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On Krawtchouk polynomials

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Krawtchouk polynomials play an important role in coding theory and are also useful in graph theory and number theory. Although the basic properties of these polynomials are known to some extent, there is, to my knowledge, no detailed development available. My aim in writing this article is to fill in this gap.

Notation In the following we will use capital letters for (algebraic) polynomials, for example P or $P(X)$; for the polynomial function associated with a polynomial P, we will use small letters in the parentheses, for example $P(x)$.

1 Definition and first properties

To begin with, we define a collection of real polynomials P_j, for $j \geq 0$, which we will use in the definition of Krawtchouk polynomials. We set $P_0 = 1$ and, for $j \geq 1$,

$$P_j(X) = \frac{1}{j!} X(X-1)\dots(X-j+1).$$

Clearly, for $i \geq j \geq 0$, $P_j(i) = \binom{i}{j}$. Now we take $n, m, s \in \mathbb{N}$, such that $m \leq n$ and $s \geq 2$, and set

$$K_{m,n,s}(X) = \sum_{j=0}^{m} (-1)^j P_j(X) P_{m-j}(n-X)(s-1)^{m-j}.$$

The real polynomial $K_{m,n,s}$ is called a Krawtchouk polynomial. For every fixed pair $(n,s) \in \mathbb{N}\times\mathbb{N}\setminus\{0,1\}$, there is a set of $n+1$ Krawtchouk polynomials $K_{m,n,s}$. Clearly $K_{0,n,s}$ is the constant polynomial 1. To simplify the notation, when the indices n and s are understood, we will write K_m for $K_{m,n,s}$.

Proposition 1.1 *K_m is a polynomial of degree m, whose leading coefficient is $\frac{-s^m}{m!}$.*

PROOF We first notice that

$$P_j(X)P_{m-j}(n-X) = \frac{(-1)^{m-j}}{j!(m-j)!} X^m \ + \text{ terms of degree smaller than } m,$$

which implies that

$$K_m(X) = \sum_{j=0}^{m} \frac{(-1)^m}{j!(m-j)!} (s-1)^{m-j} X^m \ + \text{ terms of degree smaller than } m.$$

However

$$\sum_{j=0}^{m} \frac{(-1)^m}{j!(m-j)!}(s-1)^{m-j} = \frac{(-1)^m}{m!}\sum_{j=0}^{m}\binom{m}{j}(s-1)^{m-j} = \frac{(-s)^m}{m!},$$

and hence the result. □

We have a simple expression for the value of the polynomial $K_{m,n,s}$ at certain integers.

Proposition 1.2 *For $i \in \{0,\ldots,n\}$, $K_{m,n,s}(i)$ is the coefficient of X^m in the product*

$$(1+(s-1)X)^{n-i}(1-X)^i.$$

PROOF First we have

$$K_{m,n,s}(0) = \sum_{j=0}^{m}(-1)^j P_j(0)P_{m-j}(n-0)(s-1)^{m-j} = P_m(n)(s-1)^m = \binom{n}{m}(s-1)^m,$$

which is the coefficient of X^m in the expression $(1+(s-1)X)^n$. It is also easy to see that

$$K_{m,n,s}(n) = (-1)^m P_m(n) = (-1)^m\binom{n}{m},$$

which is the coefficient of X^m in the expression $(1-X)^n$. So the result is true for $i=0$ and $i=n$.

Now let us consider the case where $1 \le i \le n-1$. We have

$$(1+(s-1)X)^{n-i} = 1 + \binom{n-i}{1}(s-1)X + \binom{n-i}{2}(s-1)^2X^2 + \ldots + (s-1)^{n-i}X^{n-i}$$

and

$$(1-X)^i = 1 - \binom{i}{1}X + \binom{i}{2}X^2 - \ldots + (-1)^i X^i.$$

It follows that the coefficient of X^m in the product of $(1+(s+1)X)^{n-1}$ and $(1-X)^i$ is

$$\sum_{j=0}^{m}(-1)^j\binom{i}{j}\binom{n-i}{m-j}(s-1)^{m-j} = K_{m,n,s}(i).$$

This ends the proof. □

Remark Further on we will use the expressions for $K_{m,n,s}(0)$ and $K_{m,n,s}(n)$ found here, namely

$$K_{m,n,s}(0) = \binom{n}{m}(s-1)^m \quad \text{and} \quad K_{m,n,s}(n) = (-1)^m\binom{n}{m}.$$

Corollary 1.1 *The following statement follows directly from the proposition:*

$$K_{1,n,s}(i) = (n-i)(s-1) - i = (s-i)n - si.$$

It is very easy to find an explicit expression for the polynomials $K_{1,n,s}$.

Proposition 1.3 *We have*

$$K_{1,n,s}(X) = n(s-1) - sX.$$

PROOF Using Corollary 1.1, we obtain $K_{1,n,s}(0) = (s-1)n$ and $K_{1,n,s}(1) = (s-1)n - s$, from which we deduce the two coefficients of $K_{1,n,s}$. □

From what we have seen, we may obtain a useful recurrence relation.

Proposition 1.4 *For $1 \leq m, i \leq n$, we have*

$$K_{m,n,s}(i) = K_{m,n,s}(i-1) - K_{m-1,n,s}(i-1) - (s-1)K_{m-1,n,s}(i).$$

PROOF We first notice the identity

$$(1+(s-1)X)^{n-i}(1-X)^i(1+(s-1)X) = \Big((1+(s-1)X)^{n-(i-1)}(1-X)^{i-1}\Big)(1-X).$$

The coefficient of X^m on the left-hand side is

$$K_{m,n,s}(i) + (s-1)K_{m-1,n,s}(i)$$

and on the right-hand side is

$$K_{m,n,s}(i-1) - K_{m-1,n,s}(i-1).$$

This gives us the result. □

Corollary 1.2 *We have*

$$K_{m,n,s}(X) = K_{m,n,s}(X-1) - K_{m-1,n,s}(X-1) - (s-1)K_{m-1,n,s}(X).$$

PROOF The polynomials $K_{m,n,s}(X)$ and $K_{m,n,s}(X-1)$ have the same degree and leading coefficient, so the degree of their difference is at most $m-1$. As the polynomials

$$K_{m,n,s}(X) - K_{m,n,s}(X-1) \quad \text{and} \quad -(K_{m-1,n,s}(X-1) + (s-1)K_{m-1,n,s}(X))$$

have the same value at n points and $m-1 \leq n$, these polynomials are the same and we have the result we were looking for. □

Remark Using the recurrence relation of the proposition, we may successively calculate $K_{m,n,s}(1)$, $K_{m,n,s}(2)$ and so on up to $K_{m,n,s}(m)$ and so determine the polynomial $K_{m,n,s}$.

Example To calculate $K_{2,n,s}$, it is sufficient to obtain $K_{2,n,s}(0)$, $K_{2,n,s}(1)$ and $K_{2,n,s}(2)$. To find $K_{2,n,s}(0)$, we can use the the expression for $K_{m,n,s}(0)$ found above. We can calculate $K_{2,n,s}(1)$ from the recurrence relation of the proposition: we need $K_{2,n,s}(0)$ (already calculated), $K_{1,n,s}(0)$ and $K_{1,n,s}(1)$, which can be found from the expression for $K_{1,n,s}$ (Proposition 1.3). We may also determine $K_{2,n,s}(2)$ from the recurrence relation of the proposition: we need $K_{2,n,s}(1)$ (already calculated) and $K_{1,n,s}(1)$ and $K_{1,n,s}(2)$, which can be found using the expression for $K_{1,n,s}$.

2 A summation formula

The recurrence relation found in the last section enables us to find a formula for the sum of successive Krawtchouk polynomials. We need a preliminary result.

Lemma 2.1 *For $j \geq 0$, we have*

$$P_j(X) + P_{j+1}(X) = P_{j+1}(X+1).$$

PROOF Let

$$Q(X) = P_j(X) + P_{j+1}(X) - P_{j+1}(X+1).$$

For any $i \geq j+1$, we have

$$Q(i) = P_j(i) + P_{j+1}(i) - P_{j+1}(i+1) = \binom{i}{j} + \binom{i}{j+1} - \binom{i+1}{j+1} = 0.$$

Therefore $Q = 0$. □

We now turn to the summation formula.

Theorem 2.1 *For $1 \leq m \leq n$, we have*

$$\sum_{k=0}^{m} K_{k,n,s}(X) = K_{m,n-1,s}(X-1).$$

PROOF We will prove this result by induction on m. First

$$K_{0,n,s}(X) + K_{1,n,s}(X) = 1 + n(s-1) - sX = (n-1)(s-1) - s(X-1) = K_{1,n-1,s}(X-1),$$

therefore the theorem is correct for $m = 1$.

Suppose now that the result is true for $m-1$ and consider the case m. Using Corollary 1.2, we obtain

$$\sum_{k=1}^{m} K_{k,n,s}(X) = K_{m,n,s}(X-1) - K_{0,n,s}(X-1) - (s-1)\sum_{k=0}^{m-1} K_{k,n,s}(X),$$

which implies that

$$\sum_{k=0}^{m} K_{k,n,s}(X) = K_{m,n,s}(X-1) - (s-1)\sum_{k=0}^{m-1} K_{k,n,s}(X),$$

because $K_{0,n,s}(X-1) = 1 = K_{0,n,s}(X)$. Now

$$K_{m,n,s}(X-1) = \sum_{j=0}^{m} (-1)^j P_j(X-1) P_{m-j}(n-X+1)(s-1)^{m-j}$$

and, by hypothesis,

$$\begin{aligned}
(s-1)\sum_{k=0}^{m-1} K_{k,n,s}(X) &= (s-1)K_{m-1,n-1,s}(X-1) \\
&= (s-1)\sum_{j=0}^{m-1} (-1)^j P_j(X-1) P_{m-1-j}((n-1)-(X-1))(s-1)^{m-1-j} \\
&= \sum_{j=0}^{m-1} (-1)^j P_j(X-1) P_{m-1-j}(n-X)(s-1)^{m-j}.
\end{aligned}$$

Using Lemma 2.1, we obtain

$$\begin{aligned}
\sum_{k=0}^{m} K_{k,n,s}(X) &= \sum_{j=0}^{m-1} (-1)^j P_j(X-1) P_{m-j}(n-X)(s-1)^{m-j} + (-1)^m P_m(X-1) \\
&= \sum_{j=0}^{m} (-1)^j P_j(X-1) P_{m-j}(n-X)(s-1)^{m-j} \\
&= K_{m,n-1,s}(X-1).
\end{aligned}$$

Therefore the result is true for m. This finishes the induction step. □

3 Inner products

For $n \in \mathbb{N}$, let us write $\mathbb{R}_n[X]$ for the set of real polynomials of degree not greater than n. $\mathbb{R}_n[X]$ is a real vector space of dimension $n+1$. Fixing $s \geq 2$, we define an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}_n[X]$ in the following way:

$$\langle A, B \rangle = \sum_{i=0}^{n} \binom{n}{i} (s-1)^i A(i) B(i).$$

If $0 \leq m \leq n$, then the polynomial $K_{m,n,s} \in \mathbb{R}_n[X]$. In the next proposition, we will drop the second and third parameters to simplify the notation.

Proposition 3.1 *For $0 \leq k, l \leq n$, we have*

$$\langle K_k, K_l \rangle = \begin{cases} 0 & k \neq l \\ s^n (s-1)^k \binom{n}{k} & k = l \end{cases}$$

PROOF Let us consider the real polynomial in two variables

$$P(X,Y) = \sum_{k=0}^{n} \sum_{l=0}^{n} \left(\sum_{i=0}^{n} \binom{n}{i} (s-1)^i K_k(i) K_l(i) \right) X^k Y^l.$$

Then

$$\begin{aligned} P(X,Y) &= \sum_{i=0}^{n} \binom{n}{i} (s-1)^i \sum_{k=0}^{n} K_k(i) X^k \sum_{l=0}^{n} K_l(i) Y^l \\ &= \sum_{i=0}^{n} \binom{n}{i} (s-1)^i (1+(s-1)X)^{n-i} (1-X)^i (1+(s-1)Y)^{n-i} (1-Y)^i \\ &= \sum_{i=0}^{n} \binom{n}{i} \left((s-1)(1-X)(1-Y)\right)^i \left((1+(s-1)X)(1+(s-1)Y)\right)^{n-i} \\ &= \left((s-1)(1-X)(1-Y) + (1+(s-1)X)(1+(s-1)Y)\right)^n. \end{aligned}$$

After simplification, we obtain

$$P(X,Y) = s^n (1+(s-1)XY)^n = \sum_{i=0}^{n} s^n \binom{n}{i} (s-1)^i X^i Y^i.$$

Therefore, if $k \neq l$, then

$$\sum_{i=0}^{n} \binom{n}{i} (s-1)^i K_k(i) K_l(i) = 0$$

and, if $k = l$, then

$$\sum_{i=0}^{n} \binom{n}{i} (s-1)^i K_k(i) K_l(i) = s^n \binom{n}{k} (s-1)^k.$$

This ends the proof. □

Corollary 3.1 *If A is a real polynomial such that $\deg(A) = d < m$, then*

$$\langle K_m, A \rangle = 0.$$

PROOF The polynomials K_k, $0 \leq k \leq n$, form a basis of $\mathbb{R}_n[X]$. If $\deg(A) = d$, then $A = \sum_{k=0}^{d} \lambda_k K_k$ and

$$\langle K_m, A \rangle = \sum_{k=0}^{d} \lambda_k \langle K_m, K_k \rangle = 0.$$

This ends the proof. □

4 Another recurrence relation

We have already seen one recurrence relation involving Krawtchouk polynomials. In this section we present another such relation, which will be useful when studying the roots of Krawtchouk polynomials. Once again, we will drop the parameters n and s from $K_{m,n,s}$ to simplify the notation.

Theorem 4.1 *For $1 \leq m \leq n-1$, we have the relation*

$$(m+1)K_{m+1} = (m + (s-1)(n-m) - sX)K_m - (s-1)(n-m+1)K_{m-1}.$$

PROOF We have seen that

$$\sum_{m=0}^{n} K_m(i)X^m = (1+(s-1)X)^{n-i}(1-X)^i.$$

If we differentiate both sides of the equation, then we obtain

$$\sum_{m=0}^{n-1}(m+1)K_{m+1}(i)X^m = (s-1)(n-i)(1+(s-1)X)^{n-i-1}(1-X)^i - i(1+(s-1)X)^{n-i}(1-X)^{i-1}.$$

We now multiply both sides of this expression by $(1+(s-1)X)(1-X)$. Writing A and B for the left- and right-hand sides, we have

$$(1+(s-1)X)(1-X)A = \sum_{m=0}^{n-1}(m+1)K_{m+1}(i)X^m + (s-2)\sum_{m=1}^{n} mK_m(i)X^m - (s-1)\sum_{m=2}^{n+1}(m-1)K_{m-1}(i)X^m$$

and

$$\begin{aligned}(1+(s-1)X)(1-X)B &= ((1-X)(s-1)(n-i) - i(1+(s-1)X))\,(1+(s-1)X)^{n-i}(1-X)^i \\ &= ((sn-n-is)+(n-sn)X)\sum_{m=0}^{n} K_m(i)X^m \\ &= (sn-n-is)\sum_{m=0}^{n} K_m(i)X^m + (n-sn)\sum_{m=1}^{n+1} K_{m-1}(i)X^m.\end{aligned}$$

Taking the difference of $(1+(s-1)X)(1-X)A$ and $(1+(s-1)X)(1-X)B$, we obtain

$$\sum_{m=0}^{n-1}(m+1)K_{m+1}(i)X^m - \sum_{m=0}^{n}(m+(s-1)(n-m)-si)K_m(i)X^m + \sum_{m=1}^{n}(s-1)(n-m+1)K_{m-1}(i)X^m = 0.$$

Hence, for $1 \leq m \leq n-1$ and $0 \leq i \leq n$,

$$(m+1)K_{m+1}(i) - (m+(s-1)(n-m)-si)K_m(i) + (s-1)(n-m+1)K_{m-1}(i) = 0.$$

However, the polynomial

$$C_m = (m+1)K_{m+1} - (m+(s-1)(n-m)-sX)K_m + (s-1)(n-m+1)K_{m-1}$$

is of degree not greater than n and has $n+1$ roots. It follows that $C_m = 0$, which ends the proof. □

Remark This recurrence relation also allows us to find the polynomials K_m successively. For example, K_2 can be obtained from K_1 and K_0, which we already know. After some calculation, we find

$$K_2 = \frac{1}{2}\left((s-1)^2 n(n-1) - s(2ns-2n-s+2)X + s^2X^2\right).$$

When $s=2$, the expression is much simpler:

$$K_2 = \frac{(n-2X)^2 - n}{2}.$$

5 Roots of Krawtchouk polynomials

In this section we will see that all the roots of a Krawtchouk polynomial are real and distinct. We obtain an interesting relation between the roots of successive Krawtchouk polynomials. To simplify the notation, we set

$$a_m = m + (s-1)(n-m) \qquad \text{and} \qquad b_m = (s-1)(n-m+1).$$

Notice that $a_m > 0$ and $b_m > 0$.

Proposition 5.1 *For $1 \le m \le n$, the polynomial K_m has m distinct real roots in the interval $(0, n)$.*

PROOF As

$$0 = \langle K_0, K_m \rangle = \sum_{i=0}^{n} \binom{n}{i} (s-1)^i K_m(i)$$

and $K_m(0) > 0$, the real-valued polynomial function $K_m(x)$ changes sign in the interval $(0, n)$. Suppose that $K_m(x)$ changes sign at the points $x_1 < \ldots < x_d$. These points are roots of K_m. As $\deg(K_m) = m$, we have $d \le m$. Suppose that $d < m$ and let

$$S = \prod_{i=1}^{d} (X - x_i).$$

The polynomial S has $x_1, \ldots, x_d$ as roots. As these roots are simple, the derivative S' does not have any of these points as roots. It follows that the real-valued polynomial function $S(x)$ changes sign at $x_1, \ldots, x_d$. Thus the polynomial functions $K_m(x)$ and $S(x)$ change signs at the same points. Hence the product $S(x)K_m(x)$ is strictly positive (or strictly negative), except at the points $x_1, \ldots, x_d$, where its value is 0. Therefore $\langle S, K_m \rangle > 0$ (or < 0). However, this contradicts Corollary 3.1 and so $d = m$, i.e. K_m has m distinct roots in the interval $(0, n)$. □

Remark As $\deg(K_m) = m$, the roots of K_m are necessarily simple.

We will now look at the 'interlacing' property of the roots of Krawtchouk polynomials.

Theorem 5.1 *If the roots of K_m are $x_1 < \ldots < x_m$ and those of K_{m+1} are $y_1 < \ldots < y_{m+1}$, then*

$$0 < y_1 < x_1 < y_2 < x_2 < \ldots < x_m < y_{m+1} < n.$$

PROOF We will prove the result by induction on m. We first notice that, from Theorem 4.1, we have

$$(m+1)K_{m+1} = (a_m - sX)K_m - b_m K_{m-1}.$$

We begin with the case $m = 1$. Let x_1 be the unique root of K_1. As $K_0 = 1$ and $b_1 > 0$, $K_2(x_1) < 0$. Also, $K_2(0) = (s-1)^2\binom{n}{2} > 0$ and $K_2(n) = (-1)^2\binom{n}{2} > 0$, and so K_2 has a root in each of the intervals $(0, x_1)$ and (x_1, n). This proves the result for $m = 1$.

Suppose now that the property is true for $m-1$ and consider the case m. Let $x_1 < \ldots < x_m$ be the roots of K_m. Then $K_{m-1}(x_1) > 0$, $K_{m-1}(x_2) < 0$, ... This implies that $K_{m+1}(x_1) < 0$, $K_{m+1}(x_2) > 0$, ... Therefore K_{m+1} has a root in the interval (x_i, x_{i+1}), for $i \in \{1, \ldots, m-1\}$. Also, $K_{m+1}(0) > 0$ and $K_{m+1}(x_1) < 0$, therefore K_{m+1} has a root in the in the interval $(0, x_1)$. We claim that K_{m+1} has a root in the interval (x_m, n). We observe that $K_{m+1}(n) = (-1)^{m+1}\binom{n}{m+1}$ and that $K_{m+1}(x_m) > 0$, if m is even, and $K_{m+1}(x_m) < 0$, if m is odd. This implies that K_{m+1} has a root in the interval (x_m, n). Thus the result is true for m. This finishes the proof. □

In the case where $s = 2$, we can say a little more about the roots. This follows from a simple symmetry relation.

Lemma 5.1 *For $x \in \mathbb{R}$, we have*

$$K_{m,n,2}(n-x) = (-1)^m K_{m,n,2}(x).$$

PROOF We have

$$\begin{aligned} K_{m,n,2}(n-x) &= \sum_{j=0}^{m} (-1)^j P_j(n-x) P_{m-j}(x) \\ &= \sum_{t=0}^{m} (-1)^{m-t} P_{m-t}(n-x) P_t(x) \\ &= (-1)^m \sum_{t=0}^{m} (-1)^t P_{m-t}(n-x) P_t(x) \\ &= (-1)^m K_{m,n,2}(x). \end{aligned}$$

This ends the proof. □

Proposition 5.2 *The roots of $K_{m,n,2}$ are symmetric with respect to $\frac{n}{2}$. In particular, if m is odd, then $\frac{n}{2}$ is a root of $K_{m,n,2}$.*

PROOF It is sufficient to notice that, if x_1 is a root, then so is $n - x_1$. □

Remark It is particularly interesting to notice that, if m is odd and n even, then $K_{m,n,2}$ has an integer root, namely $\frac{n}{2}$.