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# Synthesis of a global asymptotic stabilizing feedback law for a system satisfying two different sector conditions

Vincent Andrieu, Christophe Prieur, Sophie Tarbouriech and Denis Arzelier

**Abstract**—Global asymptotic stabilization for a class of nonlinear systems is addressed. The dynamics of these systems are composed of a linear part to which is added some nonlinearities which satisfy two different sector bound conditions depending whether the state is closed or distant from the origin. The approach described here is based on the uniting of control Lyapunov functions as introduced in [2]. The stabilization problem may be recast as an LMI optimization problem for which powerful semidefinite programming softwares exist. This is illustrated by a numerical example.

## I. INTRODUCTION

There is an extensive literature on the design of nonlinear stabilizers providing numerous techniques which apply on specific classes of nonlinear systems. The class of system under interest in this paper is the one described by nonlinear functions satisfying sector bound conditions. This class of nonlinearity includes many different memoryless functions (see e.g., [9, Chapter 6] for an introduction on this topic) such as saturations (see e.g. [17], [8], [7] for design techniques of control system with such nonlinearities). To oppose to what has been done in these papers, two different sector conditions are considered: one sector condition when the state is near the equilibrium and one other sector condition when the state is distant from the equilibrium. This distinction between small and large values of the distance from the state to the equilibrium allows us to better describe the nonlinear system. Moreover we remark that encompassing both sector conditions into one global sector condition may lead to a too conservative synthesis problem which may not have a solution (see the example of Section V-A below).

This motivates us to consider the local sector condition and the non-local one separately and to design successively 1) a local stabilizer with a basin of attraction containing a compact set and 2) a non-local controller such that the previous compact set is globally attractive. After that, to design a global stabilizer, we suggest to piece together both controllers. Many different techniques exist to unit two different feedback laws. Let us cite the use of hybrid controllers to unit them (see [12], [13], [14], [19]). In the present paper we apply the technique introduced in [2] where a continuous solution to the uniting problem is given through the construction of a uniting control Lyapunov function<sup>1</sup>.

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<sup>1</sup>see [4] for a definition of CLF.

In [2] some conditions are given to provide a global stabilizer from a local control Lyapunov function and a non-local one<sup>2</sup>. In the following, we succeed to rewrite these sufficient conditions in terms of LMIs and we introduce a global stabilizer for the control of systems satisfying two different sector conditions.

To be more precise, consider the system defined by its state-space equation:

$$\dot{x} = Ax + Bu + G\phi(Lx) \quad (1)$$

where the state vector  $x$  is in  $\mathbb{R}^n$ .  $(A, B, G, L)$  are matrices respectively in  $\mathbb{R}^{n \times n}$ ,  $\mathbb{R}^{n \times m}$ ,  $\mathbb{R}^{n \times p}$  and  $\mathbb{R}^{q \times n}$ . Moreover  $u$  in  $\mathbb{R}^m$  is the control input and  $\phi(z) : \mathbb{R}^q \rightarrow \mathbb{R}^p$  is a nonlinear locally Lipschitz function such that  $\phi(0) = 0$ .

One way to design a global stabilizer for system (1) is to use circle and Popov criteria (see [3]) under the assumption that the nonlinear function  $\phi$  satisfies some sector bound conditions<sup>3</sup>:

$$(\phi(Lx) - MLx)'(\phi(Lx) - NLx) \leq 0, \forall x \in \mathbb{R}^n, \quad (2)$$

where  $M$  and  $N$  are two given matrices in  $\mathbb{R}^{q \times p}$ . Following<sup>4</sup> [5], [11], a constructive LMI condition allowing to design a state feedback control law solving the stabilizing problem may be exhibited.

The aim of this paper is to study the case in which the function  $\phi$  satisfies two different sector conditions depending on the distance between  $x$  and the origin. The idea of the design is then to apply techniques inspired by [2] to unite local and non-local controllers and to provide a global stabilizer.

Assumptions on the nonlinear function  $\phi$  introduced in (1) can be given as follows:

**Assumption 1: Local sector condition.** *There exist a positive real number  $v_0$ , two matrices  $(M_0, N_0)$  in  $\mathbb{R}^{p \times q} \times \mathbb{R}^{p \times q}$  such that, for all  $|x| \leq v_0$ , we have:*

$$(\phi(Lx) - M_0 Lx)'(\phi(Lx) - N_0 Lx) \leq 0. \quad (3)$$

<sup>2</sup>see [2] for definitions of local and non-local CLFs.

<sup>3</sup>Note that other classes of sector conditions are possible. In particular we may consider the generalized sector conditions written as

$$(\phi(z) - Mz)'D(\phi(z) - Nz) \leq 0,$$

where  $D$  is any given diagonal positive definite matrix (as in [6], [18]). Despite the fact that considering such generalized sector conditions is possible, we restrict our attention to sector bound condition as (2) to ease the exposition of our results.

<sup>4</sup>Note that in [5] is addressed a more involved problem since some saturations on the input are considered.

**Assumption 2: Non-local sector condition.** *There exist a positive real number  $v_\infty < v_0$ , two matrices  $(M_\infty, N_\infty)$  in  $\mathbb{R}^{p \times q} \times \mathbb{R}^{p \times q}$  such that, for all  $|x| \geq v_\infty$ , we have:*

$$(\phi(Lx) - M_\infty Lx)' (\phi(Lx) - N_\infty Lx) \leq 0. \quad (4)$$

System satisfying both Assumptions 1 and 2 is of interest since local and non-local approximations of nonlinear global dynamics may be found in the literature (see for instance [1]). Moreover, as shown in the example introduced in Section V, it might be useful to split a global sector condition in two pieces (a local and a non-local one) in order to get a solution where the usual LMI-based sufficient conditions obtained from [5], [11] are too conservative.

In this paper, we address the following problem:

**Problem:** *Under Assumptions 1 and 2, can we design a nonlinear control law  $u = \alpha(x)$  with  $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^m$  a continuous function ensuring global asymptotic stabilization of the origin for the system (1) ?*

Before considering this global stabilization problem, we first consider each sector separately in Section II and we design a local (resp. a non-local) controller  $u = \alpha_0(x)$  (resp.  $u = \alpha_\infty(x)$ ) using the local (resp. non-local) sector condition of Assumption 1 (resp. of Assumption 2). After this step, a new controller, which is equal to the local controller  $u = \alpha_0(x)$  on a neighborhood of the origin and equal to the non-local controller  $u = \alpha_\infty(x)$  outside a compact set, is designed. This construction is based on [2] and is considered in Section III. We then formalize a sufficient condition, expressed in terms of the existence of solutions to LMIs constraints, allowing us to address the global stabilization problem in one step in Section IV. Two numerical examples illustrate the previous results in Section V, and Section VI contains some concluding remarks.

Note that all proofs have been removed due to space limitation and can be obtained from the authors upon request.

**Notation.** The Euclidian norm is denoted by  $|\cdot|$ . For a positive real number  $n$ ,  $I_n$  (resp.  $0_{n,m}$ ) denotes the identity matrix (resp. the null matrix) in  $\mathbb{R}^{n \times n}$  (resp. in  $\mathbb{R}^{n \times m}$ ). The subscripts may be omitted when there is no ambiguity. Moreover for a vector  $x$  the diagonal matrix defined by the entries of  $x$  is noted  $\text{Diag}(x)$ , and for a matrix  $M$ ,  $\text{Sym}(M) = M + M'$ .

## II. DESIGN OF LOCAL AND OF NON-LOCAL CONTROLLERS

### A. Local case

In this section, we consider Assumption 1 and we design a state feedback ensuring local asymptotic stabilization of the origin for the system (1). Note that if we introduce:

$A_0 = A + GM_0L$ ,  $\phi_0(z) = \phi(z) - M_0z$ ,  $S_0 = N_0 - M_0$ , system (1) can be rewritten as:

$$\dot{x} = A_0 x + B u + G \phi_0(Lx), \quad (5)$$

and the local sector condition becomes

$$\phi_0(Lx)' (\phi_0(Lx) - S_0 Lx) \leq 0, \quad \forall |x| \leq v_0. \quad (6)$$

Hence, inspired by [5], [11], we can state a sufficient condition to get local asymptotic stabilization of the origin:

**Proposition 2.1:** *Suppose Assumption 1 is satisfied (hence (6) holds). If there exists a symmetric positive definite matrix  $W_0$  in  $\mathbb{R}^{n \times n}$ , two matrices  $H_0$  in  $\mathbb{R}^{m \times n}$ , and  $J_0$  in  $\mathbb{R}^{m \times p}$  satisfying the LMI:*

$$\begin{bmatrix} \text{Sym}(A_0 W_0 + B H_0) & \star \\ J_0' B' + G' + S_0 L W_0 & -2I_p \end{bmatrix} < 0, \quad (7)$$

then the control law  $u = \alpha_0(x)$  where

$$\alpha_0(x) = K_0 x + J_0 \phi_0(Lx), \quad (8)$$

with  $K_0 = H_0 W_0^{-1}$  makes the origin of the system a locally asymptotically stable equilibrium, with basin of attraction containing the set

$$\mathcal{E}(W_0^{-1}, R_0) = \{x \in \mathbb{R}^n, x' W_0^{-1} x \leq R_0\}$$

where  $R_0$  is any positive real number satisfying

$$R_0 W_0 - v_0^2 I_n \leq 0. \quad (9)$$

### B. Non-local case

A result similar to Proposition 2.1 can be obtained when considering Assumption 2. Indeed, if we introduce:

$$A_\infty = A + G M_\infty L, \quad \phi_\infty(z) = \phi(z) - M_\infty z, \\ S_\infty = N_\infty - M_\infty,$$

system (1) becomes:

$$\dot{x} = A_\infty x + B u + G \phi_\infty(Lx), \quad (10)$$

and the non-local sector condition (i.e. inequality (4)) becomes:

$$\phi_\infty(Lx)' (\phi_\infty(Lx) - S_\infty Lx) \leq 0, \quad \forall |x| \geq v_\infty. \quad (11)$$

We can state a sufficient condition to get global asymptotic stabilization of a set containing the origin:

**Proposition 2.2:** *Suppose Assumption 2 is satisfied (hence (11) holds). If there exists a symmetric positive definite matrix  $W_\infty$  in  $\mathbb{R}^{n \times n}$ , two matrices  $H_\infty$  in  $\mathbb{R}^{m \times n}$ , and  $J_\infty$  in  $\mathbb{R}^{m \times p}$  satisfying the LMI:*

$$\begin{bmatrix} \text{Sym}(A_\infty W_\infty + B H_\infty) & \star \\ J_\infty' B' + G' + S_\infty L W_\infty & -2I_p \end{bmatrix} < 0, \quad (12)$$

then the control law  $u = \alpha_\infty(x)$  where

$$\alpha_\infty(x) = K_\infty x + J_\infty \phi_\infty(Lx), \quad (13)$$

with  $K_\infty = H_\infty W_\infty^{-1}$  makes the solutions of the closed-loop system complete and the set

$$\mathcal{E}(W_\infty^{-1}, r_\infty) = \{x \in \mathbb{R}^n, x' W_\infty^{-1} x \leq r_\infty\},$$

globally and asymptotically stable where  $r_\infty$  is any positive real number such that

$$v_\infty^2 I_n - W_\infty r_\infty \leq 0. \quad (14)$$

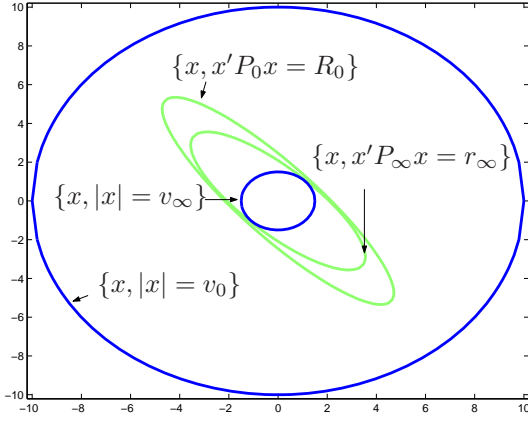


Fig. 1. Illustration of the Covering Assumption

### III. DESIGN OF A GLOBALLY AND ASYMPTOTICALLY STABILIZING CONTROLLER

In this section, we assume that we have solved the local stabilization problem and the non-local one following Propositions 2.1 and 2.2. Hence, we have in hand the matrices  $P_0$ ,  $P_\infty$ ,  $K_0$ ,  $K_\infty$ ,  $J_0$  and  $J_\infty$  such that the LMI (7) and (12) are satisfied (with  $P_0 = W_0^{-1}$  and  $P_\infty = W_\infty^{-1}$ ) and both controllers  $\alpha_0$  and  $\alpha_\infty$  defined by (8) and (13) respectively. We wish now to unite these two controllers to get a controller making the origin a global and asymptotic stable equilibrium.

To solve this problem, we follow the uniting strategy introduced in [2]. Following this procedure, the first step is to unite the local CLF  $x \mapsto x'P_0x$  and the non-local one  $x \mapsto x'P_\infty x$ . In order to do this, we need an extra Assumption expressing the fact that the two sets, in which we have a stability property, overlap.

**Assumption 3: Covering Assumption.** *There exist two positive real numbers  $R_0$  and  $r_\infty$  such that (9) and (14) are satisfied and such that*

$$r_\infty P_0 - R_0 P_\infty < 0. \quad (15)$$

In Figure 1, an illustration of the covering Assumption is presented (it is used the numerical values of Section V-A). This assumption implies that we have the following inclusions:

$$\begin{aligned} \{x, |x| \leq v_\infty\} &\subseteq \\ &\mathcal{E}(P_\infty, r_\infty) \subset \mathcal{E}(P_0, R_0) \\ &\subseteq \{x, |x| \leq v_0\}. \end{aligned}$$

To get a global stabilizing control law, we have the following result:

**Theorem 3.1:** *Assume that Assumptions 1, 2 and 3 hold. If there exist two matrices  $K_m$  in  $\mathbb{R}^{m \times n}$  and  $J_m$  in  $\mathbb{R}^{m \times p}$ , and four positive real numbers  $(\mu_1, \mu_2, \mu_3, \mu_4)$  such that the following LMIs are satisfied:*

$$\begin{bmatrix} \text{Sym}(P_0 [A + BK_m]) & \star \\ (J'_m B' + G')P_0 & 0 \end{bmatrix} - \mu_1 Q_0 - \mu_2 Q_\infty < 0, \quad (16)$$

$$\begin{bmatrix} \text{Sym}(P_\infty [A + BK_m]) & \star \\ (J'_m B' + G')P_\infty & 0 \end{bmatrix} - \mu_3 Q_0 - \mu_4 Q_\infty < 0, \quad (17)$$

where

$$\begin{aligned} Q_0 &= \begin{bmatrix} \text{Sym}(L'M'_0 N_0 L) & \star \\ -(M_0 + N_0)L & 2I_p \end{bmatrix}, \\ Q_\infty &= \begin{bmatrix} \text{Sym}(L'M'_\infty N_\infty L) & \star \\ -(M_\infty + N_\infty)L & 2I_p \end{bmatrix}, \end{aligned} \quad (18)$$

then there exists a continuous function  $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^m$  such that the control law  $u = \alpha(x)$  makes the origin a globally asymptotically stable equilibrium for system (1).

A possible control law ensuring a global asymptotic stabilization of the origin of system (1) is given in [2, Proposition 4] and is expressed as:

$$\alpha(x) = \mathcal{H}(x) - k c(x) \frac{\partial V}{\partial x}(x) B \quad (19)$$

with  $\mathcal{H}$  a continuous function such that

$$\mathcal{H}(x) = \begin{cases} \alpha_0(x) & \text{if } V_\infty(x) \leq r_\infty, \\ \alpha_\infty(x) & \text{if } V_0(x) \geq R_0. \end{cases}$$

The function  $c$  is any continuous function such that<sup>5</sup>

$$c(x) \begin{cases} = 0 & \text{if } V_0(x) \geq R_0 \text{ or } V_\infty(x) \leq r_\infty, \\ > 0 & \text{if } V_0(x) < R_0 \text{ and } V_\infty(x) > r_\infty. \end{cases}$$

$k$  is a positive real number sufficiently large and finally  $V$  is a global CLF (CLF) for system (1) obtained following the procedure introduced in [2, Theorem 1] which enables to unite both CLFs  $V_0$  and  $V_\infty$ .

### IV. DESIGN IN ONE STEP

Following the design strategy exposed in the previous sections, a stabilizing control law for system (1) satisfying Assumptions 1 and 2 can be performed if we succeed in solving the following algorithm:

*Design separately a local and a non-local CLF (i.e.  $P_0 = W_0^{-1}$  and  $P_\infty = W_\infty^{-1}$ ) via the LMIs (7) and (12), and check if*

- 1) *they satisfy the covering Assumption (15) with  $R_0$  and  $r_\infty$  satisfying (9) and (14);*
- 2) *they satisfy the LMIs feasibility conditions (16) and (17) to be united.*

In this section, we investigate the possibility of solving this problem in one shot. In other words, we wish to find an LMI formulation to prove the existence of matrices  $P_0$  and  $P_\infty$  satisfying the conditions in items 1) and 2) of the previous algorithm.

<sup>5</sup>For instance, with Assumption 3 we can take:

$$c(x) = \max\{0, (R_0 - V_0(x))(V_\infty(x) - r_\infty)\}.$$

$$\begin{bmatrix} \text{Sym}(AS + BH_m) & \star & \star \\ S - W_0 + AS + BH_m & -2W_0 & BJ_m + G - W_0 \\ M_0LS + M_\infty LS + J'_m B' + G' + AS + BH_m & \star & -4I_n - \text{Sym}(BJ_m + G) \end{bmatrix} < 0 \quad (20)$$

$$\begin{bmatrix} \text{Sym}(AS + BH_m) & \star & \star \\ S - W_\infty + AS + BH_m & -2W_\infty & BJ_m + G - W_\infty \\ M_0LS + M_\infty LS + J'_m B' + G' + AS + BH_m & \star & -4I_n - \text{Sym}(BJ_m + G) \end{bmatrix} < 0 \quad (21)$$

### A. About the covering Assumption

Assumption 3 may fail when considering an arbitrary pair of matrices  $P_0$  and  $P_\infty$  computed using Propositions 2.1 and 2.2.

Moreover, note that inequality (15), combined with Propositions 2.1 and 2.2, is not linear in  $P_0$  or  $P_\infty$  since  $R_0$  and  $r_\infty$  depend on  $P_0$  and  $P_\infty$  through the constraints (9) and (14) in which  $W_0 = P_0^{-1}$  and  $W_\infty = P_\infty^{-1}$ .

Nevertheless, note that, when  $R_0 = r_\infty$ , the covering Assumption can be easily defined as the following LMI:

$$W_0 - W_\infty > 0. \quad (22)$$

Moreover, note that the two matrix inequality constraints (9) and (14) can be recast as the following LMI constraints:

$$\rho v_\infty^2 I_n - W_\infty \leq 0, \quad W_0 - \rho v_0^2 I_n \leq 0 \quad (23)$$

where  $\rho$  is a positive real number such that  $R_0 = r_\infty = \frac{1}{\rho}$ . Consequently, this feasibility constraint can be added easily in the design of  $W_0$  and  $W_\infty$  (i.e. of  $P_0$  and  $P_\infty$ ).

To summarize, we have:

**Proposition 4.1:** *Suppose there exist two positive definite matrices  $W_0, W_\infty$  in  $\mathbb{R}^{n \times n}$  and a real number  $\rho > 0$  such that inequalities (22) and (23) are satisfied, then the covering Assumption (i.e. inequality (15)) is also satisfied with  $R_0 = r_\infty = \frac{1}{\rho}$ .*

### B. About the second feasibility constraint

Now to include the feasibility constraints (16) and (17) into the design of a global asymptotic stabilizer, we need to restrict ourselves to a specific class of matrices  $W_0, H_0, J_0, W_\infty, H_\infty$ , and  $J_\infty$  solutions of (7), (12), (22), (23).

To be more precise we consider the subclass of solutions such that the conditions (16) and (17) are satisfied, by particularizing these conditions as LMI conditions. To do that we use elimination lemma [15], and we get the following result.

**Proposition 4.2:** *If the local and the non-local conditions (3) and (4) are such that  $N_0 = N_\infty = 0$  and if there exist two symmetric positive definite matrices  $W_0$  and  $W_\infty$  in  $\mathbb{R}^{n \times n}$ , two matrices  $J_m$  in  $\mathbb{R}^{m \times p}$  and  $H_m$  in  $\mathbb{R}^{m \times n}$ , an invertible matrix  $S$  in  $\mathbb{R}^{n \times n}$ , satisfying the LMI constraints (20) and (21), then inequalities (16) and (17) hold with  $P_0 = W_0^{-1}$ ,  $P_\infty = W_\infty^{-1}$ ,  $K_m = H_m S^{-1}$  and  $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1$ .*

Note that employing a change of matrix similar to the one introduced in Section II-A and II-B, we can also deal with the case where  $N_0 = N_\infty \neq 0$ .

The key point of the previous result is that the constraints (20) and (21) obtained are linear in the unknown  $W_0$  and

$W_\infty$ . Consequently with Propositions 4.1 we are able to give a complete LMI formulation allowing us to design a state feedback control law for system (1) making the origin a globally and asymptotically stable equilibrium. This can be summarized as follows.

**Theorem 4.3:** *If  $N_0 = N_\infty = 0$  and if there exist two symmetric positive definite matrices  $W_0$  and  $W_\infty$  in  $\mathbb{R}^{n \times n}$ , three matrices  $J_0, J_\infty$  and  $J_m$  in  $\mathbb{R}^{m \times p}$  three matrices  $H_0, H_\infty$  and  $H_m$  in  $\mathbb{R}^{m \times n}$ , an invertible matrix  $S$  in  $\mathbb{R}^{n \times n}$ , and a real number  $\rho > 0$  satisfying the matrix inequality (7), (12), (20), (21), (22), and (23), then the control law  $u = \alpha(x)$  with  $\alpha$  defined in (19) makes the origin a globally asymptotically stable equilibrium for system (1).*

## V. TWO EXAMPLES

To illustrate the proposed procedure and to motivate it, we present two examples. The first one exhibits the interest of splitting in two pieces a global sector bound condition, the other one is an illustration of Theorem 4.3.

### A. Example 1

In a first example, we consider system (1) described by the following data:

$$\dot{x} = \phi(x) + Bu, \quad x \in \mathbb{R}^3, \quad (24)$$

with

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

To complete the definition of system (1), it remains to introduce the function  $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  representing the nonlinearity of the system. First consider both matrices  $M_\infty$  in  $\mathbb{R}^{3 \times 3}$  and  $N_0$  in  $\mathbb{R}^{3 \times 3}$  defined as

$$M_\infty = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad N_0 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The nonlinear function  $\phi$  is defined as a locally Lipschitz continuous path interpolating  $M_\infty$  and  $N_0$ <sup>6</sup>, i.e:

$$\phi(x) = \lambda(|x|)M_\infty x + (1 - \lambda(|x|))N_0 x, \quad x \in \mathbb{R}^3, \quad (25)$$

where  $\lambda : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is an increasing locally Lipschitz function such that<sup>7</sup>:

$$\lambda(0) = 0, \quad \lim_{s \rightarrow +\infty} \lambda(s) = 1.$$

<sup>6</sup>This is typically the case when a system is modeled as an interpolation of several linear systems.

<sup>7</sup>For instance  $\lambda$  can be defined by  $\lambda(s) = \frac{2}{\pi} \arctan(s)$  for all  $s \geq 0$ .



We wish to find a controller guaranteeing global asymptotic stabilization of the origin.

1) *Employing a unique and global sector condition:* A first strategy to address the stabilization problem for system (24) is to check if the solvability conditions inspired by [5], [11] are satisfied. For this purpose, let us first prove that the nonlinear function  $\phi$  satisfies a (global) sector condition:<sup>8</sup>

**Proposition 5.1:** *The function  $\phi$  defined in (25) satisfies the sector condition:*

$$(\phi(x) - M_\infty x)'(\phi(x) - N_0 x) \leq 0 \quad , \quad \forall x \in \mathbb{R}^3 \quad .$$

So following the computations of (the proof of) Proposition 2.1 (or applying [5] without saturation or [11]), we may try to compute a global nonlinear feedback law by solving the following LMI optimization problem:

**Proposition 5.2:** *If there exists a symmetric positive definite matrices  $W$  in  $\mathbb{R}^{3 \times 3}$ , and two matrices  $H$  in  $\mathbb{R}^{1 \times 3}$ ,  $J$  in  $\mathbb{R}$  satisfying the LMI:*

$$\begin{bmatrix} \text{Sym}(M_\infty W + BH) & \star \\ J'B' + I_3 + (N_0 - M_\infty)W & -2I_1 \end{bmatrix} < 0 \quad ,$$

then the control law:

$$u(x) = Kx + J\phi(x) \quad , \quad (26)$$

with  $K = HW^{-1}$  makes the origin a globally and asymptotically stable equilibrium for System (24).

Using parser YALMIP [10] and LMI solver SeDuMi [16], this problem is found to be unfeasible on this particular instance. This approach is clearly too conservative and cannot be used in this specific case.

2) *Employing the uniting controller approach:* Note however that our uniting controller provides another approach to solve this stabilizing problem. First we show that the function  $\phi$  introduced in (25) fits in the context of Assumptions 1 and 2.

**Proposition 5.3:** *Given two positive real numbers  $0 < \lambda_\infty < \lambda_0 < 1$ , Assumptions 1 and 2 are satisfied with*

$$v_0 = \lambda^{-1}(\lambda_0) \quad , \quad v_\infty = \lambda^{-1}(\lambda_\infty) \quad ,$$

$M_0 = \lambda_0 M_\infty + (1 - \lambda_0)N_0$ , and

$N_\infty = \lambda_\infty M_\infty + (1 - \lambda_\infty)N_0$ .

Consequently, we are in the framework of the uniting sector condition.

Note that for this system we have  $N_0 \neq 0$  and  $N_\infty \neq 0$  (and  $N_0 \neq N_\infty$ ), and thus Theorem 4.3 cannot be applied. Therefore we follow the procedure developed in Sections III and IV-A, and we apply Propositions 2.1, 2.2, and 4.1. Therefore we have to check if the LMIs (7), (12), (22) and (23) have a solution in  $W_0$ ,  $W_\infty$ , and  $\rho$  (among others variables), and we get that Assumption 3 holds for  $P_0 = W_0^{-1}$ ,  $P_\infty = W_\infty^{-1}$ , and  $R_0 = r_\infty = \frac{1}{\rho}$ .

Choosing  $\lambda_0 = 0.6$  and  $\lambda_\infty = 0.4$ , and assuming that  $\lambda$  is a continuous function such that:

$$v_0 = \lambda^{-1}(0.6) = 10 \quad , \quad v_\infty = \lambda^{-1}(0.4) = 1.5 \quad ,$$

<sup>8</sup>Note that the proof of Proposition 5.1 does not use the expression of the system (1) but only properties of the nonlinearity  $\phi$  as introduced in (25).

we get the following solutions

$$P_0 = \begin{bmatrix} 0.3635 & 0.7820 & 0.6798 \\ 0.7820 & 3.4710 & 2.7149 \\ 0.6798 & 2.7149 & 2.6948 \end{bmatrix}$$

$$P_\infty = \begin{bmatrix} 0.4085 & 0.8650 & 0.8252 \\ 0.8650 & 3.7691 & 3.0466 \\ 0.8252 & 3.0466 & 3.7480 \end{bmatrix}$$

$$\rho^{-1} = R_0 = r_\infty = 16.3666$$

of the LMIs (7), (12), (22) and (23). The fact that the covering Assumption is satisfied for the matrix  $P_0$  and  $P_\infty$  with the positive real number  $v_0 = 10$  and  $v_\infty = 1.5$  is guaranteed by Proposition 4.1 and is illustrated in Figure 1.

Hence we are in the context of Theorem 3.1 and we can check that, for the two previous matrices  $P_0$  and  $P_\infty$ , there exist two matrices  $J_m$ ,  $K_m$  and four scalars  $\mu_i$  satisfying the sufficient conditions (16) and (17). This is indeed the case with

$$J_m = [-1.0000 \quad -5.5737 \quad -5.3139] \quad ,$$

and

$$K_m = [1.0491 \quad -0.7931 \quad -0.6987] \quad .$$

Consequently, the conclusion of Theorem 3.1 holds, we get that the control law (19) makes the origin a globally and asymptotic equilibrium for the system (24).

## B. Example 2

We consider now a second order system defined as:

$$\begin{cases} \dot{x}_1 = x_2 + x_1 \eta_1(x) \quad , \\ \dot{x}_2 = u + x_2 \eta_2(x) \quad , \end{cases} \quad (27)$$

where the functions  $\eta_1$  and  $\eta_2$  are any continuous functions such that:

$$\eta_1(x) \begin{cases} \leq 1 & \text{if } |x| \leq v_0 \\ \leq 2 & \text{if } |x| \geq v_0 \end{cases} \quad , \quad \eta_2(x) \begin{cases} \leq 1 & \text{if } |x| \leq v_\infty \\ \leq 0.5 & \text{if } |x| \geq v_\infty \end{cases} \quad .$$

Note that this system can be written in the form (1) with:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad ,$$

with  $G = L = I_2$ , and with the function

$$\phi(x) = \begin{bmatrix} x_1 \eta_1(x) \\ x_2 \eta_2(x) \end{bmatrix} \quad .$$

The function  $\phi(x)$  satisfies Assumptions 1 and 2 with

$$N_0 = N_\infty = 0 \quad , \quad M_0 = I_2 \quad , \quad M_\infty = \text{Diag}(2, 0.5) \quad .$$

Hence we are in the context of Theorem 4.3. Employing YALMIP [10] with the solver SEDUMI [16], we check the solvability of LMIs (7), (12), (22), (23), (20) and (21). These LMIs are solvable with:

$$W_0 = \begin{bmatrix} 0.6483 & -1.0998 \\ -1.0998 & 4.3131 \end{bmatrix} \quad ,$$

$$W_\infty = \begin{bmatrix} 0.3610 & -0.8647 \\ -0.8647 & 3.3607 \end{bmatrix}$$

$$\begin{aligned}
J_0 &= \begin{bmatrix} -1.0998 & 3.3131 \end{bmatrix}, \\
J_\infty &= \begin{bmatrix} -1.7293 & 0.6804 \end{bmatrix}, \\
H_0 &= \begin{bmatrix} -2.1135 & -6.0105 \end{bmatrix}, \\
H_\infty &= \begin{bmatrix} -1.1991 & -3.3778 \end{bmatrix}, \\
H_m &= \begin{bmatrix} 0.8534 & -3.6833 \end{bmatrix}, \\
J_m &= \begin{bmatrix} -1.2871 & 0.0213 \end{bmatrix}, \\
S &= \begin{bmatrix} 0.7296 & -0.4451 \\ -0.9518 & 1.2088 \end{bmatrix},
\end{aligned}$$

and finally  $\rho = 0.0681$ . Consequently, the conclusion of Theorem 4.3 holds and we obtain a control law making the origin a globally and asymptotic stable equilibrium for the system (27).

## VI. CONCLUSION

In this paper we have introduced the synthesis problem of a nonlinear feedback law for a class of control systems. The control systems under consideration are those with a nonlinearity satisfying a sector condition when the state is close to the equilibrium and a (maybe) different sector condition when the state is distant from the equilibrium. We noted that encompassing both sector conditions into a unique global one may lead to a too conservative synthesis problem. This motivates us to consider both properties of the nonlinearities separately and to design successively 1) a local asymptotic stabilizing nonlinear controller whose basin of attraction contains some compact set and 2) a non-local controller which makes the previous compact set globally attractive. Then we compute a nonlinear controller which piece together the local controller with the non-local one, and we obtain a global asymptotic stabilizing controller. We emphasize that the sufficient conditions to solve this design problem are written in terms of LMIs. Two numerical examples motivate and illustrate this approach.

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