



**HAL**  
open science

## Dispersive wave runup on non-uniform shores

Denys Dutykh, Theodoros Katsaounis, Dimitrios Mitsotakis

► **To cite this version:**

Denys Dutykh, Theodoros Katsaounis, Dimitrios Mitsotakis. Dispersive wave runup on non-uniform shores. 2011. hal-00553762v1

**HAL Id: hal-00553762**

**<https://hal.science/hal-00553762v1>**

Preprint submitted on 9 Jan 2011 (v1), last revised 5 Mar 2011 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Dispersive wave runup on non-uniform shores

Denys Dutykh and Theodoros Katsaounis and Dimitrios Mitsotakis

**Abstract** Historically the finite volume methods have been developed for the numerical integration of conservation laws. In this study we present some recent results on the application of such schemes to dispersive PDEs. Namely, we solve numerically a representative of Boussinesq type equations in view of important applications to the coastal hydrodynamics. Snapshots of a moderate wave amplitude runup onto a complex beach are presented along with great lines of the employed numerical method (see D. Dutykh *et al.* (2011) [6] for more details).

## 1 Introduction

The simulation of water waves in realistic and complex environments is a very challenging problem. Most of the applications arise from the areas of coastal and naval engineering, but also from natural hazards assessment. These applications may require the computation of the wave generation [5, 12], propagation [17], interaction with solid bodies, the computation of long wave runup [16, 18] and even the extraction of the wave energy [15]. Issues like wave breaking, robustness of the numerical algorithm in wet-dry processes along with the validity of the mathematical models in the near-shore zone are some basic problems in this direction [11]. During past several decades the classical Nonlinear Shallow Water Equations (NSWE) have been

---

Denys Dutykh  
LAMA, UMR 5127 CNRS, Université de Savoie, Campus Scientifique, 73376 Le Bourget-du-Lac  
Cedex, France, e-mail: Denys.Dutykh@univ-savoie.fr

Theodoros Katsaounis  
Department of Applied Mathematics, University of Crete, Heraklion, 71409 Greece Inst. of App.  
and Comp. Math. (IACM), FORTH, Heraklion, 71110, Greece e-mail: thodoros@tem.uoi.gr

Dimitrios Mitsotakis  
IMA, University of Minnesota, Minneapolis MN 55455, USA e-mail: dmitsot@gmail.com

essentially employed to face these problems [1, 7]. Mathematically, these equations represent a system of conservation laws describing the propagation of infinitely long waves with a hydrostatic pressure assumption. The wave breaking phenomenon is commonly assimilated to the formation of shock waves (or hydraulic jumps) which is a common feature of hyperbolic PDEs. Consequently, the finite volume (FV) method has become the method of choice for these problems due to its excellent intrinsic conservative and shock-capturing properties [1, 3, 7].

In the present article we report on recent results concerning the extension of the finite volume method to dispersive wave equations stemming essentially from water wave modeling [14, 4, 6].

## 2 Mathematical model and numerical methods

Consider a cartesian coordinate system in two space dimensions  $(x, z)$  to simplify notations. The  $z$ -axis is taken vertically upwards and the  $x$ -axis is horizontal and coincides traditionally with the still water level. The fluid domain is bounded below by the bottom  $z = -h(x)$  and above by the free surface  $z = \eta(x, t)$ . Below we will also need the total water depth  $H(x, t) := h(x) + \eta(x, t)$ . The flow is supposed to be incompressible and the fluid is inviscid. An additional assumption of the flow irrotationality is made as well.

In the pioneering work of D.H. Peregrine (1967) [14] the following system of Boussinesq type equations has been derived:

$$\eta_t + ((h + \eta)u)_x = 0, \quad (1)$$

$$u_t + uu_x + g\eta_x - \frac{h}{2}(hu)_{xx} + \frac{h^2}{6}u_{xxt} = 0, \quad (2)$$

where  $u(x, t)$  is the depth averaged fluid velocity,  $g$  is the gravity acceleration and underscripts  $(u_x, \eta_t)$  denote partial derivatives.

In our recent study [6] we proposed an improved version of this system which contains higher order nonlinear terms which should be neglected from asymptotic point of view:

$$H_t + Q_x = 0, \quad (3)$$

$$\left( \left( 1 + \frac{1}{3}H_x^2 - \frac{1}{6}HH_{xx} \right) Q - \frac{1}{3}H^2Q_{xx} - \frac{1}{3}HH_xQ_x \right)_t + \left( \frac{Q^2}{H} + \frac{g}{2}H^2 \right)_x = gHh_x. \quad (4)$$

The last system is also written in conservative variables  $(H, Q) = (H, Hu)$ .

Obviously the linear characteristics of both systems (1), (2) and (3), (4) coincide since they differ only by nonlinear terms.

However, this modification has several important implications onto structural properties of the obtained system. First of all, the magnitude of dispersive terms tends to zero when we approach the shoreline  $H \rightarrow 0$ . This property corresponds to our physical representation of the wave shoaling and runup process. On the other hand, the resulting system becomes invariant under vertical translations (subgroup  $G_5$  in Theorem 4.2, T. Benjamin & P. Olver (1982) [2]):

$$z \leftarrow z + d, \quad \eta \leftarrow \eta - d, \quad h \leftarrow h + d, \quad u \leftarrow u, \quad (5)$$

where  $d$  is some constant. This property is straightforward to check since we use only the total water depth variable  $H = h + \eta$  which remains invariant under transformation (5).

**Remark 1** *In this contribution we will consider the initial-boundary value problem posed in a bounded domain  $I = [b_1, b_2]$  with reflective boundary conditions. In this case one needs to impose boundary conditions only in one of the two dependent variables, cf. [8]. In the case of reflective boundary conditions it is sufficient to take  $u(b_1, t) = u(b_2, t) = 0$ .*

## 2.1 Finite volume discretization

Let  $\mathcal{T} = \{x_i\}$ ,  $i \in \mathbb{Z}$  denotes a partition of  $\mathbb{R}$  into cells  $C_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$  where  $x_i = (x_{i+\frac{1}{2}} + x_{i-\frac{1}{2}})/2$  denotes the midpoint of  $C_i$ . Let  $\Delta x_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$  be the length of the cell  $C_i$ ,  $\Delta x_{i+\frac{1}{2}} = x_{i+1} - x_i$ .

The governing equations (3), (4) can be recast in the following vector form:

$$[\mathbf{D}(\mathbf{v}_t)] + [\mathbf{F}(\mathbf{v})]_x = \mathbf{S}(\mathbf{v}),$$

where

$$\mathbf{D}(\mathbf{v}_t) = \left( \left( 1 + \frac{1}{3}H_x^2 - \frac{1}{6}HH_{xx} \right) Q_t - \frac{1}{3}H^2 Q_{xt} - \frac{1}{3}HH_x Q_{xt} \right), \quad (6)$$

$$\mathbf{F}(\mathbf{v}) = \left( \frac{Q^2}{H} + \frac{g}{2}H^2 \right), \quad \mathbf{S}(\mathbf{v}) = \begin{pmatrix} 0 \\ gHD_x \end{pmatrix}. \quad (7)$$

We denote by  $H_i$ ,  $U_i$  and  $D_i$  the corresponding cell averages. To discretize the dispersive terms in (6) we consider the following approximations:

$$\begin{aligned} \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left[ 1 + \frac{1}{3}(H_x)^2 - \frac{1}{6}HH_{xx} \right] Q \, dx \approx \\ \left( 1 + \frac{1}{3} \left( \frac{H_{i+1} - H_{i-1}}{2\Delta x} \right)^2 - \frac{1}{6}H_i \frac{H_{i+1} - 2H_i + H_{i-1}}{\Delta x^2} \right) Q_i, \end{aligned}$$

$$\frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{1}{3} H H_x Q_x dx \approx \frac{1}{3} H_i \frac{H_{i+1} - H_{i-1}}{2\Delta x} \frac{Q_{i+1} - Q_{i-1}}{2\Delta x},$$

$$\frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{1}{3} H^2 Q_{xx} dx \approx \frac{1}{3} H_i^2 \frac{Q_{i+1} - 2Q_i + Q_{i-1}}{\Delta x^2}.$$

We note that we approximate the reflective boundary conditions by taking the cell averages of  $u$  on the first and the last cell to be  $u_0 = u_{N+1} = 0$ . We don't impose explicitly boundary conditions on  $H$ . The reconstructed values on the first and the last cell are computed using neighboring ghost cells and taking odd and even extrapolation for  $u$  and  $H$  respectively. These specific boundary conditions appeared to reflect incident waves on the boundaries while conserving the mass.

This discretization leads to a linear system with tridiagonal matrix denoted by  $\mathbf{L}$  that can be inverted efficiently by a variation of Gauss elimination for tridiagonal systems with computational complexity  $O(d)$ ,  $d$ -being the dimension of the system. We note that on the dry cells the matrix becomes diagonal since  $H_i$  is zero on dry cells. For the time integration the explicit third-order TVD-RK method is used. In the numerical experiments we observed that the fully discrete scheme is stable and preserves the positivity of  $H$  during the runup under a mild restriction on the time step  $\Delta t$ .

Therefore, the semidiscrete problem of (6) - (7) is written as a system of ODEs in the form:

$$\mathbf{L}_i \mathbf{v}_i + \frac{1}{\Delta x} (\mathcal{F}_{i+\frac{1}{2}} - \mathcal{F}_{i-\frac{1}{2}}) = \frac{1}{\Delta x} \mathbf{S}_i,$$

where  $\mathbf{L}_i$  is the  $i$ -th row of matrix  $\mathbf{L}$  and  $\mathcal{F}_{i+\frac{1}{2}}$  can be chosen as one of the numerical flux functions [6] (in computations presented below we choose the FVCF flux [9]). In the sequel we will use the KT and the CF numerical fluxes. In this case the Jacobian of  $\mathbf{F}$  is given by the matrix

$$A = \begin{pmatrix} 0 & 1 \\ gH - (Q/H)^2 & 2Q/H \end{pmatrix},$$

and the eigenvalues are  $\lambda_{1,2} = Q/H \pm \sqrt{gH}$ . Therefore, the characteristic numerical flux [9] takes the form

$$\mathcal{F}_{i+\frac{1}{2}} = \frac{\mathbf{F}(\mathbf{V}_{i+\frac{1}{2}}^L) + \mathbf{F}(\mathbf{V}_{i+\frac{1}{2}}^R)}{2} - \mathbf{U}(\boldsymbol{\mu}) \frac{\mathbf{F}(\mathbf{V}_{i+\frac{1}{2}}^R) - \mathbf{F}(\mathbf{V}_{i+\frac{1}{2}}^L)}{2},$$

where  $\boldsymbol{\mu} = (\mu_1, \mu_2)^T$  are the Roe average values,

$$\mu_1 = \frac{H_{i+\frac{1}{2}}^L + H_{i+\frac{1}{2}}^R}{2}, \quad \mu_2 = \frac{\sqrt{H_{i+\frac{1}{2}}^L} U_{i+\frac{1}{2}}^L + \sqrt{H_{i+\frac{1}{2}}^R} U_{i+\frac{1}{2}}^R}{\sqrt{H_{i+\frac{1}{2}}^L} + \sqrt{H_{i+\frac{1}{2}}^R}}$$

and

$$\mathbf{U}(\boldsymbol{\mu}) = \begin{pmatrix} \frac{s_2(\mu_2+c)-s_1(\mu_2-c)}{2c} & \frac{s_1-s_2}{2c} \\ \frac{(s_2-s_1)(\mu_2^2-c^2)}{2c} & \frac{s_1(\mu_2+c)-s_2(\mu_2-c)}{2c} \end{pmatrix}, \quad c = \sqrt{g\mu_1}, \quad s_i = \text{sign}(\lambda_i).$$

For more details on the discretization and reconstruction procedures we refer to our complete work on this subject [6].

### 3 Numerical results

In the present section we show a numerical simulation of a solitary wave runup onto a non-uniform sloping beach. More precisely, we add a small pond along the slope. As our results indicate, this small complication is already sufficient to develop some instabilities which remain controlled in our simulations.

As the initial condition we used an approximate solitary wave solution of the following form:

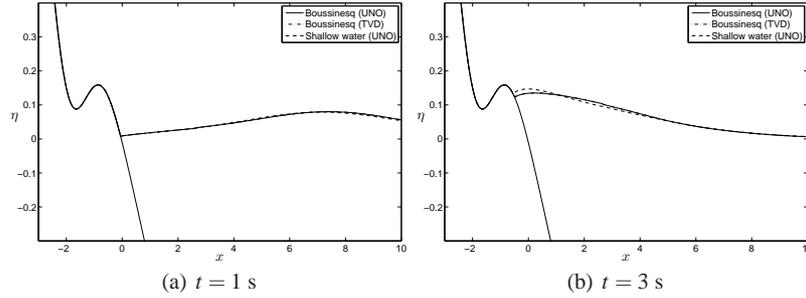
$$\eta_0(x) = A_s \text{sech}^2(\lambda(x-x_0)), \quad u_0(x) = -c_s \frac{\eta_0(x)}{1+\eta_0(x)},$$

where  $A_s$  is the amplitude relative to the constant water depth  $D_0$  taken to be 1 in our study. The solitary wave speed  $c_s$  along with the wavelength  $\lambda$  are given here:

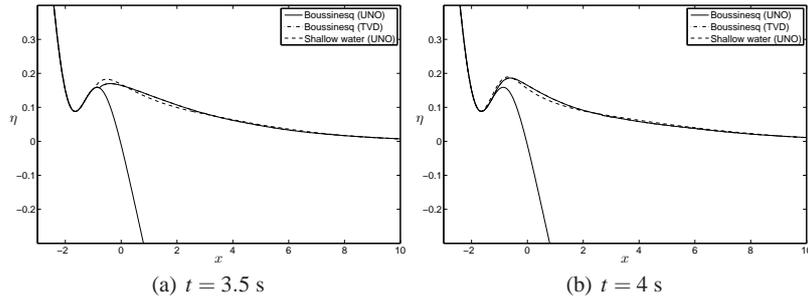
$$\lambda = \sqrt{\frac{3A_s}{4(1+A_s)}}, \quad c_s = \sqrt{g} \frac{\sqrt{6}(1+A_s)}{\sqrt{3+2A_s}} \cdot \frac{\sqrt{(1+A_s)\log(1+A_s)} - A_s}{A_s}.$$

The solitary wave is centered initially at  $x_0 = 10.62$  and has the amplitude  $A_s = 0.08$ . The constant slope  $\beta$  is equal to  $2.88^\circ$ . The sketch of the computational domain can be found in [6].

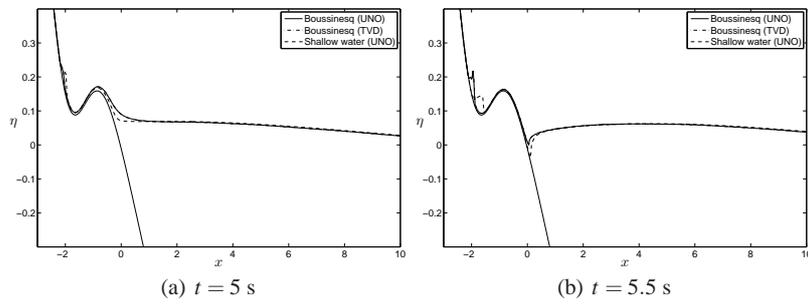
In numerical simulations presented below we used a uniform space discretization with  $\Delta x = 0.025$  and very fine time step  $\Delta t = \Delta x/100$  to guarantee the accuracy and stability during the whole simulation.



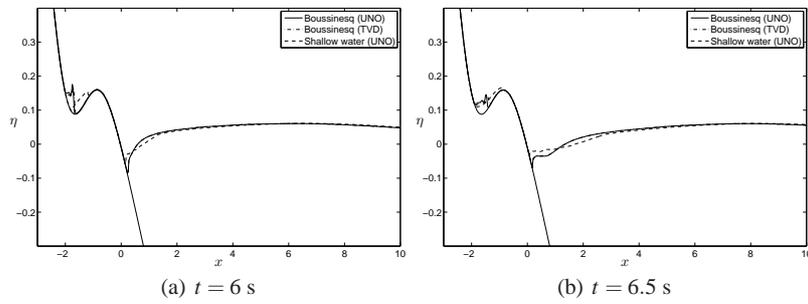
**Fig. 1** Solitary wave approaching a sloping beach with a pond.



**Fig. 2** Beginning of the pond inundation.

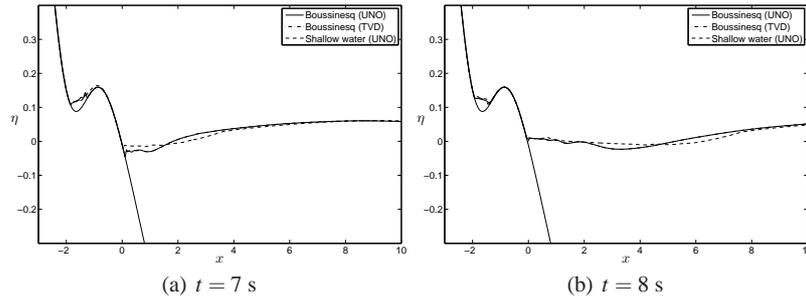


**Fig. 3** A part of the wave mass is trapped in the pond volume.

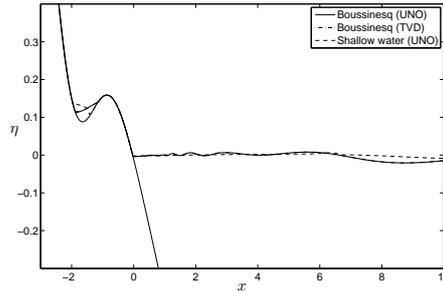


**Fig. 4** Wave oscillations in the pond.

Snapshots of numerical results are presented on Figures 1 – 6. We present simultaneously three different computational results:



**Fig. 5** Stabilization of wave oscillations.



**Fig. 6** The whole system is tending to the rest position (snapshot taken at  $t = 10$  s).

- Modified Peregrine system solved with UNO2 reconstruction [10]
- The same system with classical MUSCL TVD2 scheme [13]
- Nonlinear Shallow Water Equations (NSWE) with UNO2 scheme [10]

Surprisingly good agreement was obtained among all three numerical models. Presumably, the complex runup process under consideration is governed essentially by nonlinearity. However, on Figures 1(b) and 2(a) the amplitude predicted by NSWE is slightly overestimated.

On Figures 3(b) – 4(b) some oscillations can be seen. However, their amplitude remains small for all times and does not produce any blow up phenomena. Later these oscillations decay tending gradually to the “lake at the rest” state (see Figures 5, 6).

In the specific experiment a friction term could be beneficial to reduce the amplitude of oscillations (or damp them out completely). However, we prefer to present the computational results of our model without adding any ad-hoc term to show its original performance.

## 4 Conclusions

In this study we presented an improved version of the Peregrine system which is particularly suited for the simulation of dispersive waves runup. This system allows for the description of higher amplitude waves due to improved nonlinear characteristics. Better numerical stability properties have been obtained since most of the dispersive terms tend to zero when we approach the shoreline. Consequently, our model naturally degenerates to classical Nonlinear Shallow Water Equations (NSWE) for which the runup simulation technology is completely mastered nowadays. However we underline that there is no artificial parameter to turn off dispersive terms. Their importance is naturally governed by the underlying physical process.

Moreover we presented some numerical results on the wave runup onto a complex beach containing a pond. Even in this stiff case our numerical model produced stable and physical results, thus validating modification of the Peregrine system.

**Acknowledgements** D. Dutykh acknowledges the support from French Agence Nationale de la Recherche, project MathOcean (Grant ANR-08-BLAN-0301-01) and Ulysses Program of the French Ministry of Foreign Affairs under the project 23725ZA. The work of Th. Katsaounis was partially supported by European Union FP7 program Capacities(Regpot 2009-1), through ACMAC (<http://acmac.tem.uoc.gr>).

## References

1. Anastasiou, K., Chan, C.T.: Solution of the 2d shallow water equations using the finite volume method on unstructured triangular meshes. *International Journal for Numerical Methods in Fluids* **24**, 1225–1245 (1999)
2. Benjamin, T., Olver, P.: Hamiltonian structure, symmetries and conservation laws for water waves. *J. Fluid Mech* **125**, 137–185 (1982)
3. Delis, A.I., Katsaounis, T.: Relaxation schemes for the shallow water equations. *Int. J. Numer. Meth. Fluids* **41**, 695–719 (2003)
4. Dutykh, D., Dias, F.: Dissipative Boussinesq equations. *C. R. Mecanique* **335**, 559–583 (2007)
5. Dutykh, D., Dias, F.: Water waves generated by a moving bottom. In: A. Kundu (ed.) *Tsunami and Nonlinear waves*. Springer Verlag (Geo Sc.) (2007)
6. Dutykh, D., Katsaounis, T., Mitsotakis, D.: Finite volume schemes for dispersive wave propagation and runup. Accepted to *Journal of Computational Physics* <http://hal.archives-ouvertes.fr/hal-00472431/> (2011)
7. Dutykh, D., Poncet, R., Dias, F.: Complete numerical modelling of tsunami waves: generation, propagation and inundation. Submitted <http://arxiv.org/abs/1002.4553> (2010)
8. Fokas, A.S., Pelloni, B.: Boundary value problems for Boussinesq type systems. *Math. Phys. Anal. Geom.* **8**, 59–96 (2005)
9. Ghidaglia, J.M., Kumbaro, A., Coq, G.L.: On the numerical solution to two fluid models via cell centered finite volume method. *Eur. J. Mech. B/Fluids* **20**, 841–867 (2001)
10. Harten, A., Osher, S.: Uniformly high-order accurate nonoscillatory schemes, I. *SIAM J. Numer. Anal.* **24**, 279–309 (1987)
11. Hibberd, S., Peregrine, D.: Surf and run-up on a beach: a uniform bore. *J. Fluid Mech.* **95**, 323–345 (1979)
12. Kervella, Y., Dutykh, D., Dias, F.: Comparison between three-dimensional linear and nonlinear tsunami generation models. *Theor. Comput. Fluid Dyn.* **21**, 245–269 (2007)

13. van Leer, B.: Towards the ultimate conservative difference scheme V: a second order sequel to Godunov' method. *J. Comput. Phys.* **32**, 101–136 (1979)
14. Peregrine, D.H.: Long waves on a beach. *J. Fluid Mech.* **27**, 815–827 (1967)
15. Simon, M.: Wave-energy extraction by a submerged cylindrical resonant duct. *Journal of Fluid Mechanics* **104**, 159–187 (1981)
16. Tadepalli, S., Synolakis, C.E.: The run-up of N-waves on sloping beaches. *Proc. R. Soc. Lond. A* **445**, 99–112 (1994)
17. Titov, V., González, F.: Implementation and testing of the method of splitting tsunami (MOST) model. Tech. Rep. ERL PMEL-112, Pacific Marine Environmental Laboratory, NOAA (1997)
18. Titov, V.V., Synolakis, C.E.: Numerical modeling of tidal wave runup. *J. Waterway, Port, Coastal, and Ocean Engineering* **124**, 157–171 (1998)