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# Sufficient conditions for dissipativity on Duhem hysteresis model

Bayu Jayawardhana, Vincent Andrieu

**Abstract**— This paper presents sufficient conditions for dissipativity on the Duhem hysteresis model. The result of this paper describes the dissipativity property of several standard hysteresis models, including the backlash and Prandtl operator. It also allows the curve in the hysteresis diagram (the phase plot between the input and the output) to have negative gradient.

## I. INTRODUCTION

Hysteresis is a nonlinear element with memory and is a common phenomenon in physical systems. Although there are hysteretic elements that can be explained and understood based on its underlying physical law (such as, gear train or switch), many mathematical models of hysteresis are based on phenomenological modeling. Numerous models have been proposed to describe hysteresis, see for example [1], [11], [12], [13]. Based on specific properties inherent in these models, stability analysis of systems with hysteretic components has been carried out (e.g., [7], [10]) and controller designs have been proposed for such systems (see, for example, [2], [4], [5], [6]).

Hysteretic phenomenon in an electrical inductor and in an engine is known to dissipate energy by heat emission. If the hysteretic element completes a loop (in the phase plot), then the dissipated energy is defined by the area enclosed by the loop [1]. The energy loss can be described by constructing an 'energy' function whose rate is less than or equal to the quantity of the power transferred from an energy source [1], [3]. The constructions are not unique (c.f., the hysteresis potential function in [1] and the storage function in [3]). In the systems theory literature, the 'energy' function is called storage function [15], [16].

The existence of the storage function for a hysteretic component can be useful in the stability analysis of systems which contains such element. In Gorbet *et al* [3], a storage function is constructed for Preisach operator with non-negative weighting function, and is employed to show the stability of systems that use a hysteretic actuator. For relay and backlash operator, the corresponding storage function has been proposed in Brokate and Sprekels [1].

In nonlinear systems theory, dissipative nonlinear systems are characterized by the existence of a storage function [16]. More precisely, nonlinear systems defined by

$$\dot{x} = f(x, u), \quad y = h(x), \quad x(0) = x_0 \in \mathbb{R}^n$$

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with a locally Lipschitz  $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  and  $h : \mathbb{R}^n \rightarrow \mathbb{R}^p$ , is called dissipative with supply rate  $(y, u) \mapsto s(y, u)$ , if there exists a continuously differentiable function  $H : \mathbb{R}^n \rightarrow \mathbb{R}_+$  such that  $\frac{\partial H(x)}{\partial x} f(x, u) \leq s(y, u)$ . This characterization of dissipativity is interesting since the storage function provides an appropriate Lyapunov function candidate for the stability analysis of nonlinear systems. Moreover it is an efficient tool in nonlinear control design [8], [14], [15].

In this article, we present sufficient conditions for dissipativity on the Duhem hysteresis operator  $\Phi : u \mapsto \Phi(u), C^1(\mathbb{R}_+) \rightarrow C^1(\mathbb{R}_+)$  with supply rate  $\langle \widehat{\Phi(u)}, u \rangle$  (the precise definition of the Duhem hysteresis operator is given in Section II). In particular, we show the existence of a storage function  $(\Phi(u), u) \mapsto H(\Phi(u), u)$  such that

$$\frac{dH((\Phi(u))(t), u(t))}{dt} \leq \langle \widehat{(\Phi(u))}(t), u(t) \rangle. \quad (1)$$

The motivation of this dissipativity property stems from the physical law governing an electrical inductor. The magnetic flux  $\phi$  and the electric current  $I$  in an inductor can be related by an operator  $\Phi$ , i.e.,  $\phi = \Phi(I)$  (for instance, with a linear inductor model,  $\Phi(I) = LI$  where  $L$  is the inductance). Basic electrical law yields that  $\dot{\phi} = V$  where  $V$  is the voltage across the inductor. Moreover, the electrical power (defined by  $\langle V(t), I(t) \rangle$ ) transferred to the inductor is equal to  $\langle \widehat{(\Phi(I))}(t), I(t) \rangle$ . Since inductor is a passive electrical element and there is energy loss due to hysteresis, the power being stored in the inductor has to be less than or equal to the amount of power being transferred into the inductor. In this case, (1) holds with  $u = I$ .

## II. DUHEM HYSTERESIS OPERATOR

We denote by  $C^1(\mathbb{R}_+)$  the space of continuously differentiable functions  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ . The Duhem operator  $\Phi : C^1(\mathbb{R}_+) \rightarrow C^1(\mathbb{R}_+)$ ,  $u \mapsto \Phi(u) =: y$  is described by [11], [13], [17]

$$\dot{y}(t) = f_1(y(t), u(t))\dot{u}_+(t) + f_2(y(t), u(t))\dot{u}_-(t), \quad y(0) = y_0, \quad (2)$$

where  $\dot{u}_+(t) := \max\{0, \dot{u}(t)\}$ ,  $\dot{u}_-(t) := \min\{0, \dot{u}(t)\}$ . The functions  $f_1$  and  $f_2$  are defined appropriately according to the hysteresis curve obtained from experimental data.

Oh and Bernstein have shown that the Duhem model described by (2) is rate-independent ([13, Proposition 3.1]). This characterizes the fact that for every function  $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  continuous, non-decreasing and such that  $\lim_{t \rightarrow \infty} \rho(t) =$

$\infty$ , the Duhem operator  $\Phi$  satisfies

$$(\Phi(u \circ \rho))(t) = (\Phi(u))(\rho(t)), \quad \forall u \in C^1(\mathbb{R}_+), \quad \forall t \in \mathbb{R}_+.$$

An operator  $\Psi : C(\mathbb{R}_+) \rightarrow C(\mathbb{R}_+)$  is said to be *causal* if, for all  $\tau \geq 0$  and all  $v_1, v_2 \in C(\mathbb{R}_+)$ ,  $v_1 = v_2$  on  $[0, \tau]$  implies that  $\Psi(v_1) = \Psi(v_2)$  on  $[0, \tau]$ . With this definition, the Duhem model is causal if the solutions of ODE in (2) are unique for every  $u \in C^1(\mathbb{R}_+)$  and for every initial conditions. This is guaranteed, for example, if  $f_1$  and  $f_2$  are locally Lipschitz functions.

Following [10], the operator  $\Phi : C(\mathbb{R}_+) \rightarrow C(\mathbb{R}_+)$  is said to be a *hysteresis operator* if  $\Phi$  is causal and rate independent. The Duhem operator  $\Phi : C^1(\mathbb{R}_+) \rightarrow C^1(\mathbb{R}_+)$  is called *Duhem hysteresis operator* if (2) has unique solution for every  $u \in C^1(\mathbb{R}_+)$  and for every initial conditions  $y_0, u_0 \in \mathbb{R}^2$ .

In the following subsections, we describe several standard hysteresis operators that can be described by (2).

### A. Backlash operator

The backlash (or play) operator is widely used in mechanical models, for example, gear trains or of hydraulic servovalves. The mathematical analysis of backlash operator can be found in [1], [9], [12].

In order to relate the model used in these articles with the Duhem model (2), we describe the Backlash operator used in [1], [9], [12]. For all  $h \in \mathbb{R}_+$  and all  $\xi \in \mathbb{R}$ , we introduce a backlash operator  $\mathcal{B}_{h,\xi}$  defined on the space  $C_{\text{pm}}(\mathbb{R}_+)$  of piecewise monotone functions, by defining, for every  $u \in C_{\text{pm}}(\mathbb{R}_+)$ ,

$$\left. \begin{aligned} (\mathcal{B}_{h,\xi}(u))(0) &:= b_h(u(0), \xi) \\ (\mathcal{B}_{h,\xi}(u))(t) &:= b_h(u(t), (\mathcal{B}_{h,\xi}(u))(t_i)) \\ &\quad t \in (t_{i-1}, t_i], \quad i \in \mathbb{N} \end{aligned} \right\} \quad (3)$$

where  $0 = t_0 < t_1 < t_2 < \dots$  is a partition of  $\mathbb{R}_+$ , such that  $u$  is monotone on each of the intervals  $[t_{i-1}, t_i]$ ,  $i \in \mathbb{N}$  and where for each  $h \in \mathbb{R}_+$ , the function  $b_h : \mathbb{R}^2 \rightarrow \mathbb{R}$  is defined by

$$b_h(v, w) := \max\{v - h, \min\{v + h, w\}\}.$$

Here  $\xi$  plays the role of an ‘‘initial state’’. It is well known, see, for example, [1, page 42], that the operator  $\mathcal{B}_{h,\xi} : C_{\text{pm}}(\mathbb{R}_+) \rightarrow C(\mathbb{R}_+)$  can be extended uniquely to an operator  $\mathcal{B}_{h,\xi} : C(\mathbb{R}_+) \rightarrow C(\mathbb{R}_+)$ . The action of a backlash operator is illustrated in Figure 1.

The backlash operator  $\mathcal{B}_{h,\xi} : C^1(\mathbb{R}_+) \rightarrow C^1(\mathbb{R}_+)$  can be defined by the Duhem hysteresis operator (2) with

$$f_1(a, b) = \begin{cases} 1 & \text{if } a = b - h \\ 0 & \text{elsewhere,} \end{cases} \quad (4)$$

$$f_2(a, b) = \begin{cases} 1 & \text{if } a = b + h \\ 0 & \text{elsewhere,} \end{cases} \quad (5)$$

and with  $y_0 = \max\{u(0) - h, \min\{u(0) + h, \xi\}\}$ .

The Duhem model of backlash operator can also be easily extended to a generalized backlash operator. For instance, the generalized backlash operator  $\mathcal{B}_{\mu_1, \mu_2, h, \xi} : C^1(\mathbb{R}_+) \rightarrow$

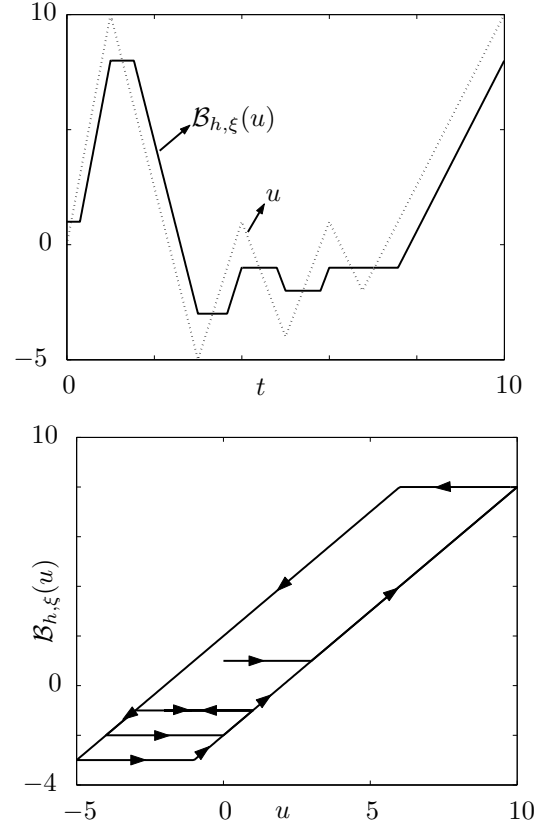


Fig. 1. Backlash operator  $\mathcal{B}_{h,\xi}$  ( $h = 2$ ,  $\xi = 1$ )

$C^1(\mathbb{R}_+)$  with  $\mu_1 > \mu_2 \geq 0$ ,  $h > 0$  and  $\xi \in \mathbb{R}$  can be defined by (2) where

$$f_1(a, b) = \begin{cases} \mu_1 & \text{if } a = \mu_1(b - h) \\ \mu_2 & \text{elsewhere} \end{cases} \quad (6)$$

$$f_2(a, b) = \begin{cases} \mu_1 & \text{if } a = \mu_1(b + h) \\ \mu_2 & \text{elsewhere} \end{cases} \quad (7)$$

and  $y_0$  is defined properly inside the hysteresis domain, i.e.  $\mu_1(u_0 - h) \leq y_0 \leq \mu_1(u_0 + h)$ .

### B. Prandtl operator

The Prandtl operator represents a more general type of hysteresis which, for certain input functions, exhibits nested loops in the corresponding input-output characteristics. Let  $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}$  be a compactly supported and globally Lipschitz function with Lipschitz constant 1. and let  $\mu$  be a signed Borel measure on  $\mathbb{R}_+$  such that  $|\mu|(K) < \infty$  for all compact sets  $K \subset \mathbb{R}_+$ , where  $|\mu|$  denotes the total variation of  $\mu$ . The Prandtl operator can be defined by [1]

$$(\mathcal{P}_\zeta(u))(t) = \int_0^\infty (\mathcal{B}_{h, \zeta(h)}(u))(t) \mu(dh), \quad \forall u \in C(\mathbb{R}_+), \quad \forall t \in \mathbb{R}_+. \quad (8)$$

In this case, the Duhem model of backlash operator can be used in (8).

To provide a concrete example of a Prandtl operator, we consider the Prandtl operator (8) with  $\zeta = 0$  and defined by

$$\begin{aligned} (\mathcal{P}_0(u))(t) &= \int_0^\infty (\mathcal{B}_{h,0}(u))(t) \chi_{[0,l]} dh \\ &= \int_0^l (\mathcal{B}_{h,0}(u))(t) dh, \quad \forall u \in C(\mathbb{R}_+), \forall t \in \mathbb{R}_+, \end{aligned} \quad (9)$$

where  $l > 0$  is a positive constant and  $\chi_{[0,l]}$  is the indicator function of the interval  $[0, l]$ . This operator exhibits nested loops as depicted in Figure 2.

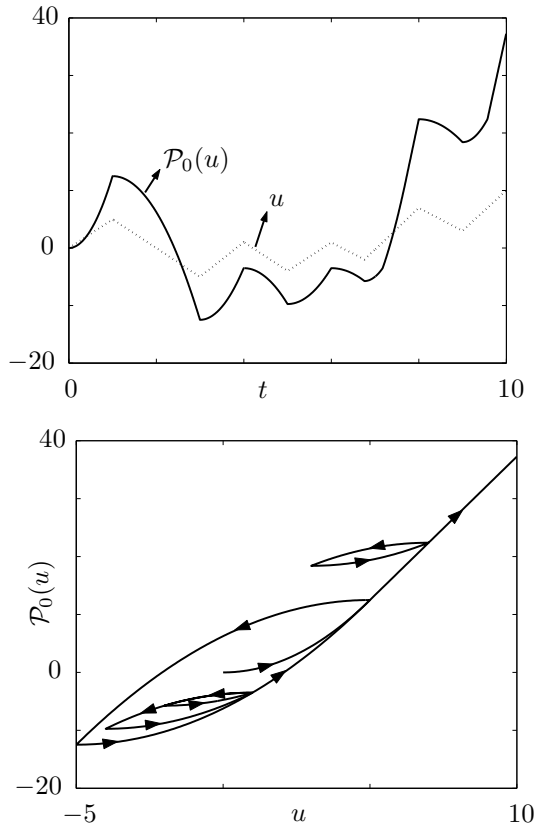


Fig. 2. Behaviour of the hysteresis operator  $\mathcal{P}_0$  with  $l = 5$ .

### III. MAIN RESULT

In the following sections, we consider the Duhem hysteresis operator  $\Phi$  defined in (2) with  $f_1, f_2 : \mathbb{R}^2 \rightarrow [0, \alpha]$ ,  $\alpha > 0$ .

We will describe the hysteresis curve that originates from a point in the hysteresis diagram. For every pair  $(y_0, u_0) \in \mathbb{R}^2$  in the hysteresis diagram, let  $w_{\Phi,1}(\cdot, y_0, u_0) : [u_0, \infty) \rightarrow \mathbb{R}$  be the solution of

$$\begin{aligned} x(\tau) - x(u_0) &= \int_{u_0}^\tau f_1(x(\sigma), \sigma) d\sigma, \\ x(u_0) &= y_0, \quad \forall \tau \in [u_0, \infty), \end{aligned}$$

and let  $w_{\Phi,2}(\cdot, y_0, u_0) : (-\infty, u_0] \rightarrow \mathbb{R}$  be the solution of

$$\begin{aligned} x(\tau) - x(u_0) &= \int_{u_0}^\tau f_2(x(\sigma), \sigma) d\sigma, \\ x(u_0) &= y_0, \quad \forall \tau \in (-\infty, u_0]. \end{aligned}$$

Using the above definitions, for every pair  $(y_0, u_0) \in \mathbb{R}^2$  in the hysteresis diagram, the curve  $w_\Phi(\cdot, y_0, u_0) : \mathbb{R} \rightarrow \mathbb{R}$  is defined by the concatenation of  $w_{\Phi,2}(\cdot, y_0, u_0)$  and  $w_{\Phi,1}(\cdot, y_0, u_0)$ :

$$w_\Phi(\tau, y_0, u_0) = \begin{cases} w_{\Phi,2}(\tau, y_0, u_0) & \forall \tau \in (-\infty, u_0) \\ w_{\Phi,1}(\tau, y_0, u_0) & \forall \tau \in [u_0, \infty) \end{cases} \quad (10)$$

The curve  $w_\Phi(\cdot, y_0, u_0)$  is the (unique) hysteresis curve where the curve defined in  $(-\infty, u_0]$  is obtained by applying a monotone decreasing  $u \in C^1(\mathbb{R}_+)$  to  $\Phi$  with  $u(0) = u_0$ ,  $\lim_{t \rightarrow \infty} u(t) = -\infty$ ,  $\Phi(u)(0) = y_0$  and, similarly, the curve defined in  $[u_0, \infty)$  is produced by introducing a monotone increasing  $u \in C^1(\mathbb{R}_+)$  to  $\Phi$  with  $u(0) = u_0$ ,  $\lim_{t \rightarrow \infty} u(t) = \infty$  and  $\Phi(u)(0) = y_0$ .

We define the storage function  $H : \mathbb{R}^2 \rightarrow \mathbb{R}$  for the hysteresis operator  $\Phi$  by

$$H(\sigma, \xi) = \sigma \xi - \int_0^\xi w_\Phi(\tau, \sigma, \xi) d\tau. \quad (11)$$

**Theorem 3.1:** Consider the Duhem hysteresis operator  $\Phi$  defined in (2) with locally Lipschitz functions  $f_1, f_2 : \mathbb{R}^2 \rightarrow [0, \alpha]$ ,  $\alpha > 0$ . Suppose that the following condition holds:

- (A)  $f_1(a, b) \geq f_2(a, b)$  for all  $(a, b) \in \mathbb{R} \times [0, \infty)$  and  $f_1(a, b) \leq f_2(a, b)$  for all  $(a, b) \in \mathbb{R} \times (-\infty, 0)$ .

Then for every  $u \in C^1(\mathbb{R}_+)$  and for every  $y_0 \in \mathbb{R}$ , the function  $t \mapsto H((\Phi(u))(t), u(t))$  with  $H$  as in (11) is differentiable and satisfies (1).

**PROOF.** Let  $u \in C^1(\mathbb{R}_+)$  and  $y_0 \in \mathbb{R}$ . First, we would prove that for all  $t \in \mathbb{R}_+$ ,  $H((\Phi(u))(t), u(t))$  exists. Using (11) and with Leibniz derivative rule and denoting  $y = (\Phi(u))$ , we have (if it exists)

$$\begin{aligned} \frac{dH(y(t), u(t))}{dt} &= y(t)\dot{u}(t) + \dot{y}(t)u(t) \\ &\quad - \frac{d}{dt} \int_0^{u(t)} w_\Phi(\tau, y(t), u(t)) d\tau \\ &= \dot{y}(t)u(t) - \int_0^{u(t)} \frac{d}{dt} w_\Phi(\tau, y(t), u(t)) d\tau, \end{aligned} \quad (12)$$

where the last equation is due to  $w_\Phi(u(t), y(t), u(t)) = y(t)$ . The first term in the RHS of (12) exist for all  $t \geq 0$  since  $y(t)$  satisfies (2). Therefore, in order to get (1), it remains to check whether the last term exists, is finite and satisfies

$$\int_0^{u(t)} \frac{d}{dt} w_\Phi(\tau, y(t), u(t)) d\tau \geq 0. \quad (13)$$

Let  $t \geq 0$ . In order to show the existence of the integrand and to compute (13), it suffices to show that, for every  $\tau \in \mathbb{R}$ ,

the following limit exists:

$$\lim_{\epsilon \searrow 0^+} \frac{1}{\epsilon} [w_\Phi(\tau, y(t+\epsilon), u(t+\epsilon)) - w_\Phi(\tau, y(t), u(t))], \quad (14)$$

and the limit is greater or equal to zero for every  $\tau \in \mathbb{R}$ .

For any  $\epsilon \geq 0$ , let us introduce the continuous function  $w_\epsilon : \mathbb{R} \rightarrow \mathbb{R}$  by

$$w_\epsilon(\tau) = w_\Phi(\tau, y(t+\epsilon), u(t+\epsilon)).$$

More precisely, for every  $\epsilon \geq 0$ ,  $w_\epsilon$  is the unique solution of

$$w_\epsilon(\tau) = \begin{cases} y(t+\epsilon) + \int_{u(t+\epsilon)}^\tau f_1(w_\epsilon(s), s) ds & \forall \tau \geq u(t+\epsilon), \\ y(t+\epsilon) + \int_{u(t+\epsilon)}^\tau f_2(w_\epsilon(s), s) ds & \forall \tau \leq u(t+\epsilon), \end{cases} \quad (15)$$

Note that  $w_\epsilon$  is  $C^1$  on  $\mathbb{R} \setminus u(t+\epsilon)$ . Moreover, we have  $w_0(\tau) = w_\Phi(\tau, y(t), u(t))$  for all  $\tau \in \mathbb{R}$  and

$$w_\epsilon(u(t+\epsilon)) = y(t+\epsilon) \quad , \quad \forall \epsilon \in \mathbb{R}_+ . \quad (16)$$

In order to show the existence of (14) and its limit being greater or equal to zero, we consider several case. First, we assume that  $\dot{u}(t) > 0$ . This implies that there exists a sufficiently small  $\gamma > 0$  such that for every  $\epsilon \in (0, \gamma]$ , we have  $u(t+\epsilon) > u(t)$  and

$$w_0(u(t+\epsilon)) = y(t) + \int_{u(t)}^{u(t+\epsilon)} f_1(w_0(s), s) ds.$$

Moreover, with the change of integration variable  $s = u(v)$ <sup>1</sup> we obtain

$$w_0(u(t+\epsilon)) = y(t) + \int_t^{t+\epsilon} f_1(w_0(u(v)), u(v)) \dot{u}(v) dv$$

for all  $\epsilon \in [0, \gamma]$ .

The functions  $\epsilon \mapsto w_0(u(t+\epsilon))$  and  $\epsilon \mapsto y(t+\epsilon)$  with  $\epsilon \in (0, \gamma]$  are two  $C^1$  functions which are solutions of the same locally Lipschitz ODE and with the same initial value. By uniqueness of solution, we get  $w_0(u(t+\epsilon)) = y(t+\epsilon)$ . This fact together with (16) shows that

$$w_\epsilon(u(t+\epsilon)) = w_0(u(t+\epsilon)) \quad \forall \epsilon \in [0, \gamma].$$

Since for every  $\epsilon \in (0, \gamma]$  the two functions  $w_\epsilon(\tau)$  and  $w_0(\tau)$  satisfy the same ODE for<sup>2</sup>  $\tau > u(t+\epsilon)$ , we have

$$w_\epsilon(\tau) = w_0(\tau) \quad , \quad \forall \tau \geq u(t+\epsilon),$$

for all  $\epsilon \in [0, \gamma]$ . This implies that

$$\lim_{\epsilon \searrow 0^+} \frac{1}{\epsilon} [w_\epsilon(\tau) - w_0(\tau)] = 0, \quad (17)$$

<sup>1</sup>This change is allowed since for every  $\epsilon \in [0, \gamma]$ ,  $u$  is a strictly increasing function from  $[t, t+\epsilon]$  toward  $[u(t), u(t+\epsilon)]$ .

<sup>2</sup>We have for all  $\tau > u(t+\epsilon)$  :

$$\frac{dw_\epsilon(\tau)}{d\tau} = f_1(w_\epsilon(\tau), \tau) \quad , \quad \frac{dw_0(\tau)}{d\tau} = f_1(w_0(\tau), \tau)$$

for all  $\tau > u(t)$ .

It remains to check (14) for  $\tau \leq u(t)$ . Since  $\dot{u}(t) > 0$ , there exists  $\gamma > 0$  such that we have  $\tau \leq u(t) < u(s) < u(t+\epsilon)$  and  $\dot{u}(s) > 0$  for all  $s$  in  $(t, t+\epsilon)$ , and all  $\epsilon$  in  $(0, \gamma)$ . It follows from (15) and Assumption (A) that for every  $\epsilon \in (0, \gamma)$ :

$$\begin{aligned} \frac{dw_\epsilon(u(s))}{ds} &= f_2(w_\epsilon(u(s)), u(s)) \dot{u}(s) \quad \forall s \in (t, t+\epsilon), \\ &\leq f_1(w_\epsilon(u(s)), u(s)) \dot{u}(s) \quad \forall s \in (t, t+\epsilon), \end{aligned} \quad (18)$$

and the function  $y$  satisfies

$$\frac{dy(s)}{ds} = f_1(y(s), u(s)) \dot{u}(s) \quad , \quad \forall s \in (t, t+\epsilon).$$

Since  $w_\epsilon(u(t+\epsilon)) = y(t+\epsilon)$  and using the comparison principle (in reverse direction), we get that for every  $\epsilon \in [0, \gamma]$ :

$$w_\epsilon(u(s)) \geq y(s) \quad , \quad \forall s \in [t, t+\epsilon].$$

Since the two functions  $w_\epsilon(\tau)$  and  $w_0(\tau)$  for  $\tau \leq u(t)$  are two solutions of the same ODE, it follows that<sup>3</sup>  $w_\epsilon(\tau) \geq w_0(\tau)$  for all  $\tau \leq u(t)$  and we get that (if exists) :

$$\lim_{\epsilon \searrow 0^+} \frac{1}{\epsilon} [w_\epsilon(\tau) - w_0(\tau)] \geq 0 \quad , \quad \forall \tau \leq u(t). \quad (19)$$

In the following, we compute the bound of (19) in order to show the existence of (19). Note that for every  $\epsilon \in [0, \gamma]$ ,

$$\begin{aligned} |w_\epsilon(\tau) - w_0(\tau)| &\leq |y(t+\epsilon) - y(t)| \\ &+ \left| \int_{u(t+\epsilon)}^{u(t)} f_1(w_\epsilon(s), s) ds \right| \\ &+ \left| \int_{u(t)}^\tau f_1(w_\epsilon(s), s) - f_1(w_0(s), s) ds \right| \\ &\leq |y(t+\epsilon) - y(t)| + \int_{u(t)}^{u(t+\epsilon)} |f_1(w_\epsilon(s), s)| ds \\ &+ \int_\tau^{u(t)} |f_1(w_\epsilon(s), s) - f_1(w_0(s), s)| ds, \end{aligned}$$

for all  $\tau \leq u(t)$ . By the locally Lipschitz property of  $f_1$ , by the boundedness of  $f_1$  and by the boundedness of  $w_\epsilon$  on  $[\tau, u(t)]$  for all  $\epsilon \in [0, \gamma]$ , we obtain

$$\begin{aligned} |w_\epsilon(\tau) - w_0(\tau)| &\leq |y(t+\epsilon) - y(t)| \\ &+ \int_\tau^{u(t)} L |w_\epsilon(s) - w_0(s)| ds + \alpha |u(t+\epsilon) - u(t)| , \end{aligned}$$

where  $L$  is the Lipschitz constant of  $f_1$  on  $[w_{\min}, w_{\max}] \times [\tau, u(t)]$  with

$$\begin{aligned} w_{\min} &= \min_{(c,s) \in [0,\gamma] \times [\tau, u(t)]} w_c(s) \\ w_{\max} &= \max_{(c,s) \in [0,\gamma] \times [\tau, u(t)]} w_c(s) . \end{aligned}$$

<sup>3</sup>Otherwise there exist  $\tau_1 < \tau_2$  such that  $w_\epsilon(\tau_1) = w_0(u(\tau_1))$  and  $w_\epsilon(\tau_2) > w_0(u(\tau_2))$  which contradict the uniqueness of the solution of the locally Lipschitz ODE.

With Gronwall's lemma, this implies that for every  $\epsilon \in [0, \gamma]$

$$|w_\epsilon(\tau) - w_0(\tau)| \leq \exp((u(t) - \tau)L) \left[ |y(t + \epsilon) - y(t)| + \alpha |u(t + \epsilon) - u(t)| \right],$$

for all  $\tau \leq u(t)$ . Hence

$$\begin{aligned} \lim_{\epsilon \searrow 0^+} \frac{1}{\epsilon} |w_\epsilon(\tau) - w_0(\tau)| \\ \leq \exp((u(t) - \tau)L) \left[ |f_1(y(t), u(t))| + \alpha \right] \dot{u}(t), \end{aligned}$$

for all  $\tau \leq u(t)$ .

We can use similar arguments to prove the case when  $\dot{u}(t) < 0$ .

Finally, when  $\dot{u}(t) = 0$ , we simply get

$$\lim_{\epsilon \searrow 0^+} \frac{1}{\epsilon} |w_\epsilon(\tau) - w_0(\tau)| = 0,$$

by continuity of the above bound.  $\square$

*Remark 3.2:* The storage function  $H$  in the Theorem 3.1 is non-negative and  $H(y, 0) = 0$  for all  $y \in \mathbb{R}$ . Indeed, without loss of generality, let us consider the case when  $u \geq 0$ . In this case, it can be checked that for every  $y \in \mathbb{R}$ ,  $w_\Phi(\tau, y, u) \leq y$  for all  $\tau \in [0, u]$ . It follows that

$$\begin{aligned} H(y, u) &= \int_0^u y - w_\Phi(\tau, y, u) \, d\tau \\ &\geq 0, \end{aligned}$$

for all  $u \geq 0$  and  $y \in \mathbb{R}$ .  $\triangle$

*Remark 3.3:* The non-negativity assumption imposed on functions  $f_1, f_2$  in Theorem 3.1 can be relaxed into locally Lipschitz function  $f_1, f_2 : \mathbb{R}^2 \rightarrow [-\alpha, \alpha]$ . Using the same proof of the theorem and using the same storage function  $H$ , we can obtain the same result. However, other conditions need to be imposed on  $f_1, f_2$  if we want a lower bounded  $H$ .  $\triangle$

*Remark 3.4:* Related to the dissipativity concepts by Willems [16], the storage function as constructed in the Theorem 3.1 is equal to the available storage function as defined in [16]. In order to show this, given  $(y_0, u_0)$  in  $\mathbb{R}^2$ , let us integrate (12) from 0 to  $T > 0$ , along the solution  $y$  of (2) with  $y(0) = y_0$  and  $u \in C^1(\mathbb{R}_+)$  with  $u(0) = u_0$ :

$$\begin{aligned} H(y(T), u(T)) - H(y_0, u_0) &= \int_0^T \dot{y}(\tau) u(\tau) \, d\tau \\ &\quad - \int_0^T \int_0^{u(\sigma)} \frac{d}{d\sigma} w_\Phi(\tau, y(\sigma), u(\sigma)) \, d\tau \, d\sigma, \end{aligned}$$

or equivalently,

$$\begin{aligned} - \int_0^T \dot{y}(\tau) u(\tau) \, d\tau &= H(y_0, u_0) - H(y(T), u(T)) \\ &\quad - \int_0^T \int_0^{u(\sigma)} \frac{d}{d\sigma} w_\Phi(\tau, y(\sigma), u(\sigma)) \, d\tau \, d\sigma. \end{aligned}$$

Taking the supremum in both sides of this equation with arguments  $T$  and  $u \in C^1(\mathbb{R}_+)$  with  $u(0) = u_0$ , we get

$$\begin{aligned} \sup_{\substack{T > 0 \\ u \in C^1(\mathbb{R}_+), u(0) = u_0}} - \int_0^T \dot{y}(\tau) u(\tau) \, d\tau &= H(y_0, u_0) \\ &+ \sup_{\substack{T > 0 \\ u \in C^1(\mathbb{R}_+), u(0) = u_0}} \left[ - H(y(T), u(T)) \right. \\ &\quad \left. - \int_0^T \int_0^{u(\sigma)} \frac{d}{d\sigma} w_\Phi(\tau, y(\sigma), u(\sigma)) \, d\tau \, d\sigma \right]. \end{aligned}$$

Note that  $H$  is non-negative according to Remark 3.2 and the integrand on the RHS is also non-negative according to (13). Therefore, we only need to check whether there exist  $T > 0$  and  $u \in C^1(\mathbb{R}_+)$  with  $u(0) = u_0$  such that the supremum value in the RHS is equal to zero. Following the proof of Theorem 3.1, we can choose arbitrary  $T > 0$  and arbitrary monotone function  $u \in C^1(\mathbb{R}_+)$  with  $u(T) = 0$  which give the desired result. In other words,

$$\sup_{\substack{T > 0 \\ u \in C^1(\mathbb{R}_+), u(0) = u_0}} - \int_0^T \dot{y}(\tau) u(\tau) \, d\tau = H(y_0, u_0). \quad (20)$$

In [16], the LHS of (20) is called the available storage function with respect to the supply rate  $\dot{y}(t)u(t)$ .  $\triangle$

*Remark 3.5:* As described in Remark 3.4, the storage function  $H$  as in the Theorem 3.1 corresponds to the maximum available energy that can be extracted from the system. On the other hand, the supplied energy is given by

$$\int_0^t \left\langle \overbrace{\Phi(u)}^{\cdot}(\tau), u(\tau) \right\rangle \, d\tau.$$

Thus (12) shows that the rate of the available energy at time  $t \geq 0$  is equal to the rate of supplied energy minus  $\int_0^{u(t)} \frac{d}{dt} w_\Phi(\tau, y(t), u(t)) \, d\tau$ . The latter component can have physical interpretation as the rate of dissipated energy at time  $t$ .  $\triangle$

#### IV. PASSIVITY FOR THE BACKLASH OPERATOR

The following theorem is used to describe passivity for the backlash operator as described in Section II-A. Note that the condition on  $f_1$  and  $f_2$  which is assumed in Theorem 3.1 (i.e. Assumption (A)) excludes the Duhem model for backlash operator.

*Theorem 4.1:* Consider the Duhem hysteresis operator  $\Phi$  defined in (2) with  $f_1, f_2$  as in (4)-(5). Then for every  $u \in C^1(\mathbb{R}_+)$  and for every  $y_0 \in \mathbb{R}$ , the function  $t \mapsto H((\Phi(u))(t), u(t))$  with  $H$  as in (11) is differentiable and satisfies (1).

**PROOF.** The proof is similar to that of Theorem 3.1. Notice that  $H$  can be given explicitly as follows:

$$H(y(t), u(t)) = \begin{cases} 0 & y(t) \in [-h, h] \\ \frac{1}{2}(y(t) - h)^2 & \text{elsewhere.} \end{cases}$$

Taking the time derivative of  $H$ , we get

$$\dot{H}(y(t), u(t)) = \begin{cases} 0 & y(t) \in [-h, h] \\ (y(t) - h)\dot{y}(t) & \text{elsewhere.} \end{cases} \quad (21)$$

It can be checked that when  $y(t) \in [-h, h]$ , we have the following two cases: 1.  $u(t) < 0$  and  $\dot{y}(t) \leq 0$ ; or 2.  $u(t) \geq 0$  and  $\dot{y}(t) \geq 0$ . In both scenario,  $u(t)\dot{y}(t) \geq 0$ . Hence  $\dot{H}(y(t), u(t)) = 0 \leq u(t)\dot{y}(t)$  whenever  $y(t) \in [-h, h]$ .

For the case when  $y(t) > h$  and  $u(t) = y(t) - h$  (i.e.,  $(y(t), u(t))$  is located at the leftmost curve in the backlash diagram), then (21)  $\Rightarrow \dot{H}(y(t), u(t)) = u(t)\dot{y}(t)$ . If  $y(t) > h$  and  $u(t) > y(t) - h$ , then it can be checked that  $\dot{y}(t) \geq 0$  for all  $\dot{u}(t) \in \mathbb{R}$ . This implies that (21)  $\Rightarrow \dot{H}(y(t), u(t)) \leq u(t)\dot{y}(t)$ . Note that when  $y(t) > h$  and  $u(t) < y(t) - h$ , the point  $(y(t), u(t))$  is not in the backlash diagram. These arguments show that  $\dot{H}(y(t), u(t)) \leq u(t)\dot{y}(t)$  for all  $y(t) > h$ .

Following the same arguments as above for  $y(t) < -h$ , we obtain  $\dot{H}(y(t), u(t)) \leq u(t)\dot{y}(t)$  for all  $y(t) < -h$ . This concludes the proof.  $\square$

## V. CONCLUSION

In this note, we have presented a possible characterization of dissipativity of some hysteresis operators. Based on the Duhem model, we have given sufficient conditions guaranteeing dissipativity of the operator.

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