



**HAL**  
open science

# Discrete approximation to solution flows of Tanaka SDE related to Walsh Brownian motion

Hatem Hajri

► **To cite this version:**

Hatem Hajri. Discrete approximation to solution flows of Tanaka SDE related to Walsh Brownian motion. 2011. hal-00553691v1

**HAL Id: hal-00553691**

**<https://hal.science/hal-00553691v1>**

Preprint submitted on 8 Jan 2011 (v1), last revised 2 Oct 2011 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# DISCRETE APPROXIMATION TO SOLUTION FLOWS OF TANAKA SDE RELATED TO WALSH BROWNIAN MOTION

Hatem Hajri  
 Université Paris Sud 11  
 Hatem.Hajri@math.u-psud.fr

## Abstract

In a previous work, we have defined a Tanaka SDE related to Walsh Brownian motion which depends on kernels. It was shown that there are only one Wiener solution and only one flow of mappings solving this equation. In the terminology of Le Jan and Raimond, these are respectively the stronger and the weaker among all solutions. In this paper, we obtain these solutions as limits of discrete models.

## 1 Introduction and main result.

Consider Tanaka equation for given  $s \geq 0, x \in \mathbb{R}$ :

$$dX_t = \text{sgn}(X_t)dW_t, \quad t \geq s, X_s = x \quad (1.1)$$

where  $\text{sgn}(x) = \mathbb{1}_{\{x>0\}} - \mathbb{1}_{\{x \leq 0\}}$  and  $W$  is a Brownian motion on  $\mathbb{R}$  that is,  $W_t = W_{0,t}\mathbb{1}_{\{t>0\}} - W_{t,0}\mathbb{1}_{\{t \leq 0\}}$  and  $(W_{s,t}, s \leq t)$  is a real white noise. This is an example of a stochastic differential equation which admits a weak solution but has no strong solution. In [6], an extension to kernels of (1.1) in the case of Walsh Brownian motion was defined.

Fix  $N \in \mathbb{N}^*$ ,  $\alpha_1, \dots, \alpha_N > 0$  such that  $\sum_{i=1}^N \alpha_i = 1$  and consider the graph  $G$  consisting of  $N$  half lines  $(D_i)_{1 \leq i \leq N}$  emanating from 0 (see Figure 1). Let  $\vec{e}_i$  be a vector of modulus 1 such that  $D_i = \{h\vec{e}_i, h \geq 0\}$  and define for all  $z \in G$ ,  $\vec{e}(z) = \vec{e}_i$  if  $z \in D_i, z \neq 0$  (convention  $\vec{e}(0) = \vec{e}_N$ ). We denote by  $|\cdot|$  the natural distance on  $G$ . Let  $G^* = G \setminus \{0\}$  and  $C_b^2(G^*)$  be the space of all  $f : G \rightarrow \mathbb{R}$ ,  $f$  is continuous on  $G$  and has bounded first and second derivatives ( $f'$  and  $f''$ ) on  $G^*$  (here  $f'(z)$  is the derivative of  $f$  at  $z$  in the direction  $\vec{e}(z)$ ) for all  $z \neq 0$ , both  $\lim_{z \rightarrow 0, z \in D_i, z \neq 0} f'(z)$  and  $\lim_{z \rightarrow 0, z \in D_i, z \neq 0} f''(z)$  exist for all  $i \in [1, N]$ . Define

$$D(\alpha_1, \dots, \alpha_N) = \left\{ f \in C_b^2(G^*) : \sum_{i=1}^N \alpha_i \lim_{z \rightarrow 0, z \in D_i, z \neq 0} f'(z) = 0 \right\}.$$

Now, Tanaka SDE on  $G$  extended to kernels is the following (see Remark 2.7 [6] for a discussion of its origin).

**Definition 1.1.** (*Equation (T)*).

On a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ , let  $W$  be a real white noise and  $K$  be a stochastic flow of kernels. We say that  $(K, W)$  solves (T) if for all  $s \leq t, f \in D(\alpha_1, \dots, \alpha_N), x \in G$ ,

$$K_{s,t}f(x) = f(x) + \int_s^t K_{s,u}f'(x)W(du) + \frac{1}{2} \int_s^t K_{s,u}f''(x)du \quad a.s.$$

If  $K = \delta_\varphi$  is a solution of (T), we just say that  $(\varphi, W)$  solves (T).

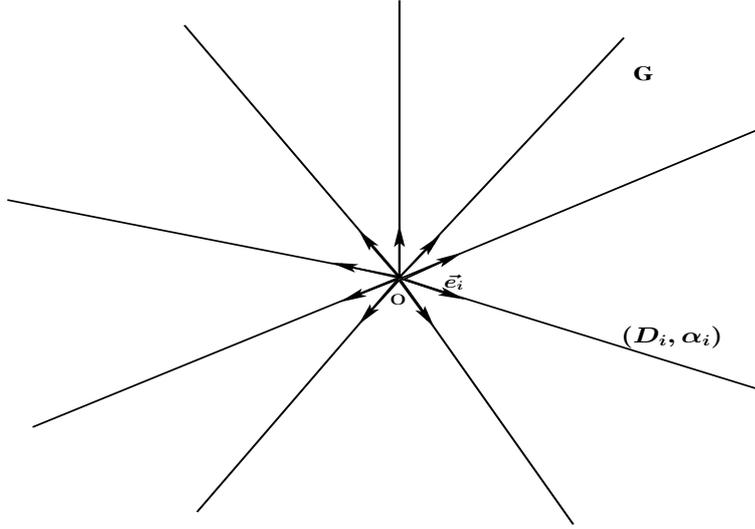


Figure 1: Graph  $G$ .

Equation  $(T)$  is a particular case of equation  $(E)$  which was defined and studied in [6]. It was shown (see Corollary 4.1 [6]) that if  $(K, W)$  solves  $(T)$ , then  $\sigma(W) \subset \sigma(K)$  and therefore one can just say that  $K$  solves  $(T)$ . We also recall

**Theorem 1.2.** [6] *There exists a unique Wiener flow  $K^W$  (resp. flow of mappings  $\varphi$ ) which solves  $(T)$ .*

As described in Theorem 1.4 [6], the unique Wiener solution of  $(T)$  is simply

$$K_{s,t}^W(x) = \delta_{x+\vec{e}_i W_{s,t}} \mathbf{1}_{\{t \leq \tau_{s,x}\}} + \sum_{i=1}^N \alpha_i \delta_{\vec{e}_i W_{s,t}^+} \mathbf{1}_{\{t > \tau_{s,x}\}}$$

where

$$\tau_{s,x} = \inf\{r \geq s : W_{s,r} = -|x|\}, \quad W_{s,t}^+ := W_t - \inf_{u \in [s,t]} W_u.$$

However, the expression of the unique (see [6] Remark 4.7) flow of mappings  $\varphi$  is more complicated. A continuous time construction of  $\varphi$  using Kolmogorov extension theorem can be derived from Section 4.1 [8]. Another construction by flipping Brownian excursions was given also in Section 3.2.1 [6] in a more general context. Here, we restrict our attention to discrete models.

The one point motion associated to any solution of  $(T)$  is the Walsh Brownian motion  $W(\alpha_1, \dots, \alpha_N)$  on  $G$  (see Proposition 2.8 [6]) which we define as a strong Markov process with state space  $G$  and Feller semigroup  $(P_t)_{t \geq 0}$  as given in Section 2.2 [6]. When  $N = 2$ , this is the famous skew Brownian motion (SBM) [7]. From now on, for any metric spaces  $E, F$ , let  $C(E, F)$  (resp.  $\mathcal{P}(E)$ ) be the space of all continuous functions from  $E$  into  $F$  (resp. probability measures on  $E$ ). In this paper, we first prove the following Donsker approximation of  $W(\alpha_1, \dots, \alpha_N)$

**Proposition 1.3.** *Let  $M = (M_n)_{n \geq 0}$  be a Markov chain on  $G$  started at 0 with stochastic matrix  $Q$  given by:*

$$Q(0, \vec{e}_i) = \alpha_i, \quad Q(n\vec{e}_i, (n+1)\vec{e}_i) = Q(n\vec{e}_i, (n-1)\vec{e}_i) = \frac{1}{2} \quad \forall i \in [1, N], \quad n \in \mathbb{N}^*. \quad (1.2)$$

Let  $t \mapsto M(t)$  be the linear interpolation of  $(M_n)_{n \geq 0}$  and define  $M_t^n = \frac{1}{\sqrt{n}}M(nt)$ ,  $n \geq 1$ . Then

$$(M_t^n)_{t \geq 0} \xrightarrow[n \rightarrow +\infty]{law} (Z_t)_{t \geq 0}$$

in  $C([0, +\infty[, G)$  where  $Z$  is an  $W(\alpha_1, \dots, \alpha_N)$  started at 0.

This result extends that of [4] who treated the case  $\alpha_1 = \dots = \alpha_N = \frac{1}{N}$  and of course the Donsker theorem for the SBM which may be found for example in [3]. We show in fact that Proposition 1.3 can be deduced immediately from the case  $N = 2$ . We also notice that throughout this paper stochastic flows are defined as in [6]. We now come to the discrete description of  $(\varphi, K^W)$  and first introduce

**Definition 1.4.** (Discrete flows) Let  $G_{\mathbb{N}} = \{x \in G; |x| \in \mathbb{N}\}$ . We say that a process  $\psi_{p,q}(x)$  (resp.  $N_{p,q}(x)$ ) indexed by  $\{p \leq q \in \mathbb{Z}, x \in G_{\mathbb{N}}\}$  with values in  $G_{\mathbb{N}}$  (resp.  $\mathcal{P}(G_{\mathbb{N}})$ ) is a discrete flow of mappings (resp. kernels) if:

(i)  $\forall p_1 < \dots < p_n; \{\psi_{p_{i-1}, p_i}; 2 \leq i \leq n\}$  (resp.  $\{N_{p_{i-1}, p_i}; 2 \leq i \leq n\}$ ) is independent.

(ii)  $\forall p \leq n, x \in G_{\mathbb{N}}; \psi_{p, n+1}(x) = \psi_{n, n+1}(\psi_{p, n}(x))$  a.s

$$\text{(resp. } N_{p, n+1}(x)(A) = \int_{\mathbb{Z}} N_{n, n+1}(y)(A) N_{p, n}(x)(dy) \text{ for all } A \subset G_{\mathbb{N}} \text{ a.s)}$$

Let  $S = (S_n)_{n \in \mathbb{Z}}$  be a simple random walk on  $\mathbb{Z}$ , that is  $(S_n)_{n \in \mathbb{N}}$  and  $(S_{-n})_{n \in \mathbb{N}}$  are two independent simple random walks on  $\mathbb{Z}$ . Let  $(\vec{\eta}_i)_{i \in \mathbb{Z}}$  be a sequence of i.i.d random variables with law  $\sum_{i=1}^N \alpha_i \delta_{\vec{e}_i}$  which is also independent of  $S$ . For  $p \leq n, x \in G_{\mathbb{N}}$ , define

$$\vec{\eta}_{p, n} = \vec{\eta}_i \text{ if } \min_{k \in [p, n]} S_k = i, \quad T_{p, x} = \inf\{q \geq p : S_q - S_p = -|x|\};$$

$$\Psi_{p, n}(x) := (x + \vec{e}(x)(S_n - S_p)) \mathbb{1}_{\{n \leq T_{p, x}\}} + \vec{\eta}_{p, n}(S_n - \min_{k \in [p, n]} S_k) \mathbb{1}_{\{n > T_{p, x}\}};$$

$$K_{p, n}(x) = E[\delta_{\Psi_{p, n}(x)} | \sigma(S)] = \delta_{x + \vec{e}(x)(S_n - S_p)} \mathbb{1}_{\{n \leq T_{p, x}\}} + \sum_{i=1}^N \alpha_i \delta_{(S_n - \min_{k \in [p, n]} S_k) \vec{e}_i} \mathbb{1}_{\{n > T_{p, x}\}}.$$

Consider the following topology on  $\mathcal{P}(G)$

$$\beta(P, Q) = \sup \left\{ \left| \int g dP - \int g dQ \right|, \|g\|_{\infty} + \sup_{x \neq y} \frac{|g(x) - g(y)|}{|x - y|} \leq 1, g(0) = 0 \right\}.$$

In this paper we construct  $(\varphi, K^W)$  starting from  $(\Psi, K)$  and in particular we show the following

**Theorem 1.5.** (1)  $\Psi$  (resp.  $K$ ) is a discrete stochastic flow of mappings (resp. kernels).  
(2) There exists a joint realization  $(\psi, N, \varphi, K^W)$  on a common probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  such that

(i)  $(\psi, N) \stackrel{law}{=} (\Psi, K)$ .

(ii)  $(\varphi, W)$  (resp.  $(K^W, W)$ ) is the unique flow of mappings (resp. Wiener flow) which solves (T).

(iii) For all  $s \in \mathbb{R}, T > 0, x \in G, x^n \in \frac{1}{\sqrt{n}}G_{\mathbb{N}}$  such that  $\lim_{n \rightarrow \infty} x^n = x$  we have

$$\lim_{n \rightarrow \infty} \sup_{s \leq t \leq s+T} \left| \frac{1}{\sqrt{n}} \psi_{[ns], [nt]}(\sqrt{n}x^n) - \varphi_{s, t}(x) \right| = 0 \text{ a.s}$$

and

$$\lim_{n \rightarrow \infty} \sup_{s \leq t \leq s+T} \beta(K_{[ns],[nt]}(\sqrt{n}x^n)(\sqrt{n}\cdot), K_{s,t}^W(x)) = 0 \quad a.s. \quad (1.3)$$

This theorem implies also the following

**Corollary 1.6.** *For all  $s \in \mathbb{R}, x \in G_{\mathbb{N}}$ , let  $t \mapsto \Psi(t)$  be the linear interpolation of  $(\Psi_{[ns],k}(x), k \geq [ns])$  and  $\Psi_{s,t}^n(x) := \frac{1}{\sqrt{n}}\Psi(nt)$ ,  $K_{s,t}^n(x) = E[\delta_{\Psi_{s,t}^n(x)}|\sigma(S)]$ ,  $t \geq s, n \geq 1$ . For all  $1 \leq p \leq q$ ,  $(x_i)_{1 \leq i \leq q} \subset G$ , let  $x_i^n \in \frac{1}{\sqrt{n}}G_{\mathbb{N}}$  such that  $\lim_{n \rightarrow \infty} x_i^n = x_i$ . Define*

$$Y^n = (\Psi_{s_1, \cdot}^n(\sqrt{n}x_1^n), \dots, \Psi_{s_p, \cdot}^n(\sqrt{n}x_p^n), K_{s_{p+1}, \cdot}^n(\sqrt{n}x_{p+1}^n), \dots, K_{s_q, \cdot}^n(\sqrt{n}x_q^n)).$$

Then

$$Y^n \xrightarrow[n \rightarrow +\infty]{law} Y \quad \text{in} \quad \prod_{i=1}^p C([s_i, +\infty[, G) \times \prod_{j=p+1}^q C([s_j, +\infty[, \mathcal{P}(G))$$

where

$$Y = (\varphi_{s_1, \cdot}(x_1), \dots, \varphi_{s_p, \cdot}(x_p), K_{s_{p+1}, \cdot}^W(x_{p+1}), \dots, K_{s_q, \cdot}^W(x_q)).$$

Our proof of Theorem 1.5 is based on a remarkable transformation introduced by Csaki and Vincze [9] which is strongly linked with Tanaka SDE. We will study this transformation in Section 2 and then extend the result of Csaki and Vincze to Walsh Brownian motion (Proposition 2.3); Let  $S = (S_n)_{n \in \mathbb{N}}$  be a simple random walk on  $\mathbb{Z}$  (SRW) and associate to  $S$  the process  $Y_n := S_n - \min_{k \leq n} S_k$ , flip independently every “excursion” of  $Y$  to each ray  $D_i$  with probability  $\alpha_i$ , then the resulting process is not far from a random walk on  $G$  whose law is given by (1.2). In Section 3, we prove Proposition 1.3 and study the scaling limits of  $\Psi, K$ .

## 2 Csaki-Vincze transformation and consequences.

### 2.1 Csaki-Vincze transformation.

**Theorem 2.1.** [9](page 109) *Let  $S = (S_n)_{n \geq 0}$  be a SRW. Then, there exists a SRW  $\bar{S} = (\bar{S}_n)_{n \geq 0}$  such that:*

$$\bar{Y}_n := \max_{k \leq n} \bar{S}_k - \bar{S}_n \Rightarrow |\bar{Y}_n - |S_n|| \leq 2 \quad \forall n \in \mathbb{N}.$$

**Sketch of the proof.** Here, we just give the expression of  $\bar{S}$  with some useful comments (see also the figures below). We **insist** that a careful reading of the pages 109 and 110 [9] is recommended for the sequel. Let  $X_i = S_i - S_{i-1}, i \geq 1$  and define

$$\tau_1 = \min \{i > 0 : S_{i-1}S_{i+1} < 0\}, \quad \tau_{l+1} = \min \{i > \tau_l : S_{i-1}S_{i+1} < 0\} \quad \forall l \geq 1.$$

Set

$$\bar{X}_j = \begin{cases} -X_1 X_{j+1} & \text{if } 1 \leq j \leq \tau_1 \\ X_1 X_{j+1} & \text{if } \tau_1 + 1 \leq j \leq \tau_2 \\ \dots\dots\dots & \\ (-1)^{l+1} X_1 X_{j+1} & \text{if } \tau_l + 1 \leq j \leq \tau_{l+1} \end{cases}$$

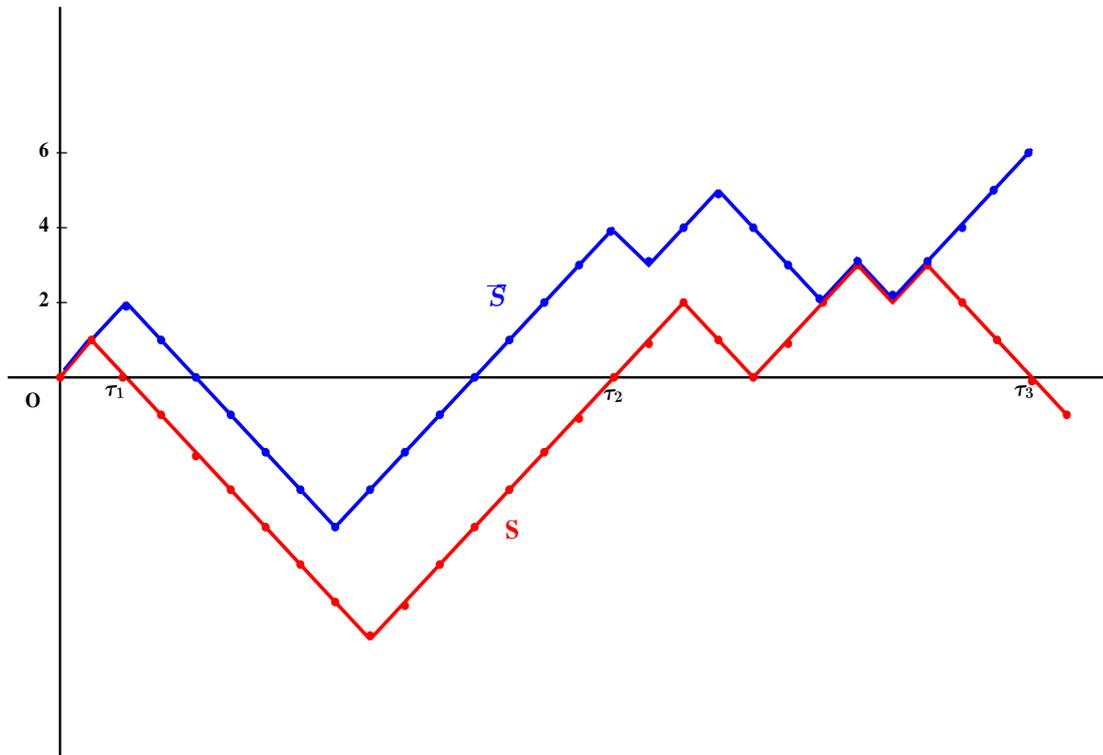


Figure 2:  $S$  and  $\bar{S}$ .

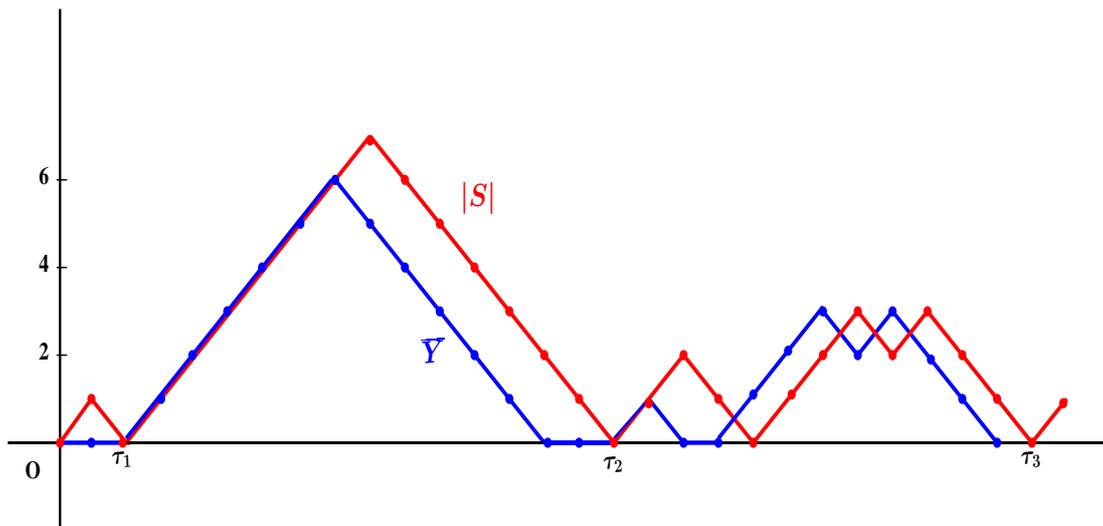


Figure 3:  $|S|$  and  $\bar{Y}$ .

Let  $\bar{S}_0 = 0$ ,  $\bar{S}_j = \bar{X}_1 + \dots + \bar{X}_j$ ,  $j \geq 1$ . Then,  $\bar{S}$  satisfies the theorem. We call  $T(S) = \bar{S}$  the Csaki-Vincze transformation of  $S$ .

As a consequence of (iii) and (iv) [9] (page 110), we have

$$\tau_l = \min \{n \geq 0, \bar{S}_n = 2l\} \quad \forall l \geq 1. \quad (2.1)$$

Note that  $T$  is an even function, that is  $T(S) = T(-S)$ . Combining this with (2.1), we get the following

**Corollary 2.2.** *Let  $\bar{S} = (\bar{S}_k)_{k \geq 0}$  be a SRW. Then*

- *There exists a SRW  $S$  such that:*

$$\bar{Y}_n := \max_{k \leq n} \bar{S}_k - \bar{S}_n \Rightarrow |\bar{Y}_n - |S_n|| \leq 2 \quad \forall n \in \mathbb{N}.$$

- $T^{-1}\{\bar{S}\}$  *is reduced to exactly two elements  $S$  and  $-S$  where  $S$  is obtained by adding information to  $\bar{S}$ .*

*Proof.* Set  $\bar{X}_j = \bar{S}_j - \bar{S}_{j-1}$ ,  $j \geq 1$  and  $\tau_l = \min \{n \geq 0, \bar{S}_n = 2l\}$  for all  $l \geq 1$ . Let  $\varepsilon$  be a random variable independent of  $\bar{S}$  such that:

$$\mathbb{P}(\varepsilon = 1) = \mathbb{P}(\varepsilon = -1) = \frac{1}{2}.$$

Define

$$X_{j+1} = \begin{cases} \varepsilon & \text{if } j = 0 \\ -\varepsilon \bar{X}_j & \text{if } 1 \leq j \leq \tau_1 \\ \dots\dots\dots & \\ (-1)^{l+1} \varepsilon \bar{X}_j & \text{if } \tau_l + 1 \leq j \leq \tau_{l+1} \end{cases}$$

and  $S_0 = 0$ ,  $S_j = X_1 + \dots + X_j$ ,  $\forall j \geq 1$ . Then, it is not hard to see that the sequence of the random times  $\tau_i(S)$ ,  $i \geq 1$  defined from  $S$  as in Theorem 2.1 is exactly  $\tau_i$ ,  $i \geq 1$  and therefore  $T(S) = \bar{S}$ .  $\square$

## 2.2 The link with Tanaka equation.

Let  $S$  be a SRW,  $\bar{S} = -T(S)$  and  $t \mapsto S(t)$  (resp.  $\bar{S}(t)$ ) be the linear interpolation of  $S$  (resp.  $\bar{S}$ ) on  $\mathbb{R}$ . Define for all  $n \geq 1$ ,  $S_t^{(n)} = \frac{1}{\sqrt{n}} S(nt)$ ,  $\bar{S}_t^{(n)} = \frac{1}{\sqrt{n}} \bar{S}(nt)$ . Then, it can be easily checked that (Proposition 2.4 in [5] page 107)

$$(\bar{S}_t^{(n)}, S_t^{(n)})_{t \geq 0} \xrightarrow[n \rightarrow +\infty]{\text{law}} (B_t, W_t)_{t \geq 0}$$

where the previous convergence holds in  $C([0, \infty[, \mathbb{R}^2)$ . In particular  $B$  and  $W$  are two standard Brownian motions. On the other hand by Theorem 2.1,  $|Y_n^+ - |S_n|| \leq 2 \quad \forall n \in \mathbb{N}$  with  $Y_n^+ := \bar{S}_n - \min_{k \leq n} \bar{S}_k$  which implies that  $|W_t| = B_t - \min_{0 \leq u \leq t} B_u$ . Itô-Tanaka formula ([10] page 222) gives

$$|W_t| = \int_0^t \text{sgn}(W_u) dW_u + L_t(W) = B_t - \min_{0 \leq u \leq t} B_u$$

where  $L_t(W)$  is the local time at 0 of  $W$  and so

$$dW_u = \text{sgn}(W_u)dB_u. \quad (2.2)$$

We deduce that for each SRW  $S$  the couple  $(-T(S), S)$ , suitably normalized and time scaled converges in law towards  $(B, W)$  satisfying (2.2). The fact that (2.2) has no strong solution for a given Brownian motion  $B$  indicates again it is impossible to construct  $T^{-1}(S)$  from  $S$  without adding information.

Finally, note that  $-T(S) = \bar{S} \Rightarrow -T(-S) = \bar{S}$  is the analogue of  $W$  solves (2.2)  $\Rightarrow -W$  solves (2.2). We have seen how to construct solutions to (2.2) by means of  $T$ . In the sequel, we will use this approach to construct a stochastic flow of mappings which solves equation (T) in general.

### 2.3 Extensions.

Let  $S = (S_n)_{n \geq 0}$  be a SRW and set  $Y_n := \max_{k \leq n} S_k - S_n$ . For  $0 \leq p < q$ , we say that  $E = [p, q]$  is an excursion for  $Y$  if the following conditions are satisfied (with the convention  $Y_{-1} = 0$ ):

- $Y_p = Y_{p-1} = Y_q = Y_{q+1} = 0$ .
- $\forall p \leq j < q, Y_j = 0 \Rightarrow Y_{j+1} = 1$ .

For example in Figure 3,  $[2, 14], [16, 18]$  are excursions for  $\bar{Y}$ . If  $E = [p, q]$  is an excursion for  $Y$ , define  $e(E) := p, f(E) := q$ .

Let  $(E_i)_{i \geq 1}$  be the random set of all excursions of  $Y$  ordered such that:  $e(E_i) < e(E_j) \forall i < j$ . From now on, we call  $E_i$  the  $i$ th excursion of  $Y$ . Then, we have

**Proposition 2.3.** *On a probability space  $(\Omega, \mathcal{A}, P)$ , consider the following jointly independent processes:*

- a sequence  $\vec{\eta} = (\vec{\eta}_i)_{i \geq 1}$  of i.i.d random variables distributed according to  $\sum_{i=1}^N \alpha_i \delta_{\vec{e}_i}$ .
- $(\bar{S}_n)_{n \in \mathbb{N}}$  a SRW.

Then, there exists, on an extension of  $(\Omega, \mathcal{A}, P)$  a Markov chain  $(M_n)_{n \in \mathbb{N}}$  started at 0 with stochastic matrix given by (1.2) such that:

$$\bar{Y}_n := \max_{k \leq n} \bar{S}_k - \bar{S}_n \Rightarrow |M_n - \vec{\eta}_i \bar{Y}_n| \leq 2$$

on the  $i$ th excursion of  $\bar{Y}$ .

*Proof.* Fix  $S \in T^{-1}\{\bar{S}\}$ . Then, by Corollary 2.2, we have  $|\bar{Y}_n - |S_n|| \leq 2 \forall n \in \mathbb{N}$ . Consider a sequence  $(\vec{\beta}_i)_{i \geq 1}$  of i.i.d random variables distributed according to  $\sum_{i=1}^N \alpha_i \delta_{\vec{e}_i}$  which is

independent of  $(\bar{S}, \vec{\eta})$ . Denote by  $(\tau_l)_{l \geq 1}$  the sequence of random times constructed in the proof of Theorem 2.1 from  $S$ . It is sufficient to look to what happens at each interval  $[\tau_l, \tau_{l+1}]$  (with the convention  $\tau_0 = 0$ ).

Using (2.1), we see that in  $[\tau_l, \tau_{l+1}]$  there are two jumps of  $\max_{k \leq n} \bar{S}_k$ ; from  $2l$  to  $2l + 1$  ( $J_1$ ) and from  $2l + 1$  to  $2l + 2$  ( $J_2$ ). The last jump ( $J_2$ ) occurs always at  $\tau_{l+1}$  by (2.1). Consequently there are only 3 possible cases:

- (i) There is no excursion of  $\bar{Y}$  ( $J_1$  and  $J_2$  occur respectively at  $\tau_l + 1$  and  $\tau_l + 2$ , see  $[0, \tau_l]$  in Figure 3).
- (ii) There is just one excursion of  $\bar{Y}$  (see  $[\tau_l, \tau_{l+1}]$  in Figure 3).

(iii) There are 2 excursions of  $\bar{Y}$  (see  $[\tau_2, \tau_3]$  in Figure 3).

Note that:  $\bar{Y}_{\tau_l} = \bar{Y}_{\tau_{l+1}} = S_{\tau_l} = S_{\tau_{l+1}} = 0$ . In the case (i), we have necessarily  $\tau_{l+1} = \tau_l + 2$ . Set  $M_n = \vec{\beta}_l \cdot |S_n| \quad \forall n \in [\tau_l, \tau_{l+1}]$ .

To treat other cases, the following remarks may be useful: from the expression of  $\bar{S}$ , we have  $\forall l \geq 0$

(a) If  $k \in [\tau_l + 2, \tau_{l+1}]$ ,  $\bar{S}_{k-1} = 2l + 1 \iff S_k = 0$ .

(b) If  $k \in [\tau_l, \tau_{l+1}]$ ,  $\bar{Y}_k = 0 \Rightarrow |S_{k+1}| \in \{0, 1\}$  and  $S_{k+1} = 0 \Rightarrow \bar{Y}_k = 0$ .

In the case (ii), let  $E_l^1$  be the unique excursion of  $\bar{Y}$  in the interval  $[\tau_l, \tau_{l+1}]$ . Then, we have two subcases:

(ii1)  $f(E_l^1) = \tau_{l+1} - 2$  ( $J_1$  occurs at  $\tau_{l+1} - 1$ ).

If  $\tau_l + 2 \leq k \leq f(E_l^1) + 1$ , then  $k - 1 \leq f(E_l^1)$ , and so  $\bar{S}_{k-1} \neq 2l + 1$ . Using (a), we get:  $S_k \neq 0$ . Thus, in this case the first zero of  $S$  after  $\tau_l$  is  $\tau_{l+1}$ . Set:  $M_n = \vec{\eta}_{N(E_l^1)} \cdot |S_n|$ , where  $N(E)$  is the number of the excursion  $E$ .

(ii2)  $f(E_l^1) = \tau_{l+1} - 1$  ( $J_1$  occurs at  $\tau_l + 1$  and so  $\bar{Y}_{\tau_{l+1}} = 0$ ). In this case, using (b) and the figure below we see that the first zero  $\tau_l^*$  of  $S$  after  $\tau_l$  is  $e(E_l^1) + 1 = \tau_l + 2$ .

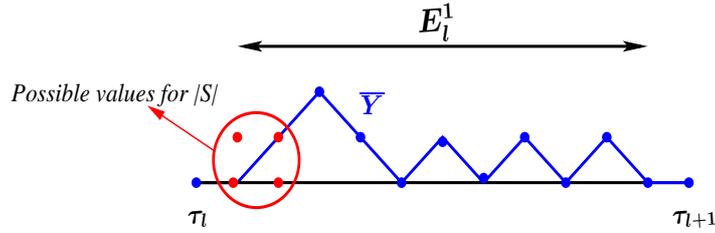


Figure 4: The case (ii2).

Set

$$M_n = \begin{cases} \vec{\beta}_l \cdot |S_n| & \text{if } n \in [\tau_l, \tau_l^* - 1] \\ \vec{\eta}_{N(E_l^1)} \cdot |S_n| & \text{if } n \in [\tau_l^*, \tau_{l+1}] \end{cases}$$

In the case (iii), let  $E_l^1$  and  $E_l^2$  denote respectively the first and the 2nd excursion of  $\bar{Y}$  in  $[\tau_l, \tau_{l+1}]$ . We have,  $\tau_l + 2 \leq k \leq e(E_l^2) \Rightarrow k - 1 \leq e(E_l^2) - 1 = f(E_l^1) \Rightarrow \bar{S}_{k-1} \neq 2l + 1 \Rightarrow S_k \neq 0$  by (a). Hence, the first zero of  $S$  after  $\tau_l$  is  $\tau_l^* := e(E_l^2) + 1$  using  $\bar{Y}_k = 0 \Rightarrow |S_{k+1}| \in \{0, 1\}$  in (b). Set:

$$M_n = \begin{cases} \vec{\eta}_{N(E_l^1)} \cdot |S_n| & \text{if } n \in [\tau_l, \tau_l^* - 1] \\ \vec{\eta}_{N(E_l^2)} \cdot |S_n| & \text{if } n \in [\tau_l^*, \tau_{l+1}] \end{cases}$$

Let  $(M_n)_{n \in \mathbb{N}}$  be the process constructed above. Then clearly  $|M_n - \vec{\eta}_i \bar{Y}_n| \leq 2$  on the  $i$ th excursion of  $\bar{Y}$ .

To complete the proof, it suffices to show that the law of  $(M_n)_{n \in \mathbb{N}}$  is given by (1.2). The only point to verify is  $\mathbb{P}(M_{n+1} = \vec{e}_i | M_n = 0) = \alpha_i$ . For this, consider on another probability space the jointly independent processes  $(S, \vec{\gamma}, \vec{\lambda})$  such that  $S$  is a SRW, and  $\vec{\gamma}, \vec{\lambda}$  have the same law as  $\vec{\eta}$ . Let  $(\tau_l)_{l \geq 1}$  be the sequence of random times defined from  $S$  as in

Theorem 2.1. For all  $l \in \mathbb{N}$ , denote by  $\tau_l^*$  the first zero of  $S$  after  $\tau_l$  and set

$$V_n = \begin{cases} \vec{\gamma}_l \cdot |S_n| & \text{if } n \in [\tau_l, \tau_l^* - 1] \\ \vec{\lambda}_l \cdot |S_n| & \text{if } n \in [\tau_l^*, \tau_{l+1}] \end{cases}$$

It is clear, by construction, that  $M \stackrel{law}{=} V$ . We can write:

$$\{\tau_0, \tau_0^*, \tau_1, \tau_1^*, \tau_2, \dots\} = \{T_0, T_1, T_2, \dots\} \text{ with } T_0 = 0 < T_1 < T_2 < \dots$$

For all  $k \geq 0$ , let  $\vec{\zeta}_k := \sum_{j=0}^N \vec{e}_j \mathbb{1}_{\{V|_{[T_k, T_{k+1}]} \in D_j\}}$ . Obviously,  $S$  and  $\vec{\zeta}_k$  are independent and

$$\vec{\zeta}_k \stackrel{law}{=} \sum_{i=1}^N \alpha_i \delta_{\vec{e}_i}. \text{ Furthermore}$$

$$\begin{aligned} \mathbb{P}(V_{n+1} = \vec{e}_i | V_n = 0) &= \frac{1}{\mathbb{P}(S_n = 0)} \sum_{k=0}^{+\infty} \mathbb{P}(V_{n+1} = \vec{e}_i, S_n = 0, n \in [T_k, T_{k+1}]) \\ &= \frac{1}{\mathbb{P}(S_n = 0)} \sum_{k=0}^{+\infty} \mathbb{P}(\vec{\zeta}_k = \vec{e}_i, S_n = 0, n \in [T_k, T_{k+1}]) \\ &= \alpha_i \end{aligned}$$

This completes the proof of the proposition.  $\square$

### 3 Proof of main results.

#### 3.1 Proof of Proposition 1.3.

Let  $(Z_t)_{t \geq 0}$  be a  $W(\alpha_1, \dots, \alpha_N)$  on  $G$  started at 0. For all  $i \in [1, N]$ , define  $Z_t^i = |Z_t| \mathbb{1}_{Z_t \in D_i} - |Z_t| \mathbb{1}_{Z_t \notin D_i}$ . Then  $Z_t^i = \Phi^i(Z_t)$  where  $\Phi^i(x) = |x| \mathbb{1}_{x \in D_i} - |x| \mathbb{1}_{x \notin D_i}$ . Let  $Q^i$  be the semigroup of the skew Brownian motion of parameter  $\alpha_i$  ( $SBM(\alpha_i)$ ) (see [10] page 87). Then the following relation is easy to check:  $P_t(f \circ \Phi^i) = Q_t^i f \circ \Phi^i$  for all bounded measurable function  $f$  defined on  $\mathbb{R}$ . This shows that  $Z^i$  is a  $SBM(\alpha_i)$  started at 0. For  $n \geq 1$ ,  $i \in [1, N]$  define

$$T_0^n = 0, \quad T_{k+1}^n = \inf\{r \geq 0 : |Z_r - Z_{T_k^n}| = \frac{1}{\sqrt{n}}\}, k \geq 0.$$

$$T_0^{n,i} = 0, \quad T_{k+1}^{n,i} = \inf\{r \geq 0 : |Z_r^i - Z_{T_k^{n,i}}^i| = \frac{1}{\sqrt{n}}\}, k \geq 0.$$

Remark that  $T_{k+1}^n = T_{k+1}^{n,i} = \inf\{r \geq 0 : ||Z_r| - |Z_{T_k^n}|| = \frac{1}{\sqrt{n}}\}$ . Furthermore if  $Z_t \in D_i$ , then obviously  $|Z_t - Z_s| = |Z_t^i - Z_s^i|$  for all  $s \geq 0$  and consequently

$$|Z_t - Z_s| \leq \sum_{i=1}^N |Z_t^i - Z_s^i|. \quad (3.1)$$

Now define  $Z_k^n = \sqrt{n} Z_{T_k^n}^n, Z_k^{n,i} = \sqrt{n} Z_{T_k^{n,i}}^i$ . Then  $(Z_k^n, k \geq 0) \stackrel{law}{=} M$  (see the proof of Proposition 2.5 in [6]). For all  $T > 0$ , we have

$$\sup_{t \in [0, T]} |Z_t - \frac{1}{\sqrt{n}} Z_{[nt]}^n| \leq \sum_{i=1}^N \sup_{t \in [0, T]} |Z_t^i - \frac{1}{\sqrt{n}} Z_{[nt]}^{n,i}| \xrightarrow{n \rightarrow +\infty} 0 \text{ in probability}$$

by Lemma 4.4 [3] which proves our result.

**Remark 3.1.** By (3.1), Proposition 2.1 [2] and Theorem 1.1.8 [10], we see that  $W(\alpha_1, \dots, \alpha_N)$  admits a modification having continuous paths. We will always choose continuous versions of Walsh Brownian motion.

### 3.2 Scaling limits of $(\Psi, K)$ .

By discussing the two cases  $S_{n+1} - S_n = 1$  or  $-1$ , we easily show (1) in Theorem 1.5. For all  $s \in \mathbb{R}$ , let  $d_s$  (resp.  $d_\infty$ ) be the distance of uniform convergence on every compact subset of  $C([s, +\infty[, G)$  (resp.  $C(\mathbb{R}, \mathbb{R})$ ). Denote by  $\mathbb{D} = \{s_n, n \in \mathbb{N}\}$  the set of all dyadic numbers of  $\mathbb{R}$  and define  $\tilde{C} = C(\mathbb{R}, \mathbb{R}) \times \prod_{n=0}^{+\infty} C([s_n, +\infty[, G)$  equipped with the metric:

$$d(x, y) = d_\infty(x', y') + \sum_{n=0}^{+\infty} \frac{1}{2^n} \inf(1, d_{s_n}(x_n, y_n)) \quad \text{where } x = (x', x_{s_0}, \dots), y = (y', y_{s_0}, \dots).$$

Let  $t \mapsto S(t)$  be the linear interpolation of  $S$  on  $\mathbb{R}$  and define  $S_t^{(n)} = \frac{1}{\sqrt{n}} S(nt)$ ,  $n \geq 1$ . If  $u \leq 0$ , we define  $\lfloor u \rfloor = -\lceil -u \rceil$ . Then, we have

$$S_t^{(n)} = S_t^n + o\left(\frac{1}{\sqrt{n}}\right), \quad \text{with } S_t^n := \frac{1}{\sqrt{n}} S_{\lfloor nt \rfloor}.$$

Let  $\Psi_{p,n} = \Psi_{p,n}(0)$  and  $\Psi_{s,t}^n = \Psi_{s,t}^n(0)$  (defined in Corollary 1.6). Then  $\Psi_{s,t}^{(n)} := \frac{1}{\sqrt{n}} \Psi_{\lfloor ns \rfloor, \lfloor nt \rfloor} + o\left(\frac{1}{\sqrt{n}}\right)$ . Remark also that  $\Psi_{p,\cdot}$  is constructed by flipping independently the excursions of  $(S_n - S_p - \inf_{k \in [p,n]}(S_k - S_p))_{n \geq p}$  which entails the following

**Lemma 3.1.** Let  $\mathbb{P}_n$  be the law of  $Z^n = (S_{\cdot}^{(n)}, (\Psi_{s_i, \cdot}^{(n)})_{i \in \mathbb{N}})$  in  $\tilde{C}$ . Then  $(\mathbb{P}_n, n \geq 1)$  is tight.

*Proof.* By Donsker theorem  $\mathbb{P}_{S^{(n)}} \rightarrow \mathbb{P}_W$  in  $C(\mathbb{R}, \mathbb{R})$  as  $n \rightarrow \infty$  where  $\mathbb{P}_W$  is the law of any Brownian motion on  $\mathbb{R}$ . Let  $\mathbb{P}_{Z_{s_i}}$  be the law of any  $W(\alpha_1, \dots, \alpha_N)$  started at 0 at time  $s_i$ . By Propositions 1.3 and 2.3, for all  $i \in \mathbb{N}$ ,  $\mathbb{P}_{\Psi_{s_i, \cdot}^{(n)}} \rightarrow \mathbb{P}_{Z_{s_i}}$  in  $C([s_i, +\infty[, G)$  as  $n \rightarrow \infty$ . Since the law of each coordinate of  $Z^n$  is tight, the lemma holds using Proposition 2.4 [5] (page 107).  $\square$

Now we fix a sequence  $(n_k, k \in \mathbb{N})$  such that  $Z^{n_k} \xrightarrow[k \rightarrow +\infty]{\text{law}} Z$  in  $\tilde{C}$ . In the next paragraph, we will describe the law of  $Z$ . Notice that  $(\Psi_{p,n})_{p \leq n}$  and  $S$  can be recovered from  $(Z^{n_k})_{k \in \mathbb{N}}$ . As  $\vec{\eta}_{p,n}$  will not play any particular role when  $S_n - \inf_{k \in [p,n]} S_k = 0$ , by defining  $\vec{\eta}_{p,n} = \vec{e}(\Psi_{p,n})$  and using Skorokhod representation theorem, we may assume that  $Z$  is defined on the original probability space and the preceding convergence holds almost surely. Write  $Z = (W, \psi_{s_1, \cdot}, \psi_{s_2, \cdot}, \dots)$ . Then,  $(W_t)_{t \in \mathbb{R}}$  is a Brownian motion on  $\mathbb{R}$  and

$$\forall s \in \mathbb{D}, T > 0 : \lim_{k \rightarrow +\infty} \sup_{s \leq t \leq s+T} \left| \frac{1}{\sqrt{n_k}} \Psi_{\lfloor n_k s \rfloor, \lfloor n_k t \rfloor} - \psi_{s,t} \right| = 0 \quad a.s. \quad (3.2)$$

From Proposition 2.3,  $(\psi_{s,t})_{t \geq s}$  is an  $W(\alpha_1, \dots, \alpha_N)$  for all  $s \in \mathbb{D}$  and moreover:  $|\psi_{s,t}| = W_t - \inf_{u \in [s,t]} W_u$ .

#### 3.2.1 Description of the limit process.

Set  $\vec{\gamma}_{s,t} = \vec{e}(\psi_{s,t})$ ,  $s \in \mathbb{D}$ ,  $s < t$  and define  $\min_{u,v} = \min_{r \in [u,v]} W_r$ ,  $u \leq v \in \mathbb{R}$ . Then, we have

**Proposition 3.2.**  $\forall s < t, u < v, s, u \in \mathbb{D}$ ,

$$\mathbb{P}(\vec{\gamma}_{s,t} = \vec{\gamma}_{u,v} \mid \min_{s,t} = \min_{u,v}) = 1 \text{ if } \mathbb{P}(\min_{s,t} = \min_{u,v}) > 0.$$

*Proof.* We first show that a.s

$$\{\min_{s,t} = \min_{u,v}\} \subset \{\exists k_0, \vec{\eta}_{[n_k s], [n_k t]} = \vec{\eta}_{[n_k u], [n_k v]} \text{ for all } k \geq k_0\}. \quad (3.3)$$

There are four cases to discuss, (i)  $s \leq u \leq v \leq t$ , (ii)  $s \leq u \leq t \leq v$ , (iii)  $u \leq s \leq v \leq t$ , (iv)  $u \leq s \leq t \leq v$  (in any other case  $\mathbb{P}(\min_{s,t} = \min_{u,v}) = 0$ ). As all cases are similar, we just treat (i). We have

$$\{\min_{s,t} = \min_{u,v}\} = \{\min_{u,v} < \min_{s,u} \wedge \min_{v,t}\} \text{ a.s.}$$

By uniform convergence the last set is contained in

$$\{\exists k_0, \min_{[n_k u] \leq j \leq [n_k v]} S_j < \min_{[n_k s] \leq j \leq [n_k u]} S_j \wedge \min_{[n_j v] \leq j \leq [n_k t]} S_j \text{ for all } k \geq k_0\}$$

which is a subset of

$$\{\exists k_0, \min_{[n_k s] \leq j \leq [n_k t]} S_j = \min_{[n_k u] \leq j \leq [n_k v]} S_j \text{ for all } k \geq k_0\}.$$

This gives (3.3). Since  $x \rightarrow \vec{e}(x)$  is continuous on  $G^*$ , we arrive at

$$\vec{\gamma}_{s,t} = \lim_{k \rightarrow \infty} \vec{e}\left(\frac{1}{\sqrt{n_k}} \Psi_{[n_k s], [n_k t]}\right) = \lim_{k \rightarrow \infty} \vec{e}\left(\frac{1}{\sqrt{n_k}} \Psi_{[n_k u], [n_k v]}\right) = \vec{\gamma}_{u,v} \text{ a.s.}$$

which completes the proof.  $\square$

**Proposition 3.3.** For all  $s < t, s \in \mathbb{D}, n \geq 1$  and  $\{(s_i, t_i); 1 \leq i \leq n\}$  with  $s_i < t_i, s_i \in \mathbb{D}$ ,  $\vec{\gamma}_{s,t}$  is independent of  $\sigma(W)$  and the law of  $\vec{\gamma}_{s,t}$  knowing  $(\vec{\gamma}_{s_i, t_i})_{1 \leq i \leq n}$  and  $W$  is given by  $\sum_{i=1}^n \alpha_i \delta_{\vec{e}_i}$  when  $\min_{s,t} \notin \{\min_{s_i, t_i}; 1 \leq i \leq n\}$  and is given by

$$\sum_{i=1}^n \delta_{\vec{\gamma}_{s_i, t_i}} \frac{\mathbb{1}_{\{\min_{s,t} = \min_{s_i, t_i}\}}}{\text{Card}\{i; \min_{s,t} = \min_{s_i, t_i}\}} \text{ otherwise.}$$

This entirely describes the law of  $(W, \psi_{s, \cdot}, s \in \mathbb{D})$  in  $\tilde{C}$  independently of  $(n_k, k \in \mathbb{N})$  and consequently  $Z^n \xrightarrow[n \rightarrow +\infty]{\text{law}} Z$  in  $\tilde{C}$ .

*Proof.* Clearly  $\vec{\gamma}_{s,t}$  is independent of  $\sigma(W)$ . We will establish the second affirmation by induction on  $n$ . For  $n = 1$ , this follows immediately from Proposition 3.2. Let  $A_k = \{\min_{v \in [s,t]} S_{[n_k v]} \neq \min_{v \in [s_1, t_1]} S_{[n_k v]}\}$  and  $A = \{\min_{s,t} \neq \min_{s_1, t_1}\}$ . For all continuous bounded functions  $f, g : G \rightarrow \mathbb{R}, h : C(\mathbb{R}, \mathbb{R}) \rightarrow \mathbb{R}$ , we have

$$E[f(\vec{\gamma}_{s,t})g(\vec{\gamma}_{s_1, t_1})h(W)\mathbb{1}_{\{\min_{s,t} \neq \min_{s_1, t_1}\}}] = \lim_{k \rightarrow +\infty} E[f(\vec{\eta}_{[n_k s], [n_k t]})g(\vec{\eta}_{[n_k s_1], [n_k t_1]})h(W)\mathbb{1}_{A_k \cap A}].$$

Since  $\vec{\eta}_{[n_k s], [n_k t]}$  and  $\vec{\eta}_{[n_k s_1], [n_k t_1]}$  are conditionally independent with respect to  $S$  on  $A_k$ , we get

$$E[f(\vec{\gamma}_{s,t})g(\vec{\gamma}_{s_1, t_1})h(W)\mathbb{1}_{\{\min_{s,t} \neq \min_{s_1, t_1}\}}] = E[f(\vec{\gamma}_{s,t})]E[g(\vec{\gamma}_{s_1, t_1})]E[h(W)\mathbb{1}_{\{\min_{s,t} \neq \min_{s_1, t_1}\}}].$$

This argument easily extends for all  $n \geq 1$ .  $\square$

In the sequel, we still assume that all processes are defined on the same probability space and that (3.2) holds with  $n_k = k; \forall s \in \mathbb{D}, T > 0$ ,

$$\lim_{k \rightarrow +\infty} \sup_{s \leq t \leq s+T} \left| \frac{1}{\sqrt{k}} \Psi_{[ks], [kt]} - \psi_{s,t} \right| = 0 \text{ a.s.} \quad (3.4)$$

### 3.2.2 Extension of the limit process.

For a fixed  $s < t$ ,  $\min_{s,t}$  is attained in  $]s, t[$  a.s. By Proposition 3.2, on a measurable set  $\Omega_{s,t}$  with probability 1,  $\lim_{s' \rightarrow s+, s' \in \mathbb{D}} \vec{\gamma}_{s',t}$  exists. Define  $\vec{\varepsilon}_{s,t} = \lim_{s' \rightarrow s+, s' \in \mathbb{D}} \vec{\gamma}_{s',t}$  on  $\Omega_{s,t}$  and give an arbitrary value to  $\vec{\varepsilon}_{s,t}$  on  $\Omega_{s,t}^c$ . Now, let  $\varphi_{s,t} = \vec{\varepsilon}_{s,t}(W_t - \min_{s,t})$ . Then for all  $s \in \mathbb{D}, t > s$ ,  $\vec{\varepsilon}_{s,t}$  is a modification of  $\vec{\gamma}_{s,t}$  and  $\varphi_{s,t}$  is a modification of  $\psi_{s,t}$ . For all  $s \in \mathbb{R}, t > s$ ,  $\varphi_{s,t} = \lim_{n \rightarrow \infty} \varphi_{s_n,t}$  a.s where  $s_n = \frac{\lfloor 2^n s \rfloor + 1}{2^n}$  and therefore  $(\varphi_{s,t})_{t \geq s}$  is an  $W(\alpha_1, \dots, \alpha_N)$  started at 0. Using Proposition 3.2, it is easy to see that:

$$\forall s < t, u < v, \mathbb{P}(\vec{\varepsilon}_{s,t} = \vec{\varepsilon}_{u,v} | \min_{s,t} = \min_{u,v}) = 1 \text{ if } \mathbb{P}(\min_{s,t} = \min_{u,v}) > 0. \quad (3.5)$$

Define:  $\varphi_{s,t}(x) = (x + \vec{e}(x)W_{s,t})\mathbb{1}_{\{t \leq \tau_{s,x}\}} + \varphi_{s,t}\mathbb{1}_{\{t > \tau_{s,x}\}}$  where  $W_{s,u} = W_u - W_s$ ,  $\tau_{s,x} = \inf\{r \geq s : W_{s,r} = -|x|\}$ .

**Proposition 3.4.** *Let  $x \in G, x_n \in \frac{1}{\sqrt{n}}G_{\mathbb{N}}, \lim_{n \rightarrow \infty} x_n = x, s \in \mathbb{R}, T > 0$ . Then, we have*

$$\lim_{n \rightarrow +\infty} \sup_{s \leq t \leq s+T} \left| \frac{1}{\sqrt{n}} \Psi_{[ns],[nt]}(\sqrt{n}x_n) - \varphi_{s,t}(x) \right| = 0 \text{ a.s.}$$

*Proof.* Let  $s'$  be a dyadic number such that  $s < s' < s + T$ . Then, by (3.5) for  $t > s'$ :

$$\{\min_{s,t} = \min_{s',t}\} \subset \{\varphi_{s,t} = \varphi_{s',t}\} \text{ a.s.}$$

and so, a.s.

$$\forall t > s', t \in \mathbb{D}; \{\min_{s,t} = \min_{s',t}\} \subset \{\varphi_{s,t} = \varphi_{s',t}\}.$$

By continuity, a.s.

$$\forall t > s'; \{\min_{s,t} = \min_{s',t}\} \subset \{\varphi_{s,t} = \varphi_{s',t}\}$$

and consequently

$$\forall s' \in \mathbb{D} \cap ]s, s+T[, \forall t > s'; \{\min_{s,t} = \min_{s',t}\} \subset \{\varphi_{s,t} = \varphi_{s',t}\}.$$

By standard properties of Brownian paths, a.s  $\min_{s,s+T} \notin \{W_s, W_{s+T}\}$  and

$$\forall p \in \mathbb{N}^*; \min_{s, s+\frac{1}{p}} < W_s, \min_{s, s+\frac{1}{p}} \neq W_{s+\frac{1}{p}}, \exists! u_p \in ]s, s+\frac{1}{p}[ : \min_{s, s+\frac{1}{p}} = W_{u_p}.$$

From (3.4), a.s.  $\forall u \in \mathbb{D}, \lim_{n \rightarrow +\infty} \sup_{u \leq t \leq u+T} \left| \frac{1}{\sqrt{n}} \Psi_{[nu],[nt]} - \varphi_{u,t} \right| = 0$ . The reasoning below holds almost surely: Take  $p \geq 1, \min_{s, s+\frac{1}{p}} > \min_{s, s+T}$ . Let  $\mathcal{S}_p \in ]s, s+\frac{1}{p}[ : \min_{s, s+\frac{1}{p}} = W_{\mathcal{S}_p}$  and  $s'$  be a dyadic number in  $]s, \mathcal{S}_p[$ . Then  $\forall \mathcal{S}_p \leq t \leq s+T, \min_{s, s'} > \min_{s', t}$ . By uniform convergence:

$$\exists n_0 \in \mathbb{N} : \forall n \geq n_0, \forall \mathcal{S}_p \leq t \leq s+T, \min_{u \in [s, s']} S_{[nu]} > \min_{u \in [s', t]} S_{[nu]} \text{ and so } \Psi_{[ns'], [nt]} = \Psi_{[ns], [nt]}.$$

For  $n \geq n_0$ , we have

$$\begin{aligned} & \sup_{s \leq t \leq s+T} \left| \frac{1}{\sqrt{n}} \Psi_{[ns], [nt]} - \varphi_{s,t} \right| \leq \sup_{s \leq t \leq \mathcal{S}_p} \left| \frac{1}{\sqrt{n}} \Psi_{[ns], [nt]} - \varphi_{s,t} \right| + \sup_{\mathcal{S}_p \leq t \leq s+T} \left| \frac{1}{\sqrt{n}} \Psi_{[ns], [nt]} - \varphi_{s,t} \right| \\ & \leq \sup_{s \leq t \leq s+\frac{1}{p}} \left| \frac{1}{\sqrt{n}} (S_{[nt]} - \min_{v \in [s, t]} S_{[nv]}) \right| + \sup_{s \leq t \leq s+\frac{1}{p}} |W_t - \min_{s,t}| + \sup_{\mathcal{S}_p \leq t \leq s+T} \left| \frac{1}{\sqrt{n}} \Psi_{[ns'], [nt]} - \varphi_{s',t} \right| \end{aligned}$$

$$\leq \sup_{s \leq t \leq s + \frac{1}{p}} \left| \frac{1}{\sqrt{n}} (S_{[nt]} - \min_{v \in [s,t]} S_{[nv]}) \right| + \sup_{s \leq t \leq s + \frac{1}{p}} |W_t - \min_{s,t}| + \sup_{s' \leq t \leq s' + T} \left| \frac{1}{\sqrt{n}} \Psi_{[ns'], [nt]} - \varphi_{s',t} \right|.$$

By letting  $n$  go to  $+\infty$  and then  $p$  go to  $+\infty$ , we obtain

$\lim_{n \rightarrow \infty} \sup_{s \leq t \leq s+T} \left| \frac{1}{\sqrt{n}} \Psi_{[ns], [nt]} - \varphi_{s,t} \right| = 0$  (i). We now show that  $\lim_{n \rightarrow +\infty} \frac{1}{n} T_{[ns], \sqrt{nx_n}} = \tau_{s,x}$  a.s (ii). We have

$$\frac{1}{n} T_{[ns], \sqrt{nx_n}} = \inf \left\{ r \geq \frac{[ns]}{n} : S_r^n - S_s^n = -|x_n| \right\}.$$

For  $\varepsilon > 0$ , from

$$\lim_{n \rightarrow \infty} \sup_{u \in [\tau_{s,x}, \tau_{s,x} + \varepsilon]} |(S_u^n - S_s^n + |x_n|) - (W_u - W_s + |x|)| = 0,$$

we get

$$\lim_{n \rightarrow \infty} \inf_{u \in [\tau_{s,x}, \tau_{s,x} + \varepsilon]} (S_u^n - S_s^n + |x_n|) = \inf_{u \in [\tau_{s,x}, \tau_{s,x} + \varepsilon]} (W_u - W_s + |x|) < 0$$

which implies that  $\frac{1}{n} T_{[ns], \sqrt{nx_n}} < \tau_{s,x} + \varepsilon$  for  $n$  large. If  $x = 0$ ,  $\frac{1}{n} T_{[ns], \sqrt{nx_n}} \geq \frac{[ns]}{n}$  entails obviously (ii). If  $x \neq 0$ , then working in  $[s, \tau_{s,x} - \varepsilon]$  as before and using  $\inf_{u \in [s, \tau_{s,x} - \varepsilon]} (W_u - W_s + |x|) > 0$ , we prove that  $\frac{1}{n} T_{[ns], \sqrt{nx_n}} \leq \tau_{s,x} - \varepsilon$  for  $n$  large which establishes (ii).

Now

$$\sup_{s \leq t \leq s+T} \left| \frac{1}{\sqrt{n}} \Psi_{[ns], [nt]}(\sqrt{nx_n}) - \varphi_{s,t}(x) \right| \leq \sup_{s \leq t \leq s+T} Q_{s,t}^{1,n} + \sup_{s \leq t \leq s+T} Q_{s,t}^{2,n} \quad (3.6)$$

where

$$Q_{s,t}^{1,n} = |(x_n + \tilde{e}(x_n)(S_t^n - S_s^n)) \mathbb{1}_{\{[nt] \leq T_{[ns], \sqrt{nx_n}}\}} - (x + \tilde{e}(x)W_{s,t}) \mathbb{1}_{\{t \leq \tau_{s,x}\}}|,$$

$$Q_{s,t}^{2,n} = \left| \frac{1}{\sqrt{n}} \Psi_{[ns], [nt]} \mathbb{1}_{\{[nt] > T_{[ns], \sqrt{nx_n}}\}} - \varphi_{s,t} \mathbb{1}_{\{t > \tau_{s,x}\}} \right|.$$

By (i), (ii) and the convergence of  $\frac{1}{\sqrt{n}} S_{[n \cdot]}$  towards  $W$  on compact sets, the right-hand side of (3.6) converges to 0 when  $n \rightarrow +\infty$ .  $\square$

**Remark 3.5.** *It follows from Proposition 3.4 that for all  $s_i \leq t_i, 1 \leq i \leq n$ , the law of  $(\tilde{e}_{s_1, t_1}, \dots, \tilde{e}_{s_n, t_n}, W)$  is the same given in Proposition 3.3.*

**Proposition 3.6.**  *$\varphi$  is the unique stochastic flow of mappings which solves (T).*

*Proof.* The cocycle property is a consequence of (3.5) (the proof is similar to Lemma 4.3 [8]). If  $p \geq 2$ ,  $t_1 < \dots < t_p$ , then  $\min_{t_i, t_{i+1}}, 1 \leq i \leq p-1$  are distinct a.s. Using the fact that  $\tilde{\eta}_{[nt_i], [nt_{i+1}]}, 1 \leq i \leq p-1$  are conditionally independent with respect to  $S$  when  $\min_{v \in [t_i, t_{i+1}]} S_{[nv]}, 1 \leq i \leq p-1$  are distinct and taking the limit as  $n \rightarrow \infty$ , this justifies the independence of increments of the flow. Now,  $(\varphi_{0,t}, t \geq 0)$  is an  $W(\alpha_1, \dots, \alpha_N)$  started at 0 and therefore satisfies Freidlin-Sheu formula (Theorem 2.6 [6]); for all  $t \geq 0, f \in D(\alpha_1, \dots, \alpha_N)$ ,

$$f(\varphi_{0,t}) = f(0) + \int_0^t f'(\varphi_{0,u}) dB_u + \frac{1}{2} \int_0^t f''(\varphi_{0,u}) du \quad a.s.$$

where  $B_t = |\varphi_{0,t}| - \tilde{L}_t(|\varphi_{0,\cdot}|)$  and  $\tilde{L}_t(|\varphi_{0,\cdot}|)$  is the symmetric local time at 0 of  $|\varphi_{0,\cdot}|$ . Since  $|\varphi_{0,t}| = W_t - \min_{0,t}$ , we get  $B_t = W_t$ . For all  $x \in G$ ,  $(\varphi_{0,t+\tau_{0,x}})_{t \geq 0}$  is an  $W(\alpha_1, \dots, \alpha_N)$

started at 0. By distinguishing the cases  $t \leq \tau_{0,x}$  and  $t > \tau_{0,x}$ , we easily obtain that for all  $t \geq 0$ ,  $f \in D(\alpha_1, \dots, \alpha_N)$ ,

$$f(\varphi_{0,t}(x)) = f(x) + \int_0^t f'(\varphi_{0,u}(x)) dW_u + \frac{1}{2} \int_0^t f''(\varphi_{0,u}(x)) du \quad a.s. \quad (3.7)$$

This proves the proposition since unicity has been discussed in Theorem 1.2.  $\square$

### 3.2.3 The Wiener flow.

Remark that  $K_{s,t}^W(x) = E[\delta_{\varphi_{s,t}(x)} | \sigma(W)]$  and therefore by conditioning with respect to  $\sigma(W)$  in (3.7), we easily see that  $(K^W, W)$  solves (T). In order to finish the proof of Theorem 1.5 and Corollary 1.6, we need only check the following lemma (the proof of (1.3) is similar)

**Lemma 3.2.** *Under the hypothesis of Proposition 3.4, we have*

$$\sup_{t \in [s, s+T]} \beta(K_{s,t}^W(x), K_{s,t}^n(\sqrt{n}x_n)) \xrightarrow{n \rightarrow +\infty} 0 \quad a.s.$$

*Proof.* Let  $g : G \rightarrow \mathbb{R}$  such that  $\|g\|_\infty + \sup_{x \neq y} \frac{|g(x) - g(y)|}{|x - y|} \leq 1$ ,  $g(0) = 0$ . Then,

$$\left| \int_G g(y) K_{s,t}^W(x)(dy) - \int_G g(y) K_{s,t}^n(\sqrt{n}x_n)(dy) \right| \leq V_{s,t}^{1,n} + V_{s,t}^{2,n}$$

where

$$V_{s,t}^{1,n} = \left| g(x_n + \vec{e}(x_n) S_{s,t}^n) \mathbf{1}_{\{\lfloor nt \rfloor \leq T_{\lfloor ns \rfloor, \sqrt{n}x_n}\}} - g(x + \vec{e}(x) W_{s,t}) \mathbf{1}_{\{t \leq \tau_{s,x}\}} \right|,$$

$$V_{s,t}^{2,n} = \sum_{j=1}^N \alpha_j \left| g(\vec{e}_j W_{s,t}^+) \mathbf{1}_{\{t > \tau_{s,x}\}} - g(\vec{e}_j S_{n,s,t}^+ + o_n) \mathbf{1}_{\{\lfloor nt \rfloor > T_{\lfloor ns \rfloor, \sqrt{n}x_n}\}} \right|$$

and  $o_n \in G$  is a  $\sigma(S)$  measurable random variable such that  $|o_n| \leq \frac{1}{\sqrt{n}}$ ,  $S_{s,t}^n = S_t^n - S_s^n$ ,  $S_{n,s,t}^+ = \frac{1}{\sqrt{n}}(S_{\lfloor nt \rfloor} - \inf_{s \leq u \leq t} S_{\lfloor nu \rfloor})$ . As  $\lfloor x \rfloor - 1 \leq x \leq \lfloor x \rfloor + 1$  for all  $x \in \mathbb{R}$ , we get

$$V_{s,t}^{1,n} \leq \sup_{t \in I_{n,s,x}} |x_n + \vec{e}(x_n) S_{s,t}^{(n)} - x - \vec{e}(x) W_{s,t}| + \sup_{t \in J_{n,s,x}} |g(x_n + \vec{e}(x_n) S_{s,t}^{(n)})| + \sup_{t \in K_{n,s,x}} |g(x + \vec{e}(x) W_{s,t})|$$

with

$$I_{n,s,x} = [s, \tau_{s,x} \vee (\frac{1}{n} + \frac{1}{n} T_{\lfloor ns \rfloor, \sqrt{n}x_n})],$$

$$J_{n,s,x} = [\tau_{s,x}, (\frac{1}{n} T_{\lfloor ns \rfloor, \sqrt{n}x_n} + \frac{1}{n}) \vee \tau_{s,x}], \quad K_{n,s,x} = [\tau_{s,x} \wedge (\frac{1}{n} T_{\lfloor ns \rfloor, \sqrt{n}x_n} - \frac{1}{n}), \tau_{s,x}].$$

Using  $|g(y)| \leq |y|$ , we obtain

$$\sup_{t \in J_{n,s,x}} |g(x_n + \vec{e}(x_n) S_{s,t}^{(n)})| + \sup_{t \in K_{n,s,x}} |g(x + \vec{e}(x) W_{s,t})| \leq \sup_{t \in J_{n,s,x}} |x_n + S_{s,t}^{(n)}| + \sup_{t \in K_{n,s,x}} |x + W_{s,t}|.$$

Since  $\lim_{n \rightarrow +\infty} \frac{1}{n} T_{\lfloor ns \rfloor, \sqrt{n}x_n} = \tau_{s,x}$  a.s., the right-hand side converges to 0. By discussing the cases  $x = 0$ ,  $x \neq 0$ , we easily see that  $\lim_{n \rightarrow \infty} \sup_{t \in I_{n,s,x}} |x_n + \vec{e}(x_n) S_{s,t}^{(n)} - x - \vec{e}(x) W_{s,t}| = 0$  and

therefore  $\lim_{n \rightarrow \infty} \sup_{t \in [s, s+T]} V_{s,t}^{1,n} = 0$ . By the same manner, we arrive at  $\lim_{n \rightarrow \infty} \sup_{t \in [s, s+T]} V_{s,t}^{2,n} =$

0 which proves the lemma.  $\square$

**Acknowledgements.** I sincerely thank Yves Le Jan, Olivier Raimond and Sophie Lemaire for very useful discussions and comments on this paper.

## References

- [1] Patrick Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1999. A Wiley-Interscience Publication.
- [2] Krzysztof Burdzy and Haya Kaspı. Lenses in skew Brownian flow. *Ann. Probab.*, 32(4):3085–3115, 2004.
- [3] A. S. Cherny, A. N. Shiryaev, and M. Yor. Limit behaviour of the “horizontal-vertical” random walk and some extensions of the Donsker-Prokhorov invariance principle. *Teor. Veroyatnost. i Primenen.*, 47(3):498–517, 2002.
- [4] Nathanaël Enriquez and Yuri Kifer. Markov chains on graphs and Brownian motion. *J. Theoret. Probab.*, 14(2):495–510, 2001.
- [5] Stewart N. Ethier and Thomas G. Kurtz. *Markov processes*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, 1986. Characterization and convergence.
- [6] Hatem Hajri. Stochastic flows related to Walsh Brownian motion. Available at <http://hal.archives-ouvertes.fr/hal-00553630/fr/>, 2011.
- [7] J. M. Harrison and L. A. Shepp. On skew Brownian motion. *Ann. Probab.*, 9(2):309–313, 1981.
- [8] Yves Le Jan and Olivier Raimond. Flows associated to Tanaka’s SDE. *ALEA Lat. Am. J. Probab. Math. Stat.*, 1:21–34, 2006.
- [9] Pál Révész. *Random walk in random and non-random environments*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, second edition, 2005.
- [10] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.
- [11] S. Watanabe. The stochastic flow and the noise associated to Tanaka’s stochastic differential equation. *Ukrain. Mat. Zh.*, 52(9):1176–1193, 2000.