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# Initial trace of positive solutions of a class of degenerate heat equation with absorption

Tai Nguyen Phuoc      Laurent Véron

Laboratoire de Mathématiques et Physique Théorique,  
Université François Rabelais, Tours, FRANCE

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## Abstract

We study the initial value problem with unbounded nonnegative functions or measures for the equation  $\partial_t u - \Delta_p u + f(u) = 0$  in  $\mathbb{R}^N \times (0, \infty)$  where  $p > 1$ ,  $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$  and  $f$  is a continuous, nondecreasing nonnegative function such that  $f(0) = 0$ . In the case  $p > \frac{2N}{N+1}$ , we provide a sufficient condition on  $f$  for existence and uniqueness of the solutions satisfying the initial data  $k\delta_0$  and we study their limit when  $k \rightarrow \infty$ , according  $f^{-1}$  and  $F^{-1/p}$  are integrable or not at infinity, where  $F(s) = \int_0^s f(\sigma) d\sigma$ . We also give new results dealing with non uniqueness for the initial value problem with unbounded initial data. If  $p > 2$ , we prove that, for a

large class of nonlinearities  $f$ , any positive solution admits an initial trace in the class of positive Borel measures. As a model case we consider the case  $f(u) = u^\alpha \ln^\beta(u+1)$ , where  $\alpha > 0$  and  $\beta \geq 0$ .

## 1 Introduction

The aim of this article is to study some qualitative properties of the positive solutions of

$$\partial_t u - \Delta_p u + f(u) = 0 \tag{1.1}$$

in  $Q_\infty := \mathbb{R}^N \times (0, \infty)$  where  $p > 1$ ,  $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$  and  $f$  is a continuous, nondecreasing function such that  $f(0) = 0 = f^{-1}(0)$ . The properties we are interested in are mainly: (a) the existence of fundamental solutions i.e. solutions with  $k\delta_0$  as initial data and the behaviour of these solutions when  $k \rightarrow \infty$ ; (b) the existence of an initial trace and its properties; (c) uniqueness and non-uniqueness results for the Cauchy problem. This type of questions have been considered in a previous paper of the authors [15] in the semilinear case  $p = 2$ . The breadcrumbs of this study lies in the existence of two types of specific solutions of (1.1). The first ones are the solutions  $\phi := \phi_a$  of the ODE

$$\phi' + f(\phi) = 0 \tag{1.2}$$

defined on  $[0, \infty)$  and subject to  $\phi(0) = a \geq 0$ ; it is given by

$$\int_{\phi(t)}^a \frac{ds}{f(s)}. \tag{1.3}$$

The second ones are the solutions of the elliptic equation

$$-\Delta_p w + f(w) = 0, \tag{1.4}$$

defined in  $\mathbb{R}^N$  or in  $\mathbb{R}^N \setminus \{0\}$ . It is well-known that the structure of the set of solutions of (1.2) depends whether the following quantity

$$J := \int_1^\infty \frac{ds}{f(s)} \tag{1.5}$$

is finite or infinite. If  $J < \infty$  there exists a maximal solution  $\phi_\infty$  to (1.2) defined on  $(0, \infty)$  while no such solution exists if  $J = \infty$  since  $\lim_{a \rightarrow \infty} \phi_a(t) = \infty$ . This maximal solution plays an important role since, by the maximum principle, it dominates any solution  $u$  of (1.1) which satisfies

$$\lim_{|x| \rightarrow \infty} u(x, t) = 0 \tag{1.6}$$

for all  $t > 0$ , locally uniformly on  $(0, \infty)$ . Concerning (1.4) we associate the quantity

$$K := \int_1^\infty \frac{ds}{F(s)^{1/p}}. \quad (1.7)$$

It is a consequence of the Vázquez's extension of the Keller-Osserman condition (see [17], [12]) that if  $K < \infty$ , equation (1.4) admits a maximal solution  $W_{\mathbb{R}_*^N}$  in  $\mathbb{R}^N \setminus \{0\}$ . This solution is constructed as the limit, when  $R \rightarrow \infty$  and  $\epsilon \rightarrow 0$  of the solution  $W := W_{\epsilon, R}$  of (1.4) in  $\Gamma_{\epsilon, R} := B_R \setminus \overline{B_\epsilon}$ , subject to the conditions  $\lim_{|x| \downarrow \epsilon} W_{\epsilon, R}(x) = \infty$  and  $\lim_{|x| \uparrow R} W_{\epsilon, R}(x) = \infty$ . On the contrary, if  $K = \infty$ , such functions  $W_{\epsilon, R}$  and  $W_{\mathbb{R}_*^N}$  do not exist, a situation which will be exploited in Section 3 for proving existence of global solutions of (1.4) in  $\mathbb{R}^N$ . An additional natural growth assumption of  $f$  that will be often made is the *super-additivity*

$$f(s + s') \geq f(s) + f(s') \quad \forall s, s' \geq 0, \quad (1.8)$$

which, combined with the monotonicity of  $f$ , implies a minimal linear growth at infinity

$$\liminf_{s \rightarrow \infty} \frac{f(s)}{s} > 0. \quad (1.9)$$

If  $p \geq 2$ ,  $K < \infty$  jointly with (1.8) implies  $J < \infty$ , but this does not hold when  $1 < p < 2$ . When  $p > 2$  and  $f$  satisfies  $J < \infty$  and  $K < \infty$ , Kamin and Vázquez proved universal estimates for solutions which vanish on  $\mathbb{R}^N \times \{0\} \setminus \{(0, 0)\}$  (see [11]). By a slight modification of the proof in [15, Proposition 2.3 and Proposition 2.6], it is possible to extend their result to the case  $p > 1$ .  $\square$

**Proposition** (Universal estimates) *Assume  $p > 1$  and  $f$  satisfies  $K < \infty$ . Let  $u \in C(\overline{Q_\infty} \setminus \{(0, 0)\})$  be a solution of (1.1) in  $Q_\infty$ , which vanishes on  $\mathbb{R}^N \times \{0\} \setminus \{(0, 0)\}$ . Then*

$$u(x, t) \leq W_{\mathbb{R}_*^N}(x) \quad \forall (x, t) \in Q_\infty. \quad (1.10)$$

*If we suppose moreover  $J < \infty$  and that (1.8) holds, then*

$$u(x, t) \leq \min \left\{ \phi_\infty(t), W_{\mathbb{R}_*^N}(x) \right\} \quad \forall (x, t) \in Q_\infty. \quad (1.11)$$

When  $K = \infty$ , no such estimate exists since the function  $w_a$  solution of (1.16) is a stationary solution of (1.1) with unbounded initial data.

In Section 2 we study the existence of the fundamental solutions  $u_k$  and their behaviour when  $k \rightarrow \infty$ . Kamin and Vázquez proved in [11, Lemma 2.3 and Lemma 2.4], that if  $p > 2$  and

$$\int_1^\infty s^{-p-\frac{p}{N}} f(s) ds < \infty, \quad (1.12)$$

then for any  $k > 0$ , there exists a unique positive solution  $u := u_k$  to problem

$$\begin{cases} \partial_t u - \Delta_p u + f(u) = 0 & \text{in } Q_\infty \\ u(\cdot, 0) = k\delta_0 & \text{in } \mathbb{R}^N. \end{cases} \quad (1.13)$$

Furthermore the mapping  $k \mapsto u_k$  is increasing. Their existence proof heavily relies on the fact that, if we denote by  $v := v_k$  the fundamental (or Barenblatt-Prattle) solution of

$$\begin{cases} \partial_t v - \Delta_p v = 0 & \text{in } Q_\infty \\ v(\cdot, 0) = k\delta_0 & \text{in } \mathbb{R}^N, \end{cases} \quad (1.14)$$

then  $v_k(\cdot, t)$  is compactly supported in some ball  $B_{\delta_k(t)}$ , where  $\delta_k(t)$  is explicit. Since  $v_k$  is a natural supersolution for (1.13), condition (1.12) states that  $f(v_k) \in L^1_{loc}(\overline{Q_\infty})$ . When  $2N/(N+1) < p \leq 2$ ,  $v_k(x, t) > 0$  for all  $(x, t) \in Q_\infty$ . It is already proved in [14] that, when  $p = 2$ , condition (1.12) yields to  $f(v_k) \in L^1(Q_T)$ . We prove here that this result also holds when  $2N/(N+1) < p \leq 2$  and more precisely,

**Theorem 1.1** *Assume  $p > \frac{2N}{N+1}$  and  $f$  satisfies (1.12). Then there exists a unique positive solution  $u := u_k$  to problem (1.13).*

In view of this result and the *a priori* estimates (1.10) and (1.11), it is natural to study the limit of  $u_k$  when  $k \rightarrow \infty$ . We denote by  $\mathcal{U}_0$  the set of positive  $u \in C(\overline{Q_\infty} \setminus \{(0, 0)\})$  which are solutions of (1.1) in  $Q_\infty$ , vanishes on the set  $\{(x, 0) : x \neq 0\}$  and satisfies

$$\lim_{t \rightarrow 0} \int_{B_\epsilon} u(x, t) dx = \infty \quad \forall \epsilon > 0.$$

**Theorem 1.2** *Assume  $p > 2N/(N+1)$ ,  $J < \infty$ ,  $K < \infty$  and (1.12) holds. Then  $\underline{U} = \lim_{k \rightarrow \infty} u_k$  exists and it is the smallest element of  $\mathcal{U}_0$ .*

When one, at least, of the above properties on  $J$  and  $K$  fails, the situation is much more complicated and fairly well understood only in the case where  $f$  has a power-like or a logarithmic-power-like growth. We first note that

(A) If  $f(s) \sim s^\alpha$  ( $\alpha > 0$ ), then  $J < \infty$  if and only if  $\alpha > 1$ , while  $K < \infty$  if and only if  $\alpha > p - 1$ . Moreover (1.12) holds if and only if  $\alpha < p(1 + \frac{1}{N}) - 1$ .

(B) If  $f(s) \sim s^\alpha \ln^\beta(s+1)$  ( $\alpha, \beta > 0$ ), then  $J < \infty$  if and only if  $\alpha > 1$  and  $\beta > 0$ , or  $\alpha = 1$  and  $\beta > 1$  while  $K < \infty$  if and only if  $\alpha > p - 1$  and  $\beta > 0$ , or  $\alpha = p - 1$  and  $\beta > p$ . Moreover (1.12) holds if and only if  $\alpha < p(1 + \frac{1}{N}) - 1$  and  $\beta > 0$ .

**Theorem 1.3** *Assume  $p > 2$  and  $f(s) = s^\alpha \ln^\beta(s+1)$  where  $\alpha \in (1, p-1)$  and  $\beta > 0$ . Let  $u_k$  be the solution of (1.13). Then  $\lim_{k \rightarrow \infty} u_k(x, t) = \phi_\infty(t)$  for every  $(x, t) \in Q_\infty$ .*

When  $\alpha = 1$  the following phenomenon occurs.

**Theorem 1.4** *Assume  $p > 2$  and  $f(s) = s \ln^\beta(s + 1)$  with  $\beta > 0$ . Let  $u_k$  be the solution of (1.13). Then*

(i) *If  $\beta > 1$  then  $\lim_{k \rightarrow \infty} u_k(x, t) = \phi_\infty(t)$  for every  $(x, t) \in Q_\infty$ ,*

(ii) *If  $0 < \beta \leq 1$  then  $\lim_{k \rightarrow \infty} u_k(x, t) = \infty$  for every  $(x, t) \in Q_\infty$ .*

Section 3 is devoted to study non-uniqueness of solutions of (1.1) with unbounded initial data. The starting observation is the following global existence result for solutions of (1.4):

**Theorem 1.5** *Assume  $p > 1$ ,  $f$  is locally Lipschitz continuous and  $K = \infty$ . Then for any  $a > 0$ , there exists a unique solution  $w := w_a$  to the problem*

$$-(r^{N-1}|w_r|^{p-2}w_r)_r + r^{N-1}f(w) = 0 \quad (1.15)$$

defined on  $[0, \infty)$  and satisfying  $w(0) = a$ ,  $w_r(0) = 0$ . It is given by

$$w_a(r) = a + \int_0^r H_p \left( s^{1-N} \int_0^s \tau^{N-1} f(w_a(\tau)) d\tau \right) ds \quad (1.16)$$

where  $H_p$  is the inverse function of  $t \mapsto |t|^{p-2}t$ .

This result extends to the general case  $p > 1$  a previous theorem of Vázquez and Véron [18] obtained in the case  $p = 2$ . The next theorem extends to the case  $p \neq 2$  a previous result of the authors in the case  $p = 2$ .

**Theorem 1.6** *Assume  $p > 2N/(N + 1)$ ,  $f$  is locally Lipschitz continuous,  $J < \infty$  and  $K = \infty$ . For any function  $u_0 \in C(Q_\infty)$  which satisfies*

$$w_a(|x|) \leq u_0(x) \leq w_b(|x|) \quad \forall x \in \mathbb{R}^N \quad (1.17)$$

for some  $0 < a < b$ , there exist at least two solutions  $\underline{u}, \bar{u} \in C(\overline{Q_\infty})$  of (1.1) with initial value  $u_0$ . They satisfy respectively

$$0 \leq \underline{u}(x, t) \leq \min\{w_b(|x|), \phi_\infty(t)\} \quad \forall (x, t) \in Q_\infty,$$

thus  $\lim_{t \rightarrow \infty} \underline{u}(x, t) = 0$ , uniformly with respect to  $x \in \mathbb{R}^N$ , and

$$w_a(|x|) \leq \bar{u}(x, t) \leq w_b(|x|) \quad \forall (x, t) \in Q_\infty$$

thus  $\lim_{|x| \rightarrow \infty} \bar{u}(x, t) = \infty$ , uniformly with respect to  $t \geq 0$ .

In section 4 we prove an existence and stability result for the initial value problem

$$\begin{cases} \partial_t u - \Delta_p u + f(u) = 0 & \text{in } Q_\infty \\ u(\cdot, 0) = \mu & \text{in } \mathbb{R}^N \end{cases} \quad (1.18)$$

where  $\mu \in \mathfrak{M}_+^b(\mathbb{R}^N)$ , the set of positive and bounded Radon measures in  $\mathbb{R}^N$ .

**Theorem 1.7** *Assume  $p > \frac{2N}{N+1}$  and  $f$  satisfies (1.12). Then for any  $\mu \in \mathfrak{M}_+^b(\mathbb{R}^N)$  the problem (1.18) admits a weak solution  $u_\mu$ . Moreover, if  $\{\mu_n\}$  is a sequence of functions in  $L_+^1(\mathbb{R}^N)$  with compact support, which converges to  $\mu \in \mathfrak{M}_+^b(\mathbb{R}^N)$  in the weak sense of measures, then the corresponding solutions  $\{u_{\mu_n}\}$  of (1.18) with initial data  $\mu_n$  converge to some solution  $u_\mu$  of (1.18), strongly in  $L_{loc}^1(\overline{Q_T})$  and locally uniformly in  $Q_T := \mathbb{R}^N \times (0, T)$ . Furthermore  $\{f(u_{\mu_n})\}$  converges strongly to  $f(u_\mu)$  in  $L_{loc}^1(\overline{Q_T})$ .*

In Section 5, we discuss the *initial trace* of positive weak solution of (1.1). The power case  $f(u) = u^q$  with  $q > 0$  was investigated by Bidaut-Véron, Chasseigne and Véron in [2]. They proved the existence of an initial trace in the class of positive Borel measures according to the different values of  $p-1$  and  $q$ . Accordingly they studied the corresponding Cauchy problem with a given Borel measure as initial data. However their method was strongly based upon the fact that the nonlinearity was a power, which enabled to use Hölder inequality in order to show the domination of the absorption term over the other terms. In the present paper, we combine the ideas in [2] and [15] with a stability result for the Cauchy problem and Harnack's inequality in the form of [5] to establish the following dichotomy result which is new even in the case  $p = 2$ .

**Theorem 1.8** *Assume  $p \geq 2$  and (1.12) holds. Let  $u \in C(Q_T)$  be a positive weak solution of (1.1) in  $Q_T$ . Then for any  $y \in \mathbb{R}^N$  the following alternative holds*

(i) *either*

$$u(x, t) \geq \lim_{k \rightarrow \infty} u_k(x - y, t) \quad \forall (x, t) \in Q_T, \quad (1.19)$$

(ii) *or there exist an open neighborhood  $U$  of  $y$  and a Radon measure  $\mu_U \in \mathfrak{M}_+(U)$  such that*

$$\lim_{t \rightarrow 0} \int_U u(x, t) \zeta(x) dx = \int_U \zeta d\mu_U \quad \forall \zeta \in C_c(U). \quad (1.20)$$

Actually, since (1.12) is verified, (1.19) is equivalent to the fact that, for any open neighborhood  $U$  of  $y$ , there holds

$$\limsup_{t \rightarrow 0} \int_U u(x, t) dx = \infty. \quad (1.21)$$

However, if (1.12) is not verified, there only holds (1.19)  $\implies$  (1.21).

The set of points  $y$  such that (1.20) (resp. (1.21)) holds is clearly open (resp. closed) and denoted by  $\mathcal{R}(u)$  (resp  $\mathcal{S}(u)$ ). Using a partition of unity, there exists a unique Radon measure  $\mu \in \mathfrak{M}_+(\mathcal{R}(u))$  such that

$$\lim_{t \rightarrow 0} \int_{\mathcal{R}(u)} u(x, t) \zeta(x) dx = \int_{\mathcal{R}(u)} \zeta d\mu \quad \forall \zeta \in C_c(\mathcal{R}(u)). \quad (1.22)$$

Owing to the above result we define the *initial trace* of a positive solution  $u$  (1.1) in  $Q_T$  as the couple  $(\mathcal{S}(u), \mu)$  for which (1.20) and (1.21) holds and we denote it by  $tr_{\mathbb{R}^N}(u)$ . The set  $\mathcal{S}(u)$  is the *set of singular points* of  $tr_{\mathbb{R}^N}(u)$ , while  $\mu$  is the *regular part* of  $tr_{\mathbb{R}^N}(u)$ . It is classical that any  $\nu \in \mathfrak{B}^{reg}(\mathbb{R}^N)$ , the set of positive outer regular Borel measures in  $\mathbb{R}^N$ , can be represented by a couple  $(\mathcal{S}, \mu)$  where  $\mathcal{S}$  is a closed subset of  $\mathbb{R}^N$  and  $\mu \in \mathfrak{M}_+(\mathcal{R})$ , where  $\mathcal{R} = \mathbb{R}^N \setminus \mathcal{S}$ , in the following way

$$\nu(A) = \begin{cases} \infty & \text{if } A \cap \mathcal{S} \neq \emptyset, \\ \mu(A) & \text{if } A \subset \mathcal{R}, \end{cases} \quad \forall A \text{ Borel.}$$

Therefore Theorem 1.8 means that  $tr_{\mathbb{R}^N}(u) \in \mathfrak{B}^{reg}(\mathbb{R}^N)$ .

The initial trace can be made more precise when the Keller-Osserman-Vázquez condition does not hold, and if we know whether  $\lim_{k \rightarrow \infty} u_k$  is equal to  $\phi_\infty$  or is infinite.

**Theorem 1.9** *Assume  $p > 2$  and (1.12) holds and  $u$  is a positive solution of (1.1) in  $Q_\infty$ .*

*I- If  $J < \infty$  and  $K = \infty$  are verified and  $\lim_{k \rightarrow \infty} u_k = \phi_\infty$ . Then either  $tr_{\mathbb{R}^N}(u)$  is the Borel measure infinity  $\nu_\infty$  which satisfies  $\nu_\infty(\mathcal{O}) = \infty$  for any non-empty open subset  $\mathcal{O} \subset \mathbb{R}^N$ , or is a positive Radon measure  $\mu$  on  $\mathbb{R}^N$ .*

*II- If  $J = \infty$  and  $K = \infty$  are verified and  $\lim_{k \rightarrow \infty} u_k = \infty$ . Then  $tr_{\mathbb{R}^N}(u)$  is a positive Radon measure  $\mu$  on  $\mathbb{R}^N$ .*

As a consequence of *I*, there exist infinitely many positive solutions  $u$  of (1.1) in  $Q_\infty$  such that  $tr_{\mathbb{R}^N}(u) = \nu_\infty$ . By Theorem 1.3, Theorem 1.4, the previous results apply in particular if  $f(s) = s^\alpha \ln^\beta(s+1)$ .

## 2 Isolated singularities

Throughout the article  $c_i$  denote positive constants depending on  $N, p, f$  and sometimes other quantities such as test functions or particular exponents, the value of which may change from one occurrence to another.



## 2.1 The semigroup approach

We refer to [9, p 117] for the detail of the Banach space framework for the construction of solutions of (1.1) in  $Q_\infty$  with initial data in  $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ . We set

$$J(u) = \int_{\mathbb{R}^N} \left( \frac{1}{p} |\nabla u|^p + F(u) \right) dx \quad (2.1)$$

when  $u$  belongs to the domain  $D(J)$  of  $J$  which is the set of  $u \in L^2(\mathbb{R}^N)$  such that  $\nabla u \in L^p(\mathbb{R}^N)$  and  $F(u) \in L^1(\mathbb{R}^N)$ , and  $J(u) = \infty$  if  $u \notin D(J)$ . Then  $J$  is a proper convex lower semicontinuous function in  $L^2(\mathbb{R}^N)$ . Its sub-differential  $A$  is defined by its domain  $D(A)$  which is the set of  $u \in L^2(\mathbb{R}^N)$  such that  $\nabla u \in L^p(\mathbb{R}^N)$  and  $F(u) \in L^1(\mathbb{R}^N)$  with the property that  $-\Delta_p u + f(u) \in L^2(\mathbb{R}^N)$  and

$$-\int_{\mathbb{R}^N} v \Delta_p u dx = \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx \quad \forall v \in D(J), \quad (2.2)$$

and by its expression

$$Au = -\Delta_p u + f(u) \quad \forall u \in D(A). \quad (2.3)$$

Notice that (2.2) implies that  $vf(u) \in L^1(\mathbb{R}^N)$  for all  $v \in D(J)$ . The restriction of the operator  $A$  is accretive in  $L^1(\mathbb{R}^N)$  and in  $L^\infty(\mathbb{R}^N)$ , hence in every  $L^q(\mathbb{R}^N)$ . The operator  $A_q$  defined in  $L^q(\mathbb{R}^N)$  is the closure in  $L^q(\mathbb{R}^N)$  of the restriction of  $A$  to  $L^q(\mathbb{R}^N)$ . It is a m-accretive operator, with domain  $D(A_q)$ . Since  $C_0^\infty(\mathbb{R}^N) \subset D(A_q)$ ,  $D(A_q)$  is dense in  $L^q(\mathbb{R}^N)$ . If  $u_0 \in L^q$  the generalized solution  $u$  to

$$\begin{cases} \frac{du}{dt} + A_q u = 0 & \text{in } (0, \infty) \\ u(0) = u_0 \end{cases} \quad (2.4)$$

is obtained by the Crandall-Liggett scheme

$$\frac{u_i - u_{i-1}}{h} + A_q u_i = 0 \quad \text{in } i = 0, 1, \dots \quad (2.5)$$

when we let  $h \rightarrow 0$ , in the sense that the continuous piecewise linear function  $U_h$  defined by  $U_h(ih) = u_i$  converges to  $u$  in the  $C([0, T], L^q(\mathbb{R}^N))$ -topology, for every  $T > 0$ . Furthermore, if  $q = 2$  and  $u_0 \in D(A_2)$  (resp.  $u_0 \in L^2(\mathbb{R}^N)$ ), then  $\frac{dU_h}{dt}$  converges to  $\frac{du}{dt}$  in  $L^2([0, T], L^2(\mathbb{R}^N))$  (resp.  $L^2([0, T], L^2(\mathbb{R}^N); tdt)$ ), see [20]. We shall denote by  $\{S^{A_q}(t)\}_{t>0}$  the semigroup of contractions of  $L^q(\mathbb{R}^N)$  generated by  $-A_q$  thru the Crandall-Liggett Theorem [4].

An important property [9, Lemma 2] is that if  $w \in L^1(\mathbb{R}^N)$  satisfies

$$A_1 w + \sigma w = h \quad (2.6)$$

where  $\sigma > 0$  and  $h \in L^1(\mathbb{R}^N)$ , then

$$\int_{\mathbb{R}^N} A_1 w dx = 0. \quad (2.7)$$

**Definition 2.1** (i) A function  $u \in C([\delta, \infty); L^1(\mathbb{R}^N))$  where  $\delta \geq 0$  is a semigroup solution (1.1) on  $(\delta, \infty)$  if for any  $t \geq \delta$  there holds  $u(\cdot, t) = S^{A_1}(t - \delta)[u(\cdot, \delta)]$ .

(ii) A function  $u \in C((\delta, \infty); L^1(\mathbb{R}^N))$  is an extended semigroup solution of (1.1) on  $(\delta, \infty)$  if for any  $t \geq \tau > \delta$ , there holds  $u(\cdot, t) = S^{A_1}(t - \tau)[u(\cdot, \tau)]$ .

## 2.2 The Barenblatt-Prattle solutions

We recall the explicit expression, due to Barenblatt and Prattle, of the solution  $v = v_k$  of problem (1.14). If  $p = 2$

$$v_k(x, t) = k(4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}}, \quad (2.8)$$

and if  $\frac{2N}{N+1} < p \neq 2$ ,

$$v_k(x, t) = t^{-\lambda} V\left(\frac{x}{t^{\frac{\lambda}{N}}}\right), \text{ where } V(\xi) = \left(C_k - d|\xi|^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}} \quad (2.9)$$

with

$$\lambda = \frac{N}{N(p-2) + p} \quad \text{and} \quad d = \frac{p-2}{p} \left(\frac{\lambda}{N}\right)^{\frac{1}{p-1}}, \quad (2.10)$$

and where  $C_k$  is connected to the mass  $k$  by

$$C_k = c(N, p)k^\ell \quad \text{with} \quad \ell = \frac{p(p-2)\lambda}{(p-1)N}. \quad (2.11)$$

The condition  $p > \frac{2N}{N+1}$  appears in order  $\lambda$  be positive. Notice that, if  $p > 2$  then  $d > 0$ , therefore the support of  $v_k(\cdot, t)$  is the ball  $B_{\delta_k(t)}$  where  $\delta_k(t) = \left(\frac{C_k}{d}\right)^{\frac{p-1}{p}} t^{\frac{\lambda}{N}}$ , while  $v_k(x, t) > 0$  for all  $(x, t) \in Q_\infty$  if  $\frac{2N}{N+1} < p < 2$  (and also  $p = 2$  although the expression of  $v_k$  is different). Furthermore, if  $\frac{2N}{N+1} < p < 2$ , the limit of  $v_k$  when  $k \rightarrow \infty$  is explicit

$$v_\infty(x, t) = \Lambda_N \left(\frac{t}{|x|^p}\right)^{\frac{1}{2-p}}, \quad (2.12)$$

where  $\Lambda_N = (-d)^{\frac{p-1}{p-2}}$ . This type of singular solution which is singular on the whole axis  $(0, t) \subset Q_\infty$ , is called a *razor blade* (see [19] for some examples). To this solution corresponds a universal estimate.

**Lemma 2.2** Assume  $1 < p < 2$  and let  $v \in C(\overline{Q_\infty} \setminus B_{R_0} \times \{0\})$  be a semigroup solution positive of (1.1)

$$\partial_t v - \Delta_p v = 0 \quad \text{in } Q_\infty \quad (2.13)$$

which satisfies

$$\lim_{t \rightarrow 0} \int_K v(x, t) dx = 0, \quad (2.14)$$

for any compact set  $K \subset \mathbb{R}^N \setminus B_{R_0}$ . Then there exists  $c_1 = c_1(N, p) > 0$  such that

$$\sup_{0 \leq \tau \leq t} \int_{\{x: |x| > R\}} v(x, \tau) dx \leq c_1 \left( \frac{t}{(R - R_0)^{\frac{N}{\lambda}}} \right)^{\frac{1}{2-p}} \quad \forall R > R_0, t > 0. \quad (2.15)$$

If we assume moreover that  $\lim_{|x| \rightarrow \infty} v(t, x) = 0$  locally uniformly with respect to  $t \geq 0$ , then

$$v(x, t) \leq \Lambda_1 \left( \frac{t}{(|x| - R_0)^p} \right)^{\frac{1}{2-p}} \quad \forall (x, t) \in Q_\infty, |x| > R_0, \quad (2.16)$$

where  $\Lambda_1$  is the value of the constant in (2.12) when  $N = 1$ .

*Proof.* The first estimate is a consequence of

$$\sup_{0 \leq \tau \leq t} \int_{B_\rho(a)} v(x, t) dx \leq c_2 \left( \int_{B_{2\rho}(a)} v(x, 0) dx + \left( \frac{t}{\rho^{\frac{N}{\lambda}}} \right)^{\frac{1}{2-p}} \right) \quad (2.17)$$

in [6, Lemma III.3.1] under the assumption that  $v(\cdot, 0)$  is continuous with compact support. Actually this assumption is not used. In this proof the first step is the following estimate obtained by a suitable choice of test function:

$$\sup_{0 \leq \tau \leq t} \int_{B_R(a)} v(x, t) dx \leq \int_{B_{2R}(a)} v(x, 0) dx + \frac{c_3}{R} \int_0^t \int_{B_R(a)} |\nabla v|^{p-1} dx d\tau \quad (2.18)$$

valid for any  $a \in \mathbb{R}^N \setminus \{(0, 0)\}$  and  $R \leq |a|/2$ . The second step to get (2.17) is to estimate the integral on the right-hand side by relation (I.4.2) in [6, Lemma I.4.1] with the same choice of  $\epsilon$ . We apply estimate (2.17) with a sequence of points in a fixed direction  $\mathbf{e}$  (with  $|\mathbf{e}| = 1$ )  $a = a_k = (2^k(R - R_0) + R_0) \mathbf{e}$  and  $\rho = \rho_k = 2^{k-1}(R - R_0)$  (actually we start with  $\rho < \rho_k$  and let it grow up to  $\rho_k$ ). Then we get

$$\sup_{0 \leq \tau \leq t} \int_{B_{\rho_k}(a_k)} v(x, t) dx \leq c_4 2^{-\frac{N(k-1)}{\lambda(2-p)}} \left( \frac{t}{(R - R_0)^{\frac{N}{\lambda}}} \right)^{\frac{1}{2-p}}. \quad (2.19)$$

Since the ball  $B_{\rho_k}(a_k)$  and  $B_{\rho_{k+1}}(a_{k+1})$  are overlapping there exist a finite number of points  $\{\mathbf{e}_j\}_{j=1}^{d_1}$  and  $\{\mathbf{e}'_j\}_{j=1}^{d_2}$  ( $d_1$  and  $d_2$  depend only on  $N$ ) on the unit sphere such that

$$\{x \in \mathbb{R}^N : |x| \geq R\} \subset \left( \bigcup_{j=1}^{d_1} \bigcup_{k=1}^{\infty} B_{\rho_k}(2\rho_k \mathbf{e}_j) \right) \cup \left( \bigcup_{j=1}^{d_2} B_{\frac{R-R_0}{2}}(R\mathbf{e}'_j) \right).$$

Therefore

$$\sup_{0 \leq \tau \leq t} \int_{\{x: |x| > R\}} v(x, \tau) dx \leq c_4 \left[ d_1 \sum_{k=0}^{\infty} 2^{-\frac{Nk}{\lambda(2-p)}} + d_2 2^{\frac{N}{\lambda(2-p)}} \right] \left( \frac{t}{(R-R_0)^{\frac{N}{\lambda}}} \right)^{\frac{1}{2-p}}$$

which is (2.15).

Estimate (2.16) follows from comparison with the 1-dim form of  $v_{\infty}$

$$v_{\infty}(s, t) = \Lambda_1 \left( \frac{t}{s^p} \right)^{\frac{1}{2-p}} \quad \forall s, t > 0. \quad (2.20)$$

For  $\epsilon > 0$ , the function

$$(x, t) \mapsto v_{\infty}(x_1 - R_0 - \epsilon, t) + \epsilon$$

where  $x = (x_1, \dots, x_N) = (x_1, x')$ , is a solution of (2.13) in  $H_{1, R_0+\epsilon} \times (0, \infty)$  where  $H_{1, m} = \{x \in \mathbb{R}^N : x_1 > m\}$ . For  $R$  large enough  $v(x, t) \leq v_{\infty}(x_1 + R_0 + \epsilon, t) + \epsilon$  on the set  $((H_{1, R_0+\epsilon} \cap \partial B_R) \cup (\partial H_{1, R_0+\epsilon} \cap B_R)) \times [0, T]$  for any  $T > 0$ , and for  $t = 0$ . By the maximum principle  $v(x, t) \leq v_{\infty}(x_1 - R_0 - \epsilon, t) + \epsilon$  in  $(H_{1, R_0+\epsilon} \cap B_R) \times (0, T]$ . Letting successively  $R \rightarrow \infty$ ,  $T \rightarrow \infty$  and  $\epsilon \rightarrow 0$  and using the invariance of the equation by rotation implies (2.16).  $\square$

**Proposition 2.3** *Let  $p > \frac{2N}{N+1}$  and  $\{v^n\} \subset C([0, \infty); L^1(\mathbb{R}^N))$  be a sequence of positive semigroup solutions of (2.13) on  $(0, \infty)$  such that  $v^n(\cdot, 0)$  has support in  $B_{\epsilon_n}$  where  $\epsilon_n \rightarrow 0$ . If*

$$\int_{\mathbb{R}^N} v^n(x, 0) dx = k_n \rightarrow k \quad \text{as } n \rightarrow \infty$$

then  $v^n \rightarrow v_k$  locally uniformly in  $Q_{\infty}$ .

*Proof.* We first give the proof in the case  $\frac{2N}{N+1} < p < 2$ . By *a priori* estimates, up to a subsequence  $v^n$  converges locally uniformly in  $Q_{\infty}$  to a solution  $v$  of (2.13) in  $Q_{\infty}$ . By Herrero-Vazquez mass conservation property [9, Theorem 2] (valid if  $p > \frac{2N}{N+1}$ )

$$\int_{\mathbb{R}^N} v^n(x, t) dx = \int_{\mathbb{R}^N} v^n(x, 0) dx = k_n.$$

By (2.16)

$$v^n(x, t) \leq \Lambda_1 \left( \frac{t}{(|x| - \epsilon_n)^p} \right)^{\frac{1}{2-p}} \quad \forall t > 0, \forall |x| > \epsilon_n.$$

Since  $\frac{p}{2-p} > N$ , the function

$$x \mapsto \left( \frac{t}{(|x| - \epsilon_n)^p} \right)^{\frac{1}{2-p}}$$

belongs to  $L^1(\mathbb{R}^N \setminus B_\delta)$ , for any  $\delta > \epsilon_n$ . Since  $v^n(x, t) \rightarrow v(x, t)$  uniformly in  $B_\delta$ , it follows by the dominated convergence theorem

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} v^n(x, t) dx = \int_{\mathbb{R}^N} v(x, t) dx = k. \quad (2.21)$$

Because  $v$  is a positive solution with isolated singularity at  $(0, 0)$ , it follows from [3] that  $v = v_k$ , solution of (1.14).

When  $p \geq 2$ , the function  $v_k(\cdot, t)$  has a compact support  $D_{k_n}(t)$  for any  $t > 0$  and  $D_{k_n}(t) \subset B_{R_n(t)}$  where

$$R_n(t) = \epsilon_n + c_5 k_n^{\frac{p-2}{p}} t^{\frac{1}{N(p-2)+p}} \leq \epsilon^* + c_5 k_*^{\frac{p-2}{p}} t^{\frac{1}{N(p-2)+p}} \quad (2.22)$$

where  $c_5 = c_5(N, p) > 0$ ,  $\epsilon^* = \sup\{\epsilon_n; n \in \mathbb{N}\}$  and  $k_* = \sup\{k_n; n \in \mathbb{N}\}$ . Using Lebesgue dominating theorem we obtain again (2.21).  $\square$

### 2.3 Fundamental solutions

The following lemma is fundamental.

**Lemma 2.4** *Assume  $p > \frac{2N}{N+1}$  and  $f$  is a continuous nondecreasing function defined on  $\mathbb{R}$  such that  $f(0) = 0$ . Then, for any  $k, R, T > 0$ ,*

$$\int_1^\infty f(s) s^{-\frac{p(N+1)}{N}} ds < \infty \implies f(v_k) \in L^1(B_R \times (0, T)). \quad (2.23)$$

*Proof.* The result is already proved in [10] in the case  $p > 2$ . It is probably known in the case  $p = 2$ , but we have not found any reference. It appears to be new in the case  $\frac{2N}{N+1} < p < 2$ . Without any loss of generality we can assume  $R = T = 1$ .

*Case 1:  $p = 2$ .* By linearity we can assume that  $k = (4\pi)^{\frac{N}{2}}$ . Let

$$I = \int \int_{B_1 \times (0, 1)} f(v_k) dx dt = \omega_N \int_0^1 \int_0^1 f \left( t^{-\frac{N}{2}} e^{-\frac{r^2}{4t}} \right) r^{N-1} dr dt.$$

Set  $s = t^{-\frac{N}{2}} e^{-\frac{r^2}{4t}}$ , then

$$\begin{aligned} I &= 2^{N-1} \omega_N \int_0^1 \int_{t^{-\frac{N}{2}} e^{-\frac{1}{4t}}}^{t^{-\frac{N}{2}}} \left[ -\ln s - \ln \left( t^{\frac{N}{2}} \right) \right]^{\frac{N-2}{2}} f(s) s^{-1} ds t^{\frac{N}{2}} dt \\ &\leq 2^{N-1} \omega_N \int_0^1 \int_{e^{-\frac{1}{4t}}}^{t^{-\frac{N}{2}}} \left[ -\ln s - \ln \left( t^{\frac{N}{2}} \right) \right]^{\frac{N-2}{2}} f(s) s^{-1} ds t^{\frac{N}{2}} dt \leq 2^{N-1} \omega_N (I_1 + I_2) \end{aligned}$$

where,

$$\begin{aligned} I_1 &= \int_0^{e^{-\frac{1}{4}}} \int_0^{-\frac{1}{4 \ln s}} \left[ -\ln s - \ln \left( t^{\frac{N}{2}} \right) \right]^{\frac{N-2}{2}} t^{\frac{N}{2}} dt s^{-1} f(s) ds \\ &= \frac{2}{N} \int_0^{e^{-\frac{1}{4}}} \int_0^{\frac{s}{(-4 \ln s)^{\frac{N}{2}}}} (-\ln \tau)^{\frac{N-2}{2}} \tau^{\frac{2}{N}} d\tau s^{-2-\frac{2}{N}} f(s) ds, \end{aligned}$$

by setting  $\tau = st^{\frac{N}{2}}$ . But

$$\begin{aligned} \int_0^{\frac{s}{(-4 \ln s)^{\frac{N}{2}}}} (-\ln \tau)^{\frac{N-2}{2}} \tau^{\frac{2}{N}} d\tau &\leq c_6 \left[ (-\ln \tau)^{\frac{N-2}{2}} \tau^{\frac{N+2}{N}} \right]_0^{\frac{s}{(-4 \ln s)^{\frac{N}{2}}}} \\ &\leq c_6 s^{1+\frac{2}{N}} (-\ln s)^{-2} \left( 1 + \frac{N \ln(-4 \ln s)}{2(-\ln s)} \right)^{\frac{N-2}{2}} \\ &\leq c_7 s^{1+\frac{2}{N}} (-\ln s)^{-2}, \end{aligned}$$

thus

$$I_1 \leq c_8 \int_0^{e^{-\frac{1}{4}}} s^{-1} (-\ln s)^{-2} f(s) ds < \infty$$

by Duhamel's rule. Further

$$\begin{aligned} I_2 &\leq \int_{e^{-\frac{1}{4}}}^{\infty} \int_0^{s^{-\frac{2}{N}}} \left[ -\ln s - \ln \left( t^{\frac{N}{2}} \right) \right]^{\frac{N-2}{2}} t^{\frac{N}{2}} dt s^{-1} f(s) ds \\ &\leq \frac{2}{N} \int_{e^{-\frac{1}{4}}}^{\infty} \int_0^1 (-\ln \tau)^{\frac{N-2}{2}} \tau^{\frac{2}{N}} d\tau s^{-2-\frac{2}{N}} f(s) ds \\ &\leq c_9 \int_{e^{-\frac{1}{4}}}^{\infty} s^{-2-\frac{2}{N}} f(s) ds, \end{aligned}$$

for some  $c_9 = c_9(N) > 0$ . This implies the claim when  $p = 2$ .

*Case 2:*  $\frac{2N}{N+1} < p < 2$ . We set  $d^* = -d$ . By rescaling we can assume that  $C_k = d^* = 1$ . Therefore

$$I = \int \int_{B_1 \times (0,1)} f(v_k) dx dt = \omega_N \int_0^1 \int_0^1 f \left( t^{-\lambda} \left[ 1 + \left( \frac{r}{t^{\frac{\lambda}{N}}} \right)^{\frac{p}{p-1}} \right]^{\frac{p-1}{p-2}} \right) r^{N-1} dr dt.$$

Set  $s = t^{-\lambda} \left[ 1 + \left( \frac{r}{t^{\frac{\lambda}{N}}} \right)^{\frac{p}{p-1}} \right]^{\frac{p-1}{p-2}}$ , then  $r = t^{\frac{\lambda}{N}} \left[ (t^\lambda s)^{\frac{p-2}{p-1}} - 1 \right]^{\frac{p-1}{p}}$  and

$$\begin{aligned} I &= \frac{2-p}{p} \omega_N \int_0^1 \int_{t^{-\lambda}(1+t^{-\frac{\lambda p}{p-1}})^{\frac{p-1}{p-2}}}^{t^{-\lambda}} (t^\lambda s)^{-\frac{1}{p-1}} \left( (t^\lambda s)^{\frac{p-2}{p-1}} - 1 \right)^{\frac{N(p-1)}{p}-1} f(s) ds t^{2\lambda} dt \\ &= \frac{2-p}{p} \omega_N (I_1 + I_2) \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_{2^{\frac{p-1}{p-2}}}^1 \int_{a(s)}^1 (t^\lambda s)^{-\frac{1}{p-1}} \left( (t^\lambda s)^{\frac{p-2}{p-1}} - 1 \right)^{\frac{N(p-1)}{p}-1} t^{2\lambda} dt f(s) ds \\ I_2 &= \int_1^\infty \int_{a(s)}^{s^{-\frac{1}{\lambda}}} (t^\lambda s)^{-\frac{1}{p-1}} \left( (t^\lambda s)^{\frac{p-2}{p-1}} - 1 \right)^{\frac{N(p-1)}{p}-1} t^{2\lambda} dt f(s) \end{aligned}$$

and  $a(s)$  is the inverse function of  $t \mapsto t^{-\lambda} (1 + t^{-\frac{\lambda p}{p-1}})^{\frac{p-1}{p-2}}$ . Clearly

$$t^{-\lambda} (1 + t^{-\frac{\lambda p}{p-1}})^{\frac{p-1}{p-2}} \leq t^{\frac{2\lambda(p-1)}{2-p}} \implies a(s) \geq s^{\frac{2-p}{2\lambda(p-1)}}.$$

Therefore

$$\begin{aligned} I_1 &\leq \int_{2^{\frac{p-1}{p-2}}}^1 \int_{s^{\frac{2-p}{2\lambda(p-1)}}}^1 (t^\lambda s)^{-\frac{1}{p-1}} \left( (t^\lambda s)^{\frac{p-2}{p-1}} - 1 \right)^{\frac{N(p-1)}{p}-1} t^{2\lambda} dt f(s) ds \\ &\leq \frac{1}{\lambda} \int_{2^{\frac{p-1}{p-2}}}^1 \int_{s^{\frac{p}{2(p-1)}}}^s (1 - \tau^{\frac{2-p}{p-1}})^{\frac{N(p-1)}{p}-1} \tau^{\frac{1}{\lambda} + \frac{N(p-2)}{p}} d\tau s^{-2-\frac{1}{\lambda}} f(s) ds. \end{aligned}$$

Since  $\frac{1}{\lambda} + \frac{N(p-2)}{p} > -1$  and  $\frac{N(p-1)}{p} - 1 > -1$ ,

$$\int_{s^{\frac{p}{2(p-1)}}}^s (1 - \tau^{\frac{2-p}{p-1}})^{\frac{N(p-1)}{p}-1} \tau^{\frac{1}{\lambda} + \frac{N(p-2)}{p}} d\tau < \int_0^1 (1 - \tau^{\frac{2-p}{p-1}})^{\frac{N(p-1)}{p}-1} \tau^{\frac{1}{\lambda} + \frac{N(p-2)}{p}} d\tau < \infty.$$

Furthermore  $-2 - \frac{1}{\lambda} = -p - \frac{p}{N}$  thus

$$I_1 \leq c_{10} \int_{2^{\frac{p-1}{p-2}}}^1 f(s) s^{-\frac{p(N+1)}{N}} ds.$$

We perform the same change of variable with  $I_2$

$$\begin{aligned} I_2 &\leq \int_1^\infty \int_{s^{\frac{2-p}{2\lambda(p-1)}}}^{s^{-\frac{1}{\lambda}}} (t^\lambda s)^{-\frac{1}{p-1}} \left( (t^\lambda s)^{\frac{p-2}{p-1}} - 1 \right)^{\frac{N(p-1)}{p}-1} t^{2\lambda} dt f(s) ds \\ &\leq \frac{1}{\lambda} \int_1^\infty \int_{s^{\frac{p}{2(p-1)}}}^1 (1 - \tau^{\frac{2-p}{p-1}})^{\frac{N(p-1)}{p}-1} \tau^{\frac{1}{\lambda} + \frac{N(p-2)}{p}} d\tau s^{-2-\frac{1}{\lambda}} f(s) ds. \end{aligned}$$

Again

$$\int_s^1 \frac{p}{2^{(p-1)}} (1 - \tau^{\frac{2-p}{p-1}})^{\frac{N(p-1)}{p}-1} \tau^{1+\frac{N(p-2)}{p}} d\tau < \int_0^1 (1 - \tau^{\frac{2-p}{p-1}})^{\frac{N(p-1)}{p}-1} \tau^{\frac{1}{\lambda}+\frac{N(p-2)}{p}} d\tau < \infty,$$

then

$$I_2 \leq c_{11} \int_1^\infty f(s) s^{-\frac{p(N+1)}{N}} ds.$$

Therefore (2.23) holds.  $\square$

Notice that the assumption implies that  $v_k \in C(Q_\infty) \cap L^\infty(\delta, \infty; L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))$  for every  $\delta > 0$ .

**Proof of Theorem 1.1. Existence.** Let  $\epsilon > 0$ ,  $Q_{\epsilon, \infty} = \mathbb{R}^N \times (\epsilon, \infty)$  and denote by  $u_\epsilon$  the solution of

$$\begin{cases} \partial_t u - \Delta_p u + f(u) = 0 & \text{in } Q_{\epsilon, \infty} \\ u(\cdot, \epsilon) = v_k(\cdot, \epsilon) & \text{in } \mathbb{R}^N. \end{cases} \quad (2.24)$$

Since  $v_k(\cdot, \epsilon)$  is a smooth positive function belonging to  $L^1(\mathbb{R}^N)$  the function  $u_\epsilon$  is constructed by truncation. By the maximum principle

$$u_\epsilon(x, t + \epsilon) \leq v_k(x, t + \epsilon) \quad \forall (x, t) \in Q_{\epsilon, \infty}. \quad (2.25)$$

For  $0 < \epsilon' < \epsilon$ ,  $u_{\epsilon'}(x, \epsilon) \leq v_k(x, \epsilon) = u_\epsilon(x, \epsilon)$ , thus  $u_{\epsilon'}(x, t + \epsilon) \leq u_\epsilon(x, t + \epsilon)$  in  $Q_{\epsilon, \infty}$ . Set  $\tilde{u} = \lim_{\epsilon \rightarrow 0} u_\epsilon$ , then  $\tilde{u} \leq v_k$  in  $Q_\infty$ . By the standard local regularity theory for degenerate equations,  $\nabla u_\epsilon$  remains locally compact in  $(C_{loc}^1(Q_\infty))^N$ , thus  $\tilde{u}$  satisfies (1.1) in  $Q_\infty$ .

In order to prove that

$$\frac{d}{dt} \int_{\mathbb{R}^N} u_\epsilon(x, s) dx + \int_{\mathbb{R}^N} f(u_\epsilon(x, s)) dx = 0$$

we recall that  $u_\epsilon$  can be obtained as the limit of thru the iterative implicit scheme (2.4) with  $q \in [1, \infty]$  is arbitrary since  $u_{\epsilon, 0} \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ . For  $h > 0$  we can write it under the form

$$u_{\epsilon, i} - h \Delta_p u_{\epsilon, i} = -h f(u_{\epsilon, i}) + u_{\epsilon, i-1}.$$

By (2.7), and denoting by  $\tilde{U}_{\epsilon, h}$  the piecewise constant function such that  $\tilde{U}_{\epsilon, h}(jh) = u_{\epsilon, j}$ , we obtain since  $u_{\epsilon, 0} = v_k(\epsilon)$

$$\int_{\mathbb{R}^N} (u_{\epsilon, i} - v_k(\epsilon))(x) dx = - \int_\epsilon^{ih} \int_{\mathbb{R}^N} f(\tilde{U}_{\epsilon, h}(x)) dx dt. \quad (2.26)$$

Letting  $h \rightarrow 0$  and  $i \rightarrow \infty$  such that  $ih = t > \epsilon$  and using the uniform convergence, we obtain

$$\int_{\mathbb{R}^N} u_\epsilon(x, t) dx - \int_{\mathbb{R}^N} v_k(x, \epsilon) dx = - \int_\epsilon^t \int_{\mathbb{R}^N} f(u_\epsilon(x, s)) dx dt. \quad (2.27)$$



Since  $0 \leq u_\epsilon \leq v_k$  and  $v_k(\cdot, t)$  has constant mass equal to  $k$ , we derive

$$\left| \int_{\mathbb{R}^N} u_\epsilon(x, t) dx - k \right| \leq \int_\epsilon^t \int_{\mathbb{R}^N} f(v_k(x, s)) dx ds. \quad (2.28)$$

Because  $f(v_k) \in L^1(\mathbb{R}^N \times (0, T))$ , we can let  $\epsilon \rightarrow 0$ , using the monotone convergence theorem, in order to get

$$\left| \int_{\mathbb{R}^N} u(x, t) dx - k \right| \leq \int_0^t \int_{\mathbb{R}^N} f(v_k(x, s)) dx ds. \quad (2.29)$$

This implies that

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} u(x, t) dx = k. \quad (2.30)$$

If  $\phi \in C_c(\mathbb{R}^N)$ , let  $\zeta \in C_c^\infty(\mathbb{R}^N)$  such that  $0 \leq \zeta \leq 1$ ,  $\zeta = 1$  on the support of  $\phi$  and  $\zeta(0) = 1$ . Then

$$\begin{aligned} \int_{\mathbb{R}^N} u(x, t) \phi(x) dx &= \int_{\mathbb{R}^N} u(x, t) \phi(x) \zeta(x) dx \\ &= \phi(0) \int_{\mathbb{R}^N} u(x, t) dx + \int_{\mathbb{R}^N} u(x, t) (\phi(x) \zeta(x) - \phi(0)) dx. \end{aligned}$$

Thus

$$\left| \int_{\mathbb{R}^N} u(x, t) \phi(x) dx - \phi(0) \int_{\mathbb{R}^N} u(x, t) dx \right| \leq \int_{\mathbb{R}^N} v_k(x, t) |\phi(x) \zeta(x) - \phi(0)| dx.$$

Because  $|\phi(x) \zeta(x) - \phi(0)|$  is continuous and vanishes at zero and  $v_k(\cdot, 0) = k\delta_0$ , it follows from (2.30)

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} u(x, t) \phi(x) dx = k\phi(0). \quad (2.31)$$

*Uniqueness.* The proof uses some ideas from [10, Th 2.4]. Assume  $\tilde{u}$  is any nonnegative solution of problem (1.13), then, for any  $\epsilon > 0$  we denote by  $\tilde{v}_\epsilon$  the solution of

$$\begin{cases} \partial_t v - \Delta_p v = 0 & \text{in } Q_{\epsilon, \infty} \\ v(\cdot, \epsilon) = \tilde{u}(\cdot, \epsilon) & \text{in } \mathbb{R}^N. \end{cases} \quad (2.32)$$

By the maximum principle  $\tilde{v}_\epsilon \geq \tilde{u}$  in  $Q_{\epsilon, \infty}$ . When  $\epsilon \rightarrow 0$ ,  $\tilde{v}_\epsilon$  converges locally uniformly to a solution  $\tilde{v}$  of the same equation in  $Q_\infty$ . Furthermore, using again [9, Lemma 2],

$$\int_{\mathbb{R}^N} \tilde{v}_\epsilon(x, t + \epsilon) dx = \int_{\mathbb{R}^N} \tilde{u}(x, \epsilon) dx.$$

By Fatou's Lemma and using the fact that

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N} \tilde{u}(x, \epsilon) dx = k,$$

we derive

$$\int_{\mathbb{R}^N} \tilde{v}(x, t) dx \leq k. \quad (2.33)$$

Since  $\tilde{v} \geq \tilde{u}$ , equality holds in (2.33). Since the fundamental solution is unique [3, Th 4.1], it implies  $\tilde{v} = v_k$  and  $\tilde{u} \leq v_k$ . We end the proof as in [3, Th 4.1], using the  $L^1$ -contraction mapping principle and the fact that any solution of (1.13) is smaller than  $v_k$ : for  $t > s > 0$ , there holds

$$\begin{aligned} \int_{\mathbb{R}^N} |u(x, t) - \tilde{u}(x, t)| dx &\leq \int_{\mathbb{R}^N} |u(x, s) - \tilde{u}(x, s)| dx \\ &\leq \int_{\mathbb{R}^N} |u(x, s) - v_k(x, s)| dx + \int_{\mathbb{R}^N} |v_k(x, s) - \tilde{u}(x, s)| dx \\ &\leq \int_{\mathbb{R}^N} (v_k(x, s) - u(x, s)) dx + \int_{\mathbb{R}^N} (v_k(x, s) - \tilde{u}(x, s)) dx. \end{aligned} \quad (2.34)$$

When  $s \rightarrow 0$  the right-hand side of the last line goes to 0. This implies the claim.  $\square$

The next result shows some geometric properties of the  $u_k$ .

**Proposition 2.5** *The solution  $u = u_k$  of problem (1.15) is radial and nonincreasing with respect to  $|x|$ .*

*Proof.* It is sufficient to prove the result with the approximation  $u_\epsilon(\cdot, t)$ . By (2.9),  $v_k(\cdot, t)$  is radial and decreasing, therefore  $u_\epsilon(\cdot, t)$  is radial too by uniqueness. We notice that  $u_\epsilon$  is the increasing limit, when  $R \rightarrow \infty$ , of the solution  $u_{\epsilon, R}$  of

$$\begin{cases} \partial_t u - \Delta_p u + f(u) = 0 & \text{in } Q_{\epsilon, \infty}^{B_R} \\ u = 0 & \text{in } \partial B_R \times (\epsilon, \infty) \\ u(\cdot, \epsilon) = v_k(\cdot, \epsilon) & \text{in } B_R. \end{cases} \quad (2.35)$$

For  $\lambda \in (0, R)$ , we set  $\Sigma_\lambda = B_R \cap \{x = (2\lambda - x_1, x') : x_1 > \lambda\} \cap B_R$  and define  $w_\lambda$  by

$$w_\lambda(x, t) = w_\lambda(x_1, x', t) := u_{\lambda, \epsilon, R}(x) - u_{\epsilon, R}(x) = u_{\epsilon, R}(2\lambda - x_1, x', t) - u_{\epsilon, R}(x_1, x', t).$$

If  $Q_{\epsilon, \infty}^{\Sigma_\lambda} = \Sigma_\lambda \times (\epsilon, \infty)$ , there holds

$$\begin{cases} \partial_t w_\lambda + \mathcal{A}w_\lambda + d(x)w_\lambda = 0 & \text{in } Q_{\epsilon, \infty}^{\Sigma_\lambda} \\ w_\lambda \geq 0 & \text{in } \partial \Sigma_\lambda \times (\epsilon, \infty) \\ w_\lambda(\cdot, \epsilon) \geq 0 & \text{in } \Sigma_\lambda. \end{cases} \quad (2.36)$$

where

$$d(x) = \begin{cases} \frac{f(u_{\lambda,\epsilon,R}) - f(u_{\epsilon,R})}{u_{\lambda,\epsilon,R} - u_{\epsilon,R}} & \text{if } u_{\lambda,\epsilon,R} \neq u_{\epsilon,R} \\ 0 & \text{if } u_{\lambda,\epsilon,R} = u_{\epsilon,R} \end{cases}$$

and

$$\mathcal{A}w_\lambda = -\Delta_p u_{\lambda,\epsilon,R} + \Delta_p u_{\epsilon,R}.$$

Notice that  $d \geq 0$  since  $f$  is nondecreasing and  $\mathcal{A}$  is elliptic [7, Lemma 1.3]. Furthermore the boundary data are continuous, therefore  $w_\lambda \geq 0$ . Letting  $\lambda \rightarrow 0$ , changing  $\lambda$  in  $-\lambda$  and replacing the  $x_1$  direction, by any direction going thru 0, we derive that  $u_{\epsilon,R}(\cdot, t)$  is radially decreasing. Letting  $R \rightarrow \infty$  yields to  $u_\epsilon(\cdot, t)$  is radially decreasing too.  $\square$

In the next result we characterize positive solutions of (1.1) with an isolated singularity at  $t = 0$

**Proposition 2.6** *Assume  $p > \frac{2N}{N+1}$  and  $f$  is continuous nondecreasing function vanishing only at 0 and satisfying (1.12). If  $u \in C(\overline{Q_\infty} \setminus \{(0,0)\})$  is a positive semigroup solution of (1.1) in  $Q_\infty$  such that  $u(x,0) = 0$ , for all  $x \neq 0$  and*

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} u(x,t) dx < \infty,$$

then there exists  $k \geq 0$  such that  $u = u_k$ .

*Proof.* Using [11, Lemma 2.2] when  $p \geq 2$ , or the proof of Theorem 1.1 when  $\frac{2N}{N+1} < p < 2$  jointly with the fact that

$$t \mapsto \int_{\mathbb{R}^N} u(x,t) dx$$

is decreasing, we derive that  $u \leq v_m$  for some  $m \geq 0$  and there exists  $k \geq 0$  such that

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} u(x,t) dx = k.$$

Since  $u(\cdot,0)$  vanishes if  $x \neq 0$ , it implies

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} u(x,t) \phi(x) dx = k\phi(0) \quad \forall \phi \in C_c(\mathbb{R}^N).$$

Therefore  $u$  satisfies (1.13). By uniqueness,  $u = u_k$ .  $\square$

## 2.4 Strong singularities

This section is devoted to study the limit of the sequence of the solutions  $u_k$  to (1.13) as  $k \rightarrow \infty$  with  $f(s) = s^\alpha \ln^\beta(s+1)$  where  $p > 2$ ,  $\alpha \in [1, p-1)$  and  $\beta > 0$ .

**Proof of Theorem 1.3.** By the comparison principle,

$$u_k(x, t) \leq v_k(x, t) \leq c_{12} k^{\frac{(p-1)\ell}{p-2}} t^{-\lambda},$$

where  $v_k$  is the solution of (1.14) in  $Q_\infty$  and  $c_{12} = c_{12}(N, p) > 0$  in (2.11). We set

$$\theta_k(t) = c_{12}^{\alpha-1} k^{\frac{\ell(\alpha-1)(p-1)}{p-2}} t^{-\lambda(\alpha-1)} \ln^\beta(c_{12} k^{\frac{(p-1)\ell}{p-2}} t^{-\lambda} + 1) \quad (2.37)$$

then

$$\partial_t u_k - \Delta_p u_k + u_k \theta_k(t) \geq 0. \quad (2.38)$$

Next we write  $u_k(x, t) = b_k(t) w_k(x, s_k(t))$  (the functions  $b_k$  and  $s_k$  will be defined later). For simplicity, we drop the subscript  $k$  in  $b_k$  and  $s_k$ . Inserting in (2.38), we get

$$b^{2-p}(t) s'(t) \partial_s w_k(x, s) - \Delta_p w_k(x, s) + b^{1-p}[b'(t) + b(t)\theta_k(t)] w_k(x, s) \geq 0. \quad (2.39)$$

We choose the functions  $b$  and  $s$  such that

$$b^{2-p}(t) s'(t) = 1 \quad \text{and} \quad b'(t) + b(t)\theta_k(t) = 0,$$

which implies

$$b(t) = \exp\left(-\int_0^t \theta_k(\tau) d\tau\right) \quad \text{and} \quad s(t) = \int_0^t \exp\left(-\int_0^\sigma \theta_k(\tau) d\tau\right) d\sigma. \quad (2.40)$$

Then  $\partial_s w_k - \Delta_p w_k \geq 0$  in  $\mathbb{R}^N \times (0, s_{k,0})$  with some  $s_{k,0} > 0$  and  $w_k(\cdot, 0) = k\delta_0$ . It follows by comparison principle that  $w_k \geq v_k$  in  $\mathbb{R}^N \times (0, s_{k,0})$ . Hence

$$u_k(x, t) \geq b(t) v_k(x, s) = \exp\left(-\int_0^t \theta_k(\tau) d\tau\right) s^{-\lambda} (c_{13} k^\ell - c_{14} s^{\frac{-p\lambda}{(p-1)N}} |x|^{\frac{p}{p-1}})_+^{\frac{p-1}{p-2}}. \quad (2.41)$$

Let  $\delta_1 > \frac{\ell(\alpha-1)(p-1)}{p-2}$  and  $0 < \delta_2 < 1 - \lambda(\alpha-1)$ . Using (2.37) there exists  $t_0 > 0$  depending on  $\delta_1$ ,  $\delta_2$  and  $k$  large enough, such that, for any  $t \in (0, t_0)$  there holds

$$\int_0^t \theta_k(\tau) d\tau \leq c_{15} k^{\delta_1} t^{\delta_2} \quad \forall t \in (0, t_0) \quad (2.42)$$

with  $c_{15} = c_{15}(c_i, \alpha, \beta, p, N) > 0$ . It follows from (2.40) and (2.42) that

$$t \exp\left[-(p-2)c_{15} k^{\delta_1} t^{\delta_2}\right] \leq s(t) \leq t. \quad (2.43)$$

Since  $J < \infty$  holds, there exists the solution  $\phi_\infty$  of (1.2). The sequence  $\{u_k\}$  is increasing and is bounded from above by  $\phi_\infty$ , then the function  $\underline{U}(x, t) := \lim_{k \rightarrow \infty} u_k(x, t)$  satisfies  $\underline{U}(x, t) \leq \phi_\infty(t)$  for every  $(x, t) \in Q_\infty$ . We restrict  $x \in B_1$  and we choose  $t$  such that

$$c_{13}k^\ell - c_{14}s(t)^{\frac{-p\lambda}{(p-1)N}} > \frac{1}{2}c_{13}k^\ell \iff k > \left(\frac{2c_{14}}{c_{13}}\right)^{\frac{1}{\ell}} s(t)^{-\frac{1}{p-2}}. \quad (2.44)$$

By (2.43), we only need to choose  $t$  such that

$$k \geq \left(\frac{2c_{14}}{c_{13}}\right)^{\frac{1}{\ell}} t^{\frac{-1}{p-2}} \exp\left(c_{15}k^{\delta_1}t^{\delta_2}\right). \quad (2.45)$$

We choose  $t$  under the form

$$t = k^{-\frac{1}{\gamma}} \text{ with } \gamma > 0, \quad (2.46)$$

then (2.45) becomes

$$t^{-\gamma} \geq \left(\frac{2c_{14}}{c_{13}}\right)^{\frac{1}{\ell}} t^{\frac{-1}{p-2}} \exp\left(c_{15}t^{\delta_2-\delta_1\gamma}\right). \quad (2.47)$$

In order to obtain (2.47), it is sufficient to choose  $\gamma$  such that

$$\frac{1}{p-2} < \gamma < \frac{\delta_2}{\delta_1}. \quad (2.48)$$

Indeed, since  $\alpha < p-1$ , we may choose  $\delta_1$  and  $\delta_2$  close enough  $\frac{\ell(\alpha-1)(p-1)}{p-2}$  and  $1 - \lambda(\alpha-1)$  respectively such that (2.48) holds true. When  $t$  has the form (2.46) where  $\gamma$  satisfies (2.48), from (2.41), (2.42)-(2.44) and the fact that  $\underline{U} \geq u_k$  in  $Q_\infty$ , we deduce that

$$\underline{U}(x, t) \geq c_{16}t^{-\lambda} \exp\left[c_{17} \ln(t^{-1}) - c_{15}t^{\delta_2-\delta_1\gamma}\right] \quad (2.49)$$

for every  $(x, t) \in B_1 \times (0, t_0)$  with  $t_0$  small enough and  $c_{16} = c_{16}(N, p)$ ,  $c_{17} = c_{17}(N, p, \gamma)$ . Since  $\gamma$  satisfies (2.48),

$$c_{17} \ln(t^{-1}) - c_{15}t^{\delta_2-\delta_1\gamma} > 0$$

for every  $t \in (0, t_0)$ . Therefore  $\lim_{t \rightarrow 0} \underline{U}(x, t) = \infty$  uniformly with respect to  $x \in B_1$ . We next proceed as in [19, Lemma 3.1] to deduce that  $\underline{U}(x, t)$  is independent of  $x$  and therefore it is a solution of (1.2). Since  $J < \infty$ ,  $\underline{U}(x, t) = \phi_\infty(t)$  for every  $(x, t) \in Q_\infty$ .  $\square$

Theorem 1.4 is proved by the same arguments as Theorem 1.3, using the fact that  $\underline{U}(x, t)$  is independent of  $x$ .

### 3 Non-Uniqueness

The next result shows that  $K = \infty$  is the necessary and sufficient condition so that a local solution of

$$(r^{N-1}|w'|^{p-2}w')' = r^{N-1}f(w) \quad (3.1)$$

can be continued as a global solution. More precisely,

**Lemma 3.1** *Every positive and increasing solution of (3.1) defined in an interval  $[a, a^*]$  to the right of  $a > 0$  can be continued as a solution of (3.1) on  $[a, +\infty)$  if and only if  $f$  satisfies*

$$\int_{\alpha}^{\infty} \frac{ds}{(F(s))^{1/p}} = \infty \quad (3.2)$$

for any  $\alpha > 0$ .

*Proof.* The proof is an extension to the case  $p \neq 2$  of the one of [18, Lemma 2.1] for the case  $p = 2$ .

*Step 1.* We first assume that  $w$  is defined on a maximal interval  $[a, a^*)$  with  $a^* < \infty$  and  $\lim_{r \rightarrow a^*} w(r) = +\infty$ . Since  $w$  is a nondecreasing function,  $w' \geq 0$ . And hence we may write (3.1) under the following form

$$\frac{N-1}{r}(w')^{p-1} + (p-1)(w')^{p-2}w'' = f(w),$$

which implies that

$$(p-1)(w')^{p-2}w'' \leq f(w) \quad (3.3)$$

and hence

$$\frac{p-1}{p}(w'^p)' \leq (F(w))'.$$

Taking the integral over  $[a, r]$ , we get

$$\frac{p-1}{p}[(w')^p(r) - (w')^p(a)] \leq F(w(r)) - F(w(a)) \leq F(w(r)).$$

Since  $f$  is positive on  $(0, \infty)$ ,  $F(s) \rightarrow \infty$  when  $s \rightarrow \infty$ , thus there exists  $\tilde{a} \in (a, a^*)$  such that

$$0 < w'(r)^p \leq \frac{p}{2(p-1)}F(w(r)) \implies \frac{w'(r)}{(F(w(r)))^{1/p}} \leq \left(\frac{p}{2(p-1)}\right)^{1/p} \quad \forall r \in [\tilde{a}, a^*].$$

Taking the integral over  $[\tilde{a}, r]$ , we obtain

$$\int_{w(\tilde{a})}^{w(r)} \frac{ds}{F(s)^{1/p}} \leq \left(\frac{p}{2(p-1)}\right)^{1/p} (r - \tilde{a}).$$

Letting  $r \rightarrow a^*$  yields to

$$\int_{w(\bar{a})}^{\infty} \frac{ds}{F(s)^{1/p}} \leq \left( \frac{p}{2(p-1)} \right)^{1/p} (a^* - \bar{a}) < \infty$$

and (3.2) is not satisfied.

*Step 2.* We assume that

$$\int_{\alpha}^{\infty} \frac{ds}{F(s)^{1/p}} < \infty$$

for some  $\alpha > 0$ , and we fix  $A > a$ . By [17, Theorem 1] there exists a function  $\gamma$  defined on  $(a, A)$  such that

$$w(r) < \gamma(r) \quad \forall r \in (a, A)$$

for any solution of (3.1) on  $(a, A)$ . Moreover,  $\gamma$  can be assumed convex, and  $\lim_{t \rightarrow a} \gamma(r) = \lim_{r \rightarrow A} \gamma(r) = +\infty$ . If  $w$  is a solution of (3.1) on  $(a, a + \epsilon)$  such that  $w(a) > \min_{a < r < A} \gamma(r)$  and  $\gamma'(a) > 0$ , it is clear that  $w(r^*) = \gamma(r^*)$  for some  $r^* < A$  and  $w(r) > \gamma(r)$  for  $r \in (r^*, r^* + \epsilon)$ , so  $w$  can not be defined on the whole  $(a, A)$ , and there exists  $a^* < A$  such that  $\lim_{r \rightarrow a^*} w(r) = \infty$ .  $\square$

**Proof of Theorem 1.5** By the Picard-Lipschitz fixed point theorem in the case  $1 < p < 2$  and [8, Th 5.2] in the case  $p \geq 2$ , there exists a unique solution  $w_a$  to (1.16) defined on a maximal interval  $[0, r_a)$  and  $w_a$  is an increasing function. Since Keller-Osserman estimate does not hold, by Lemma 3.1, the solution can be continued on the whole  $[0, +\infty)$  and global uniqueness follows from the local uniqueness. The function  $r \mapsto w_a(r)$  is increasing and

$$w_a(r) \geq a + \frac{p-1}{p} \left( \frac{f(a)}{N} \right)^{\frac{1}{p-1}} r^{\frac{p}{p-1}} \quad \text{and} \quad w'_a(r) \geq \left( \frac{f(a)}{N} \right)^{\frac{1}{p-1}} r^{\frac{1}{p-1}}$$

for any  $r > 0$ .  $\square$

**Proposition 3.2** *Assume  $p > 2N/(N+1)$ ,  $f$  is locally Lipschitz continuous and  $K = \infty$  hold. For any positive function  $u_0 \in C(Q_\infty)$  which satisfies*

$$w_a(|x|) \leq u_0(x) \leq w_b(|x|) \quad \forall x \in \mathbb{R}^N \quad (3.4)$$

*for some  $0 < a < b$ , there exists a positive function  $\bar{u} \in C(\overline{Q_\infty})$  solution of (1.1) in  $Q_\infty$  and satisfying  $\bar{u}(\cdot, 0) = u_0$  in  $\mathbb{R}^N$ . Furthermore*

$$w_a(|x|) \leq \bar{u}(x, t) \leq w_b(|x|) \quad \forall (x, t) \in Q_\infty. \quad (3.5)$$

*Proof.* Clearly  $w_a$  and  $w_b$  are ordered solutions of (1.1). We denote by  $u_n$  the solution to the initial-boundary problem

$$\begin{cases} \partial_t u_n - \Delta_p u_n + f(u_n) = 0 & \text{in } Q_n := B_n \times (0, \infty) \\ u_n(x, 0) = u_0(x) & \text{in } B_n \\ u_n(x, t) = (w_a(|x|) + w_b(|x|))/2 & \text{in } \partial B_n \times (0, \infty). \end{cases} \quad (3.6)$$

By the maximum principle,  $u_n$  satisfies (3.5) in  $Q_n$ . Using locally parabolic equation regularity [5, Th 1.1, chap III] if  $p \geq 2$  or [5, Th 1.1, chap IV] if  $1 < p < 2$ , we derive that the set of functions  $\{u_n\}$  is eventually equicontinuous on any compact subset of  $\overline{Q_\infty}$ . Using a diagonal sequence, combined with Proposition 4.4, we conclude that there exists a subsequence  $\{u_{n_k}\}$  which converges locally uniformly in  $\overline{Q_\infty}$  to some weak solution  $\bar{u} \in C(\overline{Q_\infty})$  which has the desired properties.  $\square$

**Proposition 3.3** *Assume  $p > 2N/(N+1)$ ,  $f$  is locally Lipschitz continuous and  $J = \infty$  and  $K = \infty$  hold. Then for any  $u_0 \in C(\mathbb{R}^N)$  which satisfies*

$$0 \leq u_0(x) \leq w_b(|x|) \quad \forall x \in \mathbb{R}^N \quad (3.7)$$

for some  $0 < b$ , there exists a positive solution  $\underline{u} \in C(\overline{Q_\infty})$  of (1.1) in  $Q_\infty$  satisfying  $\underline{u}(\cdot, 0) = u_0$  in  $\mathbb{R}^N$  and

$$\underline{u}(x, t) \leq \min\{w_b(|x|), \phi_\infty(t)\} \quad \forall (x, t) \in Q_\infty. \quad (3.8)$$

*Proof.* For any  $R > 0$ , let  $u_R$  be the solution of

$$\begin{cases} \partial_t u_R - \Delta_p u_R + f(u_R) = 0 & \text{in } Q_\infty \\ u_R(x, 0) = u_0(x) \chi_{B_R}(x) & \text{in } \mathbb{R}^N. \end{cases}$$

The functions  $\phi_\infty$  and  $w_b$  are solutions of (1.1) in  $Q_\infty$ , which dominate  $u_R$  at  $t = 0$ , therefore, by the maximum principle,

$$\min\{\phi_\infty(t), w_b(|x|)\} \geq u_R(x, t) \quad \forall (x, t) \in Q_\infty. \quad (3.9)$$

The mapping  $R \mapsto u_R$  is increasing, jointly with (3.9) it implies that there exists a solution  $\underline{u} := \lim_{R \rightarrow \infty} u_R$  of (1.1) in  $Q_\infty$  which satisfies  $\underline{u}(x, 0) = u_0(x)$  in  $\mathbb{R}^N$ . Letting  $R \rightarrow \infty$  in (3.9) yields to (3.8).  $\square$

**Proof of Theorem 1.6.** Combining Proposition 3.2 and Proposition 3.3 we see that there exist two solutions  $\underline{u}$  and  $\bar{u}$  with the same initial data  $u_0$ , which are ordered and different since  $\lim_{|x| \rightarrow \infty} \bar{u}(x, t) = \infty$  and  $\lim_{|x| \rightarrow \infty} \underline{u}(x, t) \leq \phi_\infty(t) < \infty$  for all  $t > 0$ .  $\square$

## 4 Estimate and stability

In this section we assume that  $\Omega$  is a domain in  $\mathbb{R}^N$ , possibly unbounded,  $0 < T \leq \infty$  and set  $Q_T^\Omega := \Omega \times (0, T)$  and  $Q_T := \mathbb{R}^N \times (0, T)$ . We denote by  $\mathfrak{M}(\Omega)$  the set of Radon measures in  $\Omega$  and by  $\mathfrak{M}_+(\Omega)$  its positive cone.



**Definition 4.1** A nonnegative function  $u$  is called a weak solution of (1.1) in  $Q_T^\Omega$  if  $u, |\nabla u|^p, f(u) \in L_{loc}^1(Q_T^\Omega)$  and

$$\int_0^T \int_\Omega \left( -G(u) \partial_t \varphi + |\nabla u|^{p-2} \nabla u \cdot \nabla (g(u) \varphi) + f(u) g(u) \varphi \right) dx dt = 0 \quad (4.1)$$

for any  $\varphi \in C_c^\infty(Q_T^\Omega)$  and any function  $g \in C(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$  where  $G'(r) = g(r)$ .

The next results are obtained by adapting the proofs in [2].

#### 4.1 Regularity Properties

The following integral estimates are essentially [2, Prop 2.1] with  $u^q$  replaced by  $f(u)$ .

**Proposition 4.2** Assume  $p > 1$ . Let  $\delta < 0$ ,  $\delta \neq -1$  and  $0 < t < \theta < T$ . Let  $u$  be a nonnegative weak solution of (1.1) in  $Q_T^\Omega$ . For any nonnegative function  $\zeta \in C_c^\infty(\Omega)$  and  $\tau > p$ ,

$$\begin{aligned} & \frac{1}{\delta+1} \int_\Omega (1+u(x,t))^{1+\delta} \zeta^\tau(x) dx + \frac{|\delta|}{2} \int_t^\theta \int_\Omega (1+u)^{\delta-1} \zeta^\tau |\nabla u|^p dx dt \\ & \leq \frac{1}{\delta+1} \int_\Omega (1+u(x,\theta))^{1+\delta} \zeta^\tau(x) dx + \int_t^\theta \int_\Omega (1+u)^\delta f(u) \zeta^\tau dx dt \\ & \quad + c_{18} \int_t^\theta \int_\Omega (1+u)^{\delta+p-1} \zeta^{\tau-p} |\nabla \zeta|^p dx dt. \end{aligned} \quad (4.2)$$

and

$$\begin{aligned} & \int_\Omega (1+u(x,t)) \zeta^\tau(x) dx \leq \int_\Omega (1+u(x,\theta)) \zeta^\tau(x) dx + \int_t^\theta \int_\Omega f(u) \zeta^\tau dx dt \\ & \quad + \tau \int_t^\theta \int_\Omega (1+u)^{\delta-1} \zeta^\tau |\nabla u|^p dx dt + \tau \int_t^\theta \int_\Omega (1+u)^{(1-\delta)(p-1)} \zeta^{\tau-p} |\nabla \zeta|^p dx dt. \end{aligned} \quad (4.3)$$

Conversely,

$$\begin{aligned} & \frac{1}{4} \int_\Omega u(x,\theta) \zeta^\tau(x) dx + \frac{1}{2} \int_t^\theta \int_\Omega f(u) \zeta^\tau dx dt \\ & \leq c_{19} + \int_\Omega u(x,t) \zeta^\tau(x) dx + \tau \int_t^\theta \int_\Omega \zeta^{\tau-1} |\nabla u|^{p-1} |\nabla \zeta| dx dt \end{aligned} \quad (4.4)$$

and

$$\begin{aligned} & \frac{1}{4} \int_\Omega (1+u(x,\theta)) \zeta^\tau(x) dx + \frac{1}{2} \int_t^\theta \int_\Omega f(u) \zeta^\tau dx dt \leq \int_\Omega (1+u(x,t)) \zeta^\tau(x) dx \\ & \quad + \tau \int_t^\theta \int_\Omega (1+u)^{\delta-1} \zeta^\tau |\nabla u|^p dx dt + \tau \int_t^\theta \int_\Omega (1+u)^{(1-\delta)(p-1)} \zeta^{\tau-p} |\nabla \zeta|^p dx dt + c_{20} \end{aligned} \quad (4.5)$$

where  $c_i = c_i(p, f, \delta, \tau)$  ( $i = 18, 19, 20$ ).

The next result is the keystone for the existence of an initial trace in the class of Radon measures. It is essentially [2, Prop 2.2] with  $u^q$  replaced by  $f(u)$ , but we shall sketch its proof for the sake of completeness.

**Proposition 4.3** *Let  $u$  be a nonnegative solution of (1.1) in  $Q_T^\Omega$ . Let  $0 < \theta < T$ . Assume that two of the three following conditions holds, for any open set  $U \subset\subset \Omega$ :*

$$\sup_{t \in (0, \theta]} \int_U u(x, t) dx < \infty, \quad (4.6)$$

$$\int_0^\theta \int_U f(u) dx dt < \infty, \quad (4.7)$$

$$\int_0^\theta \int_U |\nabla u|^{p-1} dx dt < \infty. \quad (4.8)$$

Then the third one holds for any  $U \subset\subset \Omega$ . Moreover,

$$\int_0^\theta \int_U u^\sigma dx dt < \infty \quad \forall \sigma \in (0, q_c) \quad (4.9)$$

and

$$\int_0^\theta \int_U |\nabla u|^r dx dt < \infty \quad \forall r \in (0, \frac{N}{N+1} q_c) \quad (4.10)$$

where  $q_c = p - 1 + p/N$ . Finally, there exists a Radon measure  $\mu \in \mathfrak{M}_+(\Omega)$  such that for any  $\zeta \in C_c(\Omega)$ ,

$$\lim_{t \rightarrow 0} \int_\Omega u(x, t) \zeta(x) dx = \int_\Omega \zeta(x) d\mu \quad (4.11)$$

and  $u$  satisfies

$$\begin{aligned} \int_0^\theta \int_\Omega (-u \partial_t \varphi + |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi + f(u) \varphi) dx dt \\ = \int_\Omega \varphi(x, 0) d\mu - \int_\Omega u(x, \theta) \varphi(x, \theta) dx \end{aligned} \quad (4.12)$$

for any  $0 < \theta < T$  and  $\varphi \in C_c^\infty(\Omega \times [0, T])$ .

*Proof. Step 1:* Assume (4.6) and (4.8) holds. Let  $\zeta$  and  $\tau$  as in Proposition 4.2, there holds

$$\begin{aligned} \int_\Omega (1 + u(x, t)) \zeta^\tau dx &= \int_\Omega (1 + u(x, \theta)) \zeta^\tau dx + \int_t^\theta \int_\Omega f(u) \zeta^\tau dx dt \\ &\quad + \tau \int_t^\theta \int_\Omega \zeta^{\tau-1} |\nabla u|^{p-2} \nabla u \cdot \nabla \zeta dx dt. \end{aligned} \quad (4.13)$$

It follows that  $f(u) \in L^1((0, \theta), L^1_{loc}(\Omega))$ .

*Step 2:* Assume that (4.7) and (4.8) hold. Then (4.6) follows from (4.13).

*Step 3:* Assume that (4.6) and (4.7) hold. Let  $\delta \in (\max(1 - p, -1), 0)$  be fixed. From (4.2), we get for any  $0 < t < \theta$ ,

$$\begin{aligned} \frac{|\delta|}{2} \int_t^\theta \int_\Omega (1+u)^{\delta-1} |\nabla u|^p \zeta^\tau dx dt &\leq \frac{1}{\delta+1} \int_\Omega (1+u(x, \theta))^{\delta+1} \zeta^\tau dx \\ &+ \int_t^\theta \int_\Omega (1+u)^\delta f(u) \zeta^\tau dx dt + c_{18} \int_t^\theta \int_\Omega (1+u)^{\delta+p-1} \zeta^{\tau-p} |\nabla \zeta|^p dx dt. \end{aligned} \quad (4.14)$$

If  $p \leq 2$ , then  $(1+u)^{\delta+p-1} \leq 1+u$ . Consequently, by (4.6),

$$\int_0^\theta \int_\Omega (1+u)^{\delta+p-1} \zeta^{\tau-p} |\nabla \zeta|^p dx dt < \int_0^\theta \int_\Omega (1+u) \zeta^{\tau-p} |\nabla \zeta|^p dx dt < \infty, \quad (4.15)$$

which, along with (4.7) and (4.14), implies that

$$\int_t^\theta \int_\Omega (1+u)^{\delta-1} |\nabla u|^p \zeta^\tau dx dt < c_{21}. \quad (4.16)$$

If  $p > 2$ , we choose  $\delta \in (1 - p, 2 - p)$ ,  $\delta \neq -1$ ,  $\zeta$  and  $\tau$  as in Proposition 4.2, then (4.2) remains valid. From the inequality  $(1+u)^{1+\delta} < 1+u$  and (4.6), we find that

$$\frac{1}{|\delta+1|} \int_\Omega (1+u(x, t))^{1+\delta} \zeta^\tau(x) dx < \frac{1}{|\delta+1|} \int_\Omega (1+u(x, t)) \zeta^\tau(x) dx < c_{22}.$$

Hence, by (4.2),

$$\begin{aligned} \frac{|\delta|}{2} \int_t^\theta \int_\Omega (1+u)^{\delta-1} \zeta^\tau |\nabla u|^p dx dt &\leq \frac{1}{\delta+1} \int_\Omega (1+u(x, \theta))^{\delta+1} \zeta^\tau dx \\ &+ \int_t^\theta \int_\Omega (1+u)^\delta f(u) \zeta^\tau dx dt + c_{18} \int_t^\theta \int_\Omega (1+u)^{\delta+p-1} \zeta^{\tau-p} |\nabla \zeta|^p dx dt + c_{22}. \end{aligned} \quad (4.17)$$

Since  $\delta < 2 - p$ ,  $\delta + p - 1 < 1$ , hence  $(1+u)^{\delta+p-1} \leq 1+u$ . Therefore, (4.16) follows from (4.6), (4.7) and (4.17).

By applying the Gagliardo-Nirenberg inequality as in [2, Prop 2.2 (iii)], we deduce that

$$\int_0^\theta \int_U (1+u(x, t))^\sigma dx < c_{23}$$

for any  $\sigma \in (0, q_c)$  with  $q_c = p - 1 + p/N$ , which leads to (4.9). Next for  $0 < r < p$ , and any  $\delta < 0$ , we find

$$\begin{aligned} \int_0^\theta \int_U |\nabla u|^r dx &\leq \left( \int_0^\theta \int_U (1+u)^{\delta-1} |\nabla u|^p dx dt \right)^{\frac{r}{p}} \\ &\times \left( \int_0^\theta \int_U (1+u)^{\frac{(1-\delta)r}{p-r}} dx dt \right)^{\frac{p-r}{p}}. \end{aligned} \quad (4.18)$$

Thus, if  $r \in (0, Nq_c/(N+1))$ , this proves (4.10); furthermore, since  $p-1 < Nq_c/(N+1)$ , we obtain (4.8).

*Step 4: End of the proof.* Now we use (4.1) with  $g = 1$ , for any  $\zeta \in C_c^\infty(\Omega)$  and any  $0 < t < \theta < T$ ,

$$\int_\Omega u(x, t)\zeta(x)dx = \int_\Omega u(x, \theta)\zeta(x)dx + \int_t^\theta \int_\Omega \left( |\nabla u|^{p-2} \nabla u \cdot \nabla \zeta + f(u)\zeta \right) dx dt. \quad (4.19)$$

Because the right-hand side of (4.19) has a finite limit when  $t \rightarrow 0$ , the same holds with  $t \mapsto \int_\Omega u(x, t)\zeta(x)dx$ . The mapping  $\zeta \mapsto \lim_{t \rightarrow 0} \int_\Omega u(x, t)\zeta(x)dx$  is a positive linear functional  $\ell_\Omega$  on the space  $C_c^\infty(\Omega)$ . By a partition of unity it can be extended in a unique way as a Radon measure  $\mu \in \mathfrak{M}_+(\Omega)$  and (4.11) holds.

Finally, let  $0 < t < \theta$  be fixed,  $g = 1$  and  $\varphi \in C_c^\infty(Q_T^\Omega)$ , thus

$$\begin{aligned} \int_t^\theta \int_\Omega (-u \partial_t \varphi + |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi + f(u)\varphi) dx d\tau \\ = \int_\Omega u(x, t)\varphi(x, 0)dx - \int_\Omega u(x, \theta)\varphi(x, \theta)dx. \end{aligned} \quad (4.20)$$

But

$$\left| \int_\Omega u(x, t)(\varphi(x, t) - \varphi(x, 0))dx \right| \leq c_{24}t \int_\Omega u(x, t)dx.$$

By (4.11), letting  $t \rightarrow 0$  yields

$$\int_\Omega u(x, t)\varphi(x, t)dx \rightarrow \int_\Omega \varphi(x, 0)d\mu.$$

Thus, letting  $t \rightarrow 0$  in (4.20) implies (4.12).  $\square$

Next we consider the the following problems

$$\begin{cases} \partial_t u - \Delta_p u + f(u) = 0 & \text{in } Q_T^\Omega, \\ u = 0 & \text{in } \partial\Omega \times (0, T) \\ u(\cdot, 0) = \mu & \text{in } \Omega. \end{cases} \quad (4.21)$$

where  $\mu \in \mathfrak{M}_+(\Omega)$ . The solutions are considered in the entropy sense (see [16] and [13]).

We recall that for  $q \geq 1$  and  $\Theta \subset \mathbb{R}^d$  open, the Marcinkiewicz space (or weak Lebesgue space)  $M^q(\Theta)$  is the set of all locally integrable functions  $u : \Theta \mapsto \mathbb{R}$  such there exists  $C \geq 0$  with the property that for any measurable set  $E \subset \Theta$ ,

$$\int_E |u| dy \leq C|E|^{1-\frac{1}{q}}. \quad (4.22)$$

The norm of  $u$  in  $M^q(\Theta)$  is the smallest constant such that (4.22) holds for any measurable set  $E$  (see [16], [13] for more details). Here  $dy$  denotes the Lebesgue measure in  $\mathbb{R}^d$ , although any positive Borel measure can be used.

We recall the following result of Segura de Leon and Toledo [16, Th 2] and Li [13, Th 1.1] dealing with entropy solutions with initial data in  $L^1$ . However such solutions coincide with the semi-group solutions because of uniqueness.

**Proposition 4.4** *Assume  $p > \frac{2N}{N+1}$ ,  $\Omega \subset \mathbb{R}^N$  is any open subset and,  $h \in L^1(Q_T^\Omega)$  and  $\mu \in L_+^1(\Omega)$ . Let  $v \in C([0, T; L^1(\Omega)])$  be the entropy solution to problem*

$$\begin{cases} \partial_t v - \Delta_p v = h & \text{in } Q_T^\Omega \\ v = 0 & \text{in } \partial\Omega \times (0, \infty) \\ v(\cdot, 0) = \mu & \text{in } \Omega. \end{cases} \quad (4.23)$$

Then  $v \in M^{p-1+\frac{p}{N}}(Q_T^\Omega)$ ,  $\nabla v \in M^{p-\frac{N}{N+1}}(Q_T^\Omega)$  and there holds

$$\|v\|_{M^{p-1+\frac{p}{N}}(Q_T^\Omega)} + \|\nabla v\|_{M^{p-\frac{N}{N+1}}(Q_T^\Omega)} \leq c_{25}, \quad (4.24)$$

for some  $c_{25} > 0$  depending on  $p, N$ ,  $\|\mu\|_{L^1(\Omega)}$  and  $\|h\|_{L^1(Q_T^\Omega)}$ .

## 4.2 Stability

Let  $\{\mu_n\} \subset L_+^1(\mathbb{R}^N)$  be a sequence converging to  $\mu$  in weak sense of measures, then  $\|\mu_n\|_{L^1(\mathbb{R}^N)} \leq c^*$ , where  $c^*$  depends only on  $N, p$  and  $\|\mu\|_{\mathfrak{M}(\mathbb{R}^N)}$ . Denote by  $u_{\mu_n}$  (resp.  $v_{\mu_n}$ ) the solution to problem (4.21) (resp. (4.23) with  $h \equiv 0$ ) with the initial data  $\mu_n$ . Then the following estimate holds

$$0 \leq u_{\mu_n} \leq v_{\mu_n}. \quad (4.25)$$

By [9, Theorem 3],

$$\|v_{\mu_n}(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq c_{26} t^{\frac{-N}{N(p-2)+p}} \|\mu_n\|_{L^1(\mathbb{R}^N)}^{\frac{p}{N(p-2)+p}} \quad \forall t > 0,$$

where  $c_{26} = c_{26}(N, p) > 0$ . Thus

$$\begin{aligned} \|u_{\mu_n}(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} &\leq c_{26} t^{\frac{-N}{N(p-2)+p}} \|\mu_n\|_{L^1(\mathbb{R}^N)}^{\frac{p}{N(p-2)+p}} \\ &\leq c_{27} t^{\frac{-N}{N(p-2)+p}} \end{aligned} \quad (4.26)$$

for every  $t > 0$ , where  $c_{27} = c_{27}(N, p, c^*) > 0$ .

It follows from (4.24) and (4.25) that

$$\|u_{\mu_n}\|_{M^{p-1+p/N}(Q_T)} \leq c_{25} \|\mu_n\|_{L^1(\mathbb{R}^N)}^{\frac{p+N}{1+p(N-1)}} \leq c_{28}(N, p, c^*). \quad (4.27)$$

By (4.26) and the regularity theory of degenerate parabolic equations [5], we derive that the sequence  $\{u_{\mu_n}\}$  is equicontinuous in any compact subset of  $Q_T$ . As a consequence, there exist a subsequence, still denoted by  $\{u_{\mu_n}\}$  and a function  $u$  such that  $\{u_{\mu_n}\}$  converges to  $u$  locally uniformly in  $Q_T$ .

**Lemma 4.5** *The sequence  $f(u_{\mu_n})$  converges strongly to  $f(u)$  in  $L^1(Q_T)$ . Furthermore,  $\{u_n\}$  converges strongly to  $u$  in  $L^q_{loc}(Q_T)$  for every  $1 \leq q < q_c$ .*

*Proof.* Since  $u_{\mu_n} \rightarrow u$  a.e in  $Q_T$ , by Vitali's theorem, it is sufficient to show that the sequence  $\{f(u_{\mu_n})\}$  is uniformly integrable. Let  $E$  be a Borel subset of  $Q_T$  and let  $R > 0$ . Then, since  $f$  is increasing,

$$\begin{aligned} \int \int_E f(u_{\mu_n}) dx dt &= \int \int_{E \cap \{u_{\mu_n} \leq R\}} f(u_{\mu_n}) dx dt + \int \int_{E \cap \{u_{\mu_n} > R\}} f(u_{\mu_n}) dx dt \\ &\leq f(R) \int \int_E dx dt + \int \int_{E \cap \{u_{\mu_n} > R\}} f(u_{\mu_n}) dx dt. \end{aligned}$$

For  $\lambda \geq 0$ , we set  $B_n(\lambda) = \{(x, t) \in Q_T : u_{\mu_n} > \lambda\}$  and  $a_n(\lambda) = \int \int_{B_n(\lambda)} dx dt$ . Then

$$\int \int_{E \cap \{u_{\mu_n} > R\}} f(u_{\mu_n}) dx dt \leq \int \int_{\{u_{\mu_n} \geq R\}} f(u_{\mu_n}) dx dt = - \int_R^\infty f(\lambda) da_n(\lambda) \quad (4.28)$$

and

$$- \int_R^\infty f(\lambda) da_n(\lambda) \leq f(R)a_n(R) + \int_R^\infty a_n(\lambda) df(\lambda).$$

It follows from (4.27) that

$$a_n(\lambda) \leq c_{25} \|\mu_n\|_{L^1(\mathbb{R}^N)}^{\frac{p+N}{1+p(N-1)}} \lambda^{-(p-1+\frac{p}{N})} \leq c_{29} \lambda^{-(p-1+\frac{p}{N})}.$$

Plugging these estimates into (4.28) yields

$$\begin{aligned}
\int \int_{E \cap \{u_{\mu_n} > R\}} f(u_{\mu_n}) dx dt &\leq f(R)a_n(R) + c_{29} \int_R^\infty \lambda^{-(p-1+\frac{p}{N})} df(\lambda) \\
&\leq f(R)a_n(R) - c_{29}f(R)R^{-(p-1-\frac{p}{N})} \\
&\quad + c_{29} \left(p-1 + \frac{p}{N}\right) \int_R^\infty f(\lambda)\lambda^{-(p+\frac{p}{N})} d\lambda \\
&\leq c_{29} \left(p-1 + \frac{p}{N}\right) \int_R^\infty f(\lambda)\lambda^{-(p+\frac{p}{N})} d\lambda.
\end{aligned} \tag{4.29}$$

Since

$$\int_1^\infty \lambda^{-(p+\frac{p}{N})} f(\lambda) d\lambda < \infty,$$

for given  $\epsilon > 0$ , we can choose  $R > 0$  large enough such that

$$c_{29} \left(p-1 + \frac{p}{N}\right) \int_R^\infty f(\lambda)\lambda^{-(p+\frac{p}{N})} d\lambda < \epsilon/2.$$

Set  $\delta = (1 + f(R))^{-1}\epsilon/2$ , then

$$|E| < \delta \implies 0 \leq \int \int_E f(u_{\mu_n}) dx dt < \epsilon,$$

which proves the uniform integrability of the sequence  $\{f(u_{\mu_n})\}$ . The last assertion follows from the fact that  $u_{\mu_n}$  is bounded in  $M^{q_c}(Q_T)$  (remember that  $q_c = p-1 + p/N$ ) and  $M^{q_c}(Q_T) \subset L_{loc}^q(\overline{Q_T})$  with continuous imbedding, for any  $q < q_c$ . The conclusion follows again by Vitali's theorem.  $\square$

**Lemma 4.6** *Assume  $p > \frac{2N}{N+1}$ , then for any  $U \subset \subset \mathbb{R}^N$ , the sequence  $\{\nabla u_{\mu_n}\}$  converges strongly to  $\nabla u$  in  $(L^s(Q_T))^N$  for every  $1 \leq s < s_c := p - \frac{N}{N+1}$ .*

*Proof.* We set  $h_n = -f(u_{\mu_n})$  and write the equation under the form

$$\begin{cases} \partial_t u_{\mu_n} - \Delta_p u_{\mu_n} = h_n & \text{in } Q_T \\ u_{\mu_n}(\cdot, 0) = \mu_n & \text{in } \mathbb{R}^N. \end{cases} \tag{4.30}$$

We already know from the  $L^1$ -contraction principle and Proposition 4.4 that

$$\|u_{\mu_n}(\cdot, t)\|_{L^1(\mathbb{R}^N)} \leq \|\mu_n\|_{L^1(\mathbb{R}^N)} \quad \forall t \in (0, T]$$

and  $u_{\mu_n} \rightarrow u$  in  $L_{loc}^q(\overline{Q_T})$  for every  $q \in [1, q_c)$  and  $|\nabla u_{\mu_n}|$  is bounded in  $L_{loc}^s(\overline{Q_T})$  for every  $1 \leq s < s_c$ . Thus  $|\nabla u_{\mu_n}|^{p-1}$  remains bounded in  $L_{loc}^\sigma(\overline{Q_T})$  for every  $1 \leq \sigma < \sigma_c := 1 + \frac{1}{(N+1)(p-1)}$ . Furthermore,

$$\{\nabla u_{\mu_n}\} \text{ is a Cauchy sequence in measure.} \tag{4.31}$$

and the proof is similar to the one of [2, Th 5.1-step2]. Up to the extraction of a subsequence,  $\{\nabla u_{\mu_n}\}$  converges a.e. to some  $D = (D_1, \dots, D_N)$  in  $Q_T$ . Consequently,  $\{|\nabla u_{\mu_n}|^{p-2} \nabla u_{\mu_n}\}$  converges a.e. to  $|D|^{p-2} D$  in  $Q_T$  and, by Vitali's theorem,

$$\begin{aligned} \nabla u_{\mu_n} &\rightarrow D && \text{strongly in } (L_{loc}^s(\overline{Q_T}))^N, \quad \forall s \in [1, s_c), \\ \{|\nabla u_{\mu_n}|^{p-2} \nabla u_{\mu_n}\} &\rightarrow |D|^{p-2} D && \text{strongly in } (L_{loc}^\sigma(\overline{Q_T}))^N, \quad \forall \sigma \in [1, \sigma_c). \end{aligned} \quad (4.32)$$

which implies  $\nabla u = D$  and the conclusion of the lemma follows.  $\square$

**Proof of Theorem 1.7.** *Step 1.* For any  $\zeta \in C_c^\infty(\mathbb{R}^N)$  and  $t > 0$ , we have

$$\int_{\mathbb{R}^N} u_{\mu_n}(x, t) \zeta(x) dx + \int_0^t \int_{\mathbb{R}^N} (|\nabla u_{\mu_n}|^{p-2} \nabla u_{\mu_n} \nabla \zeta + f(u_{\mu_n}) \zeta) dx dt = \int_{\mathbb{R}^N} \mu_n(x) \zeta(x) dx$$

By Lemma 4.5 and Lemma 4.6, up to the extraction of a subsequence, we can pass to the limit in each term and get

$$\int_{\mathbb{R}^N} u(x, t) \zeta(x) dx + \int_0^t \int_{\mathbb{R}^N} (|\nabla u|^{p-2} \nabla u \nabla \zeta + f(u) \zeta) dx dt = \int_{\mathbb{R}^N} \zeta d\mu.$$

Letting  $t \rightarrow 0$  yields

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} u(x, t) \zeta(x) dx = \int_{\mathbb{R}^N} \zeta(x). \quad (4.33)$$

For any  $\varphi \in C_c^\infty(\mathbb{R}^N \times [0, \infty))$  and  $\theta > 0$ , we have

$$\begin{aligned} &\int_0^\theta \int_{\mathbb{R}^N} (-u_{\mu_n} \partial_t \varphi + |\nabla u_{\mu_n}|^{p-2} \nabla u_{\mu_n} \cdot \nabla \varphi + f(u_{\mu_n}) \varphi) dx dt \\ &= \int_{\mathbb{R}^N} \varphi(0, x) \mu_n(x) dx - \int_{\mathbb{R}^N} u_{\mu_n}(x, \theta) \varphi(x, \theta) dx. \end{aligned} \quad (4.34)$$

By the previous convergence results, we can pass to the limit in (4.34) to obtain

$$\begin{aligned} &\int_0^\theta \int_{\mathbb{R}^N} (-u \partial_t \varphi + |\nabla u|^{p-2} \nabla u \nabla \varphi + f(u) \varphi) dx dt \\ &= \int_{\mathbb{R}^N} \varphi(\cdot, 0) d\mu - \int_{\mathbb{R}^N} u(\cdot, \theta) \varphi(\cdot, \theta) dx. \end{aligned} \quad (4.35)$$

*Step 2:  $u$  is a weak solution.* By (4.26)

$$\sup\{\|u_{\mu_n}(\cdot, t)\|_{L^\infty(\mathbb{R}^N)}, \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)}\} \leq c_{27} t^{-\frac{N}{N(p-2)+p}} \quad \forall t \in (0, T].$$



Let  $\zeta \in C_c^\infty(\mathbb{R}^N)$ . Since  $\{u_{\mu_n}(\cdot, \theta)\}$  converges locally uniformly to  $u(\cdot, \theta)$  in  $\mathbb{R}^N$ , for any  $\theta > 0$ , there holds

$$\begin{aligned}
& \frac{1}{2} \int_{\mathbb{R}^N} (u_{\mu_n} - u_{\mu_m})^2(\cdot, T) \zeta dx dt + \int_{\theta}^T \int_{\mathbb{R}^N} (f(u_{\mu_n}) - f(u_{\mu_m}))(u_{\mu_n} - u_{\mu_m}) \zeta dx dt \\
& \quad + \int_{\theta}^T \int_{\mathbb{R}^N} (|\nabla u_{\mu_n}|^{p-2} \nabla u_{\mu_n} - |\nabla u_{\mu_m}|^{p-2} \nabla u_{\mu_m}) \cdot \nabla (u_{\mu_n} - u_{\mu_m}) \zeta dx dt \\
& \leq \frac{1}{2} \int_{\mathbb{R}^N} (u_{\mu_n} - u_{\mu_m})^2(\cdot, \theta) \zeta dx dt \\
& \quad + \int_{\theta}^T \int_{\mathbb{R}^N} \left| |\nabla u_{\mu_n}|^{p-2} \nabla u_{\mu_n} - |\nabla u_{\mu_m}|^{p-2} \nabla u_{\mu_m} \right| |u_{\mu_n} - u_{\mu_m}| |\nabla \zeta| dx dt.
\end{aligned} \tag{4.36}$$

This implies directly

$$\nabla u_{\mu_n} \rightharpoonup \nabla u \text{ in } L_{loc}^p(Q_T), \tag{4.37}$$

by Lemma 4.6 when  $p \geq 2$ . When  $1 < p < 2$ , we derive by Fatou's lemma

$$\begin{aligned}
& \frac{1}{2} \int_{\mathbb{R}^N} (u_{\mu_n} - u)^2(\cdot, T) \zeta dx dt + \int_{\theta}^T \int_{\mathbb{R}^N} (f(u_{\mu_n}) - f(u))(u_{\mu_n} - u) \zeta dx dt \\
& \quad + \int_{\theta}^T \int_{\mathbb{R}^N} (|\nabla u_{\mu_n}|^{p-2} \nabla u_{\mu_n} - |\nabla u|^{p-2} \nabla u) \cdot \nabla (u_{\mu_n} - u) \zeta dx dt \\
& \leq \frac{1}{2} \int_{\mathbb{R}^N} (u_{\mu_n} - u)^2(\cdot, \theta) \zeta dx dt \\
& \quad + \int_{\theta}^T \int_{\mathbb{R}^N} \left| |\nabla u_{\mu_n}|^{p-2} \nabla u_{\mu_n} - |\nabla u|^{p-2} \nabla u \right| |u_{\mu_n} - u| |\nabla \zeta| dx dt.
\end{aligned} \tag{4.38}$$

Using again Lemma 4.6, it implies

$$\lim_{n \rightarrow \infty} \int_{\theta}^T \int_{\mathbb{R}^N} |\nabla u_{\mu_n}|^p \zeta dx dt = \int_{\theta}^T \int_{\mathbb{R}^N} |\nabla u|^p \zeta dx dt. \tag{4.39}$$

Since  $\nabla u_{\mu_n} \rightharpoonup \nabla u$  weakly in  $L_{loc}^p(Q_T)$ , it implies again that (4.37) holds true. At end, let  $\varphi \in C_c^\infty(Q_T)$  and consider  $0 < \theta < T$  and  $U \subset \subset \mathbb{R}^N$  such that  $\text{supp } \varphi \subset (\theta, T) \times U$ . Let  $g \in C(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$  where  $G'(r) = g(r)$ . Multiplying the equation in (4.21) (with initial data  $\mu = \mu_n$ ) by  $g(u_{\mu_n})\varphi$ , we obtain

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}^N} (-G(u_{\mu_n}) \partial_t \varphi + |\nabla u_{\mu_n}|^p g'(u_{\mu_n}) \varphi) dx dt \\
& \quad + g(u_{\mu_n}) |\nabla u_{\mu_n}|^{p-2} \nabla u_{\mu_n} \cdot \nabla \varphi + \int_0^T \int_{\mathbb{R}^N} g(u_{\mu_n}) f(u_{\mu_n}) dx dt = 0.
\end{aligned} \tag{4.40}$$

By Lemma 4.5 and (4.37), we can pass to the limit in each term. As a consequence,  $u$  is a weak solution.

*Step 3: Stability.* Assume that  $\{\mu_n\}$  is a sequence of functions in  $L^1_+(\mathbb{R}^N)$  with compact support, which converges to  $\mu \in \mathfrak{M}^b_+(\mathbb{R}^N)$  in the dual sense of  $C(\mathbb{R}^N)$ , then  $\|\mu_n\|_{L^1(\mathbb{R}^N)}$  is bounded independently of  $n$ . By the same argument as in step 1 and step 2, we can pass to the limit in each term of (4.40), hence the conclusion follows.  $\square$

**Lemma 4.7** *Assume  $p > 2$ . Let  $u \in C(Q_T)$  be a positive weak solution of (1.1) in  $Q_T$ . Assume that there exists  $r > 0$  such that*

$$\int_0^T \int_{B_r} |\nabla u|^{p-1} dx dt = \infty. \quad (4.41)$$

Then

$$\sup_{\tau \in (0, T)} \int_{B_{8r}} u(x, \tau) = \infty. \quad (4.42)$$

*Proof.* By contradiction we assume that (4.42) does not hold. Then there exist  $A_1 > 0$  such that

$$\sup_{\tau \in (0, T)} \int_{B_{8r}} u(x, \tau) = A_1. \quad (4.43)$$

*Step 1: We claim that*

$$u \in L^\infty(Q_T^{B_{2r}}).$$

Since  $u$  is a positive subsolution of the equation in (2.13), by [5, Theorem 4.2, Chapter V], there exists a constant  $c_{30} = c_{30}(N, p)$  such that for every  $x_0 \in \mathbb{R}^N$ ,  $0 < \theta \leq t_0 < T$  and  $\sigma \in (0, 1)$ , there holds

$$\sup_{K_{\sigma\rho} \times (t_0 - \sigma\theta, t_0)} u \leq \frac{c_{30}\theta^{\frac{1}{2}}}{\rho^{\frac{p}{2}}(1-\sigma)^{\frac{N(p+1)+p}{2}}} \left( \sup_{0 < \tau < t} |K_\rho|^{-1} \int_{K_\rho} u(x, \tau) dx \right)^{\frac{p}{2}}, \quad (4.44)$$

where  $K_\rho(x_0)$  is the cube centered at  $x_0$  and wedge  $2\rho$ , i.e.,

$$K_\rho(x_0) = \{x \in \mathbb{R}^N : \max_{1 \leq i \leq N} |x^i - x_0^i| < \rho\}.$$

We choose  $x_0 = 0$ ,  $t_0 = \theta = t$ ,  $\sigma = 1/2$  and  $\rho = 4r$ , then (4.44) becomes

$$\sup_{K_{2r} \times (\frac{t}{2}, t)} u \leq 2^{\frac{N(p+1)+p}{2}} c_{30} t^{\frac{1}{2}} (4r)^{\frac{-p}{2}} \left( \sup_{0 < \tau < t} |K_{4r}|^{-1} \int_{K_{4r}} u(x, \tau) dx \right)^{\frac{p}{2}}. \quad (4.45)$$

Since  $B_{2r} \subset K_{2r}$  and  $K_{4r} \subset B_{8r}$ , from (4.43) and (4.45), we obtain that

$$\sup_{B_{2r} \times (0, T)} u \leq 2^{\frac{N-p(2N+1)}{2}} c_{30} T^{\frac{1}{2}} r^{\frac{-p(N+1)}{2}} A_1^{\frac{p}{2}} =: A_2, \quad (4.46)$$

which implies the claim.

*Step 2:* Let  $\zeta \in C_c^\infty(\mathbb{R}^N)$  such that  $\zeta \geq 0$  in  $\mathbb{R}^N$ ,  $\zeta = 1$  in  $B_r$  and  $|\nabla\zeta| \leq 1/r$ . We show that

$$\begin{aligned} J_1(t) &:= \int_0^t \int_{B_{2r}} (u+1)^{\frac{-2}{p}} \zeta^p |\nabla u|^p dx d\tau < \infty, \\ J_2(t) &:= \int_0^t \int_{B_{2r}} (u+1)^{\frac{2(p-1)}{p}} dx d\tau < \infty. \end{aligned} \quad (4.47)$$

Multiplying (1.1) by  $(u+1)^{\frac{p-2}{p}} \zeta^p$  and then integrating on  $\mathbb{R}^N \times [\epsilon, t]$  with  $0 < \epsilon < t$ , we get

$$\begin{aligned} & \frac{p}{2(p-1)} \int_{B_{2r}} (u(x, t) + 1)^{\frac{2(p-1)}{p}} \zeta^p dx + \frac{p-2}{p} \int_\epsilon^t \int_{B_{2r}} (u+1)^{\frac{-2}{p}} \zeta^p |\nabla u|^p dx d\tau \\ & \quad + \int_\epsilon^t \int_{B_{2r}} (u+1)^{\frac{p-2}{p}} f(u) \zeta^p dx d\tau \\ &= \frac{p}{2(p-1)} \int_{B_{2r}} (u(x, \epsilon) + 1)^{\frac{2(p-1)}{p}} \zeta^p dx \\ & \quad - p \int_\epsilon^t \int_{B_{2r}} (u+1)^{\frac{p-2}{p}} \zeta^{p-1} |\nabla u|^{p-2} \nabla u \nabla \zeta dx d\tau, \end{aligned}$$

which implies that

$$\begin{aligned} & \frac{p-2}{p} \int_\epsilon^t \int_{B_{2r}} (u+1)^{\frac{-2}{p}} \zeta^p |\nabla u|^p dx d\tau \\ & \leq \frac{p}{2(p-1)} \int_{B_{2r}} (u(x, \epsilon) + 1)^{\frac{2(p-1)}{p}} \zeta^p dx \\ & \quad - p \int_\epsilon^t \int_{B_{2r}} (u+1)^{\frac{p-2}{p}} \zeta^{p-1} |\nabla u|^{p-2} \nabla u \nabla \zeta dx d\tau. \end{aligned} \quad (4.48)$$

By Young's inequality,

$$\begin{aligned} & p \int_\epsilon^t \int_{B_{2r}} (u+1)^{\frac{p-2}{p}} \zeta^{p-1} |\nabla u|^{p-1} |\nabla \zeta| dx d\tau \\ & \leq \frac{p-2}{2p} \int_\epsilon^t \int_{B_{2r}} (u+1)^{\frac{-2}{p}} \zeta^p |\nabla u|^p dx d\tau \\ & \quad + p \left( \frac{2p^2}{p-2} \right)^{p-1} \int_\epsilon^t \int_{B_{2r}} (u+1)^{\frac{p^2-2}{p}} |\nabla \zeta|^p dx d\tau. \end{aligned} \quad (4.49)$$

It follows from (4.48) and (4.49) that

$$\begin{aligned}
& \frac{p-2}{2p} \int_{\epsilon}^t \int_{B_{2r}} (u+1)^{\frac{-2}{p}} \zeta^p |\nabla u|^p dx d\tau \\
& \leq \frac{p}{2(p-1)} \int_{B_{2r}} (u(x, \epsilon) + 1)^{\frac{2(p-1)}{p}} \zeta^p dx \\
& \quad + p \left( \frac{2p^2}{p-2} \right)^{p-1} \int_0^t \int_{B_{2r}} (u+1)^{\frac{p^2-2}{p}} |\nabla \zeta|^p dx d\tau.
\end{aligned} \tag{4.50}$$

By (4.46),

$$\sup_{\epsilon \in (0, T)} \int_{B_{2r}} (u(x, \epsilon) + 1)^{\frac{2(p-1)}{p}} \zeta^p dx \leq c_{31}(N, p, r, \zeta, A_2)$$

and

$$\int_0^t \int_{B_{2r}} (u+1)^{\frac{p^2-2}{p}} |\nabla \zeta|^p dx d\tau \leq r^{-p} \int_0^t \int_{B_{2r}} (u+1)^{\frac{p^2-2}{p}} dx dt \leq c_{32}(N, p, r, T, A_2).$$

Combining the previous two estimates with (4.50) yields

$$J_1(t) \leq c_{33}(N, p, r, T, \zeta), \quad \forall t \in (0, T). \tag{4.51}$$

By (4.46), we also find that

$$J_2(t) \leq c_{34}(N, p, r, T, A_2). \tag{4.52}$$

*Step 3: End of proof.* By Hölder's inequality, we get

$$\int_0^t \int_{B_{2r}} |\nabla u|^{p-1} \zeta^{p-1} dx d\tau \leq (J_1(t))^{\frac{p-1}{p}} (J_2(t))^{\frac{1}{p}}.$$

By step 2, we deduce that

$$\int_0^T \int_{B_{2r}} |\nabla u|^{p-1} \zeta^{p-1} dx dt < c_{35}(N, p, r, T, \zeta), \tag{4.53}$$

which contradicts (4.41). □

## 5 Initial trace

### 5.1 The dichotomy theorem

The dichotomy result Theorem 1.8 is a consequence of Proposition 4.3 and Lemma 4.7.

**Proof of Theorem 1.8** By translation we may suppose that  $y = 0$ .

*Case 1: there exists an open neighborhood  $U$  of 0 such that (4.7) and (4.8) hold true.* Then the statement (ii) follows from Proposition 4.3.

*Case 2: for any open neighborhood  $U$  of 0, (4.7) or (4.8) does not hold.* We first suppose that (4.8) does not hold. We can choose  $r > 0$  such that  $B_{8r} \subset U$  and (4.41) holds. Then the statement (i) follows from Lemma 4.7. Suppose next that (4.8) holds but (4.7) does not hold, then Proposition 4.3 implies that (4.6) does not hold and the statement (i) follows.  $\square$

**Proposition 5.1** *Assume  $p > 2$  and  $f$  is nondecreasing and satisfies (1.12). Let  $u$  is a positive weak solution of (1.1) in  $Q_\infty$  with initial trace  $(\mathcal{S}, \mu)$ . Then for every  $y \in \mathcal{S}$ ,*

$$\underline{U}_y(x, t) := \underline{U}(x - y, t) \leq u(x, t) \quad (5.1)$$

in  $Q_\infty$ .

*Proof.* By translation we may suppose that  $y = 0$ . Since  $0 \in \mathcal{S}(u)$ , for any  $\eta > 0$  small enough

$$\lim_{t \rightarrow 0} \int_{B_\eta} u(x, t) dx = \infty.$$

For  $\epsilon > 0$ , denote  $M_{\epsilon, \eta} = \int_{B_\eta} u(x, \epsilon) dx$ . For any  $m > m_\eta = \inf_{\sigma > 0} M_{\sigma, \eta}$  there exists  $\epsilon = \epsilon(m, \eta)$  such that  $m = M_{\epsilon, \eta}$  and  $\lim_{\eta \rightarrow 0} \epsilon(m, \eta) = 0$ . Let  $\tilde{u}_\eta$  be the solution to the problem

$$\begin{cases} \partial_t \tilde{u}_\eta - \Delta_p \tilde{u}_\eta + f(\tilde{u}_\eta) = 0 & \text{in } Q_\infty \\ \tilde{u}_\eta(x, 0) = u(x, \epsilon) \chi_{B_\eta} & \text{in } \mathbb{R}^N \end{cases}$$

where  $\chi_{B_\eta}$  is the characteristic function of  $B_\eta$ . By the maximum principle  $\tilde{u}_\eta \leq u$  in  $\mathbb{R}^N \times (\epsilon, \infty)$ . By Theorem 1.7  $v_\eta$  converges to  $u_k$  when  $\eta$  goes to zero. Letting  $m$  go to infinity yields (5.1).  $\square$

**Proof of Theorem 1.2** The conclusion follows directly from Proposition 5.1.  $\square$

## 5.2 The Keller-Osserman condition does not hold

**Lemma 5.2** *Assume  $p > 2$ , (1.12) and  $J < \infty$  are satisfied and  $\lim_{k \rightarrow \infty} u_k = \phi_\infty$ . If  $u$  is a positive solution of (1.1) in  $Q_\infty$  which satisfies*

$$\limsup_{t \rightarrow 0} \int_G u(x, t) dx = \infty, \quad (5.2)$$

for some bounded open subset  $G \subset \mathbb{R}^N$ , then  $u(x, t) \geq \phi_\infty(t)$ .

*Proof.* By assumption, there exists a sequence  $\{t_n\}$  decreasing to 0 such that

$$\lim_{n \rightarrow \infty} \int_G u(x, t_n) dx = \infty. \quad (5.3)$$

If (5.2) holds, we can construct a decreasing sequence of open subsets  $G_k \subset G$  such that  $\overline{G_k} \subset G_{k-1}$ ,  $\text{diam}(G_k) = \epsilon_k \rightarrow 0$  when  $k \rightarrow \infty$ , and

$$\lim_{n \rightarrow \infty} \int_{G_k} u(x, t_n) dx = \infty \quad \forall k \in \mathbb{N}. \quad (5.4)$$

Furthermore there exists a unique  $a \in \bigcap_k G_k$ . We set

$$\int_{G_k} u(x, t_n) dx = M_{n,k}.$$

Since  $\lim_{n \rightarrow \infty} M_{n,k} = \infty$ , we claim that for any  $m > 0$  and any  $k$ , there exists  $n = n(k) \in \mathbb{N}$  such that

$$\int_{G_k} u(x, t_{n(k)}) dx \geq m. \quad (5.5)$$

By induction, we define  $n(1)$  as the smallest integer  $n$  such that  $M_{n,1} \geq m$ . This is always possible. Then we define  $n(2)$  as the smallest integer larger than  $n(1)$  such that  $M_{n,2} \geq m$ . By induction,  $n(k)$  is the smallest integer  $n$  larger than  $n(k-1)$  such that  $M_{n,k} \geq m$ . Next, for any  $k$ , there exists  $\ell = \ell(k)$  such that

$$\int_{G_k} \inf\{u(x, t_{n(k)}); \ell\} dx = m \quad (5.6)$$

and we set

$$\hat{U}_k(x) = \inf\{u(x, t_{n(k)}); \ell\} \chi_{G_k}(x).$$

Let  $\hat{u}_k = u$  be the unique bounded solution of

$$\begin{cases} \partial_t u - \Delta_p u + f(u) = 0 & \text{in } Q_\infty \\ u(\cdot, 0) = \hat{U}_k & \text{in } \mathbb{R}^N. \end{cases} \quad (5.7)$$

Since  $\hat{u}_k(x, 0) \leq u(x, t_{n(k)})$ , we derive

$$u(x, t + t_{n(k)}) \geq \hat{u}_k(x, t) \quad \forall (x, t) \in Q_\infty. \quad (5.8)$$

When  $k \rightarrow \infty$ ,  $\hat{U}_k \rightarrow m\delta_a$ , thus  $\hat{u}_k \rightarrow u_{m\delta_a}$  by Theorem 1.7. Therefore  $u \geq u_{m\delta_a}$ . Since  $m$  is arbitrary and  $u_{m\delta_a} \rightarrow \phi_\infty$  when  $m \rightarrow \infty$ , it follows that  $u \geq \phi_\infty$ .  $\square$

**Lemma 5.3** *Assume  $p > 2$ , (1.12) and  $J = \infty$  are satisfied, and  $\lim_{k \rightarrow \infty} u_k = \infty$ . There exists no positive solution  $u$  of (1.1) in  $Q_\infty$  which satisfies (5.2) for some bounded open subset  $G \subset \mathbb{R}^N$ .*

*Proof.* If we assume that such a  $u$  exists, we proceed as in the proof of the previous lemma. Since Theorem 1.7 holds, we derive that  $u \geq u_{m\delta_a}$  for any  $m$ . Since  $\lim_{m \rightarrow \infty} u_{m\delta_a}(x, t) = \infty$  for all  $(x, t) \in Q_\infty$ , we are led to a contradiction.  $\square$

Thanks to these results, we can characterize the initial trace of positive solutions of (1.1) when the Keller-Osserman condition does not hold.

**Proof of Theorem 1.4.** (i) If  $\mathcal{S}(u) \neq \emptyset$ , there exists  $y \in \mathcal{S}(u)$  and an open neighborhood  $G$  of  $y$  such that (5.2) holds. By Lemma 5.2,  $u \geq \phi_\infty$  and the initial trace of  $u$  is the Borel measure  $\nu_\infty$ . Otherwise,  $\mathcal{R}(u) = \mathbb{R}^N$  and  $Tr_{\mathbb{R}^N}(u) \in \mathfrak{M}_+(\mathbb{R}^N)$ .

(ii) Using the argument as in Theorem 1.9 and because of Lemma 5.3,  $\mathcal{S}(u) = \emptyset$ . Therefore  $\mathcal{R}(u) = \mathbb{R}^N$  and  $Tr_{\mathbb{R}^N}(u) \in \mathfrak{M}_+(\mathbb{R}^N)$ .  $\square$

**Corollary 5.4** *Assume  $p > 2$ . If  $f$  is convex and satisfies (1.12),  $J < \infty$  and  $K = \infty$ , there exist infinitely many different positive solutions  $u$  of (1.1) such that  $tr_{\mathbb{R}^N}(u) = \nu_\infty$ .*

*Proof.* Let  $b > 0$  be fixed. Since  $f$  is increasing,  $(x, t) \mapsto U(x, t) = w_b(x) + \phi_\infty(t)$  is a supersolution for (1.1). Let  $V(x, t) = \max\{w_b(x), \phi_\infty(t)\}$  then  $V$ ,  $f(V)$  and  $|\nabla V|^p$  are locally integrable in  $Q_T$ ; actually  $V$  is locally Lipschitz continuous. Let  $\epsilon > 0$  and  $\rho_\epsilon$  be a smooth approximation defined by

$$\rho_\epsilon(r) = \begin{cases} 0 & \text{if } r < 0 \\ \frac{r^2}{2\epsilon} & \text{if } 0 < r < \epsilon \\ r - \frac{\epsilon}{2} & \text{if } r > \epsilon \end{cases}$$

We set  $V_\epsilon(x, t) = \phi_\infty(t) + \rho_\epsilon[w_b(x) - \phi_\infty(t)]$ . Then

$$\begin{aligned} \partial_t V_\epsilon - \Delta_p V_\epsilon + f(V_\epsilon) &= \phi'_\infty(1 - \rho'_\epsilon[w_b - \phi_\infty]) - (\rho'_\epsilon[w_b - \phi_\infty])^{p-1} \Delta_p w_b \\ &\quad - (p-1)(\rho'_\epsilon[w_b - \phi_\infty])^{p-2} \rho''_\epsilon[w_b - \phi_\infty] |\nabla w_b|^p + f(V_\epsilon) \\ &\leq f(V_\epsilon) - (1 - \rho'_\epsilon[w_b - \phi_\infty]) f(\phi_\infty) - (\rho'_\epsilon[w_b - \phi_\infty])^{p-1} f(w_b) \end{aligned}$$

If  $\phi \in C_c^\infty(Q_T)$  is nonnegative, then

$$\int \int_{Q_T} (-V_\epsilon \partial_t \phi + |\nabla V_\epsilon|^{p-2} \nabla V_\epsilon \cdot \nabla \phi + f(V_\epsilon)) \, dx \, dt \leq o(1)$$

Letting  $\epsilon \rightarrow 0$  implies

$$\int \int_{Q_T} (-V \partial_t \phi + |\nabla V|^{p-2} \nabla V \cdot \nabla \phi + f(V)) \, dx \, dt \leq 0.$$

Thus  $V$  is a subsolution, smaller than  $U$ . Therefore there exists a solution  $u_b$  such that  $V \leq u \leq U$ . This implies that  $tr_{\mathbb{R}^N}(u_b) = \nu_\infty$ . If  $b' > b$  we construct  $u_{b'}$  with  $tr_{\mathbb{R}^N}(u_{b'}) = \nu_\infty$  and  $\lim_{t \rightarrow \infty} (u_{b'}(0, t) - u_b(0, t)) > 0$ .  $\square$

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