



HAL
open science

Asymptotic Analysis of Regularized Zero-Forcing Precoding in MISO Broadcast Channels with Limited Feedback

Sebastian Wagner, Romain Couillet, Dirk Slock, Merouane Debbah

► **To cite this version:**

Sebastian Wagner, Romain Couillet, Dirk Slock, Merouane Debbah. Asymptotic Analysis of Regularized Zero-Forcing Precoding in MISO Broadcast Channels with Limited Feedback. ISIT, Jun 2010, Austin, United States. pp.ISIT 2010 - Wagner. hal-00553697

HAL Id: hal-00553697

<https://hal.science/hal-00553697>

Submitted on 8 Jan 2011

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Asymptotic Analysis of Regularized Zero-Forcing Precoding in MISO Broadcast Channels with Limited Feedback

Sebastian Wagner^{*†}, Romain Couillet^{*‡}, Dirk T. M. Slock[†], and M erouane Debbah[‡]

^{*}ST-ERICSSON, 06904 Sophia-Antipolis, France, Email: {sebastian.wagner, romain.couillet}@stericsson.com

[†]EURECOM, 06904 Sophia-Antipolis, France, Email: {sebastian.wagner, dirk.slock}@eurecom.fr

[‡]SUP ELEC, 91192 Gif sur Yvette, France, Email: {romain.couillet, merouane.debbah}@supelec.fr

Abstract—In this paper we analyse the asymptotic sum-rate of regularized zero-forcing (RZF) precoding in MISO broadcast channels with limited feedback, transmit correlation and path-loss. Our analysis assumes that the ratio of the number of transmit antennas M to the number of users K is bounded as (K, M) grow large. By applying recent results from random matrix theory we derive a deterministic equivalent of the SINR and compute the sum-rate maximizing regularization term as well as sum-rate bounds for high SNR. Numerical simulations show that the asymptotic results extend well into finite regimes.

I. INTRODUCTION

It has been shown in [1] that the capacity achieving precoding strategy of the Gaussian MIMO broadcast channel is based on the non-linear dirty-paper coding (DPC) technique. But so far no efficient practical algorithm implementing the optimal DPC scheme has been proposed. Therefore, low complexity linear precoding strategies have gained a lot of attention since they achieve a large portion of the channel capacity [2], [3].

The RZF filters have first been analysed in the MU-MIMO context in [3]. It has been observed that the RZF precoding matrix has a similar structure as the transmit Wiener filter derived in [4]. Indeed, both transmit filters are equal for asymptotically high (K, M) .

In this contribution we consider a system where both M and K grow asymptotically large but their ration $\beta = M/K$ is bounded. In particular we extend the models of [3], [5], [6] by considering imperfect CSIT, transmit correlation as well as different path-losses of the users. With the aid of recent tools from random matrix theory (RMT), we derive a *deterministic equivalent* of the signal-to-interference plus noise ratio (SINR) of RZF which is *independent* of the individual channel realizations.

Notation: In the following boldface lower-case and upper-case characters denote vectors and matrices, respectively. The operators $(\cdot)^H$ and $\text{tr}(\cdot)$ denote conjugate transpose and the trace, respectively. The expectation is $\mathbb{E}[\cdot]$ and $\text{diag}(\mathbf{x})$ is a diagonal matrix with vector \mathbf{x} on the main diagonal. The $N \times N$ identity matrix is \mathbf{I}_N . The k th column of \mathbf{X} is denoted \mathbf{x}_k .

II. MATHEMATICAL PRELIMINARIES

Definition 1: Let F be a probability distribution function. For $z \in \mathbb{C}$ outside the support of F , we define the *Stieltjes*

transform of F as the function

$$m_F(z) = \int \frac{1}{\lambda - z} dF(\lambda) \quad (1)$$

In this paper we are interested in the Stieltjes transform $m_{\mathbf{B}_K}$ of random matrices $\mathbf{B}_K \in \mathbb{C}^{M \times M}$ of the type

$$\mathbf{B}_K = \mathbf{R}_K^{1/2} \mathbf{X}_K^H \mathbf{L}_K \mathbf{X}_K \mathbf{R}_K^{1/2} + \mathbf{A}_K \quad (2)$$

where $\mathbf{R}_K \in \mathbb{C}^{M \times M}$ and $\mathbf{L}_K \in \mathbb{C}^{K \times K}$ are positive definite Hermitian matrices, $\mathbf{X}_K \in \mathbb{C}^{K \times M}$ is random with independent and identically distributed (i.i.d.) entries of zero mean and variance $1/M$, and \mathbf{A}_K is a nonnegative Hermitian matrix, with same eigenspace as \mathbf{R}_K ; denote then $a(\lambda)$ a function mapping the eigenvalues of \mathbf{R}_K to those of \mathbf{A}_K . In our derivations, we will require the following result,

Theorem 1: [7] Under the above model for \mathbf{B}_K where \mathbf{L}_K and \mathbf{R}_K have uniformly bounded spectral norm (w.r.t. M), as (K, M) grow large with ratio $\beta = \beta(M) \triangleq M/K$ such that $0 < \liminf_M \beta(M) \leq \limsup_M \beta(M) < \infty$, for $z \in \mathbb{C}^+$,

$$m_{\mathbf{B}_K}(z) - m_{\mathbf{B}_K}^\circ(z) \xrightarrow{M \rightarrow \infty} 0 \quad (3)$$

almost surely, where $m_{\mathbf{B}_K}^\circ(z)$ is defined as

$$m_{\mathbf{B}_K}^\circ(z) = \int \frac{1}{a(\lambda) + \lambda c(z) - z} dF^{\mathbf{R}_K}(\lambda) \quad (4)$$

$$\text{with } c(z) = \frac{1}{\beta} \int \frac{\nu}{1 + \nu e_K(z)} dF^{\mathbf{L}_K}(\nu) \quad (5)$$

where $e_K(z) \in \mathbb{C}^+$ is the unique solution of

$$e_K(z) = \int \frac{\lambda}{a(\lambda) + \lambda c(z) - z} dF^{\mathbf{R}_K}(\lambda) \quad (6)$$

where $m_{\mathbf{B}_K}^\circ(z)$ is referred to as a *deterministic equivalent* of $m_{\mathbf{B}_K}(z)$.

III. SYSTEM MODEL

Consider the MISO broadcast channel composed of one central transmitter equipped with M antennas and of K single-antenna receivers. Assume narrow-band communication. Denoting y_k the signal received by user k , the concatenated

received signal vector $\mathbf{y} = [y_1, \dots, y_K]^\top \in \mathbb{C}^K$ at a given time interval reads

$$\mathbf{y} = \sqrt{M}\mathbf{H}\mathbf{x} + \mathbf{n} \quad (7)$$

with transmit vector $\mathbf{x} \in \mathbb{C}^M$, channel matrix $\mathbf{H} \in \mathbb{C}^{K \times M}$ containing i.i.d. elements of zero mean and variance $1/M$ and noise vector $\mathbf{n} \sim \mathcal{CN}(0, \sigma^2 \mathbf{I}_K)$. The transmit vector \mathbf{x} is obtained by linear precoding

$$\mathbf{x} = \mathbf{G}\mathbf{s} \quad (8)$$

where $\mathbf{s} \sim \mathcal{CN}(0, \mathbf{I}_K)$ is the symbol vector and $\mathbf{G} = [\mathbf{g}_1, \dots, \mathbf{g}_K] \in \mathbb{C}^{M \times K}$ is the precoding matrix. The transmitter has a limited amount of transmit power $P > 0$, thus we have the constraint

$$\text{tr}(\mathbf{E}[\mathbf{x}\mathbf{x}^H]) = \text{tr}(\mathbf{G}\mathbf{G}^H) \leq P \quad (9)$$

In this paper we consider regularized channel inversion precoding

$$\mathbf{G} = \sqrt{M}\xi \left(M\tilde{\mathbf{H}}\tilde{\mathbf{H}} + M\alpha\mathbf{I}_M \right)^{-1} \tilde{\mathbf{H}}^H \quad (10)$$

$$= \frac{\xi}{\sqrt{M}} \left(\tilde{\mathbf{H}}^H\tilde{\mathbf{H}} + \alpha\mathbf{I}_M \right)^{-1} \tilde{\mathbf{H}}^H \triangleq \frac{\xi}{\sqrt{M}} \mathbf{W}\tilde{\mathbf{H}}^H \quad (11)$$

where $\tilde{\mathbf{H}}$ is the estimated channel matrix available at the transmitter and the scaling factor ξ is set to fulfill the power constraint (9). The regularization scalar α in (10) is scaled by M to ensure that, as (K, M) grow large, both $\text{tr}\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}$ and $\text{tr}M\alpha\mathbf{I}_M$ grow with the same order of magnitude. From (9) we obtain

$$\begin{aligned} \xi^2 &= \frac{P}{\frac{1}{M}\text{tr} \left[\tilde{\mathbf{H}}^H\tilde{\mathbf{H}} \left(\tilde{\mathbf{H}}^H\tilde{\mathbf{H}} + \alpha\mathbf{I}_M \right)^{-2} \right]} = \int \frac{P}{(\nu+\alpha)^2} dF^{\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}}(\nu) \\ &\stackrel{(a)}{=} \frac{P}{m_{\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}}(-\alpha) - \alpha m'_{\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}}(-\alpha)} \triangleq \frac{P}{\Psi(\alpha)} \end{aligned} \quad (12)$$

where (a) follows from (1) and $(\cdot)'$ denotes the derivative w.r.t. z in $z = -\alpha$. The received symbol y_k of user k is given by

$$y_k = \xi \mathbf{h}_k^H \mathbf{W} \tilde{\mathbf{h}}_k s_k + \xi \sum_{i=1, i \neq k}^K \mathbf{h}_k^H \mathbf{W} \tilde{\mathbf{h}}_i s_i + n_k \quad (13)$$

where \mathbf{h}_k^H and $\tilde{\mathbf{h}}_k^H$ denote the k^{th} row of \mathbf{H} and $\tilde{\mathbf{H}}$, respectively. The SINR γ_k of user k can be written in the form

$$\gamma_k = \frac{|\mathbf{h}_k^H \mathbf{W} \tilde{\mathbf{h}}_k|^2}{\mathbf{h}_k^H \mathbf{W} \mathbf{U}_k^H \mathbf{U}_k \mathbf{W} \mathbf{h}_k + \frac{1}{\rho} \Psi(\alpha)} \quad (14)$$

where $\mathbf{U}_k^H = [\tilde{\mathbf{h}}_1, \dots, \tilde{\mathbf{h}}_{k-1}, \tilde{\mathbf{h}}_{k+1}, \dots, \tilde{\mathbf{h}}_K] \in \mathbb{C}^{M \times (K-1)}$ and $\rho = P/\sigma^2$ denotes the SNR. The sum-rate R_{sum} is defined as

$$R_{\text{sum}} = \sum_{k=1}^K \log(1 + \gamma_k) \quad [\text{nats/s/Hz}] \quad (15)$$

Under the assumption of a rich scattering environment the correlated channel can be modeled as [8]–[10]

$$\mathbf{H} = \mathbf{L}^{1/2} \mathbf{H}_w \mathbf{\Theta}_T^{1/2} \quad (16)$$

where $\mathbf{H}_w \in \mathbb{C}^{K \times M}$ has i.i.d. zero-mean entries of variance $1/M$, $\mathbf{\Theta}_T \in \mathbb{C}^{M \times M}$ is the nonnegative definite correlation matrix at the transmitter and $\mathbf{L} = \text{diag}([l_1, \dots, l_K])$ is a diagonal matrix containing the user's channel gain. We assume $\|\mathbf{\Theta}_T\|^{-1}$ to be uniformly bounded with respect to M .

Moreover, we suppose that only $\tilde{\mathbf{H}}$, an imperfect estimate of the true channel matrix \mathbf{H} , is available at the transmitter. The channel-gain matrix \mathbf{L} as well as the transmit correlation $\mathbf{\Theta}_T$ can be estimated accurately and are assumed to be perfectly known. We therefore model $\tilde{\mathbf{H}}$ as

$$\tilde{\mathbf{H}} = \mathbf{L}^{1/2} \tilde{\mathbf{H}}_w \mathbf{\Theta}_T^{1/2} \quad (17)$$

$$\text{with } \tilde{\mathbf{H}}_w = \sqrt{1 - \tau^2} \mathbf{H}_w + \tau \mathbf{Q} \quad (18)$$

where $\mathbf{Q} \in \mathbb{C}^{K \times M}$ has i.i.d. zero-mean entries of variance $1/M$ which are not necessarily Gaussian distributed. Furthermore we suppose that \mathbf{H}_w and \mathbf{Q} are mutually independent as well as independent of the symbol vector \mathbf{s} and noise \mathbf{n} . A similar model for imperfect CSIT has been used in [11]–[13].

IV. DETERMINISTIC EQUIVALENT OF THE SINR

In the following we will derive a *deterministic equivalent* γ_k° of the SINR of user k , i.e. γ_k° is such that, almost surely,

$$\gamma_k - \gamma_k^\circ \xrightarrow{M \rightarrow \infty} 0 \quad (19)$$

That is, γ_k° is an approximation of γ_k independent of the particular realizations of \mathbf{H}_w and \mathbf{Q} . We will proceed by calculating deterministic equivalent expressions of $\Psi(\alpha)$, the signal power $|\mathbf{h}_k^H \mathbf{W} \tilde{\mathbf{h}}_k|^2$ and the power of the interference $\mathbf{h}_k^H \mathbf{W} \mathbf{U}_k^H \mathbf{U}_k \mathbf{W} \mathbf{h}_k$.

Consider $\Psi(\alpha)$ in (12), from Theorem 1, with $\mathbf{R}_K = \mathbf{\Theta}_T$ and $\mathbf{A}_K = 0$, we know that, for K large, $m_{\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}}(-\alpha)$ is close to $m_{\mathbf{H}^H\mathbf{H}}^\circ(-\alpha)$ given by Equation (3). Therefore we have

$$\Psi(\alpha) - [m_{\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}}^\circ(-\alpha) - \alpha m'_{\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}}(-\alpha)] \xrightarrow{M \rightarrow \infty} 0 \quad (20)$$

The Stieltjes transform is differentiable and $m'_{\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}}$ is well defined.

At this point we need the following results.

Corollary 1: [14] Let \mathbf{A} be a deterministic $N \times N$ complex matrix with uniformly bounded norm. Let $\mathbf{x} \in \mathbb{C}^N$ have i.i.d. complex entries of zero mean and variance $1/N$. Then, almost surely,

$$\mathbf{x}^H \mathbf{A} \mathbf{x} - \frac{1}{N} \text{tr} \mathbf{A} \xrightarrow{N \rightarrow \infty} 0 \quad (21)$$

and for vector $\mathbf{y} \in \mathbb{C}^N$ with standard i.i.d. entries, independent of \mathbf{x} ,

$$\mathbf{y}^H \mathbf{A} \mathbf{x} \xrightarrow{N \rightarrow \infty} 0 \quad (22)$$

almost surely.

In addition we will make use of the following identity

Lemma 1: [15, Lemma 2.2] Let \mathbf{A} be an $N \times N$ invertible matrix and $\mathbf{x} \in \mathbb{C}^N$, $c \in \mathbb{C}$ for which $\mathbf{A} + c\mathbf{x}\mathbf{x}^H$ is invertible. Then

$$\mathbf{x}^H (\mathbf{A} + c\mathbf{x}\mathbf{x}^H)^{-1} = \frac{\mathbf{x}^H \mathbf{A}^{-1}}{1 + c\mathbf{x}^H \mathbf{A}^{-1} \mathbf{x}} \quad (23)$$

A. Signal Power

Applying Lemma 1, we have

$$\tilde{\mathbf{h}}_k^H \mathbf{W} \mathbf{h}_k = \frac{\tilde{\mathbf{h}}_k^H (\mathbf{U}_k^H \mathbf{U}_k + \alpha \mathbf{I}_M)^{-1} \mathbf{h}_k}{1 + \tilde{\mathbf{h}}_k^H (\mathbf{U}_k^H \mathbf{U}_k + \alpha \mathbf{I}_M)^{-1} \tilde{\mathbf{h}}_k} \quad (24)$$

Together with $\tilde{\mathbf{h}}_k = \sqrt{1 - \tau^2} \mathbf{h}_k + \tau \tilde{\mathbf{q}}_k$ and $\tilde{\mathbf{q}}_k = \sqrt{l_k} \mathbf{q}_k \Theta_T^{1/2}$ we obtain

$$\tilde{\mathbf{h}}_k^H \mathbf{W} \mathbf{h}_k = \frac{\sqrt{1 - \tau^2} l_k \mathbf{h}_{wk}^H \mathbf{A}_k^{-1} \mathbf{h}_{wk}}{1 + l_k \tilde{\mathbf{h}}_{wk}^H \mathbf{A}_k^{-1} \tilde{\mathbf{h}}_{wk}} + \frac{\tau l_k \mathbf{q}_k^H \mathbf{A}_k^{-1} \mathbf{h}_{wk}}{1 + l_k \tilde{\mathbf{h}}_{wk}^H \mathbf{A}_k^{-1} \tilde{\mathbf{h}}_{wk}}$$

with $\mathbf{V}_k^H = [\tilde{\mathbf{h}}_{w1}, \dots, \tilde{\mathbf{h}}_{w(k-1)}, \tilde{\mathbf{h}}_{w(k+1)}, \dots, \tilde{\mathbf{h}}_{wK}]$, $\mathbf{L}_k = \text{diag}([l_1, \dots, l_{k-1}, l_{k+1}, \dots, l_K])$ and $\mathbf{A}_k = \mathbf{V}_k^H \mathbf{L}_k \mathbf{V}_k + \alpha \Theta_T^{-1}$. Since \mathbf{h}_{wk} and \mathbf{q}_k have i.i.d. entries of variance $1/M$ and are independent of \mathbf{A}_k we evoke Corollary 1 and obtain

$$\mathbf{h}_{wk}^H \mathbf{A}_k^{-1} \mathbf{h}_{wk} - \frac{1}{M} \text{tr} \mathbf{A}_k^{-1} \xrightarrow{M \rightarrow \infty} 0 \quad (25)$$

$$\tilde{\mathbf{h}}_{wk}^H \mathbf{A}_k^{-1} \tilde{\mathbf{h}}_{wk} - \frac{1}{M} \text{tr} \mathbf{A}_k^{-1} \xrightarrow{M \rightarrow \infty} 0 \quad (26)$$

$$\mathbf{q}_k^H \mathbf{A}_k^{-1} \mathbf{h}_{wk} \xrightarrow{M \rightarrow \infty} 0 \quad (27)$$

Consequently we have

$$\tilde{\mathbf{h}}_k^H \mathbf{W} \mathbf{h}_k - \sqrt{1 - \tau^2} \frac{l_k \frac{1}{M} \text{tr} \mathbf{A}_k^{-1}}{1 + l_k \frac{1}{M} \text{tr} \mathbf{A}_k^{-1}} \xrightarrow{M \rightarrow \infty} 0 \quad (28)$$

In [16] we prove that a rank-1 perturbation has no impact on the trace of \mathbf{A}_k^{-1} for asymptotically large K . Therefore, almost surely,

$$\frac{1}{M} \text{tr} \mathbf{A}_k^{-1} - \frac{1}{M} \text{tr} \mathbf{A}^{-1} \xrightarrow{M \rightarrow \infty} 0 \quad (29)$$

where

$$\mathbf{A} = \tilde{\mathbf{H}}_w^H \mathbf{L} \tilde{\mathbf{H}}_w + \alpha \Theta_T^{-1} \quad (30)$$

Furthermore we can write $\frac{1}{M} \text{tr} \mathbf{A}^{-1}$ in terms of the Stieltjes transform $m_{\mathbf{A}}(z)$ in $z = 0$ (which is valid since $\|\Theta_T^{-1}\|$ is bounded away from zero) which, for large K is close to

$$m_{\mathbf{A}}^{\circ}(0) = \int \frac{\lambda}{\lambda c(0) + \alpha} dF^{\Theta_T}(\lambda) \quad (31)$$

$$\text{with } c(0) = \frac{1}{\beta} \int \frac{\nu}{1 + \nu m_{\mathbf{A}}^{\circ}(0)} dF^{\mathbf{L}}(\nu) \quad (32)$$

Finally, Equation (28) implies

$$\tilde{\mathbf{h}}_k^H \mathbf{W} \mathbf{h}_k - \sqrt{1 - \tau^2} \frac{l_k m_{\mathbf{A}}^{\circ}}{1 + l_k m_{\mathbf{A}}^{\circ}} \xrightarrow{M \rightarrow \infty} 0 \quad (33)$$

B. Interference Power

After applying Lemma 1 twice, we obtain

$$\mathbf{h}_k^H \mathbf{W} \mathbf{U}_k^H \mathbf{U}_k \mathbf{W} \mathbf{h}_k = \frac{l_k \mathbf{h}_{wk}^H \mathbf{B}_k^{-1} \mathbf{V}_k^H \mathbf{L}_k \mathbf{V}_k \mathbf{B}_k^{-1} \mathbf{h}_{wk}}{(1 + l_k (1 - \tau^2) \mathbf{h}_{wk}^H \mathbf{B}_k^{-1} \mathbf{h}_{wk})^2} \quad (34)$$

where $\mathbf{B}_k = \mathbf{A}_k + l_k \tau^2 \mathbf{q}_k \mathbf{q}_k^H + l_k \tau \sqrt{1 - \tau^2} \mathbf{h}_{wk} \mathbf{q}_k^H + l_k \tau \sqrt{1 - \tau^2} \mathbf{h}_{wk}^H \mathbf{q}_k$.

Lemma 2: Let $\mathbf{A} \in \mathbb{C}^{N \times N}$ of bounded norm. Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^N$ have i.i.d. complex entries of zero mean and variance $1/N$. Then, for $c_i \in \mathbb{C}$, $i=0, 1, 2$ and $u = \frac{1}{N} \text{tr} \mathbf{A}^{-1}$

$$\mathbf{x}^H (\mathbf{A} + c_0 \mathbf{x} \mathbf{x}^H + c_1 \mathbf{y} \mathbf{y}^H + c_2 \mathbf{x} \mathbf{y}^H + c_2 \mathbf{y} \mathbf{x}^H)^{-1} \mathbf{x} - \frac{u(1 + c_1 u)}{(c_0 c_1 - c_2^2) u^2 + (c_0 + c_1) u + 1} \xrightarrow{N \rightarrow \infty} 0 \quad (35)$$

almost surely. The proof can be found in [16].

Applying Lemma 2, after some algebraic manipulations we obtain

$$\frac{l_k [m_{\mathbf{A}}^{\circ}(0) - \alpha m_{\mathbf{A}^2 \Theta_T}^{\circ}(0)] [1 + l_k \tau^2 (l_k m_{\mathbf{A}}^{\circ}(0) + 2) m_{\mathbf{A}}^{\circ}(0)]}{(1 + l_k m_{\mathbf{A}}^{\circ}(0))^2} \mathbf{h}_k^H \mathbf{W} \mathbf{U}_k^H \mathbf{U}_k \mathbf{W} \mathbf{h}_k \xrightarrow{M \rightarrow \infty} 0 \quad (36)$$

where $m_{\mathbf{A}^2 \Theta_T}^{\circ}(0) = \frac{1}{M} \text{tr} \left(\Theta_T^{1/4} \tilde{\mathbf{H}}_w^H \mathbf{L} \tilde{\mathbf{H}}_w \Theta_T^{1/4} + \alpha \Theta_T^{-1/2} \right)^{-2}$. We can express $m_{\mathbf{A}^2 \Theta_T}^{\circ}(0)$ in terms of the derivative of $m_{\mathbf{A} \Theta_T^{1/2}}^{\circ}(0)$

$$m_{\mathbf{A}^2 \Theta_T}^{\circ}(0) = \left. \frac{\partial m_{\mathbf{A} \Theta_T^{1/2}}^{\circ}(z)}{\partial z} \right|_{z=0} = m_{\mathbf{A} \Theta_T^{1/2}}^{\circ \prime}(0) \quad (37)$$

Applying Theorem 1 to matrix $\mathbf{A} \Theta_T^{1/2}$ leads to

$$m_{\mathbf{A} \Theta_T^{1/2}}^{\circ}(0) = \int \frac{\lambda^{1/2}}{\lambda c(0) + \alpha} dF^{\Theta_T}(\lambda) \quad (38)$$

$$\text{with } c(0) = \frac{1}{\beta} \int \frac{\nu}{1 + \nu e_{\mathbf{A} \Theta_T^{1/2}}(0)} dF^{\mathbf{L}}(\nu) \quad (39)$$

$$e_{\mathbf{A} \Theta_T^{1/2}}(0) = \int \frac{\lambda}{\lambda c(0) + \alpha} dF^{\Theta_T}(\lambda) \quad (40)$$

For the derivatives we obtain

$$m_{\mathbf{A} \Theta_T^{1/2}}^{\circ \prime}(0) = \int \frac{\lambda - \lambda^{3/2} c'(0)}{(\lambda c(0) + \alpha)^2} dF^{\Theta_T}(\lambda) \quad (41)$$

$$c'(0) = -\frac{1}{\beta} \int \frac{\nu^2 e'_{\mathbf{A} \Theta_T^{1/2}}(0)}{(1 + \nu e_{\mathbf{A} \Theta_T^{1/2}}(0))^2} dF^{\mathbf{L}}(\nu) \quad (42)$$

$$e'_{\mathbf{A} \Theta_T^{1/2}}(0) = \int \frac{\lambda^{3/2} - \lambda^2 c'(0)}{(\lambda c(0) + \alpha)^2} dF^{\Theta_T}(\lambda) \quad (43)$$

Finally, the deterministic equivalent γ_k° is given by (44). Note that the computation of (44) requires the solution of only *one* fixed-point equation [16].

V. SUM-RATE MAXIMIZING REGULARIZATION

To optimize the achievable sum-rate, α in (44) should be chosen to maximize (15). We then define α^* as

$$\alpha^* = \arg \max_{\alpha > 0} \sum_{k=1}^K \log(1 + \gamma_k^{\circ}) \quad (45)$$

For the general channel model (17) the optimization in (45) is very tedious and no closed-form solution for α^* exists. However, in case of a homogeneous network ($\mathbf{L} = \mathbf{I}_K$) without transmit correlation ($\Theta_T = \mathbf{I}_M$) α^* has a closed-form. In this case $m_{\mathbf{A}}^{\circ}(0) = m_{\mathbf{A} \Theta_T^{1/2}}^{\circ} = m_{\tilde{\mathbf{H}}_w^H \tilde{\mathbf{H}}_w}^{\circ}(-\alpha)$ is the Stieltjes

$$\gamma_k^\circ = \frac{l_k^2(1-\tau^2)(m_{\mathbf{A}}^\circ(0))^2}{l_k \left[m_{\mathbf{A}}^\circ(0) - \alpha m_{\mathbf{A}\Theta_{\mathbf{T}}^{1/2}}^\circ(0) \right] \left[1 + l_k \tau^2 (l_k m_{\mathbf{A}}^\circ(0) + 2) m_{\mathbf{A}}^\circ(0) \right] + \frac{1}{\rho} (1 + l_k m_{\mathbf{A}}^\circ(0))^2 \Psi(\alpha)} \quad (44)$$

transform of the Marčenko-Pastur law and has the unique solution given by [17]

$$m_{\mathbf{H}_w^H \mathbf{H}_w}^\circ(-\alpha) = \frac{\beta(1-\alpha) - 1 + d(\alpha, \beta)}{2\alpha\beta}$$

with $d(\alpha, \beta) = \sqrt{\beta^2\alpha^2 + 2\alpha\beta(1+\beta) + (1-\beta)^2}$ (46)

Substituting (46) into (44) and setting the derivative w.r.t. α to zero, we obtain

$$\alpha^* = \left(\frac{1 + \tau^2 \rho}{1 - \tau^2} \right) \frac{1}{\beta \rho} \quad (47)$$

For this α^* , γ_k° in (44) takes the surprisingly simple form

$$\gamma_k^\circ = m_{\mathbf{H}_w^H \mathbf{H}_w}^\circ(-\alpha^*) = \frac{\beta(1-\alpha^*) - 1 + d(\alpha^*, \beta)}{2\alpha^*\beta} \quad (48)$$

The sum-rate saturation level at high SNR is

$$R_{\text{sum}}^{\text{lim}} = \lim_{\rho \rightarrow \infty} K \log(1 + \gamma^\circ)$$

$$= K \log \left(1 - \frac{1 + \beta(\tau^2 - 1) + d(\beta, \tau)}{2\tau^2} \right) \quad (49)$$

$$\text{with } d(\beta, \tau) = \sqrt{\beta^2\tau^2 + \beta\tau^2(1-\tau^2)(4-\beta) + (\beta-1)^2}$$

Let's assume that the parameters involved in (47) are mutually independent and look at their asymptotic values.

First, notice that for perfect CSIT ($\tau = 0$) we have $\alpha^* = 1/(\beta\rho)$ which corresponds to the result derived in [3]. As mentioned in [3], for *large* (K, M) the RZF precoder is equal to the MMSE precoder in [4], [11].

In contrast, for $\tau > 0$ the RZF transmit filter and the MMSE transmit filter are not identical anymore, even in the large K limit. Moreover, for asymptotically high SNR, (47) becomes

$$\lim_{\rho \rightarrow \infty} \alpha^* = \frac{\tau^2}{1 - \tau^2} \frac{1}{\beta} \quad (50)$$

Thus, for $\rho \rightarrow \infty$, RZF has a finite sum-rate limit. That means, as soon as there are errors in the CSIT a sum-rate saturation effect occurs and the system becomes interference-limited. An open question is to determine how the distortion τ^2 has to scale in order to assure that the system sum-rate does not saturate at high SNR.

VI. NUMERICAL RESULTS

In our simulations all results are averaged over 10,000 independent channel realizations. Additional results can be found in [16].

Figure 1 shows the ergodic sum-rate two different precoder, RZF-1 using the sum-rate maximizing regularization term α^* in (47) and RZF-2 with $\alpha = 1/(\beta\rho)$ i.e. designed based on perfect CSIT. For comparison we also plot the performance of the MMSE filter proposed in [11] and the ZF precoder.

We observe that as soon as the error variance τ^2 dominates over the noise power σ^2 , the ergodic sum-rate of the RZF-2 filter decreases and approaches ZF precoding for high SNR. We further notice that the RZF-1 and MMSE filters achieve similar performance since they are almost identical for small values of τ^2 . But, since α^* is derived for asymptotically large (K, M), the performance advantage of the RZF-1 over the MMSE filter increases with increasing (K, M).

Figure 2 illustrates the ergodic sum-rate of RZF with the asymptotic optimal regularization α^* and the true optimal regularization α_{Opt} found by exhaustive search for $M = \{2, 4, 8\}$. As expected, we observe that for an increasing number of transmit antennas γ_k approaches γ_k° .

Figure 3 compares our deterministic results to Monte-Carlo simulations for a correlated channels with unequal user path loss. We indicate the standard deviation of the simulations by error bars.

The transmit correlation is assumed to depend only on distance d_{ij} , $i, j = 1, 2, \dots, M$ between antennas i and j placed on a uniform circular array (UCA). Thus, $(\Theta_{\mathbf{T}})_{ij} = J_0(2\pi d_{ij}/\lambda)$ [18], where J_0 is the zero-order Bessel function of the first kind and λ is the signal wavelength. To assure that $\|\Theta_{\mathbf{T}}\|^{-1}$ is bounded we suppose that the distance between adjacent antennas $d = d_{i,i+1}$ is independent of M , i.e. as M grows the radius of the UCA increases.

Furthermore, we consider a uniform density of users in a circular cell. We take K samples of the user distribution which remain constant over all channel realizations. According to [19] ("Suburban Macro") $l_k = -(31.5 + 35 \log_{10} d_k)$ dB, where d_k is the distance of user k to the transmitter.

From Figure 3 we observe, that the expressions derived for large (K, M) lie approximately within one standard deviation of the simulation results even for finite (K, M). To avoid the small divergence of the asymptotic results from the simulation results for high SNR, M has to be increased.

VII. CONCLUSION

In this paper we derived a deterministic equivalent of the SINR of RZF precoding in MISO broadcast channels by applying recent results from random matrix theory. We use the deterministic equivalent expression of the SINR to compute the sum-rate maximizing RZF precoder for large (K, M). Simulations show that the asymptotic results extend well into finite regimes.

REFERENCES

- [1] H. Weingarten, Y. Steinberg, and S. Shamai, "The Capacity Region of the Gaussian Multiple-Input Multiple-Output Broadcast Channel," *IEEE Trans. Inf. Theory*, vol. 52, no. 9, pp. 3936–3964, 2006.
- [2] G. Caire and S. Shamai, "On the Achievable Throughput of a Multi-antenna Gaussian Broadcast Channel," *IEEE Trans. Inf. Theory*, vol. 49, no. 7, pp. 1691–1706, 2003.

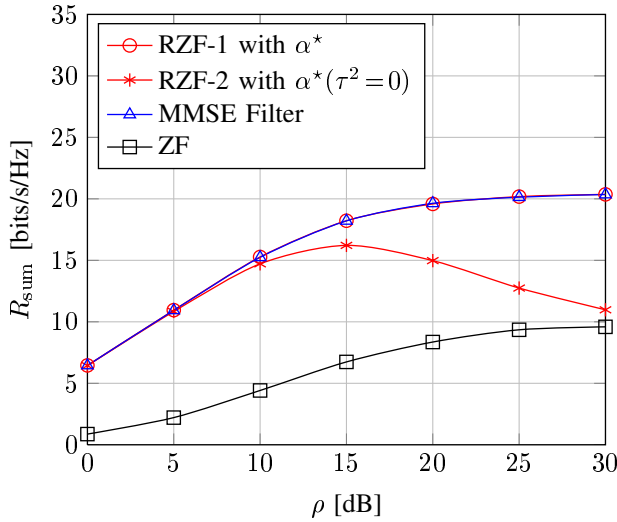


Fig. 1. Ergodic sum-rate vs. average SNR with $\Theta_T = \mathbf{I}_M$, $\mathbf{L} = \mathbf{I}_K$, $M = 10$, $\beta = 1$, $\tau^2 = 0.1$.

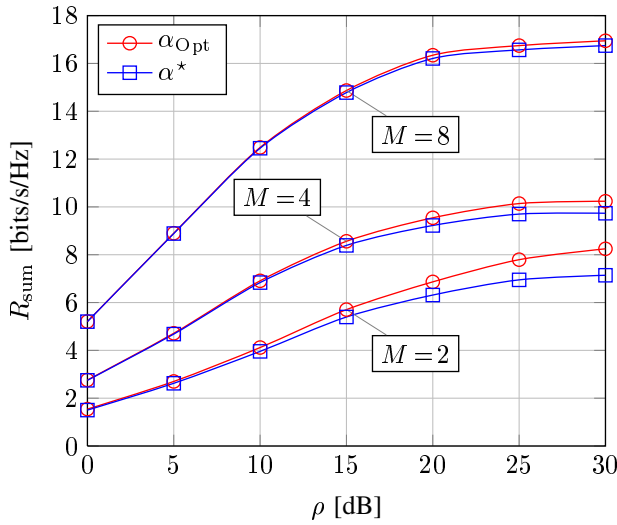


Fig. 2. Ergodic sum-rate vs. average SNR with $\Theta_T = \mathbf{I}_M$, $\mathbf{L} = \mathbf{I}_K$, $\beta = 1$, $\tau^2 = 0.1$.

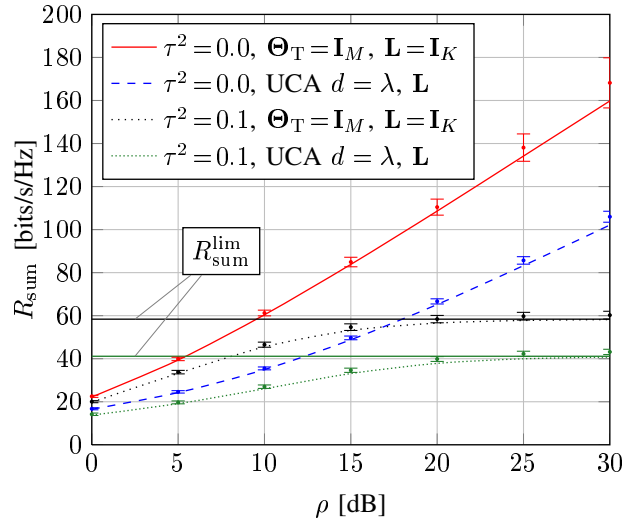


Fig. 3. Ergodic sum-rate vs. average SNR with $M = 32$, $\beta = 1$, simulation results are indicated by circle marks with error bars indicating the standard deviation.

[3] C. Peel, B. Hochwald, and A. Swindlehurst, "A Vector-Perturbation Technique for Near-Capacity Multiantenna Multiuser Communication—Part I: Channel Inversion and Regularization," *IEEE Trans. Commun.*, vol. 53, no. 1, pp. 195–202, 2005.

[4] M. Joham, K. Kusume, M. Gzara, W. Utschick, and J. Nosske, "Transmit Wiener Filter for the Downlink of TDD DS-CDMA Systems," in *Proc. ISSSTA 2002*, vol. 1, 2002, pp. 9–13.

[5] H. Viswanathan and S. Venkatesan, "Asymptotics of Sum Rate for Dirty Paper Coding and Beamforming in Multiple-Antenna Broadcast Channels," in *Proc. Allerton Conf. Communication, Control, and Computing*, Monticello, Illinois, Oct. 2003, pp. 1064–1073.

[6] R. Couillet, S. Wagner, and M. Debbah, "Asymptotic Analysis of Linear Precoding Techniques in Correlated Multi-Antenna Broadcast Channels," *IEEE Trans. Inf. Theory*, submitted for publication. [Online]. Available: <http://arxiv.org/abs/0906.3682>

[7] R. Couillet, M. Debbah, and J. W. Silverstein, "A Deterministic Equivalent for the Capacity Analysis of Correlated Multi-user MIMO Channels," *IEEE Trans. Inf. Theory*, submitted for publication. [Online]. Available: <http://arxiv.org/abs/0906.3667v3>

[8] C. Chuah, D. Tse, J. Kahn, and R. Valenzuela, "Capacity Scaling in

MIMO Wireless Systems Under Correlated Fading," *IEEE Trans. Inf. Theory*, vol. 48, no. 3, pp. 637–650, 2002.

[9] A. Tulino, A. Lozano, and S. Verdú, "Impact of Antenna Correlation on the Capacity of Multiantenna Channels," *IEEE Trans. Inf. Theory*, vol. 51, no. 7, pp. 2491–2509, 2005.

[10] H. Shin, M. Win, and M. Chiani, "Asymptotic Statistics of Mutual Information for Doubly Correlated MIMO Channels," *IEEE Trans. Wireless Commun.*, vol. 7, no. 2, pp. 562–573, 2008.

[11] A. D. Dabbagh and D. J. Love, "Multiple antenna MMSE based downlink precoding with quantized feedback or channel mismatch," *IEEE Trans. Commun.*, vol. 56, no. 11, pp. 1859–1868, 2008.

[12] T. Yoo and A. Goldsmith, "Capacity and Power Allocation for Fading MIMO Channels with Channel Estimation Error," *IEEE Trans. Inf. Theory*, vol. 52, no. 5, pp. 2203–2214, 2006.

[13] A. A. Hutter, E. de Carvalho, and J. M. Cioffi, "On the Impact of Channel Estimation for Multiple Antenna Diversity Reception in Mobile OFDM Systems," in *Conference Record of the Thirty-Fourth Asilomar Conference on Signals, Systems and Computers*, vol. 2, 2000.

[14] Z. Bai and J. W. Silverstein, "No Eigenvalues Outside the Support of the Limiting Spectral Distribution of Large Dimensional Sample Covariance Matrices," *Annals of Probability*, vol. 26, pp. 316–345, 1998.

[15] J. W. Silverstein and Z. Bai, "On the Empirical Distribution of Eigenvalues of a Class of Large Dimensional Random Matrices," *Journal of Multivariate Analysis*, vol. 54, no. 2, pp. 175–192, 1995.

[16] S. Wagner, R. Couillet, D. T. M. Slock, and M. Debbah, "Asymptotic Analysis of Linear Precoding in Vector Broadcast Channels with Limited Feedback," *IEEE Trans. Signal Process.*, submitted for publication. [Online]. Available: <http://arxiv.org/dummy>

[17] A. M. Tulino and S. Verdú, *Random Matrix Theory and Wireless Communications*. Now Publishers Inc., 2004.

[18] W. Jakes and D. Cox, *Microwave Mobile Communications*. Wiley, 1994.

[19] H. Huang, "TR 25.996 Spatial Channel Model for Multiple Input Multiple Output (MIMO) Simulations," 3GPP, Tech. Rep., Dec. 2008.