v \rangle_{f_\eta \circ f_2(\Gamma^\infty)} - \langle v \rangle_{f_\eta \circ f_1(\Gamma^\infty)}|^q \int_{f_\eta \circ f_1(\Gamma^\infty)} \int_{f_\eta \circ f_2(\Gamma^\infty)} \frac{1}{|x - y|^{d+qt}} d\mu(y) d\mu(x), \quad (56)$$

$$I_{3,\eta} = 2 \int_{x \in f_\eta \circ f_1(\Gamma^\infty)} \int_{y \in f_\eta \circ f_2(\Gamma^\infty)} \frac{|\langle v \rangle_{f_\eta \circ f_2(\Gamma^\infty)} - v(y)|^q}{|x - y|^{d+qt}} d\mu(x) d\mu(y). \quad (57)$$

Let us estimate  $I_1 + \sum_{n \geq 1} \sum_{\eta \in \mathcal{A}_n} I_{1,\eta}$ : the change of variables  $x = f_\eta(x')$  and  $y = f_\eta(y')$  yields

$$I_{1,\eta} = 2a^{n(d-qt)} \int_{x' \in f_1(\Gamma^\infty)} \int_{y' \in f_2(\Gamma^\infty)} \frac{|v \circ f_\eta(x') - \langle v \circ f_\eta \rangle_{f_1(\Gamma^\infty)}|^q}{|x' - y'|^{d+qt}} d\mu(y') d\mu(x').$$

From Lemma 10,  $I_{1,\eta} \lesssim a^{n(d-qt)} \int_{x' \in f_1(\Gamma^\infty)} \frac{|v \circ f_\eta(x') - \langle v \circ f_\eta \rangle_{f_1(\Gamma^\infty)}|^q}{d(x', \Lambda)^{qt}} d\mu(x')$ . Let  $\beta_0, (\beta_\sigma)_{\sigma \in \mathcal{A}}$

be the coefficients in the Haar basis of  $v$ :  $v = P_0 v + \beta_0 g_0 + \sum_{k \geq 1} \sum_{\sigma \in \mathcal{A}_k} \beta_\sigma g_\sigma$ . Note that for any

$\eta \in \mathcal{A}_n$ ,  $v \circ f_\eta - \langle v \circ f_\eta \rangle_{\Gamma^\infty} = 2^{\frac{n}{2}} (\beta_\eta g_0 + \sum_{k \geq 1} \sum_{\sigma \in \mathcal{A}_k} \beta_{\eta+\sigma} g_\sigma)$ , where  $\eta + \sigma \in \mathcal{A}_{n+k}$  is the sequence  $(\eta(1), \dots, \eta(n), \sigma(1), \dots, \sigma(k))$ . Thus,

$$\begin{aligned} I_{1,\eta} &\lesssim a^{n(d-qt)} 2^{\frac{nq}{2}} \int_{x \in f_1(\Gamma^\infty)} \frac{\left| \sum_{k \geq 0} \sum_{\sigma \in \mathcal{A}_k} \beta_{\eta+\sigma} g_\sigma(x) \right|^q}{d(x, \Lambda)^{qt}} d\mu(x) \\ &\simeq \sum_{i \geq 0} 2^{n(\frac{q}{2} + \frac{qt}{d} - 1)} \int_{X_i} a^{-2iqt} \left| \sum_{k \geq 0} \sum_{\sigma \in \mathcal{A}_k} \beta_{\eta+\sigma} g_\sigma(x) \right|^q d\mu(x), \end{aligned}$$

where  $X_i$  is defined in (43).

We are led to estimate

$$I_1 + \sum_{n \geq 1} \sum_{\eta \in \mathcal{A}_n} I_{1,\eta} \lesssim \sum_{n \geq 0} \sum_{\eta \in \mathcal{A}_n} \sum_{i \geq 0} 2^{n(\frac{q}{2} + \frac{qt}{d} - 1)} a^{-2iqt} \int_{X_i} \left| \sum_{k \geq 0} \sum_{\sigma \in \mathcal{A}_k} \beta_{\eta+\sigma} g_\sigma(x) \right|^q d\mu(x) \lesssim S_1 + S_2, \quad (58)$$

where

$$S_1 = \sum_{n \geq 0} \sum_{\eta \in \mathcal{A}_n} \sum_{i \geq 0} 2^{n(\frac{q}{2} + \frac{qt}{d} - 1)} a^{-2iqt} \int_{X_i} \left| \sum_{k \leq 2i} \sum_{\sigma \in \mathcal{A}_k} \beta_{\eta + \sigma} g_{\sigma}(x) \right|^q d\mu(x), \quad (59)$$

$$S_2 = \sum_{n \geq 0} \sum_{\eta \in \mathcal{A}_n} \sum_{i \geq 0} 2^{n(\frac{q}{2} + \frac{qt}{d} - 1)} a^{-2iqt} \int_{X_i} \left| \sum_{k > 2i} \sum_{\sigma \in \mathcal{A}_k} \beta_{\eta + \sigma} g_{\sigma}(x) \right|^q d\mu(x), \quad (60)$$

with the convention that if  $n = 0$ , then  $\mathcal{A}_n = \{0\}$ ,  $f_0 = Id$  and  $0 + \sigma = \sigma$ . It is convenient to rewrite  $S_2$  as follows:

$$S_2 = \sum_{n \geq 0} \sum_{\eta \in \mathcal{A}_n} \sum_{i \geq 0} a^{-(2i+n)qt} \int_{f_{\eta}(X_i)} \left| \sum_{k > 2i} \sum_{\sigma \in \mathcal{A}_k, \sigma(1)=1} \beta_{\eta + \sigma} g_{\eta + \sigma}(x) \right|^q d\mu(x). \quad (61)$$

We have

$$S_2 = \sum_{\ell \geq 0} a^{-\ell qt} \sum_{i=0}^{\lfloor \ell/2 \rfloor} \sum_{\eta \in \mathcal{A}_{\ell-2i}} \int_{f_{\eta}(X_i)} \left| \sum_{k > 2i} \sum_{\sigma \in \mathcal{A}_k, \sigma(1)=1} \beta_{\eta + \sigma} g_{\eta + \sigma}(x) \right|^q d\mu(x).$$

From the definition of  $\mathcal{Z}_{\ell}$  in (44),

$$S_2 \lesssim \sum_{\ell \geq 0} 2^{\ell \frac{qt}{d}} \sum_{Y \in \mathcal{Z}_{\ell}} \int_Y \left| \sum_{k \geq \ell} \sum_{\nu \in \mathcal{A}_k} \beta_{\nu} g_{\nu}(x) \right|^q d\mu(x) \quad (62)$$

**Remark 11** Note that in (62), the sign  $\lesssim$  has been used instead of  $=$ , because we did not prove that there exists a unique pair  $(i, \eta)$  such that  $Y \in \mathcal{Z}_{\ell}$  coincide with  $f_{\eta}(X_i)$ , but only that the number of such pairs is bounded, see Remark 10.

Then from a triangle inequality,

$$\begin{aligned} S_2 &\lesssim \sum_{\ell \geq 0} 2^{\ell \frac{qt}{d}} \left( \sum_{k \geq \ell} \left( \sum_{Y \in \mathcal{Z}_{\ell}} \int_Y \left| \sum_{\nu \in \mathcal{A}_k} \beta_{\nu} g_{\nu}(x) \right|^q d\mu(x) \right)^{\frac{1}{q}} \right)^q = \\ &= \sum_{\ell \geq 0} 2^{\ell \frac{qt}{d}} \left( \sum_{k \geq \ell} \left( \sum_{\nu \in \mathcal{A}_k} |\beta_{\nu}|^q \sum_{Y \in \mathcal{Z}_{\ell}} \int_Y |g_{\nu}(x)|^q d\mu(x) \right)^{\frac{1}{q}} \right)^q \lesssim \sum_{\ell \geq 0} 2^{\ell \frac{qt}{d}} \left( \sum_{k \geq \ell} \left( \sum_{\nu \in \mathcal{A}_k} |\beta_{\nu}|^q 2^{-k} 2^{q \frac{k}{2}} \right)^{\frac{1}{q}} \right)^q. \end{aligned}$$

The latter inequality comes from Lemma 8, because any point  $x \in \Gamma^{\infty}$  belongs to at most a finite number of sets  $Y \in \mathcal{Z}_{\ell}$  (this number is independent of  $\ell$ ).

Hardy inequality (51) in Lemma 9 can be used because  $\frac{qt}{d} > 0$ : this yields

$$S_2 \lesssim \sum_{\ell \geq 0} 2^{\ell(\frac{qt}{d} + \frac{q}{2} - 1)} \sum_{\nu \in \mathcal{A}_{\ell}} |\beta_{\nu}|^q \lesssim |v|_{JLip(t, q, q; 0; \Gamma^{\infty})}^q. \quad (63)$$

Let us turn to  $S_1$  defined in (59): we have, using a triangle inequality,

$$\begin{aligned} S_1 &\leq \sum_{i \geq 0} 2^{2i \frac{qt}{d}} \left( \sum_{k \leq 2i} \left( \sum_{n \geq 0} \sum_{\eta \in \mathcal{A}_n} 2^{n(\frac{q}{2} + \frac{qt}{d} - 1)} \int_{X_i} \left| \sum_{\sigma \in \mathcal{A}_k} \beta_{\eta + \sigma} g_{\sigma}(x) \right|^q d\mu(x) \right)^{\frac{1}{q}} \right)^q \\ &= \sum_{i \geq 0} 2^{2i \frac{qt}{d}} \left( \sum_{k \leq 2i} \left( \sum_{n \geq 0} \sum_{\eta \in \mathcal{A}_n} 2^{n(\frac{q}{2} + \frac{qt}{d} - 1)} \sum_{\sigma \in \mathcal{A}_k} |\beta_{\eta + \sigma}|^q \int_{X_i} |g_{\sigma}(x)|^q d\mu(x) \right)^{\frac{1}{q}} \right)^q, \end{aligned}$$

because the supports of  $g_\sigma$ ,  $\sigma \in \mathcal{A}_k$  are disjoint (up to a negligible set). This implies that

$$S_1 \lesssim \sum_{i \geq 0} 2^{2i \frac{qt}{d}} \left( \sum_{k \leq 2i} \left( \sum_{n \geq 0} \sum_{\eta \in \mathcal{A}_n} 2^{n(\frac{q}{2} + \frac{qt}{d} - 1)} 2^{k \frac{q}{2}} \sum_{\sigma \in \mathcal{A}_k} |\beta_{\eta+\sigma}|^q \mu(\text{supp } g_\sigma \cap X_i) \right)^{\frac{1}{q}} \right)^q.$$

From Definition 3, Lemma 3, if  $\Pi(\sigma)$  is not maximal, then  $i > 1$  and  $d(\Gamma^{\infty, \sigma}, \Lambda) > c_1 a^{2\Pi(\sigma)} > c_1 a^{2(i-1)}$  thanks to Definition 5, and  $\mu(\text{supp } g_\sigma \cap X_i) = 0$ . Hence, if  $P_k = \{\sigma \in \mathcal{A}_k, \sigma(1) = 1, \Pi(\sigma) \text{ maximal}\}$ ,

$$\begin{aligned} S_1 &\lesssim \sum_{n \geq 0} \sum_{\eta \in \mathcal{A}_n} 2^{n(\frac{q}{2} + \frac{qt}{d} - 1)} |\beta_\eta|^q + \sum_{i \geq 1} 2^{2i(\frac{qt}{d} - 1)} \left( \sum_{k \leq 2i} \left( \sum_{n \geq 0} \sum_{\eta \in \mathcal{A}_n} 2^{n(\frac{q}{2} + \frac{qt}{d} - 1)} 2^{k \frac{q}{2}} \sum_{\sigma \in P_k} |\beta_{\eta+\sigma}|^q \right)^{\frac{1}{q}} \right)^q \\ &\leq \sum_{n \geq 0} \sum_{\eta \in \mathcal{A}_n} 2^{n(\frac{q}{2} + \frac{qt}{d} - 1)} |\beta_\eta|^q + \sum_{j \geq 1} 2^{j(\frac{qt}{d} - 1)} \left( \sum_{k \leq j} \left( \sum_{n \geq 0} \sum_{\eta \in \mathcal{A}_n} 2^{n(\frac{q}{2} + \frac{qt}{d} - 1)} 2^{k \frac{q}{2}} \sum_{\sigma \in P_k} |\beta_{\eta+\sigma}|^q \right)^{\frac{1}{q}} \right)^q \\ &\lesssim \sum_{n \geq 0} \sum_{\eta \in \mathcal{A}_n} 2^{n(\frac{q}{2} + \frac{qt}{d} - 1)} |\beta_\eta|^q + \sum_{j \geq 1} \sum_{n \geq 0} 2^{(n+j)(\frac{q}{2} + \frac{qt}{d} - 1)} \sum_{\eta \in \mathcal{A}_n} \sum_{\sigma \in P_j} |\beta_{\eta+\sigma}|^q, \end{aligned}$$

by Hardy's inequality (50) in Lemma 9, because  $qt < d$ . For all  $\nu \in \mathcal{A}$ , there exist at most  $N = \mathfrak{p} + 4$  pairs  $(\eta, \sigma)$ ,  $\eta, \sigma \in \mathcal{A}$  such that  $\nu = \eta + \sigma$  and  $\Pi(\sigma)$  is maximal. Therefore, for all  $\nu \in \mathcal{A}$ ,  $\beta_\nu$  appears in the latter sum at most  $N$  times. Hence,

$$S_1 \lesssim \sum_{m \geq 0} 2^{m(\frac{q}{2} + \frac{qt}{d} - 1)} \sum_{\nu \in \mathcal{A}_m} |\beta_\nu|^q \lesssim |v|_{JLip(t, q, q; 0; \Gamma^\infty)}. \quad (64)$$

From the bounds (63) and (64), we immediately deduce that

$$I_1 + \sum_{n \geq 1} \sum_{\eta \in \mathcal{A}_n} I_{1, \eta} \lesssim |v|_{JLip(t, q, q; 0; \Gamma^\infty)}^q, \quad (65)$$

and the same argument shows that

$$I_3 + \sum_{n \geq 1} \sum_{\eta \in \mathcal{A}_n} I_{3, \eta} \lesssim |v|_{JLip(t, q, q; 0; \Gamma^\infty)}^q. \quad (66)$$

We are left with estimating  $I_2 + \sum_{n \geq 1} \sum_{\eta \in \mathcal{A}_n} I_{2, \eta}$ . From (47) in Proposition 6 and easy scaling arguments,

$$\int_{f_\eta \circ f_1(\Gamma^\infty)} \int_{f_\eta \circ f_2(\Gamma^\infty)} \frac{1}{|x - y|^{d+qt}} d\mu(y) d\mu(x) \lesssim 2^{n(\frac{qt}{d} - 1)}, \quad \forall \eta \in \mathcal{A}_n.$$

On the other hand,  $|\langle v \rangle_{f_\eta \circ f_2(\Gamma^\infty)} - \langle v \rangle_{f_\eta \circ f_1(\Gamma^\infty)}| = 2^{\frac{n}{2} + 1} |\beta_\eta|$ . Combining these two observations, we have that for all  $t < \min(d/q, 1)$ ,

$$I_2 + \sum_{n \geq 1} \sum_{\eta \in \mathcal{A}_n} I_{2, \eta} \lesssim \sum_{n \geq 0} 2^{n(\frac{qt}{d} - 1)} 2^{\frac{qn}{2}} \sum_{\eta \in \mathcal{A}_n} |\beta_\eta|^q \lesssim |v|_{JLip(t, q, q; 0; \Gamma^\infty)}^q. \quad (67)$$

From (65), (66), (67), we obtain the desired result.  $\blacksquare$

### 5.3 Proof of Theorem 7

We now consider the case when  $\mathfrak{p}\theta = \pi/2$ , with  $\mathfrak{p}$  defined in (11). The situation is more complex because  $\Xi^\infty$  is a non countable set whose Hausdorff dimension is  $d/2$ .

#### 5.3.1 Geometrical lemmas

We state several useful geometrical lemmas whose proofs are given in appendix. Here, we define the mapping  $\Pi : \mathcal{A} \rightarrow \mathbb{N}$  as follows:

**Definition 6** • if  $\sigma \in \mathcal{A}_n$  with  $n < \mathfrak{p} + 4$  then  $\Pi(\sigma) = 0$ ,

• if  $n \geq \mathfrak{p} + 4$  and  $\sigma_{\mathfrak{p}+2} \notin \{(1, 2, \dots, 2), (2, 1, \dots, 1)\}$ , then  $\Pi(\sigma) = 0$ ,

• else,  $\Pi(\sigma) = \max \left\{ k \geq 0, \forall j \in \{1, \dots, k\}, \sigma(\mathfrak{p} + 2j + 1) = \overline{\sigma(\mathfrak{p} + 2(j + 1))} \right\}$ .

In other words, with  $F_1$  and  $F_2$  defined in (13),  $\Pi(\sigma)$  is the largest integer  $k \geq 0$  such that  $f_{\sigma_{2(k+1)+\mathfrak{p}}} = f_1 \circ f_2^{\mathfrak{p}+1} \circ F_{\tau(1)} \circ \dots \circ F_{\tau(k)}$  or  $f_{\sigma_{2(k+1)+\mathfrak{p}}} = f_2 \circ f_1^{\mathfrak{p}+1} \circ F_{\tau(1)} \circ \dots \circ F_{\tau(k)}$  for some  $\tau \in \mathcal{A}_k$ .

If  $n < \mathfrak{p} + 4$  then  $\Pi(\sigma) = 0$  and if  $n \geq \mathfrak{p} + 4$ , then  $\Pi(\sigma)$  takes its values in  $\{0, \dots, [(n - \mathfrak{p} - 2)/2]\}$ . For  $\sigma \in \mathcal{A}_n$ ,  $n \geq 1$ , we say that  $\Pi(\sigma)$  is maximal if  $n < \mathfrak{p} + 4$  (in this case  $\Pi(\sigma) = 0$ ) or if  $n \geq \mathfrak{p} + 4$  and  $\Pi(\sigma) = [(n - \mathfrak{p} - 2)/2]$ .

One can estimate the distance of  $\Gamma^{\infty, \sigma}$  to  $\Lambda$  as a function of  $\Pi(\sigma)$ :

**Lemma 11** Take  $n \geq 1$  and  $\sigma \in \mathcal{A}_n$ ; with  $d_\sigma$  defined in (33), we have

$$d_\sigma \simeq a^{2\Pi(\sigma)}.$$

Estimating the distance  $d(\Gamma^{\infty, \sigma}, \Gamma^{\infty, \tau})$  for  $\sigma, \tau \in \mathcal{A}_n$   $\sigma(1) = 1$ ,  $\tau(1) = 2$  must be done more carefully than in the case when  $\mathfrak{p}\theta > \pi/2$ : indeed, in the present case, the quantity  $\max(d(\Gamma^{\infty, \sigma}, \Lambda), d(\Gamma^{\infty, \tau}, \Lambda))$  is too coarse an underestimate of  $d(\Gamma^{\infty, \sigma}, \Gamma^{\infty, \tau})$ , because  $\Gamma^{\infty, \sigma}$  and  $\Gamma^{\infty, \tau}$  may touch  $\Lambda$  without facing each other. This is why we have to make the following definition:

**Definition 7** For any  $n \geq \mathfrak{p} + 2$  and any  $k \geq 0$  such that  $\mathfrak{p} + 2 + 2k \leq n$ , let  $\mathcal{P}_n^k$  be the set containing all the pairs  $(\sigma, \tau)$  such that

$$\begin{cases} \sigma \in \mathcal{A}_n \quad \text{and} \quad \tau \in \mathcal{A}_n, \\ \sigma(1) = 1 \quad \text{and} \quad \tau(1) = 2, \\ k \text{ is the largest integer such that } \exists \eta \in \mathcal{A}_k \text{ with } \sigma_{\mathfrak{p}+2+2k} = \eta^{(1)} \text{ and } \tau_{\mathfrak{p}+2+2k} = \eta^{(2)}, \end{cases} \quad (68)$$

where  $\eta^{(1)}$  and  $\eta^{(2)}$  are defined by (14) or (15).

For example, take  $\sigma = (1, \underbrace{2, \dots, 2}_{\mathfrak{p}+1}, \underbrace{1, 2}_{\mathfrak{p}+1}, \underbrace{2, 1}_{\mathfrak{p}+1}, 2, 1, 1, 2)$  and  $\tau = (2, \underbrace{1, \dots, 1}_{\mathfrak{p}+1}, \underbrace{2, 1}_{\mathfrak{p}+1}, \underbrace{1, 2}_{\mathfrak{p}+1}, 2, 1, 2, 2)$ .

We have  $(\sigma, \tau) \in \mathcal{P}_{\mathfrak{p}+10}^2$ , with  $\eta = (1, 2)$  in (68).

**Lemma 12** For any  $n \geq \mathfrak{p} + 2$  and  $k \geq 0$  such that  $\mathfrak{p} + 2 + 2k \leq n$ , for any  $(\sigma, \tau) \in \mathcal{P}_n^k$ ,

$$d(\Gamma^{\infty, \sigma}, \Gamma^{\infty, \tau}) + a^n \simeq a^{2k}. \quad (69)$$

Finally, as in § 5.2.1, there exist two positive constants  $c_1 \leq c_2$  such that (41) holds for all  $n \geq 1$ ,  $\sigma \in \mathcal{A}_n$  and  $x \in \Gamma^{\infty, \sigma}$ ; the following lemma should be compared to Lemma 6.

**Lemma 13** For all integers  $i \geq 1$ , the sets  $X_i$  defined in (43) are such that

$$\mu(X_i) \lesssim 2^{-i}. \quad (70)$$

**Remark 12** It can be seen that the set  $X_i$  is made of  $O(2^i)$  disjoint connected components whose measure is of the order of  $2^{-2i}$ .

### 5.3.2 Sobolev regularity of the Haar wavelet $g_0$

The following proposition, which should be compared to Proposition 5, will imply regularity results for the Haar wavelet  $g_0$ :

**Proposition 6** We have

$$\int_{f_1(\Gamma^\infty)} \int_{f_2(\Gamma^\infty)} \frac{1}{|x-y|^\gamma} d\mu(y) d\mu(x) < \infty, \quad \text{if } \gamma < \frac{3d}{2}, \quad (71)$$

$$\int_{f_1(\Gamma^\infty)} \int_{f_2(\Gamma^\infty)} \frac{1}{|x-y|^\gamma} d\mu(y) d\mu(x) = \infty, \quad \text{if } \gamma > \frac{3d}{2}. \quad (72)$$

**Proof.** For any  $n \geq \mathbf{p} + 2$ , let  $\kappa$  be the largest integer such that  $\mathbf{p} + 2 + 2\kappa \leq n$ . We have

$$\begin{aligned} \int_{f_1(\Gamma^\infty)} \int_{f_2(\Gamma^\infty)} \frac{d\mu(y) d\mu(x)}{(|x-y| + a^n)^\gamma} &= \sum_{\substack{\sigma \in \mathcal{A}_n \\ \sigma(1)=1}} \sum_{\substack{\tau \in \mathcal{A}_n \\ \tau(1)=2}} \int_{\Gamma^{\infty, \sigma}} \int_{\Gamma^{\infty, \tau}} \frac{1}{(|x-y| + a^n)^\gamma} d\mu(y) d\mu(x) \\ &\simeq \sum_{\substack{\sigma \in \mathcal{A}_n \\ \sigma(1)=1}} \sum_{\substack{\tau \in \mathcal{A}_n \\ \tau(1)=2}} \frac{1}{2^{2n}} \cdot \frac{1}{(d(\Gamma^{\infty, \sigma}, \Gamma^{\infty, \tau}) + a^n)^\gamma} \\ &\simeq \sum_{k=0}^{\kappa} \sum_{(\sigma, \tau) \in \mathcal{P}_n^k} \frac{1}{2^{2n}} \cdot \frac{1}{a^{2k\gamma}}, \end{aligned}$$

from Lemma 12. It is easy to see that  $\mathcal{P}_n^k$  has  $2^k \cdot 2^{n-2k} \cdot 2^{n-2k} = 2^{2n-3k}$  elements. Therefore,

$$\int_{f_1(\Gamma^\infty)} \int_{f_2(\Gamma^\infty)} \frac{1}{(|x-y| + a^n)^\gamma} d\mu(y) d\mu(x) \simeq \sum_{k=0}^{\kappa} 2^k \left(\frac{2\gamma}{d} - 3\right).$$

Thus

$$\begin{aligned} \int_{f_1(\Gamma^\infty)} \int_{f_2(\Gamma^\infty)} \frac{1}{(|x-y| + a^n)^\gamma} d\mu(y) d\mu(x) &\lesssim \sum_{k=0}^{\infty} 2^k \left(\frac{2\gamma}{d} - 3\right) < \infty, & \text{if } \gamma < \frac{3d}{2}, \\ \int_{f_1(\Gamma^\infty)} \int_{f_2(\Gamma^\infty)} \frac{1}{(|x-y| + a^n)^\gamma} d\mu(y) d\mu(x) &\gtrsim 2^{\frac{n}{2} \left(\frac{2\gamma}{d} - 3\right)} \xrightarrow{n \rightarrow \infty} \infty, & \text{if } \gamma > \frac{3d}{2}, \end{aligned}$$

and the result follows by the monotone convergence theorem. ■

The following should be compared to Corollary 5:

**Corollary 6** For any  $q$ ,  $1 \leq q < \infty$ ,  $g_0 \in B_s^{q,q}(\Gamma^\infty)$  if  $0 \leq s < \frac{d}{2q}$  and  $g_0 \notin B_s^{q,q}(\Gamma^\infty)$  if  $s > \frac{d}{2q}$ .

### 5.3.3 Two Lemmas

**Lemma 14** For all  $\gamma > d$ , we have

$$\int_{f_1(\Gamma^\infty)} \int_{f_2(\Gamma^\infty)} \frac{|v(x)|^q}{|x-y|^\gamma} d\mu(y) d\mu(x) \lesssim \int_{f_1(\Gamma^\infty)} \frac{|v(x)|^q}{d(x, \Lambda)^{\gamma-d}} d\mu(x), \quad \forall v \in L^q_\mu.$$

**Proof.** Consider  $n \geq \mathfrak{p} + 2$ . For any  $\sigma \in \mathcal{A}_n$ , define  $\mathcal{P}_\sigma^k = \{\tau \in \mathcal{A}_n, (\sigma, \tau) \in \mathcal{P}_n^k\}$  (see Definition 7). Lemma 12 implies that

$$\begin{aligned} & \int_{f_1(\Gamma^\infty)} \int_{f_2(\Gamma^\infty)} \frac{|v(x)|^q}{(|x-y| + a^n)^\gamma} d\mu(y) d\mu(x) \\ & \leq \sum_{\substack{\sigma \in \mathcal{A}_n \\ \sigma(1)=1}} \sum_{\substack{\tau \in \mathcal{A}_n \\ \tau(1)=2}} \int_{\Gamma^{\infty, \sigma}} \int_{\Gamma^{\infty, \tau}} \frac{|v(x)|^q}{(d(\Gamma^{\infty, \sigma}, \Gamma^{\infty, \tau}) + a^n)^\gamma} d\mu(y) d\mu(x) \\ & \simeq 2^{-n} \sum_{\substack{\sigma \in \mathcal{A}_n \\ \sigma(1)=1}} \sum_{k=0}^{\Pi(\sigma)} \sum_{\tau \in \mathcal{P}_\sigma^k} \frac{1}{a^{2\gamma k}} \int_{\Gamma^{\infty, \sigma}} |v(x)|^q d\mu(x). \end{aligned}$$

It is easy to see that since  $\sigma \in \mathcal{A}_n$ ,  $\mathcal{P}_\sigma^k$  has  $2^{n-2k}$  elements. Therefore,

$$\begin{aligned} \int_{f_1(\Gamma^\infty)} \int_{f_2(\Gamma^\infty)} \frac{|v(x)|^q}{(|x-y| + a^n)^\gamma} d\mu(y) d\mu(x) & \lesssim \sum_{\substack{\sigma \in \mathcal{A}_n \\ \sigma(1)=1}} \left( \sum_{k=0}^{\Pi(\sigma)} \frac{2^{-2k}}{a^{2\gamma k}} \right) \int_{\Gamma^{\infty, \sigma}} |v(x)|^q d\mu(x) \\ & \simeq \sum_{\substack{\sigma \in \mathcal{A}_n \\ \sigma(1)=1}} \frac{1}{a^{2\Pi(\sigma)(\gamma-d)}} \int_{\Gamma^{\infty, \sigma}} |v(x)|^q d\mu(x) \\ & \simeq \sum_{\substack{\sigma \in \mathcal{A}_n \\ \sigma(1)=1}} \int_{\Gamma^{\infty, \sigma}} \frac{|v(x)|^q}{(d(x, \Lambda) + a^n)^{\gamma-d}} d\mu(x) \\ & = \int_{f_1(\Gamma^\infty)} \frac{|v(x)|^q}{(d(x, \Lambda) + a^n)^{\gamma-d}} d\mu(x). \end{aligned}$$

The desired result is obtained by letting  $n$  tend to  $\infty$ , by monotone convergence.  $\blacksquare$

**Remark 13** Although the statements of Lemma 10 and 14 are similar, the proofs differ.

We define  $X_i$  by (43) where  $c_1$  is the constant appearing in (41).

**Lemma 15** For any  $i \geq 1$ ,  $k \geq 1$  and  $\sigma \in \mathcal{A}_k$ ,

$$\int_{X_i} |g_\sigma|^q d\mu \lesssim 2^{\frac{k}{2}(q-1)} 2^{-i}, \quad \text{if } k < 2i, \quad (73)$$

$$\int_{X_i} |g_\sigma|^q d\mu \lesssim 2^{k(\frac{q}{2}-1)}, \quad \text{if } k \geq 2i, \quad (74)$$

**Proof.** Assume that  $k < 2i$ . It is easy to see that for  $\sigma \in \mathcal{A}_1$ ,  $\int_{X_i} |g_\sigma|^q d\mu \leq 2^{\frac{q}{2}} \mu(X_i)$ ; this is exactly (73) for  $k = 1$ . If  $k \geq 2$ , let  $\sigma \in \mathcal{A}_k$  and  $\tau \in \mathcal{A}_k$  be such that  $\sigma \neq \tau$ ,  $\Pi(\sigma)$  and  $\Pi(\tau)$  are maximal; then  $\Gamma^{\infty, \sigma}$  and  $\Gamma^{\infty, \tau}$  can be obtained from each other by a translation with a vertical vector (parallel to  $\Lambda$ ). Hence,

$$\Pi(\sigma) \text{ and } \Pi(\tau) \text{ are maximal} \quad \Rightarrow \quad \mu(X_i \cap \Gamma^{\infty, \sigma}) = \mu(X_i \cap \Gamma^{\infty, \tau}). \quad (75)$$



Moreover, since  $\mu(\Gamma^{\infty,\tau} \cap \Gamma^{\infty,\sigma}) = 0$  for  $\sigma, \tau \in \mathcal{A}_k$ , we get

$$\mu(X_i) \geq \sum_{\substack{\tau \in \mathcal{A}_k \\ \Pi(\tau) \text{ maximal}}} \mu(X_i \cap \Gamma^{\infty,\tau}). \quad (76)$$

On the other hand, if  $\tau \in \mathcal{A}_k$  and  $\Pi(\tau)$  is not maximal, then  $\Pi(\tau) \leq k/2 - 1$  and, from Lemma 11,

$$d(\Gamma^{\infty,\tau}, \Lambda) \geq c_1 a^{2\Pi(\tau)} \geq c_1 a^{(k-2)} > c_1 a^{2(i-1)}.$$

Therefore, if  $\tau \in \mathcal{A}_k$  and  $\Pi(\tau)$  is not maximal, then  $X_i \cap \Gamma^{\infty,\tau} = \emptyset$ . The latter observation, (75) and (76) imply that for any  $\sigma \in \mathcal{A}_k$ ,  $\mu(X_i \cap \Gamma^{\infty,\sigma}) \lesssim 2^{-k/2} \mu(X_i) \lesssim 2^{-i-k/2}$ , from Lemma 13. Therefore,

$$\int_{X_i} |g_\sigma|^q d\mu = 2^{\frac{kq}{2}} \mu(X_i \cap \text{Supp}(g_\sigma)) = 2^{\frac{kq}{2}} \mu(X_i \cap \Gamma^{\infty,\sigma}) \lesssim 2^{\frac{k}{2}(q-1)-i},$$

and we have proved (73). On the other hand,

$$\int_{X_i} |g_\sigma|^q d\mu \leq 2^{\frac{kq}{2}} \mu(\text{Supp}(g_\sigma)) = 2^{k(\frac{q}{2}-1)},$$

and we have proved (74). ■

### 5.3.4 Proof of Theorem 7

**Proof of 1.)** The result stems from the fact that  $g_0 \in J\text{Lip}(t, q, q; 0; \Gamma^\infty)$  and from Corollary 6.

**Proof of 2.)** Exactly as in the proof of Theorem 6,

$$\int_{\Gamma^\infty} \int_{\Gamma^\infty} \frac{|v(x) - v(y)|^q}{|x - y|^{d+qt}} d\mu(x) d\mu(y) \lesssim I_1 + I_2 + I_3 + \sum_{n \geq 1} \sum_{\eta \in \mathcal{A}_n} (I_{1,\eta} + I_{2,\eta} + I_{3,\eta})$$

where  $I_1, I_2, I_3, I_{1,\eta}, I_{2,\eta}$  and  $I_{3,\eta}$  are respectively given by (52) (53) (54) (55) (56) (57). As above, we get that  $I_1 + \sum_{n \geq 1} \sum_{\eta \in \mathcal{A}_n} I_{1,\eta} \lesssim S_1 + S_2$ , where  $S_1$  and  $S_2$  are given by (59) and (60).

Let us first find a bound on  $S_1$ : exactly as in the proof of Theorem 6, we see that

$$S_1 \lesssim \sum_{n \geq 0} \sum_{\eta \in \mathcal{A}_n} 2^{n(\frac{q}{2} + \frac{qt}{d} - 1)} |\beta_\eta|^q + \sum_{i \geq 1} 2^{2i \frac{qt}{d}} \left( \sum_{k \leq 2i} \left( \sum_{n \geq 0} \sum_{\eta \in \mathcal{A}_n} 2^{n(\frac{q}{2} + \frac{qt}{d} - 1)} \sum_{\sigma \in P_k} |\beta_{\eta+\sigma}|^q \int_{X_i} |g_\sigma|^q d\mu \right)^{\frac{1}{q}} \right)^q,$$

where  $P_k = \{\sigma \in \mathcal{A}_k, \sigma(1) = 1, \Pi(\sigma) \text{ maximal}\}$ . Thus, from (73),

$$\begin{aligned} S_1 &\lesssim \sum_{n \geq 0} \sum_{\eta \in \mathcal{A}_n} 2^{n(\frac{q}{2} + \frac{qt}{d} - 1)} |\beta_\eta|^q + \sum_{i \geq 1} 2^{2i \frac{qt}{d}} \left( \sum_{k \leq 2i} \left( \sum_{n \geq 0} \sum_{\eta \in \mathcal{A}_n} 2^{n(\frac{q}{2} + \frac{qt}{d} - 1)} 2^{\frac{k}{2}(q-1)} 2^{-i} \sum_{\sigma \in P_k} |\beta_{\eta+\sigma}|^q \right)^{\frac{1}{q}} \right)^q \\ &\leq \sum_{n \geq 0} \sum_{\eta \in \mathcal{A}_n} 2^{n(\frac{q}{2} + \frac{qt}{d} - 1)} |\beta_\eta|^q + \sum_{j \geq 1} 2^{j(\frac{qt}{d} - 1/2)} \left( \sum_{k \leq j} \left( \sum_{n \geq 0} \sum_{\eta \in \mathcal{A}_n} 2^{n(\frac{q}{2} + \frac{qt}{d} - 1)} 2^{\frac{k}{2}(q-1)} \sum_{\sigma \in P_k} |\beta_{\eta+\sigma}|^q \right)^{\frac{1}{q}} \right)^q \\ &\lesssim \sum_{n \geq 0} \sum_{\eta \in \mathcal{A}_n} 2^{n(\frac{q}{2} + \frac{qt}{d} - 1)} |\beta_\eta|^q + \sum_{j \geq 1} \sum_{n \geq 0} 2^{(n+j)(\frac{q}{2} + \frac{qt}{d} - 1)} \sum_{\eta \in \mathcal{A}_n} \sum_{\sigma \in P_j} |\beta_{\eta+\sigma}|^q, \end{aligned}$$

by Hardy's inequality (50) in Lemma 9. For all  $\nu \in \mathcal{A}$ , there exist at most  $N = \mathbf{p} + 4$  pairs  $(\eta, \sigma)$ ,  $\eta, \sigma \in \mathcal{A}$  such that  $\nu = \eta + \sigma$  and  $\Pi(\sigma)$  is maximal. Therefore, for all  $\nu \in \mathcal{A}$ ,  $\beta_\nu$  appears in the latter sum at most  $N$  times. Hence,

$$S_1 \lesssim \sum_{m \geq 0} 2^{m(\frac{q}{2} + \frac{qt}{d} - 1)} \sum_{\nu \in \mathcal{A}_m} |\beta_\nu|^q \lesssim |v|_{JLip(t,q,q;0;\Gamma^\infty)}^q. \quad (77)$$

We now consider  $S_2$ . Since  $\pi/\theta = 2\mathbf{p}$  is an integer, the rotation angles of the similitudes  $f_\eta$  can take only a finite number of values in  $[0, 2\pi)$ . Call  $\Theta$  the finite set of all possible angles:  $\Theta = \{i\theta, 0 \leq i < 4\mathbf{p}\}$ . It is convenient to split  $S_2$  as follows

$$S_2 = \sum_{\phi \in \Theta} S_{2,\phi}, \text{ with } S_{2,\phi} = \sum_{n \geq 0} \sum_{\eta \in \mathcal{A}_n, \phi} \sum_{i \geq 0} a^{-(2i+n)qt} \int_{f_\eta(X_i)} \left| \sum_{k > 2i} \sum_{\sigma \in \mathcal{A}_k, \sigma(1)=1} \beta_{\eta+\sigma} g_{\eta+\sigma}(x) \right|^q d\mu(x).$$

We have

$$S_{2,\phi} = \sum_{\ell \geq 0} a^{-\ell qt} \sum_{i=0}^{[\ell/2]} \sum_{\eta \in \mathcal{A}_{\ell-2i, \phi}} \int_{f_\eta(X_i)} \left| \sum_{k > 2i} \sum_{\sigma \in \mathcal{A}_k, \sigma(1)=1} \beta_{\eta+\sigma} g_{\eta+\sigma}(x) \right|^q d\mu(x).$$

We can rewrite  $S_{2,\phi}$  as follows:

$$S_{2,\phi} = \sum_{\ell \geq 0} 2^{\ell \frac{qt}{d}} \sum_{Y \in \mathcal{Z}_{\ell, \phi}} \int_Y \left| \sum_{k \geq \ell} \sum_{\nu \in \mathcal{A}_k} \beta_\nu g_\nu(x) \right|^q d\mu(x). \quad (78)$$

Thus, by the triangle inequality and the fact that the supports of  $g_\nu$ ,  $\nu \in \mathcal{A}_k$  are disjoint,

$$S_{2,\phi} \leq \sum_{\ell \geq 0} 2^{\ell \frac{qt}{d}} \left( \sum_{k \geq \ell} \left( \sum_{\nu \in \mathcal{A}_k} |\beta_\nu|^q \sum_{Y \in \mathcal{Z}_{\ell, \phi}} \int_Y |g_\nu(x)|^q d\mu(x) \right)^{\frac{1}{q}} \right)^q.$$

From Remark 9, for all  $\ell \geq 0$  and  $\phi \in \Theta$ , the sets  $Y \in \mathcal{Z}_{\ell, \phi}$  are disjoint. Therefore,

$$S_{2,\phi} \lesssim \sum_{\ell \geq 0} 2^{\ell \frac{qt}{d}} \left( \sum_{k \geq \ell} \left( \sum_{\nu \in \mathcal{A}_k} |\beta_\nu|^q 2^{-k} 2^{q \frac{k}{2}} \right)^{\frac{1}{q}} \right)^q,$$

because any  $x \in \Gamma^\infty$  belongs to at most one set  $Y \in \mathcal{Z}_{\ell, \phi}$ .

Hardy inequality (51) in Lemma 9 can be used because  $\frac{qt}{d} > 0$ : this yields

$$S_{2,\phi} \lesssim \sum_{\ell \geq 0} 2^{\ell(\frac{qt}{d} + \frac{q}{2} - 1)} \sum_{\nu \in \mathcal{A}_\ell} |\beta_\nu|^q \lesssim |v|_{JLip(t,q,q;0;\Gamma^\infty)}^q.$$

Since this is true for all  $\phi \in \Theta$  and since  $\Theta$  is a finite set, we get  $S_2 \lesssim |v|_{JLip(t,q,q;0;\Gamma^\infty)}^q$ . From this and (77), we immediately deduce (65) and the same argument yields (66). The conclusion of the proof is identical as that of Theorem 6. ■

**Remark 14** For  $s > 1 - \frac{2-d}{q}$ ,  $q < 2$ , it is interesting to construct a function  $u \in W^{1,q}(\Omega)$  whose trace  $\ell^\infty(u)$  does not belong to  $B_s^{q,q}(\Gamma^\infty)$ . One can take the following example: let  $\chi \in W^{1,q}(Y^0)$  be such that  $\chi|_{\Gamma^0} = 0$ ,  $\chi|_{f_1(\Gamma^0)} = 1$  and  $\chi|_{f_2(\Gamma^0)} = 0$ . For  $\rho > 0$ , we build  $u$  by the following iterative process:

- $u|_{Y^0} = \chi$ ;
- let the polygonal open domain  $Y^n$  be obtained by stopping the construction at step  $n + 1$ :  $Y^n = \text{Interior} \left( K^0 \cup \left( \bigcup_{p=1}^n \bigcup_{\sigma \in \mathcal{A}_p} f_\sigma(K^0) \right) \right)$ . Let us also introduce  $Y^\sigma = f_\sigma(Y^0)$ . If  $u$  is already defined in  $Y^{n-1}$ , we define  $u|_{Y^\sigma}$ ,  $\sigma \in \mathcal{A}_n$  as follows:

$$\begin{cases} u|_{Y^\sigma} = 1 + \rho u|_{Y^{n-1}} \circ f_{\sigma(n)}^{-1} & \text{if } \Pi(\sigma) \text{ is maximal,} \\ u|_{Y^\sigma} = \gamma^\sigma & \text{otherwise,} \end{cases}$$

where  $\gamma^\sigma = u|_{\Gamma^\sigma}$  for  $\sigma \in \mathcal{A}$  (note that the function  $u$  is constant on the lines  $\Gamma^\sigma$ ).

It is possible to prove that if  $\rho = 2^{\frac{2-q}{dq} - \frac{1}{2q} - \epsilon}$ , for  $\epsilon > 0$ , then  $u \in W^{1,q}(\Omega)$  and that for any  $s > 1 - \frac{2-d}{q}$ , one may choose  $\epsilon$  small enough such that  $\ell^\infty(u) \notin B_s^{q,q}(\Gamma^\infty)$ .

## A Proofs of the geometrical lemmas in the case $\mathfrak{p}\theta > \pi/2$

**Proof of Lemma 1** We have

$$\sup_{x \in f_1 \circ f_1(\Omega)} x_2 = \sup_{x \in f_2 \circ f_2(\Omega)} x_2 = \sup_{x \in f_1 \circ f_1 \circ f_2 \circ f_2(Y^0)} x_2 + a^4 h.$$

On the other hand, with  $F_1$  defined in (13),

$$h = \sup_{x \in F_1 \circ F_1(Y^0)} x_2 + a^4 h.$$

Easy algebra shows that

$$f_1^2 \circ f_2^2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\alpha + a(-\alpha \cos \theta - \beta \sin \theta) + a^2(\alpha \cos 2\theta - \beta \sin 2\theta) + a^3(\alpha \cos \theta - \beta \sin \theta) + a^4 x_1 \\ \beta + a(-\alpha \sin \theta + \beta \cos \theta) + a^2(\alpha \sin 2\theta + \beta \cos 2\theta) + a^3(\alpha \sin \theta + \beta \cos \theta) + a^4 x_2 \end{pmatrix},$$

and that

$$F_1 \circ F_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\alpha + a(\alpha \cos \theta - \beta \sin \theta) - a^2 \alpha + a^3(\alpha \cos \theta - \beta \sin \theta) + a^4 x_1 \\ \beta + a(\alpha \sin \theta + \beta \cos \theta) + a^2 \beta + a^3(\alpha \sin \theta + \beta \cos \theta) + a^4 x_2 \end{pmatrix}.$$

Thus, the desired result will be a consequence of the inequality

$$-\alpha \sin \theta + \beta \cos \theta + a(\alpha \sin 2\theta + \beta \cos 2\theta) + a^2(\alpha \sin \theta + \beta \cos \theta) < \alpha \sin \theta + \beta \cos \theta + a\beta + a^2(\alpha \sin \theta + \beta \cos \theta)$$

which is true, since  $\alpha(-2 \sin \theta + a \sin 2\theta) + a\beta(\cos 2\theta - 1) < 0$ . ■

**Proof of Lemma 2** From the definition of  $k$  in the statement of Lemma 2,

- if  $k = [n/2]$  (where we denote by  $[z]$  the integer part of  $z$ ), then at least one of the two points  $\lim_{m \rightarrow \infty} f_{\sigma_{2k}} \circ F_1^m(O)$ ,  $\lim_{m \rightarrow \infty} f_{\sigma_{2k}} \circ F_2^m(O)$  belongs to  $\Gamma^{\infty, \sigma}$ , which implies that  $d(\Gamma^{\infty, \sigma}, \mathcal{H}) = 0$ . This implies that  $d(\Gamma^{\infty, \sigma}, \mathcal{H}) + a^n \simeq a^{2k}$ .
- otherwise, let us define  $i = \sigma(2k + 1) = \sigma(2k + 2)$ , and  $\sigma' = \sigma_{2k+2}$ . We have by self-similarity that

$$d(f_{\sigma'}(\Omega), \mathcal{H}) = a^{2k} d(f_i \circ f_i(\Omega), \mathcal{H}) > 0,$$

by Lemma 1. Hence,

$$d(\Gamma^{\infty,\sigma}, \mathcal{H}) \geq d(f_{\sigma'}(\Omega), \mathcal{H}) = a^{2k} d(f_i \circ f_i(\Omega), \mathcal{H}).$$

On the other hand,

$$d(\Gamma^{\infty,\sigma}, \mathcal{H}) \leq d(f_{\sigma'}(\Omega), \mathcal{H}) + \text{Diam}(f_{\sigma'}(\Omega)) = a^{2k} (d(f_i \circ f_i(\Omega), \mathcal{H}) + a^2 \text{Diam}(\Omega)).$$

We have proved that  $d(\Gamma^{\infty,\sigma}, \mathcal{H}) \simeq a^{2k}$ , which implies (32). ■

**Proof of Lemma 3** We may suppose that  $n \geq \mathfrak{p} + 4$ . Let  $\kappa$  be the largest integer such that  $\mathfrak{p} + 2 + 2\kappa \leq n$ . Take  $\sigma \in \mathcal{A}_n$ .

If  $\Pi(\sigma) = \kappa$ , then  $d(\Gamma^{\infty,\sigma}, \Lambda) = 0$ , which yields (34).

Suppose that  $\Pi(\sigma) < \kappa$ .

Suppose first that  $\sigma_{\mathfrak{p}+2} \notin \{(1, 2, \dots, 2), (2, 1, \dots, 1)\}$ . Without restriction, we can also suppose that  $\sigma(1) = 1$ : there exists  $j$ ,  $0 \leq j \leq \mathfrak{p}$  such that  $f_{\sigma_{j+2}} = f_1 \circ f_2^j \circ f_1$ . Thus  $d(\Gamma^{\infty,\sigma}, \Lambda) \geq d(f_1 \circ f_2^j \circ f_1(\Omega), \Lambda)$ . From §2.1.2, we know that  $c \equiv \min_{0 \leq j \leq \mathfrak{p}+1} d((f_1 \circ f_2^j \circ f_1(\Omega), \Lambda) > 0$ . We have that  $d(\Gamma^{\infty,\sigma}, \Lambda) \geq c$ . On the other hand  $d(\Gamma^{\infty,\sigma}, \Lambda) \leq \text{Diam}(\Omega)$ . We have obtained (34).

We are left with considering the case when  $\Pi(\sigma) < \kappa$  and  $\sigma_{\mathfrak{p}+2} \in \{(1, 2, \dots, 2), (2, 1, \dots, 1)\}$ . Without restriction, we can also suppose that  $\sigma(1) = 1$ : there exists  $\tau \in \mathcal{A}_{n-\mathfrak{p}-2}$  and  $\tau' \in \mathcal{A}_{n-\mathfrak{p}-1}$  such that  $\Gamma^{\infty,\sigma} = f_1 \circ f_2^{\mathfrak{p}+1}(\Gamma^{\infty,\tau}) = f_1 \circ f_2^{\mathfrak{p}}(\Gamma^{\infty,\tau'})$ . We have that

$$\sigma = 1 \ 2 \ \dots \ 2 \ \underbrace{\underbrace{1 \ 2 \ \dots \ 1 \ 2}_{\tau'} \ i \ j \ \dots}_{\tau},$$

with  $i = \sigma(\mathfrak{p} + 3 + 2\pi(\sigma))$  and  $j = \sigma(\mathfrak{p} + 4 + 2\pi(\sigma))$ . The definition of  $\Pi(\sigma)$  implies that  $(i, j) \neq (1, 2)$ . We have obviously that

$$d(\Gamma^{\infty,\sigma}, f_1 \circ f_2^{\mathfrak{p}+1}(\mathcal{H})) = a^{\mathfrak{p}+2} d(\Gamma^{\infty,\tau}, \mathcal{H}), \quad \text{and} \quad d(\Gamma^{\infty,\sigma}, f_1 \circ f_2^{\mathfrak{p}}(\mathcal{H})) = a^{\mathfrak{p}+1} d(\Gamma^{\infty,\tau'}, \mathcal{H}). \quad (79)$$

Three observations will prove useful, see Figure 3 : a) by self similarity, the set  $f_1(\Gamma^{\infty})$  lies on one side of the straight lines  $f_1 \circ f_2^{\mathfrak{p}}(\mathcal{H})$  and  $f_1 \circ f_2^{\mathfrak{p}+1}(\mathcal{H})$ , whose intersection is  $\Xi^{\infty}$ . b) It is also easy to see that the line  $f_1 \circ f_2^{\mathfrak{p}}(\mathcal{H})$  makes an angle of  $\frac{\pi}{2} - (\mathfrak{p} - 1)\theta > 0$  with  $\Lambda$ . c) Similarly, the line  $f_1 \circ f_2^{\mathfrak{p}+1}(\mathcal{H})$  makes an angle of  $\mathfrak{p}\theta - \frac{\pi}{2} > 0$  with  $\Lambda$ . An elementary geometrical argument combining points a), b) and c), leads to

$$\begin{aligned} d(\Gamma^{\infty,\sigma}, \Lambda) &\geq \sin\left(\frac{\pi}{2} - (\mathfrak{p} - 1)\theta\right) d(\Gamma^{\infty,\sigma}, f_1 \circ f_2^{\mathfrak{p}+1}(\mathcal{H})) \\ d(\Gamma^{\infty,\sigma}, \Lambda) &\geq \sin\left(\mathfrak{p}\theta - \frac{\pi}{2}\right) d(\Gamma^{\infty,\sigma}, f_1 \circ f_2^{\mathfrak{p}}(\mathcal{H})). \end{aligned} \quad (80)$$

The geometrical argument for the first inequality in (80) is summarized in the right part of Figure 3.

We make out two cases:

- If  $i = j$  then  $\Pi(\sigma) = \max\{m \geq 0 : \forall \ell, 1 \leq \ell \leq m, \tau(2\ell - 1) \neq \tau(2\ell)\}$ . From Lemma 2 and since  $\Pi(\sigma) < \kappa$ , we have that  $d(\Gamma^{\infty,\tau}, \mathcal{H}) \gtrsim a^{2\Pi(\sigma)}$ . Thus

$$d(\Gamma^{\infty,\sigma}, f_1 \circ f_2^{\mathfrak{p}+1}(\mathcal{H})) \gtrsim a^{\mathfrak{p}+2+2\Pi(\sigma)}. \quad (81)$$

Combining the first inequality in (80) and (81) yields that  $d(\Gamma^{\infty,\sigma}, \Lambda) \gtrsim a^{2\Pi(\sigma)}$ .

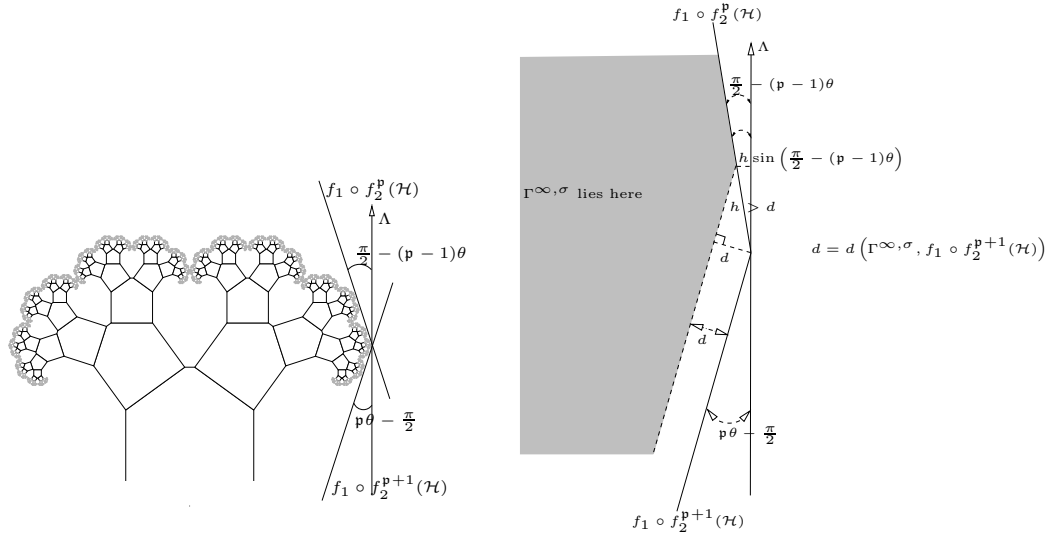


Figure 3: Left: the lines  $f_1 \circ f_2^{\mathfrak{p}}(\mathcal{H})$  and  $f_1 \circ f_2^{\mathfrak{p}+1}(\mathcal{H})$  (for  $\theta = \pi/5$ ). Right:  $d(\Gamma^{\infty, \sigma}, \Lambda) \geq \sin\left(\frac{\pi}{2} - (\mathfrak{p} - 1)\theta\right) d(\Gamma^{\infty, \sigma}, f_1 \circ f_2^{\mathfrak{p}+1}(\mathcal{H}))$ .

- If  $(i, j) = (2, 1)$ , then  $\Pi(\sigma) = \max\{m \geq 0 : \forall \ell, 1 \leq \ell \leq m, \tau'(2\ell - 1) \neq \tau'(2\ell)\}$ . From Lemma 2 and since  $\Pi(\sigma) < \kappa$ , we have that  $d(\Gamma^{\infty, \tau'}, \mathcal{H}) \gtrsim a^{2\Pi(\sigma)}$ . Thus

$$d(\Gamma^{\infty, \sigma}, f_1 \circ f_2^{\mathfrak{p}}(\mathcal{H})) \gtrsim a^{\mathfrak{p}+1+2\Pi(\sigma)}. \quad (82)$$

Combining the second inequality in (80) and (82) yields that  $d(\Gamma^{\infty, \sigma}, \Lambda) \gtrsim a^{2\Pi(\sigma)}$ .

In both cases, we have proved that  $d(\Gamma^{\infty, \sigma}, \Lambda) \gtrsim a^{2\Pi(\sigma)}$ . For the opposite inequality, recall that  $\Xi^\infty = \{\xi\}$  where the point  $\xi$  is defined by (12). There exists  $\zeta \in \Gamma^\infty$  such that  $\xi = f_1 \circ f_2^{\mathfrak{p}+1} \circ F_1^{\Pi(\sigma)}(\zeta)$ . Let  $\eta \in \mathcal{A}_{n-\mathfrak{p}-2-2\Pi(\sigma)}$  be such that  $f_\sigma = f_1 \circ f_2^{\mathfrak{p}+1} \circ F_1^{\Pi(\sigma)} \circ f_\eta$ . We have  $d(\Gamma^{\infty, \sigma}, \Lambda) = d(\Gamma^{\infty, \sigma}, \xi) = a^{\mathfrak{p}+2+2\Pi(\sigma)} d(\Gamma^{\infty, \eta}, \zeta) \leq a^{\mathfrak{p}+2+2\Pi(\sigma)} \text{Diam}(\Omega)$ , which yields that  $d(\Gamma^{\infty, \sigma}, \Lambda) \lesssim a^{2\Pi(\sigma)}$ . ■

**Proof of Lemma 4** We may suppose that  $n \geq \mathfrak{p} + 4$ . Let  $\kappa$  be the largest integer such that  $\mathfrak{p} + 2 + 2\kappa \leq n$ . If  $\Pi(\sigma) = \Pi(\tau) = \kappa$ , then  $d(\Gamma^{\infty, \sigma}, \Gamma^{\infty, \tau}) = 0$  which yields (35). Otherwise,  $\min(\Pi(\sigma), \Pi(\tau)) < \kappa$ : since  $\Gamma^{\infty, \sigma}$  and  $\Gamma^{\infty, \tau}$  are separated by  $\Lambda$ , we have

$$d(\Gamma^{\infty, \sigma}, \Gamma^{\infty, \tau}) \geq \max(d(\Gamma^{\infty, \sigma}, \Lambda), d(\Gamma^{\infty, \tau}, \Lambda)) \gtrsim a^{2\min(\Pi(\sigma), \Pi(\tau))}, \quad (83)$$

from Lemma 3.

On the other hand, defining  $k = \mathfrak{p} + 2 + 2\min(\Pi(\sigma), \Pi(\tau))$ , we see that for any  $\ell \leq k$ ,  $\sigma(\ell) = \overline{\tau(\ell)}$ . Thus,  $\Gamma^{\infty, \sigma_k}$  and  $\Gamma^{\infty, \tau_k}$  are symmetric with respect to  $\Lambda$ . Hence,

$$d(\Gamma^{\infty, \sigma_k}, \Gamma^{\infty, \tau_k}) = 2d(\Gamma^{\infty, \sigma_k}, \Lambda) \lesssim a^{2\min(\Pi(\sigma), \Pi(\tau))},$$

which implies

$$d(\Gamma^{\infty, \sigma}, \Gamma^{\infty, \tau}) \leq d(\Gamma^{\infty, \sigma_k}, \Gamma^{\infty, \tau_k}) + 2a^k \text{Diam}(\Omega). \quad (84)$$

From (83) and (84), we obtain (35). ■

**Proof of Lemma 5**

- If  $\Pi(\eta) = 0$ , then we use (37).
- If  $\Pi(\eta) > 0$ , then  $\Pi(\eta)$  cannot be maximal: indeed, if  $\Pi(\eta)$  was maximal, then  $\eta$  would be of the form  $\eta = \sigma^{(1)}$  or  $\eta = \sigma^{(1)} + 1$  or  $\eta = \sigma^{(1)} + 2$ , where  $\sigma \in \mathcal{A}$ , and  $\sigma^{(1)}$  is defined as in (14); hence, the angle of  $f_\eta$  would be  $\mathfrak{p}\theta$ ,  $(\mathfrak{p} - 1)\theta$  or  $(\mathfrak{p} + 1)\theta$ , so it would not be an integer multiple of  $2\pi$ . Since  $\Pi(\eta)$  is not maximal, the result stems from (41) and the fact that  $\Lambda$  and  $f_\eta(\Lambda)$  are parallel. ■

**Proof of Lemma 6** Take  $i \geq 1$ . Since  $\Gamma^\infty$  is symmetric w.r.t.  $\Lambda$ , we can estimate  $\mu(\{x \in \Gamma^\infty, c_1 a^{2i} < d(x, \Lambda) \leq c_1 a^{2(i-1)}\})$  instead of  $\mu(X_i)$ .

Consider  $n$ ,  $n > 2I + 3 + \mathfrak{p}$  where  $I \equiv i + \left\lceil \frac{\log\left(\frac{c_1}{c_3}\right)}{2\log a} \right\rceil$ . Let us first estimate  $\mu\{x \in \Gamma^\infty, c_1 a^{2i} + a^n \text{Diam}(\Gamma^\infty) < d(x, \Lambda) \leq c_1 a^{2(i-1)}\}$ .

Take  $x \in \Gamma^\infty$  and assume that  $c_1 a^{2i} + a^n \text{Diam}(\Gamma^\infty) < d(x, \Lambda) \leq c_1 a^{2(i-1)}$ . We know that  $\Gamma^\infty = \cup_{\sigma \in \mathcal{A}_n} \Gamma^{\infty, \sigma}$ , so there exists  $\sigma \in \mathcal{A}_n$  such that  $x \in \Gamma^{\infty, \sigma}$ .

We have  $d(\Gamma^{\infty, \sigma}, \Lambda) > c_1 a^{2i}$ . The upper bounds in (41) imply that we must have  $\Pi(\sigma) \leq I$ , which implies that  $\Pi(\sigma)$  is not maximal.

Then, the lower bound in (41) implies that  $\Pi(\sigma) \geq i$ .

Hence,

$$\begin{aligned} \mu(\{x \in \Gamma^\infty, c_1 a^{2i} + a^n \text{Diam}(\Gamma^\infty) < d(x, \Lambda) \leq c_1 a^{2(i-1)}\}) &\leq \mu\left(\bigcup_{\Pi(\sigma)=i}^I \Gamma^{\infty, \sigma}\right) \\ &\leq 2^{-n} \sum_{\ell=i}^I \#\{\sigma \in \mathcal{A}_n, \Pi(\sigma) = \ell\} \\ &\lesssim 2^{-n} \sum_{\ell=i}^I 2^{n-2\ell} \lesssim 2^{-2i}, \end{aligned}$$

which yields (46) by letting  $n$  tend to infinity (monotone convergence). ■

**Proof of Lemma 7** Let  $\kappa$  be the maximal integer  $k$  such that  $\nu_k = \eta_k$  for all  $k \leq \kappa$ . We must have either  $\kappa < \min(n, m)$  or  $(\kappa = \min(n, m) \text{ and } n \neq m)$  otherwise  $(m, j, \nu) = (n, i, \eta)$ .

- Assume that  $\kappa < \min(n, m)$ : this implies that there exist  $\sigma \in \mathcal{A}_\kappa$ ,  $\nu' \in \mathcal{A}_{m-\kappa}$ ,  $\eta' \in \mathcal{A}_{n-\kappa}$  such that  $\nu = \sigma + \nu'$ ,  $\eta = \sigma + \eta'$  and  $\nu'(1) \neq \eta'(1)$ , with the notation defined in (4):  $f_{\eta'}(\Gamma^\infty)$  and  $f_{\nu'}(\Gamma^\infty)$  lie on two different sides of  $\Lambda$ . We may assume that  $f_{\nu'}(\Gamma^\infty)$  lies on the right side of  $\Lambda$  and that  $f_{\eta'}(\Gamma^\infty)$  lies on the left side of  $\Lambda$ .

- If  $f_{\nu'}(\Gamma^\infty)$  lies strictly on the right side of  $\Lambda$ , we get the desired result. This happens in particular if  $\Pi(\nu')$  is not maximal.
- If  $f_{\eta'}(\Gamma^\infty)$  lies strictly on the left side of  $\Lambda$ , we get the desired result. This happens in particular if  $\Pi(\eta')$  is not maximal.
- Assume that  $f_{\nu'}(\Gamma^\infty) \cap \Lambda \neq \emptyset$  and  $f_{\eta'}(\Gamma^\infty) \cap \Lambda \neq \emptyset$ .

- \* if  $\Pi(\nu')$  is positive then it is maximal, and  $f_{\nu'}$  is a similitude whose angle can be  $-(\mathfrak{p}-1)\theta$ ,  $-\mathfrak{p}\theta$  or  $-(\mathfrak{p}+1)\theta$ . If the angle is  $-\mathfrak{p}\theta$  or  $-(\mathfrak{p}+1)\theta$ , then  $f_{\nu'}(X_j)$  does not intersect  $\Lambda$  (because  $f_{\nu'}(X_j)$  is on the left of  $f_{\nu'}(\Lambda)$ ), which yields the desired result. If the angle is  $-(\mathfrak{p}-1)\theta$ , then the similitude  $f_{\eta'}$  has the same angle and  $f_{\eta'}(X_i)$  does not intersect  $\Lambda$  (because  $f_{\eta'}(X_i)$  is on the left of  $f_{\eta'}(\Lambda)$ ), which yields the desired result.
  - \* Similarly, if  $\Pi(\eta')$  is positive then it is maximal, and  $f_{\eta'}$  is a similitude whose angle can be  $(\mathfrak{p}-1)\theta$ ,  $\mathfrak{p}\theta$  or  $(\mathfrak{p}+1)\theta$ . If the angle is  $(\mathfrak{p}-1)\theta$ , then  $f_{\eta'}(X_i)$  does not intersect  $\Lambda$  which yields the desired result. If the angle is  $\mathfrak{p}\theta$  or  $(\mathfrak{p}+1)\theta$ , then the similitude  $f_{\nu'}$  has the same angle and  $f_{\nu'}(X_j)$  does not intersect  $\Lambda$  which yields the desired result.
  - \* We are left with the case where  $\Pi(\eta') = 0$  and  $\Pi(\nu') = 0$ : it can be shown that there are only three pairs  $(\nu', \eta')$  such that the related similitudes have the same angle,  $\Pi(\eta') = 0$ ,  $\Pi(\nu') = 0$ ,  $f_{\nu'}(\Gamma^\infty) \cap \Lambda \neq \emptyset$  and  $f_{\eta'}(\Gamma^\infty) \cap \Lambda \neq \emptyset$ : 1)  $\eta' = (1)$  and  $\nu' = (2, 1, 1)$  2)  $\eta' = (1, 2)$  and  $\nu' = (2, 1)$ , 3)  $\eta' = (1, 2, 2)$  and  $\nu' = (2)$ . In these three cases, the desired result follows easily.
- If  $\kappa = \min(n, m)$ , for example  $\kappa = n < m$ , then  $Y = f_\eta(X_i)$  and  $Z = f_\eta \circ f_{\nu'}(X_j)$ ,  $\nu' \in \mathcal{A}_{m'}$ . We have to prove that  $X_i \cap f_{\nu'}(X_j) = \emptyset$ . The angle of the similitude  $f_{\nu'}$  is 0 and  $2i = 2j + m'$ .
    - If  $f_{\nu'}(1) = 2$ , then  $f_{\nu'}(X_j)$  lies on the right side of  $\Lambda$  and  $X_i$  strictly lies on the left side of  $\Lambda$ , which yields the result.
    - If  $f_{\nu'}(1) = 1$ , then  $d(f_{\nu'}(\Lambda), \Lambda) > c_1 a^{2\Pi(\nu')}$  from Lemma 5. Therefore, from the definition of  $X_i$ ,  $d(f_{\nu'}(\Lambda), X_i) > c_1 a^{2\Pi(\nu')} - c_1 a^{2i-2} > 0$ . ■

**Proof of Lemma 8** We can assume  $\ell > 1$ .

- Suppose first that  $\ell = 2i$ ,  $i > 0$ ,  $x \in X_i$ . Since we are interested in finding  $j < i$  and  $\eta \in \mathcal{A}_{\ell-2j}$  such that the set  $Z = f_\eta(X_j)$  contains  $x$ , we can suppose that  $\eta(1) = 1$ . If  $\Pi(\eta)$  is not maximal, then  $X_i \cap Z = \emptyset$ : indeed, from (41),  $Z \subset f_\eta(\Gamma^\infty)$  and  $d(f_\eta(\Gamma^\infty), \Lambda) > c_1 a^{2\Pi(\eta)}$ . Therefore  $d(Z, X_i) \geq d(f_\eta(\Gamma^\infty), \Lambda) - c_1 a^{2(i-1)} > c_1 (a^{2\Pi(\eta)} - a^{2(i-1)})$ . But  $2\Pi(\eta) < 2(i-1)$ , so  $d(Z, X_i) > 0$  and  $x \notin Z$ .  
We now focus on the  $Z = f_\eta(X_j) \in \mathcal{Z}_\ell$  such that  $\eta(1) = 1$  and  $\Pi(\eta)$  is maximal. Since there are a finite number, namely  $2^{\mathfrak{p}+2}$ , of  $\eta$  such that  $\Pi(\eta) = 0$  and  $\Pi(\eta)$  is maximal, we can suppose that  $\Pi(\eta) > 0$ . We make out two cases:
  - If  $\mathfrak{p}$  is even, then  $\eta \in \mathcal{A}_{\mathfrak{p}+2+2\Pi(\eta)}$ , and the angle of the similitude  $f_\eta$  is  $\mathfrak{p}\theta$ . Since the sets  $Z \in \mathcal{Z}_{\ell, \mathfrak{p}\theta}$  are pairwise disjoint, only one of them can contain  $x$ .
  - If  $\mathfrak{p}$  is odd, then  $\eta \in \mathcal{A}_{\mathfrak{p}+3+2\Pi(\eta)}$ . Since  $\eta + 1 \in \mathcal{A}_{\mathfrak{p}+4+2\Pi(\eta)}$ ,  $\Pi(\eta + 1) = \Pi(\eta)$  is not maximal. The facts that  $Z = f_\eta(X_j) \subset f_{\eta+1}(\Gamma^\infty)$  and  $d(f_{\eta+1}(\Gamma^\infty), \Lambda) > c_1 a^{2\Pi(\eta)} > c_1 a^{2(i-1)}$  imply that  $X_i \cap Z = \emptyset$ , so  $x \notin Z$ .
- Suppose  $x \in Y = f_\nu(X_i)$ ,  $Y \in \mathcal{Z}_\ell$ , with  $\nu \in \mathcal{A}_{n_Y}$ ,  $n_Y > 0$  and  $x \notin Z$ ,  $Z \in \mathcal{Z}_\ell$ , with  $n_Z < n_Y$ . On the one hand, the number of the sets  $f_{\nu+\nu'}(X_j) \in \mathcal{Z}_\ell$  containing  $x$  coincides with the number of the sets  $f_{\nu'}(X_j)$  containing  $f_\nu^{-1}(x) \in X_i$ ; this number has been estimated above. On the other hand, if a set  $Z = f_\eta(X_j) \in \mathcal{Z}_\ell$ , with  $n_Z \geq n_Y$ , is such that  $\eta \neq \nu + \nu'$ , then calling  $\kappa$  the maximal integer such that  $\nu_k = \eta_k$  for all  $k \leq \kappa$ , we know that  $\kappa < n_Y$  and that  $f_\eta(X_j)$  and  $f_\nu(X_i)$  lie on different sides of  $f_{\nu_\kappa}(\Lambda)$ , so their intersection is empty:  $Z$  does not contain  $x$ . ■

## B Proofs of the geometrical lemmas in the case $\mathfrak{p}\theta = \pi/2$

**Proof of Lemma 11** It is enough to consider  $n \geq \mathfrak{p} + 4$ .

- If  $\sigma_{\mathfrak{p}+2} \notin \{(1, 2, \dots, 2), (2, 1, \dots, 1)\}$ , then there exists a constant  $c$  independent of  $\sigma$  such that  $d(\Gamma^{\infty, \sigma}, \Lambda) > c$ . Since  $d(\Gamma^{\infty, \sigma}, \Lambda) \leq \text{Diam}(\Omega)$ , we obtain the desired result in this case.
- If  $\sigma_{\mathfrak{p}+2} \in \{(1, 2, \dots, 2), (2, 1, \dots, 1)\}$ , for example  $\sigma_{\mathfrak{p}+2} = (1, 2, \dots, 2)$ , then  $\Gamma^{\infty, \sigma} = f_1 \circ f_2^{\mathfrak{p}+1} \Gamma^{\infty, \tau}$ , for some  $\tau \in \mathcal{A}_{n-\mathfrak{p}-2}$ . Since we also have  $\Lambda = f_1 \circ f_2^{\mathfrak{p}+1}(\mathcal{H})$ ,

$$d(\Gamma^{\infty, \sigma}, \Lambda) = d(f_1 \circ f_2^{\mathfrak{p}+1}(\Gamma^{\infty, \tau}), f_1 \circ f_2^{\mathfrak{p}+1}(\mathcal{H})) = a^{\mathfrak{p}+2} d(\Gamma^{\infty, \tau}, \mathcal{H}). \quad (85)$$

We also have that  $\Pi(\sigma) = \max\{k; \forall j \leq k, \tau(2j-1) \neq \tau(2j)\}$ . Thus from Lemma 2,  $d(\Gamma^{\infty, \tau}, \mathcal{H}) + a^{n-\mathfrak{p}-2} \simeq a^{2\Pi(\sigma)}$ . Combining this and (85) yields the desired result. ■

**Proof of Lemma 12** From the definition of  $k$ , it is clear that  $\Pi(\sigma) \geq k$  and that  $\Pi(\tau) \geq k$ . Let  $\kappa$  be the largest integer such that  $\mathfrak{p} + 2 + 2\kappa \leq n$ . With the notation defined in (15), let  $\eta \in \mathcal{A}_k$  be such that  $\sigma_{\mathfrak{p}+2+2k} = \eta^{(1)}$  and  $\tau_{\mathfrak{p}+2+2k} = \eta^{(2)}$ .

- If  $k = \kappa$ , then  $\Gamma^{\infty, \eta^{(1)}}$  is the symmetric of  $\Gamma^{\infty, \eta^{(2)}}$  w.r.t.  $\Lambda$ , and  $d(\Gamma^{\infty, \eta^{(1)}}, \Gamma^{\infty, \eta^{(2)}}) = 0$ . This implies (69).
- If  $k < \kappa$ , then  $d(\Gamma^{\infty, \sigma}, \Gamma^{\infty, \tau}) \leq d(\Gamma^{\infty, \eta^{(1)}}, \Gamma^{\infty, \eta^{(2)}}) + 2\text{Diam}(\Gamma^{\infty, \eta^{(1)}}) \simeq a^{2k}$ , because  $d(\Gamma^{\infty, \eta^{(1)}}, \Gamma^{\infty, \eta^{(2)}}) = 0$ . For the opposite inequality, since  $\kappa > k$ , we can define  $\sigma' = \sigma_{\mathfrak{p}+4+2k}$  and  $\tau' = \tau_{\mathfrak{p}+4+2k}$ .

– If  $\min(\Pi(\sigma), \Pi(\tau)) = k$  then

$$d(\Gamma^{\infty, \sigma}, \Gamma^{\infty, \tau}) \geq d(\Gamma^{\infty, \sigma'}, \Gamma^{\infty, \tau'}) \geq \max\left(d(\Gamma^{\infty, \sigma'}, \Lambda), d(\Gamma^{\infty, \tau'}, \Lambda)\right) \simeq a^{2k}.$$

– Otherwise,  $\min(\Pi(\sigma), \Pi(\tau)) > k$ . This implies that

$$\sigma(\mathfrak{p} + 3 + 2k) = \tau(\mathfrak{p} + 3 + 2k) \neq \sigma(\mathfrak{p} + 4 + 2k) = \tau(\mathfrak{p} + 4 + 2k).$$

Without restriction, we may assume that  $\sigma(\mathfrak{p} + 3 + 2k) = \tau(\mathfrak{p} + 3 + 2k) = 1$  and that  $\sigma(\mathfrak{p} + 4 + 2k) = \tau(\mathfrak{p} + 4 + 2k) = 2$ , thus  $\sigma' = \eta^{(1)}12$  and  $\tau' = \eta^{(2)}12$ .

For what follows we define  $\delta = d(\Gamma^{1, \infty} \cap \mathcal{H}, \Gamma^{2, \infty} \cap \mathcal{H}) > 0$  and  $\mathcal{E}$  as the convex subset of  $\mathbb{R}^2$  located under the straight lines  $\mathcal{H}$ ,  $f_1(\mathcal{H})$  and  $f_2(\mathcal{H})$ . It is clear that  $\Omega \subset \mathcal{E}$ . Therefore,  $\Omega_{\sigma'} \subset f_{\sigma'}(\mathcal{E})$  and  $\Omega_{\tau'} \subset f_{\tau'}(\mathcal{E})$ , see Figure 4.

Elementary geometrical arguments lead to

$$d(f_{\sigma'}(\mathcal{E}), f_{\tau'}(\mathcal{E})) = \beta \sin \theta,$$

where  $\beta = d(f_{\sigma'}(\Gamma^{\infty}) \cap \Lambda, f_{\tau'}(\Gamma^{\infty}) \cap \Lambda)$ , see Figure 4. But

$$\begin{aligned} d(f_{\sigma'}(\Gamma^{\infty}) \cap \Lambda, f_{\tau'}(\Gamma^{\infty}) \cap \Lambda) &= d(f_{\eta^{(1)}}(F_1(\Gamma^{\infty})) \cap \Lambda, f_{\eta^{(2)}}(F_1(\Gamma^{\infty})) \cap \Lambda) \\ &= d(f_{\eta^{(1)}}(F_1(\Gamma^{\infty})) \cap \Lambda, f_{\eta^{(1)}}(F_2(\Gamma^{\infty})) \cap \Lambda). \end{aligned}$$

By self-similarity,  $\beta = a^{\mathfrak{p}+2+2k} \delta$ . Therefore  $d(\Gamma^{\infty, \sigma}, \Gamma^{\infty, \tau}) \geq d(\Gamma^{\infty, \sigma'}, \Gamma^{\infty, \tau'}) \geq d(f_{\sigma'}(\mathcal{E}), f_{\tau'}(\mathcal{E})) \gtrsim a^{2k}$ .



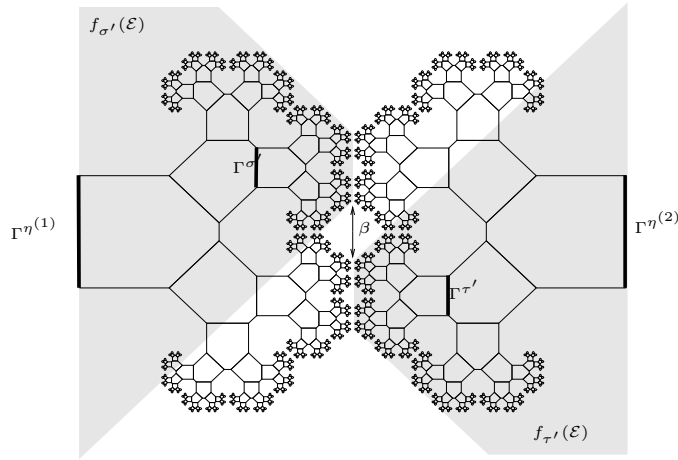


Figure 4:  $\theta = \pi/4$ : the case when  $\sigma' = \eta^{(1)}12$  and when  $\tau' = \eta^{(2)}12$

**Proof of Lemma 13** The argument is the same as the one used for Lemma 6. The only difference is that  $\#\{\sigma \in \mathcal{A}_n, \Pi(\sigma) = \ell\} \lesssim 2^{n-\ell}$ , instead of  $2^{n-2\ell}$  in the former case. Hence, with  $I$  defined as in the proof of Lemma 6,

$$\begin{aligned}
\mu(\{x \in \Gamma^\infty, c_1 a^{2i} + a^n \text{Diam}(\Gamma^\infty) < d(x, \Lambda) \leq c_1 a^{2(i-1)}\}) &\leq \mu\left(\bigcup_{\Pi(\sigma)=i}^{i+I} \Gamma^{\infty, \sigma}\right) \\
&\leq 2^{-n} \sum_{\ell=i}^{i+I} \#\{\sigma \in \mathcal{A}_n, \Pi(\sigma) = \ell\} \\
&\lesssim 2^{-n} \sum_{\ell=i}^{i+I} 2^{n-\ell} \lesssim 2^{-i},
\end{aligned}$$

and (70) is obtained by Fatou lemma. ■

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