



## Another presentation of the Craig-Martinez method

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# Technical note: another presentation of Craig-Martinez method

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## 1 Introduction

The Craig-Martinez technique is a popular component mode synthesis procedure that allows for the construction of *a priori* reduced-order basis of large finite element models and belongs to the free-interface family [11]. Similarly to its numerous counterparts [8, 9, 10, 12], this strategy relies in the definition of an interface separating substructures then yielding a mixed reduction space comprising both modal and physical coordinates with subsequent benefits and limitations [2, 3, 1]. More importantly, it provides access to user-defined physical coordinates directly within the reduced model, which may be of great interest for structural configurations featuring unilateral contact interfaces by avoiding prohibitive backward and forward mappings between the reduced and the finite element spaces to treat displacement -related non-penetration conditions.

As far as the authors know, the usual description of the Craig-Martinez method uses a pseudo-static approximation which may be stated as follows: the highest excitation frequency  $\omega_{\max}$  of the structure can be neglected in comparison with any eigenfrequency  $\omega_2$  of the free vibration modes  $\Psi_2$  eliminated from the reduction basis. This short note aims at introducing the Craig-Martinez method in an original manner by reformulating the usual pseudo-static approximation into a simpler statement: the initially truncated high frequency free vibration modes do not dynamically participate to the reduced-order model.

## 2 Foreword

### 2.1 Assumptions

The following theoretical developments are derived with the assumption that there is no rigid body motion. If the structure of interest possesses rigid body modes, its stiffness matrix  $\mathbf{K}$  is semi-positive definite and cannot be inverted and the following descriptions of the Craig-Martinez method do not directly apply. The reader may refer to [6] and to section 2.5 of [13] that detailed a procedure for the computation of residual flexibility matrix when rigid body modes are to be considered.

In our study, both the stiffness and mass matrices  $\mathbf{K}$  and  $\mathbf{M}$  are positive definite and their inverse exist. Also, the diagonal matrix  $\mathbf{\Omega}$  that contains the square of the angular eigenfrequencies of the system is composed of strictly positive terms and  $\mathbf{\Omega}^{-1}$  also exists.

Consistently with the definition of eigenmodes, the description of a component mode synthesis method is usually made without considering structural damping. Accordingly, there is no structural

damping in the system of interest. As mentioned in the introduction, the Craig-Martinez method yields a mixed reduced space with both modal and physical DoFs. The physical DoFs are typically substructures boundaries or DoFs that will be loaded during the simulation. In this technical note, there is only one structure without substructures and the physical DoFs of the reduced space are called *boundary DoFs*: they include any DoF on which a load may be applied.

## 2.2 Definition

Consider the general problem to be solved:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{F} \quad (1)$$

Computationally expensive numerical models driven by Eq. (1) may be advantageously solved using Component Mode Synthesis (CMS) procedures [5]. Reduction from the original discrete space  $\mathbf{u}$  of size  $n$  to the reduced-order space  $\mathbf{q}$  of size  $m$  with  $m \ll n$  can be achieved through a rectangular transformation matrix  $\Phi$  of size  $n \times m$  such as:

$$\mathbf{u} = \Phi \mathbf{q} \quad (2)$$

In the following,  $\mathbf{u}_i$  and  $\mathbf{u}_b$  respectively stand for the internal and boundary DoFs. The  $n$  normal modes (stored matrix  $\Psi$ ) are divided into the first  $\phi$  modes  $\Psi_1$  retained in the reduction basis and the truncated modes  $\Psi_2$  associated to their corresponding modal participations  $\mathbf{q}_1$  and  $\mathbf{q}_2$ , yielding:

$$\mathbf{u} = \Psi_1 \mathbf{q}_1 + \Psi_2 \mathbf{q}_2 \quad \Leftrightarrow \quad \begin{pmatrix} \mathbf{u}_i \\ \mathbf{u}_b \end{pmatrix} = \begin{bmatrix} \Psi_{1i} & \Psi_{2i} \\ \Psi_{1b} & \Psi_{2b} \end{bmatrix} \begin{pmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{pmatrix} \quad (3)$$

## 3 Usual formulation

Let us denote  $\Omega$ , the matrix storing the square of the angular eigenfrequencies of the system of interest:

$$\Omega = \begin{bmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{bmatrix} \quad (4)$$

associated with modes  $\Psi_1$  and  $\Psi_2$  such as:

$$\Psi^T \mathbf{K} \Psi = \Omega \quad (5)$$

Under the assumption that there is no rigid modes<sup>1</sup>, the inverse of the stiffness matrix  $\mathbf{K}$  exists and:

$$\mathbf{K}^{-1} = \Psi \Omega^{-1} \Psi^T \quad (6)$$

Expanding Eq. (6) reads as follows:

$$\mathbf{K}^{-1} = \Psi_1 \Omega_1^{-1} \Psi_1^T + \Psi_2 \Omega_2^{-1} \Psi_2^T \quad (7)$$

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<sup>1</sup>If the system contains rigid modes, several methods can be used to overcome the inherent singularity [4, 7]

and the usual pseudo-static approximation neglects the dynamical contribution  $\ddot{\mathbf{q}}_2$  of the high frequency modes  $\Psi_2$ , in other words:

$$\Omega_2 \mathbf{q}_2 = \Psi_2^T \mathbf{F} \quad (8)$$

Equation (8) is then multiplied by  $\Psi_2 \Omega_2^{-1}$  such as:

$$\Psi_2 \mathbf{q}_2 = \Psi_2 \Omega_2^{-1} \Psi_2^T \mathbf{F} \quad (9)$$

which, combined to Equation (7) implies:

$$\Psi_2 \mathbf{q}_2 = (\mathbf{K}^{-1} - \Psi_1 \Omega_1^{-1} \Psi_1^T) \mathbf{F} \quad (10)$$

Finally, by plugging Eq. (10) in Eq. (3), we find:

$$\mathbf{u} = \Psi_1 \mathbf{q}_1 + (\mathbf{K}^{-1} - \Psi_1 \Omega_1^{-1} \Psi_1^T) \mathbf{F} \quad (11)$$

or, in a contracted form:

$$\mathbf{u} = \Psi_1 \mathbf{q}_1 + \mathbf{R} \mathbf{F} \quad (12)$$

where  $\mathbf{R}$  is called the residual flexibility matrix.

The key feature of the Craig-Martinez technique is to introduce the boundary DoFs  $\mathbf{u}_b$  in the unknowns of the reduced system by modifying transformation (12). This is achieved by expressing the external forces  $\mathbf{F}_{\text{ext}}$  as a function of  $\mathbf{u}_b$  and  $\mathbf{q}_1$  of Eq. (3). Consequently, Eq. (12) becomes:

$$\begin{pmatrix} \mathbf{u}_i \\ \mathbf{u}_b \end{pmatrix} = \begin{bmatrix} \Psi_{1i} \\ \Psi_{1b} \end{bmatrix} \mathbf{q}_1 + \begin{bmatrix} \mathbf{R}_{ii} & \mathbf{R}_{ib} \\ \mathbf{R}_{bi} & \mathbf{R}_{bb} \end{bmatrix} \mathbf{F}_{\text{ext}} \quad (13)$$

As mentioned in section 2.1  $\mathbf{F}_{\text{ext}}$  only has coordinates on the boundary DoFs so that:

$$\begin{bmatrix} \mathbf{R}_{ii} & \mathbf{R}_{ib} \\ \mathbf{R}_{bi} & \mathbf{R}_{bb} \end{bmatrix} \begin{pmatrix} \mathbf{0} \\ \mathbf{F} \end{pmatrix} = \begin{bmatrix} \mathbf{R}_{ib} \\ \mathbf{R}_{bb} \end{bmatrix} \mathbf{F} \quad (14)$$

Equation (13) becomes then:

$$\begin{pmatrix} \mathbf{u}_i \\ \mathbf{u}_b \end{pmatrix} = \begin{bmatrix} \Psi_{1i} & \mathbf{R}_{ib} \\ \Psi_{1b} & \mathbf{R}_{bb} \end{bmatrix} \begin{pmatrix} \mathbf{q}_1 \\ \mathbf{F} \end{pmatrix} \quad (15)$$

The second block of Eq. (15) is:

$$\mathbf{F} = \mathbf{R}_{bb}^{-1} (\mathbf{u}_b - \Psi_{1b} \mathbf{q}_1) \quad (16)$$

This finally leads to the following transformation matrix:

$$\Phi = \begin{bmatrix} \Psi_{1i} - \mathbf{R}_{ib} \mathbf{R}_{bb}^{-1} \Psi_{1b} & \mathbf{R}_{ib} \mathbf{R}_{bb}^{-1} \\ \mathbf{0} & \mathbf{I}_{bb} \end{bmatrix} \quad (17)$$

## 4 Proposed formulation

Let us assume that the normal modes  $\Psi$  of the system of interest are  $\mathbf{K}$ -orthogonal and  $\mathbf{M}$ -orthonormal:

$$\Omega = \Psi^T \mathbf{K} \Psi \quad (18)$$

yielding:

$$\Psi^{-1} = \Omega^{-1} \Psi^T \mathbf{K} \quad (19)$$

that may be expanded in a block-matrix form as follows:

$$\Psi^{-1} = \begin{bmatrix} \Omega_1^{-1} & \mathbf{0} \\ \mathbf{0} & \Omega_2^{-1} \end{bmatrix} \begin{bmatrix} \Psi_{1b}^T & \Psi_{1i}^T \\ \Psi_{2b}^T & \Psi_{2i}^T \end{bmatrix} \begin{bmatrix} \mathbf{K}_{bb} & \mathbf{K}_{bi} \\ \mathbf{K}_{ib} & \mathbf{K}_{ii} \end{bmatrix} \quad (20)$$

$$\Psi^{-1} = \begin{bmatrix} \Omega_1^{-1}(\Psi_{1b}^T \mathbf{K}_{bb} + \Psi_{1i}^T \mathbf{K}_{ib}) & \Omega_1^{-1}(\Psi_{1b}^T \mathbf{K}_{bi} + \Psi_{1i}^T \mathbf{K}_{ii}) \\ \Omega_2^{-1}(\Psi_{2b}^T \mathbf{K}_{bb} + \Psi_{2i}^T \mathbf{K}_{ib}) & \Omega_2^{-1}(\Psi_{2b}^T \mathbf{K}_{bi} + \Psi_{2i}^T \mathbf{K}_{ii}) \end{bmatrix} \quad (21)$$

in such a way that inverting (3) becomes:

$$\begin{pmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{pmatrix} = \begin{bmatrix} \Omega_1^{-1}(\Psi_{1b}^T \mathbf{K}_{bi} + \Psi_{1i}^T \mathbf{K}_{ii}) & \Omega_1^{-1}(\Psi_{1b}^T \mathbf{K}_{bb} + \Psi_{1i}^T \mathbf{K}_{ib}) \\ \Omega_2^{-1}(\Psi_{2b}^T \mathbf{K}_{bi} + \Psi_{2i}^T \mathbf{K}_{ii}) & \Omega_2^{-1}(\Psi_{2b}^T \mathbf{K}_{bb} + \Psi_{2i}^T \mathbf{K}_{ib}) \end{bmatrix} \begin{pmatrix} \mathbf{u}_i \\ \mathbf{u}_b \end{pmatrix} \quad (22)$$

Invoking Eqs. (3) and (22) in order to eliminate  $\mathbf{q}_2$  gives:

$$\begin{aligned} \mathbf{u}_i &= \Psi_{1i} \mathbf{q}_1 + \Psi_{2i} (\Omega_2^{-1} (\Psi_{2b}^T \mathbf{K}_{bb} + \Psi_{2i}^T \mathbf{K}_{ib}) \mathbf{u}_b + \Omega_2^{-1} (\Psi_{2b}^T \mathbf{K}_{bi} + \Psi_{2i}^T \mathbf{K}_{ii}) \mathbf{u}_i) \\ \mathbf{u}_b &= \Psi_{1b} \mathbf{q}_1 + \Psi_{2b} (\Omega_2^{-1} (\Psi_{2b}^T \mathbf{K}_{bb} + \Psi_{2i}^T \mathbf{K}_{ib}) \mathbf{u}_b + \Omega_2^{-1} (\Psi_{2b}^T \mathbf{K}_{bi} + \Psi_{2i}^T \mathbf{K}_{ii}) \mathbf{u}_i) \end{aligned} \quad (23)$$

or, equivalently:

$$\begin{aligned} (\mathbf{I} - \Psi_{2i} \Omega_2^{-1} (\Psi_{2b}^T \mathbf{K}_{bi} + \Psi_{2i}^T \mathbf{K}_{ii})) \mathbf{u}_i &= \Psi_{1i} \mathbf{q}_1 + \Psi_{2i} \Omega_2^{-1} (\Psi_{2b}^T \mathbf{K}_{bb} + \Psi_{2i}^T \mathbf{K}_{ib}) \mathbf{u}_b \\ \Psi_{2i} \Omega_2^{-1} (\Psi_{2b}^T \mathbf{K}_{bi} + \Psi_{2i}^T \mathbf{K}_{ii}) \mathbf{u}_i &= -\Psi_{1b} \mathbf{q}_1 + (\mathbf{I} - \Psi_{2b} \Omega_2^{-1} (\Psi_{2b}^T \mathbf{K}_{bb} + \Psi_{2i}^T \mathbf{K}_{ib})) \mathbf{u}_b \end{aligned} \quad (24)$$

For the sake of brevity, the following matrix  $\mathbf{R}$  is introduced:

$$\mathbf{R} = \Psi_2 \Omega_2^{-1} \Psi_2^T \quad (25)$$

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_{ii} & \mathbf{R}_{ib} \\ \mathbf{R}_{bi} & \mathbf{R}_{bb} \end{bmatrix} = \begin{bmatrix} \Psi_{2i} \Omega_2^{-1} \Psi_{2i}^T & \Psi_{2i} \Omega_2^{-1} \Psi_{2b}^T \\ \Psi_{2b} \Omega_2^{-1} \Psi_{2i}^T & \Psi_{2b} \Omega_2^{-1} \Psi_{2b}^T \end{bmatrix} \quad (26)$$

The definition of matrix  $\mathbf{R}$  is analogous to the one of the residual flexibility matrix introduced in section 3. However, one may notice that the relation:

$$\mathbf{R} = \Psi_2 \Omega_2^{-1} \Psi_2^T = \mathbf{K}^{-1} - \Psi_1 \Omega_1^{-1} \Psi_1^T \quad (27)$$

is only verified when there is no rigid body motions in the system.

The introduction of matrix  $\mathbf{R}$  allows to rewrite the system of equations (24) in terms of  $\mathbf{R}$ ,  $\mathbf{K}$ ,  $\mathbf{u}_i$ ,  $\mathbf{q}_1$  and  $\mathbf{u}_b$  only:

$$\begin{aligned} (\mathbf{I} - \mathbf{R}_{ib}\mathbf{K}_{bi} - \mathbf{R}_{ii}\mathbf{K}_{ii})\mathbf{u}_i &= \mathbf{\Psi}_{1i}\mathbf{q}_1 + (\mathbf{R}_{ib}\mathbf{K}_{bb} + \mathbf{R}_{ii}\mathbf{K}_{ib})\mathbf{u}_b \\ (\mathbf{R}_{bb}\mathbf{K}_{bi} + \mathbf{R}_{bi}\mathbf{K}_{ii})\mathbf{u}_i &= -\mathbf{\Psi}_{1b}\mathbf{q}_1 + (\mathbf{I} - \mathbf{R}_{bb}\mathbf{K}_{bb} - \mathbf{R}_{bi}\mathbf{K}_{ib})\mathbf{u}_b \end{aligned} \quad (28)$$

The second equation of system (28) is now left-multiplicated by  $\mathbf{R}_{ib}\mathbf{R}_{bb}^{-1}$ :

$$\begin{aligned} (\mathbf{I} - \mathbf{R}_{ib}\mathbf{K}_{bi} - \mathbf{R}_{ii}\mathbf{K}_{ii})\mathbf{u}_i &= \mathbf{\Psi}_{1i}\mathbf{q}_1 + (\mathbf{R}_{ib}\mathbf{K}_{bb} + \mathbf{R}_{ii}\mathbf{K}_{ib})\mathbf{u}_b \\ (\mathbf{R}_{ib}\mathbf{K}_{bi} + \mathbf{R}_{ib}\mathbf{R}_{bb}^{-1}\mathbf{R}_{bi}\mathbf{K}_{ii})\mathbf{u}_i &= -\mathbf{R}_{ib}\mathbf{R}_{bb}^{-1}\mathbf{\Psi}_{1b}\mathbf{q}_1 \\ &\quad + (\mathbf{R}_{ib}\mathbf{R}_{bb}^{-1} - \mathbf{R}_{ib}\mathbf{K}_{bb} - \mathbf{R}_{ib}\mathbf{R}_{bb}^{-1}\mathbf{R}_{bi}\mathbf{K}_{ib})\mathbf{u}_b \end{aligned} \quad (29)$$

The two equations of system (29) may now be summed:

$$(\mathbf{I} - \mathbf{R}_{ii}\mathbf{K}_{ii} + \mathbf{R}_{ib}\mathbf{R}_{bb}^{-1}\mathbf{R}_{bi}\mathbf{K}_{ii})\mathbf{u}_i = (\mathbf{\Psi}_{1i} - \mathbf{R}_{ib}\mathbf{R}_{bb}^{-1}\mathbf{\Psi}_{1b})\mathbf{q}_1 + (\mathbf{R}_{ib}\mathbf{R}_{bb}^{-1} + \mathbf{R}_{ii}\mathbf{K}_{ib} - \mathbf{R}_{ib}\mathbf{R}_{bb}^{-1}\mathbf{R}_{bi}\mathbf{K}_{ib})\mathbf{u}_b \quad (30)$$

Equation (30) defines a general transformation expressing the internal physical DoFs  $\mathbf{u}_i$  with respect to  $\mathbf{u}_b$  and the retained modal coefficients  $\mathbf{q}_1$ . This transformation does not imply any condensation of the system. We may write Eq. (30) as:

$$\mathbf{u}_i = \underbrace{(\mathbf{\Psi}_{1i} - \mathbf{R}_{ib}\mathbf{R}_{bb}^{-1}\mathbf{\Psi}_{1b})\mathbf{q}_1 + (\mathbf{R}_{ib}\mathbf{R}_{bb}^{-1})\mathbf{u}_b}_{\text{Condensation}} + \underbrace{[(\mathbf{R}_{ii} - \mathbf{R}_{ib}\mathbf{R}_{bb}^{-1}\mathbf{R}_{bi})(\mathbf{K}_{ii}\mathbf{u}_i + \mathbf{K}_{ib}\mathbf{u}_b)]}_{\mathbf{T}} \quad (31)$$

Two terms explicitly appear in the decomposition of  $\mathbf{u}_i$ . The left term — namely the condensation term — is an approximation of  $\mathbf{u}_i$ . It depends on both the retained normal modes  $\mathbf{\Psi}_1$  and the suppressed normal modes  $\mathbf{\Psi}_2$  through  $\mathbf{R}$  detailed in Eq. (26). The additional term  $\mathbf{T}$  is required to retrieve  $\mathbf{u}_i$ . It is worthy to note that  $\mathbf{T}$  does not depend on the retained normal modes  $\mathbf{\Psi}_1$ .

In order to extract an approximation of  $\mathbf{u}_i$ , it is assumed that the dynamical participation of the suppressed normal modes  $\mathbf{\Psi}_2$  can be neglected, such as:

$$\ddot{\mathbf{q}}_2 = \mathbf{0} \quad (32)$$

Due to Eq. (22), this leads to:

$$\mathbf{\Omega}_2^{-1}(\mathbf{\Psi}_{2b}^T\mathbf{K}_{bb} + \mathbf{\Psi}_{2i}^T\mathbf{K}_{ib})\ddot{\mathbf{u}}_b + \mathbf{\Omega}_2^{-1}(\mathbf{\Psi}_{2b}^T\mathbf{K}_{bi} + \mathbf{\Psi}_{2i}^T\mathbf{K}_{ii})\ddot{\mathbf{u}}_i = \mathbf{0} \quad (33)$$

Because the boundary DoFs are chosen so that no effort can apply on the internal DoFs, Eq. (1) may be expanded as follows:

$$\begin{bmatrix} \mathbf{M}_{ii} & \mathbf{M}_{ib} \\ \mathbf{M}_{bi} & \mathbf{M}_{bb} \end{bmatrix} \begin{pmatrix} \ddot{\mathbf{u}}_i \\ \ddot{\mathbf{u}}_b \end{pmatrix} + \begin{bmatrix} \mathbf{K}_{ii} & \mathbf{K}_{ib} \\ \mathbf{K}_{bi} & \mathbf{K}_{bb} \end{bmatrix} \begin{pmatrix} \mathbf{u}_i \\ \mathbf{u}_b \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{F}_b \end{pmatrix} \quad (34)$$

The first block of Eq. (34) gives:

$$\mathbf{M}_{ii}\ddot{\mathbf{u}}_i + \mathbf{M}_{ib}\ddot{\mathbf{u}}_b + \mathbf{K}_{ii}\mathbf{u}_i + \mathbf{K}_{ib}\mathbf{u}_b = \mathbf{0} \quad (35)$$

The  $\mathbf{M}$ -orthonormality of the normal modes yields:

$$\mathbf{M} = \mathbf{\Psi}^{-T}\mathbf{\Psi}^{-1} \quad (36)$$

By plugging Eq. (21) in Eq. (36),  $\mathbf{M}_{ii}$  and  $\mathbf{M}_{ib}$  in Eq. (35) may be expressed explicitly in function of  $\mathbf{\Omega}$ ,  $\mathbf{\Psi}$  and  $\mathbf{K}$  such as:

$$\begin{aligned}\mathbf{M}_{ii} &= (\mathbf{K}_{ii}\mathbf{\Psi}_{1i} + \mathbf{K}_{ib}\mathbf{\Psi}_{1b})\mathbf{\Omega}_1^{-2}(\mathbf{\Psi}_{1b}^T\mathbf{K}_{bi} + \mathbf{\Psi}_{1i}^T\mathbf{K}_{ii}) + (\mathbf{K}_{ii}\mathbf{\Psi}_{2i} + \mathbf{K}_{ib}\mathbf{\Psi}_{2b})\mathbf{\Omega}_2^{-2}(\mathbf{\Psi}_{2b}^T\mathbf{K}_{bi} + \mathbf{\Psi}_{2i}^T\mathbf{K}_{ii}) \\ \mathbf{M}_{ib} &= (\mathbf{K}_{ii}\mathbf{\Psi}_{1i} + \mathbf{K}_{ib}\mathbf{\Psi}_{1b})\mathbf{\Omega}_1^{-2}(\mathbf{\Psi}_{1b}^T\mathbf{K}_{bb} + \mathbf{\Psi}_{1i}^T\mathbf{K}_{ib}) + (\mathbf{K}_{ii}\mathbf{\Psi}_{2i} + \mathbf{K}_{ib}\mathbf{\Psi}_{2b})\mathbf{\Omega}_2^{-2}(\mathbf{\Psi}_{2b}^T\mathbf{K}_{bb} + \mathbf{\Psi}_{2i}^T\mathbf{K}_{ib})\end{aligned}\quad (37)$$

Replacing  $\mathbf{M}_{ii}$  and  $\mathbf{M}_{ib}$  by their respective expression given by Eqs. (37) in Eq. (35), and taking into account the first line of Eqs. (22) differentiated twice with respect to time we obtain:

$$(\mathbf{K}_{ii}\mathbf{\Psi}_{1i} + \mathbf{K}_{ib}\mathbf{\Psi}_{1b})\mathbf{\Omega}_1^{-1}\ddot{\mathbf{q}}_1 + \mathbf{K}_{ii}\mathbf{u}_i + \mathbf{K}_{ib}\mathbf{u}_b = \mathbf{0} \quad (38)$$

As a consequence, the additional term  $\mathbf{T}$  in Eq. (31) is equal to:

$$\mathbf{T} = -(\mathbf{R}_{ii} - \mathbf{R}_{ib}\mathbf{R}_{bb}^{-1}\mathbf{R}_{bi})(\mathbf{K}_{ii}\mathbf{\Psi}_{1i} + \mathbf{K}_{ib}\mathbf{\Psi}_{1b})\mathbf{\Omega}_1^{-1}\ddot{\mathbf{q}}_1 \quad (39)$$

A more detailed expression of  $\mathbf{T}$  is obtained using Eq. (26):

$$\mathbf{T} = -(\mathbf{\Psi}_{2i}\mathbf{\Omega}_2^{-1} - \mathbf{\Psi}_{2i}\mathbf{\Omega}_2^{-1}\mathbf{\Psi}_{2b}^T(\mathbf{\Psi}_{2b}\mathbf{\Omega}_2^{-1}\mathbf{\Psi}_{2b}^T)^{-1}\mathbf{\Psi}_{2b}\mathbf{\Omega}_2^{-1})\mathbf{\Psi}_{2i}^T(\mathbf{K}_{ii}\mathbf{\Psi}_{1i} + \mathbf{K}_{ib}\mathbf{\Psi}_{1b})\mathbf{\Omega}_1^{-1}\ddot{\mathbf{q}}_1 \quad (40)$$

Due to the  $\mathbf{K}$ -orthogonality of the normal modes, it is easy to show that:

$$\mathbf{\Psi}_{2i}^T[\mathbf{K}_{ii}\mathbf{\Psi}_{1i} + \mathbf{K}_{ib}\mathbf{\Psi}_{1b}] = -\mathbf{\Psi}_{2b}^T(\mathbf{K}_{bi}\mathbf{\Psi}_{1i} + \mathbf{K}_{bb}\mathbf{\Psi}_{1b}) \quad (41)$$

As a consequence, Eq.(40) may be written as:

$$\mathbf{T} = (\mathbf{\Psi}_{2i}\mathbf{\Omega}_2^{-1} - \mathbf{\Psi}_{2i}\mathbf{\Omega}_2^{-1}\mathbf{\Psi}_{2b}^T(\mathbf{\Psi}_{2b}\mathbf{\Omega}_2^{-1}\mathbf{\Psi}_{2b}^T)^{-1}\mathbf{\Psi}_{2b}\mathbf{\Omega}_2^{-1})\mathbf{\Psi}_{2b}^T(\mathbf{K}_{bi}\mathbf{\Psi}_{1i} + \mathbf{K}_{bb}\mathbf{\Psi}_{1b})\mathbf{\Omega}_1^{-1}\ddot{\mathbf{q}}_1 \quad (42)$$

and yields:

$$\mathbf{T} = (\mathbf{\Psi}_{2i}\mathbf{\Omega}_2^{-1}\mathbf{\Psi}_{2b}^T - \mathbf{\Psi}_{2i}\mathbf{\Omega}_2^{-1}\mathbf{\Psi}_{2b}^T)(\mathbf{K}_{bi}\mathbf{\Psi}_{1i} + \mathbf{K}_{bb}\mathbf{\Psi}_{1b})\mathbf{\Omega}_1^{-1}\ddot{\mathbf{q}}_1 \quad (43)$$

greatly simplifying into:

$$\mathbf{T} = \mathbf{0} \quad (44)$$

The general transformation defined in Eq. (30) simplifies into Eq. (45) when it is assumed that the initially truncated high frequency free vibration modes do not dynamically participate to the reduced-order model ( $\ddot{\mathbf{q}}_2 = \mathbf{0}$ ) and that efforts only affect the boundary DoFs. This assumption is known as the pseudo-static approximation and the resulting condensation is usually named the Craig-Martinez method:

$$\mathbf{u}_i = (\mathbf{\Psi}_{1i} - \mathbf{R}_{ib}\mathbf{R}_{bb}^{-1}\mathbf{\Psi}_{1b})\mathbf{q}_1 + (\mathbf{R}_{ib}\mathbf{R}_{bb}^{-1})\mathbf{u}_b \quad (45)$$

This finally leads to the transformation:

$$\begin{pmatrix} \mathbf{u}_i \\ \mathbf{u}_b \end{pmatrix} = \mathbf{\Phi} \begin{pmatrix} \mathbf{q}_1 \\ \mathbf{u}_b \end{pmatrix} \text{ with } \mathbf{\Phi} = \begin{bmatrix} \mathbf{\Psi}_{1i} - \mathbf{R}_{ib}\mathbf{R}_{bb}^{-1}\mathbf{\Psi}_{1b} & \mathbf{R}_{ib}\mathbf{R}_{bb}^{-1} \\ \mathbf{0} & \mathbf{I}_{bb} \end{bmatrix} \quad (46)$$

## 5 Conclusion

The presentation of the Craig-Martinez component mode synthesis method usually involves a pseudo-static approximation. Through the details of the reduction equations, it is shown in this note that the change of variable associated with the Craig-Martinez method is obtained from a more general transformation under the assumption that the dynamic participation of the free vibration modes not retained in the reduction basis is zero.

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