Stochastic flows related to Walsh Brownian motion

Hatem Hajri

STOCHASTIC FLOWS RELATED TO WALSH BROWNIAN MOTION

Hatem Hajri
Université de Paris sud 11
Hatem.Hajri@math.u-psud.fr

Abstract

We define an equation on a simple graph which is an extension of Tanaka equation and the skew Brownian motion equation. We then apply the theory of transition kernels developed by Le Jan and Raimond and show that all the solutions can be classified by probability measures.

1 Introduction and main results.

In [9], [10] Le Jan and Raimond have extended the classical theory of stochastic flows to include flows of probability kernels. Using the Wiener chaos decomposition, it was shown that non Lipschitzian stochastic differential equations have a unique Wiener measurable solution given by random kernels. Later, the theory was applied in [11] to the study of Tanaka equation:

$$\varphi_{s,t}(x) = x + \int_s^t sgn(\varphi_{s,u}(x))W(du), \quad s \le t, x \in \mathbb{R}, \tag{1.1}$$

where $sgn(x) = 1\!\!1_{\{x>0\}} - 1\!\!1_{\{x\le 0\}}, W_t = W_{0,t}1\!\!1_{\{t>0\}} - W_{t,0}1\!\!1_{\{t\le 0\}}$ and $(W_{s,t}, s \le t)$ is a real white noise (see Definition 1.10 [10]) on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. If K is a stochastic flow of kernels (see Section 2.1 below) and W is a real white noise, then by definition, (K, W) is a solution of Tanaka SDE if for all $s \le t, x \in \mathbb{R}$, $f \in C_b^2(\mathbb{R})$ (f is C^2 on $\mathbb{R}$ and f', f'' are bounded)

$$K_{s,t}f(x) = f(x) + \int_s^t K_{s,u}(f'sgn)(x)W(du) + \frac{1}{2}\int_s^t K_{s,u}f''(x)du \quad a.s. \tag{1.2}$$

When $K = \delta_\varphi$ is a Dirac flow (flow of mappings), (1.2) is then equivalent to (1.1). It has been proved [11] that each solution flow of (1.2) can be characterized by a probability measure on $[0, 1]$ which entirely determines its law. Define

$$\tau_{s,x} = \inf\{r \ge s : W_{s,r} = -|x|\}, s, x \in \mathbb{R}.$$

Then, the unique $\mathcal{F}^W$ adapted solution (Wiener flow) of (1.2) is given by

$$K^W_{s,t}(x) = \delta_{x+sgn(x)W_{s,t}}1\!\!1_{t\le\tau_{s,x}} + \frac{1}{2}(\delta_{W^+_{s,t}} + \delta_{W^+_{s,t}})1\!\!1_{t>\tau_{s,x}}, \; W^+_{s,t} := W_{s,t} - \inf_{u\in[s,t]} W_{s,u}.$$

Among solutions of (1.2), there is only one flow of mappings which has been already studied in [19].

We now fix $\alpha \in [0,1]$ and consider the following SDE having a less obvious extension to kernels:

$$X_t^{s,x} = x + W_{s,t} + (2\alpha - 1)\tilde{L}_{s,t}^x, \quad t \geq s, x \in \mathbb{R}, \tag{1.3}$$

where

$$\tilde{L}_{s,t}^x = \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_s^t 1\!\!1_{|X_u^{s,x}| \leq \varepsilon} du \quad \text{(The symmetric local time)}.$$

Equation (1.3) was introduced in [6]. For a fixed initial condition, it has a pathwise unique solution which is distributed as the skew Brownian motion (SBM) with parameter α ($SBM(\alpha)$). It was shown in [1] that when $\alpha \neq \frac{1}{2}$, flows associated to (1.3) are coalescing and a deeper study of (1.3) was provided later in [3] and [4]. Now, consider the following generalization of (1.1):

$$X_{s,t}(x) = x + \int_s^t sgn(X_{s,u}(x))W(du) + (2\alpha - 1)\tilde{L}_{s,t}^x(X), \quad s \leq t, x \in \mathbb{R}, \tag{1.4}$$

where

$$\tilde{L}_{s,t}^x(X) = \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_s^t 1\!\!1_{|X_{s,u}(x)| \leq \varepsilon} du.$$

Each solution of (1.4) is distributed as the $SBM(\alpha)$. By Tanaka formula for symmetric local time ([16] page 234),

$$|X_{s,t}(x)| = |x| + \int_s^t \widetilde{sgn}(X_{s,u}(x))dX_{s,u}(x) + \tilde{L}_{s,t}^x(X),$$

where $\widetilde{sgn}(x) = 1\!\!1_{\{x>0\}} - 1\!\!1_{\{x<0\}}$. By combining the last identity with (1.4), we have

$$|X_{s,t}(x)| = |x| + W_{s,t} + \tilde{L}_{s,t}^x(X). \tag{1.5}$$

The uniqueness of solutions of the Skorokhod equation ([16] page 239) entails that

$$|X_{s,t}(x)| = |x| + W_{s,t} - \min_{s \leq u \leq t} [(|x| + W_{s,u}) \wedge 0]. \tag{1.6}$$

Clearly (1.5) and (1.6) imply that $\sigma(|X_{s,u}(x)|; s \leq u \leq t) = \sigma(W_{s,u}; s \leq u \leq t)$ which is strictly smaller than $\sigma(X_{s,u}(x); s \leq u \leq t)$ and so $X_{s,\cdot}(x)$ cannot be a strong solution of (1.4). For these reasons, we call (1.4) Tanaka SDE related to $SBM(\alpha)$.

From now on, for any metric space E, $C(E)$ (respectively $C_b(E)$) will denote the space of all continuous (respectively bounded continuous) $\mathbb{R}$-valued functions on E. Let

- $C_b^2(\mathbb{R}^*) = \{f \in C(\mathbb{R}) : f$ is twice derivable on $\mathbb{R}^*, f', f'' \in C_b(\mathbb{R}^*), \ f'_{|]0,+\infty[}, f''_{|]0,+\infty[}$ (resp. $f'_{|]-\infty,0[}, f''_{|]-\infty,0[}$) have right (resp. left) limit in $0\}$.
- $D_\alpha = \{f \in C_b^2(\mathbb{R}^*) : \alpha f'(0+) = (1-\alpha)f'(0-)\}$.

For $f \in D_\alpha$, we set by convention $f'(0) = f'(0-), f''(0) = f''(0-)$. By Itô-Tanaka formula ([14] page 432) or Freidlin-Sheu formula (see Lemma 2.3 [5] or Theorem 2.6 in Section 2) and Proposition 2.8 below, both extensions to kernels of (1.3) and (1.4) may be defined by

$$K_{s,t}f(x) = f(x) + \int_s^t K_{s,u}(\varepsilon f')(x)W(du) + \frac{1}{2}\int_s^t K_{s,u}f''(x)du, \quad f \in D_\alpha, \tag{1.7}$$

where $\varepsilon(x) = 1$ (respectively $\varepsilon(x) = sgn(x)$) in the first (respectively second) case, but due to the pathwise uniqueness of (1.3), the unique solution of (1.7) when $\varepsilon(x) = 1$, is $K_{s,t}(x) = \delta_{X_t^{s,x}}$ (this can be justified by the weak domination relation, see (4.5)). Our aim now is to define an extention of (1.7) related to Walsh Brownian motion in general. The latter process was introduced in [18] and will be recalled in the coming section. We begin by defining our graph.

Definition 1.1. *(Graph G)*

Fix $N \in \mathbb{N}^*$ and $\alpha_1, ..., \alpha_N > 0$ such that $\sum_{i=1}^{N} \alpha_i = 1$.

In the sequel G will denote the graph below (Figure 1) consisting of N half lines $(D_i)_{1\le i\le N}$ emanating from 0. Let $\vec{e}_i$ be a vector of modulus 1 such that $D_i = \{h\vec{e}_i, h \geqslant 0\}$ and define for all function $f : G \longrightarrow \mathbb{R}$ and $i \in [1, N]$, the mappings:

$$\begin{array}{rcccc} f_i & : & \mathbb{R}_+ & \longrightarrow & \mathbb{R} \\ & & h & \longmapsto & f(h\vec{e}_i) \end{array}$$

Denote by $|.|$ the natural distance on G and equip G with its Borel σ-field $\mathcal{B}(G)$. We also use the notation $G^* = G \setminus \{0\}$. Now, define

- *$C_b^2(G^*) = \{f \in C(G) : \forall i \in [1, N], f_i$ is twice derivable on $\mathbb{R}_+^*, f_i', f_i'' \in C_b(\mathbb{R}_+^*)$ and both have finite limits at $0+\}$.*
- *$D(\alpha_1, ..., \alpha_N) = \{f \in C_b^2(G^*) : \sum_{i=1}^{N} \alpha_i f_i'(0+) = 0\}$.*

For all $x \in G$, we define $\vec{e}(x) = \vec{e}_i$ if $x \in D_i, x \neq 0$ (convention $\vec{e}(0) = \vec{e}_N$). For $f \in C_b^2(G^*)$, $x \neq 0$, let $f'(x)$ be the derivative of f at x relatively to $\vec{e}(x)$ $(= f_i'(|x|)$ if $x \in D_i)$ and $f''(x) = (f')'(x)$ $(= f_i''(|x|)$ if $x \in D_i)$. We use the conventions $f'(0) = f_N'(0+), f''(0) = f_N''(0+)$. Now, associate to each ray D_i a sign $\varepsilon_i \in \{-1, 1\}$ and then define

$$\varepsilon(x) = \begin{cases} \varepsilon_i & \text{if } x \in D_i, x \neq 0 \\ \varepsilon_N & \text{if } x = 0 \end{cases}$$

To simplify, we suppose that $\varepsilon_1 = ... = \varepsilon_p = 1, \quad \varepsilon_{p+1} = ... = \varepsilon_N = -1$ for some $p \le N$. Set

$$G^+ = \bigcup_{1\le i\le p} D_i, \quad G^- = \bigcup_{p+1\le i\le N} D_i. \text{ Then } G = G^+ \bigcup G^-.$$

We also put $\alpha^+ = 1 - \alpha^- := \sum_{i=1}^{p} \alpha_i$.

Definition 1.2. *(Equation (E)).*

On a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, let W be a real white noise and K be a stochastic flow of kernels (a precise definition will be given in Section 2). We say that (K, W) solves (E) if for all $s \le t, f \in D(\alpha_1, ..., \alpha_N), x \in G$,

$$K_{s,t}f(x) = f(x) + \int_s^t K_{s,u}(\varepsilon f')(x)W(du) + \frac{1}{2}\int_s^t K_{s,u}f''(x)du \quad a.s.$$

If $K = \delta_\varphi$ is a solution of (E), we simply say that (φ, W) solves (E).

Remarks 1.3. *(1) If (K, W) solves (E), then $\sigma(W) \subset \sigma(K)$ (see Corollary 4.1) below. So, one can simply say that K solves (E).*
(2) The case $N = 2, p = 2, \varepsilon_1 = \varepsilon_2 = 1$ corresponds to Tanaka SDE related to SBM and includes in particular the usual Tanaka SDE [11]. In fact, let $(K^{\mathbb{R}}, W)$ be a solution of

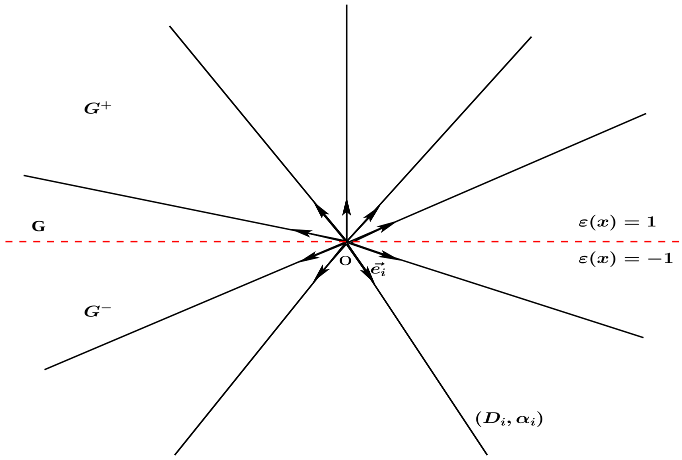

Figure 1: Graph G.

(1.7) with $\alpha = \alpha_1, \varepsilon(y) = sgn(y)$ and define $\psi(y) = |y|(\vec{e}_1 1\!\!1_{y \geq 0} + \vec{e}_2 1\!\!1_{y<0}), y \in \mathbb{R}$. For all $x \in G$, define $K^G_{s,t}(x) = \psi(K^{\mathbb{R}}_{s,t}(y))$ with $y = \psi^{-1}(x)$. Let $f \in D(\alpha_1, \alpha_2), x \in G$ and g be defined on $\mathbb{R}$ by $g(z) = f(\psi(z))$ $(g \in D_{\alpha_1})$. Since $K^{\mathbb{R}}$ satisfies(1.7) in $(g, \psi^{-1}(x))$ (g is the test function and $\psi^{-1}(x)$ is the starting point), it easily comes that K^G satisfies (E) in (f, x). Similarly, if K^G solves (E), then $K^{\mathbb{R}}$ solves (1.7).

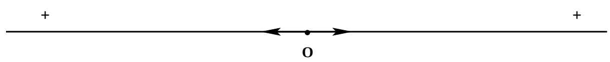

Figure 2: Tanaka SDE.

(3) As in (2), the case $N = 2, p = 1, \varepsilon_1 = 1, \varepsilon_2 = -1$ corresponds to (1.3).

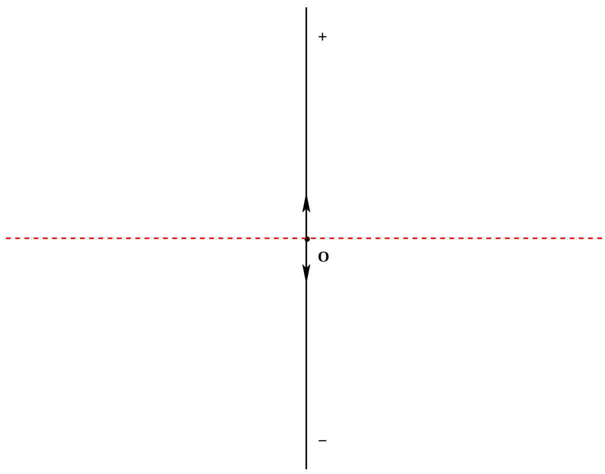

Figure 3: SBM equation.

In this paper, we classify all solutions of (E) by means of probability measures. We now state the first

Theorem 1.4. *Let W be a real white noise and $X_t^{s,x}$ be the flow associated to (1.3) with $\alpha=\alpha^+$. Define $Z_{s,t}(x)=X_t^{s,\varepsilon(x)|x|}, s\le t, x\in G$ and*

$$\begin{aligned} K^W_{s,t}(x) &= \delta_{\vec{e}(x)|Z_{s,t}(x)|}1\!\!1_{t\le\tau_{s,x}} \\ &+ \Big(\sum_{i=1}^{p}\frac{\alpha_i}{\alpha^+}\delta_{\vec{e}_i|Z_{s,t}(x)|}1\!\!1_{Z_{s,t}(x)>0}+\sum_{i=p+1}^{N}\frac{\alpha_i}{\alpha^-}\delta_{\vec{e}_i|Z_{s,t}(x)|}1\!\!1_{Z_{s,t}(x)\le 0}\Big)1\!\!1_{t>\tau_{s,x}}, \end{aligned}$$

where $\tau_{s,x}=\inf\{r\ge s: Z_{s,r}(x)=0\}$. Then, K^W is the unique Wiener solution of (E).

The proof of this theorem follows [9] (see also [15] for more details) with some modifications adapted to our case. We will use Freidlin-Sheu formula for Walsh Brownian motion to check that K^W solves (E). Unicity will be justified by means of the Wiener chaos decomposition (Proposition 4.2). Besides the Wiener flow, there are also other weak solutions associated to (E) which are fully described by the following

Theorem 1.5. *(1) Define*

$$\Delta_k=\{u=(u_1,...,u_k)\in[0,1]^k:\sum_{i=1}^{k}u_i=1\},\quad k\ge 1.$$

Suppose $\alpha^+\neq\frac{1}{2}$.

(a) Let m^+ and m^- be two probability measures respectively on Δ_p and Δ_{N-p} satisfying:

$$(+)\int_{\Delta_p}u_i m^+(du)=\frac{\alpha_i}{\alpha^+},\ (-)\int_{\Delta_{N-p}}u_j m^-(du)=\frac{\alpha_{j+p}}{\alpha^-}\ ,\ \forall 1\le i\le p, 1\le j\le N-p.$$

Then, to (m^+,m^-) is associated a stochastic flow of kernels K^{m^+,m^-} solution of (E).

- *To $(\delta_{(\frac{\alpha_1}{\alpha^+},...,\frac{\alpha_p}{\alpha^+})},\delta_{(\frac{\alpha_{p+1}}{\alpha^-},...,\frac{\alpha_N}{\alpha^-})})$ is associated a Wiener solution K^W.*
- *To $(\sum_{i=1}^{p}\frac{\alpha_i}{\alpha^+}\delta_{0,..,0,1,0,..,0},\sum_{i=p+1}^{N}\frac{\alpha_i}{\alpha^-}\delta_{0,..,0,1,0,..,0})$ is associated a coalescing stochastic flow of mappings φ.*

(b) For all stochastic flow of kernels K solution of (E) there exists a unique couple of measures (m^+,m^-) satisfying conditions $(+)$ and $(-)$ such that $K\overset{law}{=}K^{m^+,m^-}$.

(2) If $\alpha^+=\frac{1}{2}, N>2$, then there is just one solution of (E) which is a Wiener solution.

Remarks 1.6. *(1)If $\alpha^+=1$, solutions of (E) are characterized by a unique measure m^+ satisfying condition $(+)$ instead of a couple (m^+,m^-) and a similar remark applies if $\alpha^-=1$.*
(2) The case $\alpha^+=\frac{1}{2}, N=2$ does not appear in the last theorem since it corresponds to $dX_t=W(dt)$.

This paper follows ideas of [4] in a more general context and is organized as follows. In Section 2, we remind basic definitions of stochastic flows and Walsh Brownian motion. In Section 3, we use a "specific"$SBM(\alpha^+)$ flow introduced by Burdzy-Kaspi and excursion theory to construct all solutions of (E). Unicity of solutions is proved in Section 4. Section 5 is an appendix devoted to Freidlin-Sheu formula stated in [5] for a general class of diffusion processes defined by means of their generators. Here we first establish this formula using simple arguments and then deduce the characterization of Walsh Brownian motion by means of its generator (Proposition 2.8).

2 Stochastic flows and Walsh Brownian motion.

Let $\mathcal{P}(G)$ be the space of probability measures on G and $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions dense in $\{f \in C_0(G), ||f||_\infty \leq 1\}$ with $C_0(G)$ being the space of continuous functions on G which vanish at infinity. We equip $\mathcal{P}(G)$ with the distance $d(\mu,\nu) = (\sum_n 2^{-n}(\int f_n d\mu - \int f_n d\nu)^2)^{\frac{1}{2}}$ for all μ and ν in $\mathcal{P}(G)$. Thus, $\mathcal{P}(G)$ is a locally compact separable metric space. Let us recall that a kernel K on G is a measurable mapping from G into $\mathcal{P}(G)$. We denote by E the space of all kernels on G and we equip E with the σ-field $\mathcal{E}$ generated by the mappings $K \longmapsto \mu K, \mu \in \mathcal{P}(G)$, with μK the probability measure defined as $\mu K(A) = \int_G K(x,A)\mu(dx)$ for every $\mu \in \mathcal{P}(G)$. Let us recall some fundamental definitions from [10].

2.1 Stochastic flows.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space.

Definition 2.1. *(Stochastic flow of kernels.) A family of $(E,\mathcal{E})$-valued random variables $(K_{s,t})_{s\leq t}$ is called a stochastic flow of kernels if, $\forall s \leqslant t$ the mapping*

$$\begin{array}{rccc} K_{s,t} & : & (G\times\Omega, \mathcal{B}(G)\otimes\mathcal{A}) & \longrightarrow & (\mathcal{P}(G), \mathcal{B}(\mathcal{P}(G))) \\ & & (x,\omega) & \longmapsto & K_{s,t}(x,\omega) \end{array}$$

is measurable and if it satisfies the following properties:

1. *$\forall s < t < u, x \in G$ a.s. $\forall f \in C_0(G), K_{s,u}f(x) = K_{s,t}(K_{t,u}f)(x)$ (flow property).*
2. *$\forall s \leqslant t$ the law of $K_{s,t}$ only depends on $t-s$.*
3. *For all $t_1 < t_2 < ... < t_n$, the family $\{K_{t_i,t_{i+1}}, 1 \leq i \leq n-1\}$ is independent.*
4. *$\forall t \geq 0, x \in M, f \in C_0(G)$, $\lim_{y\to x} E[(K_{0,t}f(x) - K_{0,t}f(y))^2] = 0$.*
5. *$\forall t \geq 0, f \in C_0(G)$, $\lim_{x\to+\infty} E[(K_{0,t}f(x))^2] = 0$.*
6. *$\forall x \in G, f \in C_0(G)$, $\lim_{t\to 0+} E[(K_{0,t}f(x) - f(x))^2] = 0$.*

Definition 2.2. *(Stochastic flow of mappings.) A family $(\varphi_{s,t})_{s\leq t}$ is called a stochastic flow of mappings if $K = \delta_\varphi$ is a stochastic flow of kernels.*

Remark 2.3. *Let K be a stochastic flow of kernels on G and set $P^n_t = E[K^{\otimes n}_{0,t}], n \geq 1$. Then, $(P^n)_{n\geq 1}$ is a compatible family of Feller semigroups acting respectively on $C_0(G^n)$ (see Proposition 2.2 [10]).*

2.2 Walsh Brownian motion.

Recall that for all $f \in C_0(G)$, f_i is defined on $\mathbb{R}_+$. From now on, **we extend this definition on $\mathbb{R}$ by setting $f_i = 0$ on $]-\infty, 0[$.** We will introduce Walsh Brownian motion $W(\alpha_1, ..., \alpha_N)$, by giving its transition density as defined in [2]. On $C_0(G)$, consider

$$P_t f(h\vec{e}_j) = 2\sum_{i=1}^{N} \alpha_i p_t f_i(-h) + p_t f_j(h) - p_t f_j(-h), h > 0, \quad P_t f(0) = 2\sum_{i=1}^{N} \alpha_i p_t f_i(0).$$

where $(p_t)_{t>0}$ is the heat kernel of the standard one dimensional Brownian motion. Then $(P_t)_{t\geq 0}$ is a Feller semigroup on $C_0(G)$. A strong Markov process Z with state space G and semigroup P_t, and such that Z is càdlàg is by definition the Walsh Brownian motion $W(\alpha_1, ..., \alpha_N)$ on G.

2.2.1 Construction by flipping Brownian excursions.

For all $n \geq 0$, let $\mathbb{D}_n = \{\frac{k}{2^n},\ k \in \mathbb{N}\}$ and $\mathbb{D} = \cup_{n\in\mathbb{N}} \mathbb{D}_n$. For $0 \leq u < v$, define $n(u, v) = \inf\{n \in \mathbb{N} : \mathbb{D}_n \cap]u, v[\neq \emptyset\}$ and $f(u, v) = \inf \mathbb{D}_{n(u,v)} \cap]u, v[$.
Let B be a standard Brownian motion defined on $(\Omega, \mathcal{A}, \mathbb{P})$ and $(\vec{\gamma}_r, r \in \mathbb{D})$ be a sequence of independent random variables with the same law $\sum_{i=1}^{N} \alpha_i \delta_{\vec{e}_i}$ which is also independent of B. We define

$$B_t^+ = B_t - \min_{u\in[0,t]} B_u,\ g_t = \sup\{r \leq t : B_r^+ = 0\},\ d_t = \inf\{r \geq t : B_r^+ = 0\},$$

and finally $Z_t = \vec{\gamma}_r B_t^+, r = f(g_t, d_t)$ if $B_t^+ > 0$, $\quad Z_t = 0$ if $B_t^+ = 0$. Then we have the following

Proposition 2.4. *$(Z_t, t \geq 0)$ is an $W(\alpha_1, ..., \alpha_N)$ on G started at 0.*

Proof. We use these notations

$$min_{s,t} = \min_{u\in[s,t]} B_u,\ \vec{e}_{0,t} = \vec{e}(Z_t), \mathcal{F}_s = \sigma(\vec{e}_{0,u}, B_u; 0 \leq u \leq s).$$

Fix $0 \leq s < t$ and denote by $E_{s,t} = \{min_{0,s} = min_{0,t}\}(= \{g_t \leq s\}\ a.s)$. Let $h : G \longrightarrow \mathbb{R}$ be a bounded measurable function. Then

$$E[h(Z_t)|\mathcal{F}_s] = E[h(Z_t)1\!\!1_{E_{s,t}}|\mathcal{F}_s] + E[h(Z_t)1\!\!1_{E_{s,t}^c}|\mathcal{F}_s],$$

with

$$E[h(Z_t)1\!\!1_{E_{s,t}^c}|\mathcal{F}_s] = \sum_{i=1}^{N} E[h_i(B_t^+)1\!\!1_{\{g_t>s,\vec{e}_{0,t}=\vec{e}_i\}}|\mathcal{F}_s] = \sum_{i=1}^{N} \alpha_i E[h_i(B_t^+)1\!\!1_{\{g_t>s\}}|\mathcal{F}_s].$$

If $B_{s,r} = B_r - B_s$, then the density of $(\min_{r\in[s,t]} B_{s,r}, B_{s,t})$ with respect to the Lebesgue measure is given by

$$g(x, y) = \frac{2}{\sqrt{2\pi(t-s)^3}}(-2x + y)\exp(\frac{-(-2x + y)^2}{2(t - s)})1\!\!1_{\{y>x,x<0\}} \text{ ([13] page 28).}$$

Since $(B_{s,r}, r \geq s)$ is independent of $\mathcal{F}_s$, we get

$$\begin{aligned} E[h_i(B_t^+)1\!\!1_{\{g_t>s\}}|\mathcal{F}_s] &= E[h_i(B_{s,t} - \min_{r\in[s,t]} B_{s,r})1\!\!1_{\{\min_{r\in[0,s]} B_{s,r} > \min_{r\in[s,t]} B_{s,r}\}}|\mathcal{F}_s] \\ &= \int_{\mathbb{R}} 1\!\!1_{\{-B_s^+>x\}}(\int_{\mathbb{R}} h_i(y-x)g(x,y)dy)dx \\ &= 2\int_{\mathbb{R}_+} h_i(u)p_{t-s}(B_s^+, -u)du \quad (u = y-x) \end{aligned}$$

and so $E[h(Z_t)1\!\!1_{E_{s,t}^c}|\mathcal{F}_s] = 2\sum_{i=1}^{N}\alpha_i p_{t-s}h_i(-B_s^+)$. On the other hand

$$E[h(Z_t)1\!\!1_{E_{s,t}}|\mathcal{F}_s] = E[h(\vec{e}_{0,s}(B_t - min_{0,s}))1\!\!1_{E_{s,t}\cap(B_t>min_{0,s})}|\mathcal{F}_s]$$

$$= E[h(\vec{e}_{0,s}(B_t - min_{0,s}))1\!\!1_{\{B_t>min_{0,s}\}}|\mathcal{F}_s] - E[h(\vec{e}_{0,s}(B_t - min_{0,s}))1\!\!1_{E_{s,t}^c\cap(B_t>min_{0,s})}|\mathcal{F}_s].$$

Obviously on $\{\vec{e}_{0,s} = \vec{e}_k\}$, we have

$$\begin{aligned} E[h(\vec{e}_{0,s}(B_t - min_{0,s}))1\!\!1_{\{B_t>min_{0,s}\}}|\mathcal{F}_s] &= E[h_k(B_{s,t} + B_s^+)1\!\!1_{B_{s,t}+B_s^+>0}|\mathcal{F}_s] \\ &= p_{t-s}h_k(B_s^+) \end{aligned}$$

and

$$E[h(\vec{e}_{0,s}(B_t - min_{0,s}))1\!\!1_{E_{s,t}^c\cap(B_t>min_{0,s})}|\mathcal{F}_s] = E[h_k(B_{s,t}+B_s^+)1\!\!1_{-B_s^+>\min_{r\in[s,t]} B_{s,r}, B_{s,t}+B_s^+>0}|\mathcal{F}_s]$$

$$= \int_{\mathbb{R}} h_k(y+B_s^+)1\!\!1_{y+B_s^+>0}\left(\int_{\mathbb{R}} 1\!\!1_{-B_s^+>x}g(x,y)dx\right)dy = p_{t-s}h_k(-B_s^+).$$

As a result, $E[h(Z_t)|\mathcal{F}_s] = P_{t-s}h(Z_s)$ where P is the semigroup of $W(\alpha_1, ..., \alpha_N)$. □

Proposition 2.5. *Let $M = (M_n)_{n\geq 0}$ be a Markov chain on G started at 0 with stochastic matrix Q given by*

$$Q(0, \vec{e}_i) = \alpha_i, \quad Q(n\vec{e}_i, (n+1)\vec{e}_i) = Q(n\vec{e}_i, (n-1)\vec{e}_i) = \frac{1}{2} \quad \forall i \in [1, N], \quad n \in \mathbb{N}^*. \tag{2.1}$$

Then, for all $0 \leq t_1 < ... < t_p$, we have

$$(\frac{1}{2^n}M_{\lfloor 2^{2n}t_1\rfloor}, ..., \frac{1}{2^n}M_{\lfloor 2^{2n}t_p\rfloor}) \xrightarrow[n\to+\infty]{law} (Z_{t_1}, ..., Z_{t_p}),$$

where Z is an $W(\alpha_1, ..., \alpha_N)$ started at 0.

Proof. Let B be a standard Brownian motion and define for all $n \geq 1$: $T_0^n(B) = T_0^n(|B|) = 0$ and for $k \geq 0$

$$\begin{aligned} T_{k+1}^n(B) &= \inf\{r \geq T_k^n(B), |B_r - B_{T_k^n}| = \frac{1}{2^n}\}, \\ T_{k+1}^n(|B|) &= \inf\{r \geq T_k^n(|B|), ||B_r| - |B_{T_k^n}|| = \frac{1}{2^n}\}. \end{aligned}$$

Then, clearly $T_k^n(B) = T_k^n(|B|)$ and so $(T_k^n(|B|))_{k\geq 0} \overset{law}{=} (T_k^n(B))_{k\geq 0}$. It is known ([12] page 31) that $\lim_{n\to+\infty} T_{\lfloor 2^{2n}t\rfloor}^n(B) = t$ a.s uniformly on compact sets. Then, the result holds also for $T_{\lfloor 2^{2n}t\rfloor}^n(|B|)$. Now, let Z be the $W(\alpha_1, ..., \alpha_N)$ started at 0 constructed in the beginning of this section from the reflected Brownian motion B^+. Let $T_k^n = T_k^n(B^+)$ (defined analogously to $T_k^n(|B|)$) and $Z_k^n = 2^n Z_{T_k^n}$. Then obviously $(Z_k^n, k \geq 0) \overset{law}{=} M$ for all $n \geq 0$. Since a.s $t \longrightarrow Z_t$ is continuous, it comes that a.s $\forall t \geq 0, \lim_{n\to+\infty} \frac{1}{2^n} Z_{\lfloor 2^{2n}t\rfloor}^n = Z_t$. □

2.2.2 Freidlin-Sheu formula.

Theorem 2.6. *[5] Let $(Z_t)_{t\geq 0}$ be a $W(\alpha_1, ..., \alpha_N)$ on G started at z and let $X_t = |Z_t|$. Then*
(i) $(X_t)_{t\geq 0}$ is a reflecting Brownian motion started at $|z|$.
(ii) $B_t = X_t - \tilde{L}_t(X) - |z|$ is a standard Brownian motion where

$$\tilde{L}_t(X) = \lim_{\varepsilon \to 0+} \frac{1}{2\varepsilon} \int_0^t 1\!\!1_{|X_u| \leq \varepsilon} du.$$

(iii) $\forall f \in C_b^2(G^*)$,

$$f(Z_t) = f(z) + \int_0^t f'(Z_s) dB_s + \frac{1}{2} \int_0^t f''(Z_s) ds + (\sum_{i=1}^N \alpha_i f_i'(0+)) \tilde{L}_t(X). \tag{2.2}$$

Remarks 2.7. *(1) By taking $f(z) = |z|$ and applying Skorokhod lemma, we find the following analogous of (1.6)*

$$|Z_t| = |z| + B_t - \min_{s \leq u \leq t} [(|z| + B_u) \wedge 0].$$

From this observation, when $\varepsilon_i = 1$ for all $i \in [1, N]$, we call (E), Tanaka SDE related to $W(\alpha_1, ..., \alpha_N)$.
(2) For $N \geq 3$, the filtration $(\mathcal{F}_t^Z)$ has the martingale representation property with respect to B [2], but there is no Brownian motion W such that $\mathcal{F}_t^Z = \mathcal{F}_t^W$ [17].

Using this theorem, we obtain the following characterization of $W(\alpha_1, ..., \alpha_N)$ by means of its semigroup.

Proposition 2.8. *Let*

- $D(\alpha_1, ..., \alpha_N) = \{f \in C_b^2(G^*) : \sum_{i=1}^N \alpha_i f_i'(0+) = 0\}$.
- $Q = (Q_t)_{t\geq 0}$ *be a Feller semigroup satisfying*

$$Q_t f(x) = f(x) + \frac{1}{2} \int_0^t Q_u f''(x) du \quad \forall f \in D(\alpha_1, ..., \alpha_N).$$

Then, Q is the semigroup of $W(\alpha_1, ..., \alpha_N)$.

Proof. Denote by P the semigroup of $W(\alpha_1, ..., \alpha_N)$, A' and $D(A')$ being respectively its generator and its domain on $C_0(G)$. If

$$D'(\alpha_1, ..., \alpha_N) = \{f \in C_0(G) \bigcap D(\alpha_1, ..., \alpha_N), f'' \in C_0(G)\}, \tag{2.3}$$

then it is enough to prove these statements:
(i) $\forall t > 0, \quad P_t(C_0(G)) \subset D'(\alpha_1, ..., \alpha_N)$.
(ii) $D'(\alpha_1, ..., \alpha_N) \subset D(A')$ and $A' f(x) = \frac{1}{2} f''(x)$ on $D'(\alpha_1, ..., \alpha_N)$.
(iii) $D'(\alpha_1, ..., \alpha_N)$ is dense in $C_0(G)$ for $||.||_\infty$.
(iv) If R and R' are respectively the resolvents of Q and P, then

$$R_\lambda = R_\lambda' \quad \forall \ \lambda > 0 \ \text{ on } \ D'(\alpha_1, ..., \alpha_N).$$

The proof of (i) is based on elementary calculations using dominated convergence, (ii) comes from (2.2), (iii) is a consequence of (i) and the Feller property of P (approximate f by $P_{\frac{1}{n}} f$). To prove (iv), let A be the generator of Q and fix $f \in D'(\alpha_1, ..., \alpha_N)$. Then, $R_\lambda f$ is the unique element of $D(A)$ such that $(\lambda I - A)(R_\lambda f) = f$. We have $R'_\lambda f \in D'(\alpha_1, ..., \alpha_N)$ by (i), $D'(\alpha_1, ..., \alpha_N) \subset D(A)$ by hypothesis. Hence $R'_\lambda f \in D(A)$ and since $A = A'$ on $D'(\alpha_1, ..., \alpha_N)$, we deduce that $R_\lambda f = R'_\lambda f$. $\square$

3 Construction of flows associated to (E).

In this section, we prove (a) of Theorem 1.5 and we show that K^W given in Theorem 1.4 solves (E).

3.1 Flow of Burdzy-Kaspi associated to SBM.

3.1.1 Definition.

We are looking for flows associated to the SDE (1.3). The flow associated to $SBM(1)$ which solves (1.3) is the reflected Brownian motion above 0 given by

$$Y_{s,t}(x) = (x + W_{s,t})1\!\!1_{t \le \tau_{s,x}} + (W_{s,t} - \inf_{u \in [\tau_{s,x}, t]} W_{s,u})1\!\!1_{\{t > \tau_{s,x}\}},$$

where

$$\tau_{s,x} = \inf\{r \ge s : x + W_{s,r} = 0\}. \tag{3.1}$$

and a similar expression holds for the $SBM(0)$ which is the reflected Brownian motion below 0.

These flows satisfy all properties of the $SBM(\alpha), \alpha \in]0,1[$ we will mention below such that the "strong"flow property (Proposition 3.1) and the strong comparison principle (3.2). When $\alpha \in]0,1[$, we follow Burdzy-Kaspi [4]. In the sequel, we will be interested in $SBM(\alpha^+)$ and so we suppose in this paragraph that $\alpha^+ \notin \{0,1\}$.

With probability 1, for all rationals s and x simultaneously, equation (1.3) has a unique strong solution with $\alpha = \alpha^+$. Define

$$Y_{s,t}(x) = \inf_{\substack{u,y \in \mathbb{Q} \\ u < s, x < X_s^{u,y}}} X_t^{u,y} \ , \quad L_{s,t}(x) = \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_s^t 1\!\!1_{|Y_{s,u}(x)| \le \varepsilon} du.$$

Then, it is easy to see that a.s

$$Y_{s,t}(x) \le Y_{s,t}(y) \ \ \forall s \le t, x \le y. \tag{3.2}$$

This implies that $x \longmapsto Y_{s,t}(x)$ is increasing and càdlàg for all $s \le t$ a.s.

According to [4] (Proposition 1.1), $t \longmapsto Y_{s,t}(x)$ is Hölder continuous for all s, x a.s and with probability equal to 1: $\forall s, x \in \mathbb{R}, \ \ Y_{s,\cdot}(x)$ satisfies (1.3). We first check that Y is a flow of mappings and we start by the following flow property:

Proposition 3.1. *$\forall \ t \ge s$ a.s.*

$$Y_{s,u}(x) = Y_{t,u}(Y_{s,t}(x)) \ \ \forall u \ge t, x \in \mathbb{R} \ .$$

Proof. It is known, since pathwise uniqueness holds for the SDE (1.3), that for a fixed $s \le t \le u, x \in \mathbb{R}$, we have $Y_{s,u}(x) = Y_{t,u}(Y_{s,t}(x)) \ \ a.s$ ([8] page 161). Now, using the regularity of the flow, the result extends clearly as desired. $\square$

To conclude that Y is a stochastic flow of mappings, it remains to show the following

Lemma 3.1. $\forall t \geq s,\ x \in \mathbb{R},\ f \in C_0(\mathbb{R})$

$$\lim_{y \to x} E[(f(Y_{s,t}(x)) - f(Y_{s,t}(y)))^2] = 0.$$

Proof. We take $s = 0$. For $g \in C_0(\mathbb{R}^2)$, set

$$P_t^{(2)} g(x) = E[g(Y_{0,t}(x_1), Y_{0,t}(x_2))], \quad x = (x_1, x_2).$$

If $\varepsilon > 0$, $\ f_\varepsilon(x, y) = 1\!\!1_{\{|x-y| \geq \varepsilon\}}$, then by Theorem 10 in [14], $P_t^{(2)} f_\varepsilon(x, y) \xrightarrow[y \to x]{} 0$. For all $f \in C_0(\mathbb{R})$, we have

$$E[(f(Y_{0,t}(x)) - f(Y_{0,t}(y)))^2] = P_t^{(2)} f^{\otimes^2}(x, x) + P_t^{(2)} f^{\otimes^2}(y, y) - 2P_t^{(2)} f^{\otimes^2}(x, y).$$

To conclude the lemma, we need only to check that

$$\lim_{y \to x} P_t^{(2)} f(y) = P_t^{(2)} f(x),\ \forall x \in \mathbb{R}^2, \quad f \in C_0(\mathbb{R}^2).$$

Let $f = f_1 \otimes f_2$ with $f_i \in C_0(\mathbb{R})$, $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Then

$$|P_t^{(2)} f(y) - P_t^{(2)} f(x)| \leq M \sum_{k=1}^{2} P_t^{(2)} (|1 \otimes f_k - f_k \otimes 1|)(y_k, x_k),$$

where $M > 0$ is a constant. For all $\alpha > 0$, $\exists \varepsilon > 0$, $|u - v| < \varepsilon \Rightarrow \forall 1 \leq k \leq 2 :$ $|f_k(u) - f_k(v)| < \alpha$. As a result

$$|P_t^{(2)} f(y) - P_t^{(2)} f(x)| \leq 2M\alpha + 2M \sum_{k=1}^{2} ||f_k||_\infty P_t^{(2)} f_\varepsilon(x_k, y_k),$$

and we arrive at $\limsup_{y \to x} |P_t^{(2)} f(y) - P_t^{(2)} f(x)| \leq 2M\alpha$ for all $\alpha > 0$ which means that $\lim_{y \to x} P_t^{(2)} f(y) = P_t^{(2)} f(x)$. Now this easily extends by a density argument for all $f \in C_0(\mathbb{R}^2)$. □

In the coming section, we present some properties related to the coalescence of Y we will require in Section 3.2 to construct solutions of (E).

3.1.2 Coalescence of the Burdzy-Kaspi flow.

In this paragraph, we suppose $\frac{1}{2} < \alpha^+ < 1$. The analysis of the case $0 < \alpha^+ < \frac{1}{2}$ requires an application of symmetry. Define

$$T_{x,y} = \inf\{r \geq 0,\ Y_{0,r}(x) = Y_{0,r}(y)\}, \quad x, y \in \mathbb{R}.$$

By the fundamental result of [1], $T_{x,y} < \infty$ $\ a.s$ for all $x, y \in \mathbb{R}$. Due to the local time, coalescence always occurs in 0; $Y_{0,r}(x) = Y_{0,r}(y) = 0$ if $r = T_{x,y}$. Recall the definition of $\tau_{s,x}$ from (3.1). Then $T_{x,y} > \sup(\tau_{0,x}, \tau_{0,y})$ a.s ([1] page 203). Set

$$L_t^x = x + (2\alpha^+ - 1)L_{0,t}(x), \quad U(x, y) = \inf\{z \geq y : L_t^x = L_t^y = z \text{ for some } \ t \geq 0\}, y \geq x.$$

According to [3] (Theorem 1.1), there exists $\lambda > 0$ such that

$$\forall u \geq y > 0, \quad \mathbb{P}(U(0,y) \leq u) = (1 - \frac{y}{u})^{\lambda}.$$

Thus for a fixed $0 < \gamma < 1$, we get $\lim_{y\to 0+} \mathbb{P}(U(0,y) \leq y^{\gamma}) = \lim_{y\to 0+}(1 - y^{1-\gamma})^{\lambda} = 1$. From Theorem 1.1 [3], we have $U(x,y) - x \overset{law}{=} U(0, y-x)$ for all $0 < x < y$ and so

$$\lim_{y\to x+} \mathbb{P}(U(x,y) - x \leq (y-x)^{\gamma}) = 1, \ \forall x \geq 0. \tag{3.3}$$

Lemma 3.2. *For all $x \in \mathbb{R}$, we have $\lim_{y\to x} T_{x,y} = \tau_{0,x}$ in probability.*

Proof. In this proof we denote $Y_{0,t}(0)$ simply by Y_t. We first establish the result for $x = 0$. For all $t > 0$, we have

$$\mathbb{P}(t \leq T_{0,y}) \leq \mathbb{P}(L_{0,t}(0) \leq L_{0,T_{0,y}}(0)) = \mathbb{P}(L_t^0 \leq U(0,y))$$

since $(2\alpha^+ - 1)L_{0,T_{0,y}}(0) = U(0,y)$. The right-hand side converges to 0 as $y \to 0+$ by (3.3). On the other hand, by the strong Markov property at time $\tau_{0,y}$ for $y < 0$,

$$G_t(y) := \mathbb{P}(t \leq T_{0,y}) = \mathbb{P}(t \leq \tau_{0,y}) + E[1\!\!1_{\{t>\tau_{0,y}\}} G_{t-\tau_{0,y}}(Y_{\tau_y})].$$

For all $\varepsilon > 0$,

$$\begin{aligned} E[1\!\!1_{\{t>\tau_{0,y}\}} G_{t-\tau_{0,y}}(Y_{\tau_{0,y}})] &= E[1\!\!1_{\{t-\tau_{0,y}>\varepsilon\}} G_{t-\tau_{0,y}}(Y_{\tau_{0,y}})] + E[1\!\!1_{\{0<t-\tau_{0,y}\leq\varepsilon\}} G_{t-\tau_{0,y}}(Y_{\tau_{0,y}})] \\ &\leq E[G_{\varepsilon}(Y_{\tau_{0,y}})] + \mathbb{P}(0 < t - \tau_{0,y} \leq \varepsilon). \end{aligned}$$

From previous observations, we have $Y_{\tau_{0,y}} > 0$ *a.s* for all $y < 0$ and consequently $Y_{\tau_{0,y}} \longrightarrow 0+$ as $y \to 0-$. Since $\lim_{z\to 0+} G_{\varepsilon}(z) = 0$, by letting $y \to 0-$ and using dominated convergence, then $\varepsilon \to 0$, we get $\limsup_{y\to 0-} G_t(y) = 0$ as desired for $x = 0$. Now, the lemma easily holds after remarking that

$$T_{x,y} - \tau_{0,x} \overset{law}{=} T_{0,y-x} \text{ if } 0 \leq x < y, \quad T_{x,y} - \tau_{0,x} \overset{law}{=} T_{0,x-y} \text{ if } x < y \leq 0.$$

$\square$

For $s \leqslant t, x \in \mathbb{R}$, define

$$g_{s,t}(x) = \sup\{u \in [s,t] : Y_{s,u}(x) = 0\} \quad (\sup(\emptyset) = -\infty) \tag{3.4}$$

We use Lemma 3.2 to prove the

Lemma 3.3. *Fix $s, x \in \mathbb{R}$. Then, a.s for all $t > \tau_{s,x}$, there exists $(v,y) \in \mathbb{Q}^2$ such that*

$$v < g_{s,t}(x) \ \textit{and} \ Y_{s,r}(x) = Y_{v,r}(y) \ \forall \ r \geq g_{s,t}(x).$$

Proof. We prove the result for $s = 0$ and first for $x = 0$. Let $t > 0$, then for all $\varepsilon > 0$

$$\mathbb{P}(\exists \ \varepsilon > 0 : Y_{0,t}(\varepsilon) = Y_{0,t}(-\varepsilon)) \geq \mathbb{P}(T_{-\varepsilon,\varepsilon} \leq t).$$

From $\mathbb{P}(t < T_{-\varepsilon,\varepsilon}) \leq \mathbb{P}(t < T_{0,\varepsilon}) + \mathbb{P}(t < T_{0,-\varepsilon})$ and the previous lemma, we have $\lim_{\varepsilon\to 0} \mathbb{P}(t < T_{-\varepsilon,\varepsilon}) = 0$ and therefore $\mathbb{P}(\exists \ \varepsilon > 0 : Y_{0,t}(\varepsilon) = Y_{0,t}(-\varepsilon)) = 1$. Choose

$\varepsilon > 0$, such that $Y_{0,t}(\varepsilon) = Y_{0,t}(-\varepsilon)$ and let $v \in]0, T_{-\varepsilon,\varepsilon}[\cap\mathbb{Q}$. Then $Y_{0,v}(\varepsilon) > Y_{0,v}(-\varepsilon)$ and for any rational $y \in]Y_{0,v}(-\varepsilon), Y_{0,v}(\varepsilon)[$, we have by (3.2)

$$Y_{v,u}(Y_{0,v}(-\varepsilon)) \leq Y_{v,u}(y) \leq Y_{v,u}(Y_{0,v}(\varepsilon)),\ \forall u \geq v.$$

The flow property (Proposition 3.1) yields $Y_{0,u}(-\varepsilon) \leq Y_{v,u}(y) \leq Y_{0,u}(\varepsilon)$, $\forall u \geq v$. So necessarily $Y_{0,r}(0) = Y_{v,r}(y)$, $\forall r \geq g_{0,t}(0)$. For $x > 0$ and ε small enough, we have

$$\mathbb{P}(Y_{0,t}(x+\varepsilon) > Y_{0,t}(x), t > \tau_{0,x}) \leq \mathbb{P}(\tau_{0,x} < t < T_{x,x+\varepsilon}).$$

This shows that $\lim_{\varepsilon\to 0}\mathbb{P}(Y_{0,t}(x+\varepsilon) > Y_{0,t}(x)|t > \tau_{0,x}) = 0$ by Lemma 3.2. Similarly, for ε small

$$\mathbb{P}(Y_{0,t}(x-\varepsilon) < Y_{0,t}(x), t > \tau_{0,x}) \leq \mathbb{P}(\tau_{0,x} < t < T_{x-\varepsilon,x}).$$

The right-hand side converges to 0 as $\varepsilon \to 0$ by Lemma 3.2 and so $\lim_{\varepsilon\to 0}\mathbb{P}(Y_{0,t}(x) > Y_{0,t}(x-\varepsilon)|t > \tau_{0,x}) = 0$. Since

$$\{Y_{0,t}(x+\varepsilon) > Y_{0,t}(x-\varepsilon)\} \subset \{Y_{0,t}(x+\varepsilon) > Y_{0,t}(x)\} \cup \{Y_{0,t}(x) > Y_{0,t}(x-\varepsilon)\},$$

we get $\mathbb{P}(\exists\varepsilon > 0 : Y_{0,t}(x-\varepsilon) = Y_{0,t}(x+\varepsilon)|t > \tau_{0,x}) = 1$. Following the same steps as the case $x = 0$, we show the lemma for a fixed t a.s. Finally, the result easily extends almost surely for all t. □

We close this section by the

Lemma 3.4. *With probability 1, for all $(s_1, x_1) \neq (s_2, x_2) \in \mathbb{Q}^2$ simultaneously*

$$T^{x_1,x_2}_{s_1,s_2} := \inf\{r \geq sup(s_1, s_2) : Y_{s_1,r}(x_1) = Y_{s_2,r}(x_2)\} < \infty,$$

$$T^{x_1,x_2}_{s_1,s_2} > \sup(\tau_{s_1,x_1}, \tau_{s_2,x_2}),\ Y_{s_1,T^{x_1,x_2}_{s_1,s_2}}(x_1) = Y_{s_2,T^{x_1,x_2}_{s_1,s_2}}(x_2) = 0,$$

and

$$Y_{s_1,r}(x_1) = Y_{s_2,r}(x_2)\ \forall r \geq T^{x_1,x_2}_{s_1,s_2}.$$

Proof. The first assertion is a consequence of Proposition 3.1, the independence of increments and the coalescence of Y. Fix $(s_1, x_1) \neq (s_2, x_2) \in \mathbb{Q}^2$ with $s_1 \leq s_2$. By the comparison principle (3.2) and Proposition 3.1, $Y_{s_1,t}(x_1) \geq Y_{s_2,t}(x_2)$ for all $t \geq s_2$ or $Y_{s_1,t}(x_1) \leq Y_{s_2,t}(x_2)$ for all $t \geq s_2$. Suppose for example that $0 < z := Y_{s_1,s_2}(x_1) < x_2$ and take a rational $r \in]z, x_2[$. Then $T^{x_1,x_2}_{s_1,s_2} > \tau_{s_2,z} \geq \tau_{s_1,x_1}$ and $T^{x_1,x_2}_{s_1,s_2} \geq T^{r,x_2}_{s_2,s_2} > \tau_{s_2,x_2}$. The last affirmation is an immediate consequence of the pathwise uniqueness of (1.3). □

3.2 Construction of solutions associated to (E).

We now extend the notations given in Section 2.2.1. For all $n \geq 0$, let $\mathbb{D}_n = \{\frac{k}{2^n},\ k \in \mathbb{Z}\}$ and $\mathbb{D}$ be the set of all dyadic numbers: $\mathbb{D} = \cup_{n\in\mathbb{N}}\mathbb{D}_n$. For $u < v$, define $n(u, v) = \inf\{n \in \mathbb{N} : \mathbb{D}_n \cap]u, v[\neq \emptyset\}$ and $f(u, v) = \inf \mathbb{D}_{n(u,v)} \cap]u, v[$. Denote by $G_{\mathbb{Q}} = \{x \in G : |x| \in \mathbb{Q}_+\}$. We also fix a bijection $\psi : \mathbb{N} \longrightarrow \mathbb{Q} \times G_{\mathbb{Q}}$ and set $(s_i, x_i) = \psi(i)$ for all $i \geq 0$.

3.2.1 Construction of a stochastic flow of mappings φ solution of (E).

Let W be a real white noise and Y be the flow of the $SBM(\alpha^+)$ constructed from W in the previous paragraph. We first construct $\varphi_{s,.}(x)$ for all $(s,x)\in\mathbb{Q}\times G_{\mathbb{Q}}$ and then extend this definition for all $(s,x)\in\mathbb{R}\times G$. We begin by $\varphi_{s_0,.}(x_0)$, then $\varphi_{s_1,.}(x_1)$ and so on. To define $\varphi_{s_0,.}(x_0)$, we flip excursions of $Y_{s_0,.}(\varepsilon(x_0)|x_0|)$ suitably. Then let $\varphi_{s_1,t}(x_1)$ be equal to $\varphi_{s_0,t}(x_0)$ if $Y_{s_0,t}(\varepsilon(x_0)|x_0|)=Y_{s_1,t}(\varepsilon(x_1)|x_1|)$. Before coalescence of $Y_{s_0,.}(\varepsilon(x_0)|x_0|)$ and $Y_{s_1,.}(\varepsilon(x_1)|x_1|)$, we define $\varphi_{s_1,.}(x_1)$ by flipping excursions of $Y_{s_1,.}(\varepsilon(x_1)|x_1|)$ independently of what happens to $\varphi_{s_0,.}(x_0)$ and so on. In what follows, we translate this idea rigorously. Let $\vec{\gamma}^+,\vec{\gamma}^-$ be two independent random variables on any probability space such that

$$\vec{\gamma}^+ \overset{law}{=} \sum_{i=1}^{p}\frac{\alpha_i}{\alpha^+}\delta_{\vec{e}_i},\quad \vec{\gamma}^- \overset{law}{=} \sum_{j=p+1}^{N}\frac{\alpha_j}{\alpha^-}\delta_{\vec{e}_j}. \tag{3.5}$$

Let $(\Omega,\mathcal{A},\mathbb{P})$ be a probability space rich enough and $W=(W_{s,t},s\le t)$ be a real white noise defined on it. For all $s\le t, x\in G$, let $Z_{s,t}(x):=Y_{s,t}(\varepsilon(x)|x|)$ where Y is the flow of Burdzy-Kaspi constructed from W as in Section 3.1.1 if $\alpha^+\notin\{0,1\}$ (= the reflecting Brownian motion associated to (1.3) if $\alpha^+\in\{0,1\}$).
We retain the notations $\tau_{s,x}, g_{s,t}(x)$ of the previous section (see (3.1) and (3.4)). For $s\in\mathbb{R},x\in G$ define, by abuse of notations

$$\tau_{s,x}=\tau_{s,\varepsilon(x)|x|},\ g_{s,.}(x)=g_{s,.}(\varepsilon(x)|x|) \text{ and } d_{s,t}(x)=\inf\{r\ge t: Z_{s,r}(x)=0\}.$$

It will be convenient to set $Z_{s,r}(x)=\infty$ if $r<s$. For all $q\ge 1, u_0,...,u_q\in\mathbb{R}, y_0,...,y_q\in G$ define

$$T^{y_0,...,y_q}_{u_0,...,u_q}=\inf\{r\ge\tau_{u_q,y_q}: Z_{u_q,r}(y_q)\in\{Z_{u_i,r}(y_i), i\in[1,q-1]\}\}.$$

Let $\{(\vec{\gamma}^+_{s_0,x_0}(r),\vec{\gamma}^-_{s_0,x_0}(r)), r\in\mathbb{D}\cap[s_0,+\infty[\}$ be a family of independent copies of $(\vec{\gamma}^+,\vec{\gamma}^-)$ which is independent of W. We define $\varphi_{s_0,.}(x_0)$ by

$$\varphi_{s_0,t}(x_0)=\begin{cases}\vec{e}(x_0)|Z_{s_0,t}(x_0)| & \text{if } s_0\le t\le\tau_{s_0,x_0}\\ 0 & \text{if } t>\tau_{s_0,x_0}, Z_{s_0,t}(x_0)=0\\ \vec{\gamma}^+_{s_0,x_0}(f_0)|Z_{s_0,t}(x_0)|,\quad f_0=f(g_{s_0,t}(x_0),d_{s_0,t}(x_0)) & \text{if } t>\tau_{s_0,x_0}, Z_{s_0,t}(x_0)>0\\ \vec{\gamma}^-_{s_0,x_0}(f_0)|Z_{s_0,t}(x_0)|,\quad f_0=f(g_{s_0,t}(x_0),d_{s_0,t}(x_0)) & \text{if } t>\tau_{s_0,x_0}, Z_{s_0,t}(x_0)<0\end{cases}$$

Now, suppose that $\varphi_{s_0,.}(x_0),...,\varphi_{s_{q-1},.}(x_{q-1})$ are defined and let $\{(\vec{\gamma}^+_{s_q,x_q}(r),\vec{\gamma}^-_{s_q,x_q}(r)), r\in\mathbb{D}\cap[s_q,+\infty[\}$ be a family of independent copies of $(\vec{\gamma}^+,\vec{\gamma}^-)$ which is also independent of $\sigma\left(\vec{\gamma}^+_{s_i,x_i}(r),\vec{\gamma}^-_{s_i,x_i}(r), r\in\mathbb{D}\cap[s_i,+\infty[, 1\le i\le q-1, W\right)$. Since $T^{x_0,...,x_q}_{s_0,...,s_q}<\infty$, let $i\in[1,q-1]$ and (s_i,x_i) such that $Z_{s_q,t_0}(x_q)=Z_{s_i,t_0}(x_i)$ with $t_0=T^{x_0,...,x_q}_{s_0,...,s_q}$. We define $\varphi_{s_q,.}(x_q)$ by

$$\varphi_{s_q,t}(x_q)=\begin{cases}\vec{e}(x_q)|Z_{s_q,t}(x_q)| & \text{if } s_q\le t\le\tau_{s_q,x_q}\\ 0 & \text{if } t>\tau_{s_q,x_q}, Z_{s_q,t}(x_q)=0\\ \vec{\gamma}^+_{s_q,x_q}(f_q)|Z_{s_q,t}(x_q)|,\quad f_q=f(g_{s_q,t}(x_q),d_{s_q,t}(x_q)) & \text{if } t\in[\tau_{s_q,x_q},t_0], Z_{s_q,t}(x_q)>0\\ \vec{\gamma}^-_{s_q,x_q}(f_q)|Z_{s_q,t}(x_q)|,\quad f_q=f(g_{s_q,t}(x_q),d_{s_q,t}(x_q)) & \text{if } t\in[\tau_{s_q,x_q},t_0], Z_{s_q,t}(x_q)<0\\ \varphi_{s_i,t}(x_i) & \text{if } t\ge t_0\end{cases}$$

In this way, we construct $(\varphi_{s,.}(x), s\in\mathbb{Q}, x\in G_{\mathbb{Q}})$.
Now, for all $s\in\mathbb{R}, x\in G$, let $\varphi_{s,t}(x)=\vec{e}(x)|Z_{s,t}(x)|$ if $s\le t\le\tau_{s,x}$. If $t>\tau_{s,x}$, then by Lemma (3.3), there exist $v\in\mathbb{Q}, y\in G_{\mathbb{Q}}$ such that $v<g_{s,t}(x)$ and $Z_{s,r}(x)=Z_{v,r}(y)\ \forall\ r\ge g_{s,t}(x)$. In this case, we define $\varphi_{s,t}(x)=\varphi_{v,t}(y)$. Later, we will show that φ is a coalescing solution of (E).

3.2.2 Construction of a stochastic flow of kernels K^{m^+,m^-} solution of (E).

Let m^+ and m^- be two probability measures respectively on Δ_p and Δ_{N-p}. Let $\mathcal{U}^+$,$\mathcal{U}^-$ be two independent random variables on any probability space such that

$$\mathcal{U}^+ \overset{law}{=} m^+,\ \mathcal{U}^- \overset{law}{=} m^-. \tag{3.6}$$

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space rich enough and $W = (W_{s,t}, s \le t)$ be a real white noise defined on it. We retain the notation introduced in the previous paragraph for all functions of W. We consider a family $\{(\mathcal{U}^+_{s_0,x_0}(r), \mathcal{U}^-_{s_0,x_0}(r)), r \in \mathbb{D} \cap [s_0, +\infty[\}$ of independent copies of $(\mathcal{U}^+,\mathcal{U}^-)$ which is independent of W.
If $t > \tau_{s_0,x_0}$ and $Z_{s_0,t}(x_0) > 0$ (resp. $Z_{s_0,t}(x_0) < 0$), let

$$U^+_{s_0,t}(x_0) = \mathcal{U}^+_{s_0,x_0}(f_0)\ \ (\text{resp. } U^-_{s_0,t}(x_0) = \mathcal{U}^-_{s_0,x_0}(f_0)),\ \ f_0 = f(g_{s_0,t}(x_0), d_{s_0,t}(x_0)).$$

Write $U^+_{s_0,t}(x_0) = (U^{+,i}_{s_0,t}(x_0))_{1\le i\le p}$ (resp. $U^-_{s_0,t}(x_0) = (U^{-,i}_{s_0,t}(x_0))_{p+1\le i\le N}$) if $Z_{s_0,t}(x_0) > 0, t > \tau_{s_0,x_0}$ (resp. $Z_{s_0,t}(x_0) < 0, t > \tau_{s_0,x_0}$) and now define

$$K^{m^+,m^-}_{s_0,t}(x_0) = \begin{cases} \delta_{\vec{e}(x_0)|Z_{s_0,t}(x_0)|} & \text{if } s_0 \le t \le \tau_{s_0,x_0} \\ \sum_{i=1}^{p} U^{+,i}_{s_0,t}(x_0)\delta_{\vec{e}_i|Z_{s_0,t}(x_0)|} & \text{if } t > \tau_{s_0,x_0}, Z_{s_0,t}(x_0) > 0 \\ \sum_{i=p+1}^{N} U^{-,i}_{s_0,t}(x_0)\delta_{\vec{e}_i|Z_{s_0,t}(x_0)|} & \text{if } t > \tau_{s_0,x_0}, Z_{s_0,t}(x_0) < 0 \\ \delta_0 & \text{if } t > \tau_{s_0,x_0}, Z_{s_0,t}(x_0) = 0 \end{cases}$$

Suppose that $K^{m^+,m^-}_{s_0,.}(x_0), ..., K^{m^+,m^-}_{s_{q-1},.}(x_{q-1})$ are defined and let $\{(\mathcal{U}^+_{s_q,x_q}(r), \mathcal{U}^-_{s_q,x_q}(r)), r \in \mathbb{D} \cap [s_q, +\infty[\}$ be a family of independent copies of $(\mathcal{U}^+,\mathcal{U}^-)$ which is also independent of $\sigma\left(\mathcal{U}^+_{s_i,x_i}(r), \mathcal{U}^-_{s_i,x_i}(r), r \in \mathbb{D} \cap [s_i, +\infty[, 1 \le i \le q-1, W\right)$.
If $t > \tau_{s_q,x_q}$ and $Z_{s_q,t}(x_q) > 0$ (resp. $Z_{s_q,t}(x_q) < 0$), we define $U^+_{s_q,t}(x_q) = (U^{+,i}_{s_q,t}(x_q))_{1\le i\le p}$ (resp. $U^-_{s_q,t}(x_q) = (U^{-,i}_{s_q,t}(x_q))_{p+1\le i\le N}$) by analogy to $q = 0$. Let $i \in [1, q-1]$ and (s_i, x_i) such that $Z_{s_q,t_0}(x_q) = Z_{s_i,t_0}(x_i)$ with $t_0 = T^{x_0,...,x_q}_{s_0,...,s_q}$. Then, define

$$K^{m^+,m^-}_{s_q,t}(x_q) = \begin{cases} \delta_{\vec{e}(x_q)|Z_{s_q,t}(x_q)|} & \text{if } s_q \le t \le \tau_{s_q,x_q} \\ \sum_{i=1}^{p} U^{+,i}_{s_q,t}(x_q)\delta_{\vec{e}_i|Z_{s_q,t}(x_q)|} & \text{if } t_0 > t > \tau_{s_q,x_q}, Z_{s_q,t}(x_q) > 0 \\ \sum_{i=p+1}^{N} U^{-,i}_{s_q,t}(x_q)\delta_{\vec{e}_i|Z_{s_q,t}(x_q)|} & \text{if } t_0 > t > \tau_{s_q,x_q}, Z_{s_q,t}(x_q) < 0 \\ \delta_0 & \text{if } t_0 \ge t > \tau_{s_q,x_q}, Z_{s_q,t}(x_q) = 0 \\ K^{m^+,m^-}_{s_i,t}(x_i) & \text{if } t > t_0 \end{cases}$$

In this way, we construct $(K^{m^+,m^-}_{s,}(x), s \in \mathbb{Q}, x \in G_{\mathbb{Q}})$.
Now, for $s \in \mathbb{R}, x \in G$, let $K^{m^+,m^-}_{s,t}(x) = \delta_{\vec{e}(x)|Z_{s,t}(x)|}$ if $s \le t \le \tau_{s,x}$. If $t > \tau_{s,x}$, let $v \in \mathbb{Q}, y \in G_{\mathbb{Q}}$ such that $v < g_{s,t}(x)$ and $Z_{s,r}(x) = Z_{v,r}(y)\ \forall\ r \ge g_{s,t}(x)$. Then, define $K^{m^+,m^-}_{s,t}(x) = K^{m^+,m^-}_{v,t}(y)$.
In the next paragraph we will show that K^{m^+,m^-} is a stochastic flow of kernels which solves (E).

3.2.3 Construction of (K^{m^+,m^-}, φ) by filtering.

Let m^+ and m^- be two probability measures as in Theorem 1.5 and $(\vec{\gamma}^+, \mathcal{U}^+), (\vec{\gamma}^-, \mathcal{U}^-)$ be two independent random variables satisfying

$$\mathcal{U}^+ = (\mathcal{U}^{+,i})_{1\le i\le p} \overset{law}{=} m^+,\ \ \mathcal{U}^- = (\mathcal{U}^{-,j})_{p+1\le j\le N} \overset{law}{=} m^-,$$

$$\mathbb{P}(\vec{\gamma}^+ = \vec{e}_i | \mathcal{U}^+) = \mathcal{U}^{+,i}, \ \forall i \in [1,p], \tag{3.7}$$

and

$$\mathbb{P}(\vec{\gamma}^- = \vec{e}_j | \mathcal{U}^-) = \mathcal{U}^{-,j}, \ \forall j \in [p+1,N]. \tag{3.8}$$

Then, in particular $(\vec{\gamma}^+, \vec{\gamma}^-)$ and $(\mathcal{U}^+, \mathcal{U}^-)$ satisfy respectively (3.5) and (3.6).
On a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ consider the following independent processes

- $W = (W_{s,t}, s \le t)$ a real white noise.
- $\{(\vec{\gamma}^+_{s,x}(r), \mathcal{U}^+_{s,x}(r)), r \in \mathbb{D} \cap [s,+\infty[, (s,x) \in \mathbb{Q} \times G_{\mathbb{Q}}\}$ a family of independent copies of $(\vec{\gamma}^+, \mathcal{U}^+)$.
- $\{(\vec{\gamma}^-_{s,x}(r), \mathcal{U}^-_{s,x}(r)), r \in \mathbb{D} \cap [s,+\infty[, (s,x) \in \mathbb{Q} \times G_{\mathbb{Q}}\}$ a family of independent copies of $(\vec{\gamma}^-, \mathcal{U}^-)$.

Now, let φ and K^{m^+,m^-} be the processes constructed in Sections 3.2.1 and 3.2.2 respectively from $(\vec{\gamma}^+, \vec{\gamma}^-, W)$ and $(\mathcal{U}^+, \mathcal{U}^-, W)$. Let $\sigma(\mathcal{U}^+, \mathcal{U}^-, W)$ be the σ-field generated by $\{\mathcal{U}^+_{s,x}(r), \mathcal{U}^-_{s,x}(x), r \in \mathbb{D} \cap [s,+\infty[, (s,x) \in \mathbb{Q} \times G_{\mathbb{Q}}\}$ and W. We then have the

Proposition 3.2. *(i) For all measurable bounded function f on G, $s \le t \in \mathbb{R}, x \in G$:*

$$K^{m^+,m^-}_{s,t} f(x) = E[f(\varphi_{s,t}(x)) | \sigma(\mathcal{U}^+, \mathcal{U}^-, W)] \ \ a.s.$$

(ii) For all s, x, with probability 1, $\forall t \ge s$

$$|\varphi_{s,t}(x)| = |Z_{s,t}(x)|, \quad \varphi_{s,t}(x) \in G^+ \Leftrightarrow Z_{s,t}(x) \ge 0 \ \textit{and} \quad \varphi_{s,t}(x) \in G^- \Leftrightarrow Z_{s,t}(x) \le 0.$$

(iii) For all s, x, y, with probability 1

$$t_0 := \inf\{r \ge s : \varphi_{s,r}(x) = \varphi_{s,r}(y)\} = \inf\{r \ge s : Z_{s,r}(x) = Z_{s,r}(y)\}$$

and $\varphi_{s,r}(x) = \varphi_{s,r}(y), \quad \forall r \ge t_0$.

Proof. (i) comes from (3.7), (3.8) and the definiton of our flows, (ii) is clear by construction. To prove (iii), let $x \ne y$ and $t > s$ such that $Z_{s,t}(x) = Z_{s,t}(y)$. Then $g_{s,t}(x) = g_{s,t}(y)$ and therefore $\varphi_{s,t}(x) = \varphi_{s,t}(y)$. Conversely, if $\varphi_{s,r}(x) = \varphi_{s,r}(y)$ for some r, then $Z_{s,r}(x) = Z_{s,r}(y)$ by (ii). □

Next we will prove that φ is a stochastic flow of mappings. It remains to prove that properties (1) and (4) in the definition are satisfied. As in Lemma 3.1, property (4) can be derived from the following

Lemma 3.5. *$\forall t \ge s, \varepsilon > 0, x \in G$, we have*

$$\lim_{y \to x} \mathbb{P}(d(\varphi_{s,t}(x), \varphi_{s,t}(y)) \ge \varepsilon) = 0.$$

Proof. We take $s = 0$. By Lemma 3.2, if $x \in \ G^+ \setminus \{0\}$, then

$$\mathbb{P}(d(\varphi_{0,t}(x), \varphi_{0,t}(y)) \ge \varepsilon) \le \mathbb{P}(\tau_{0,|x|} < t < T_{|x|,|y|}) \to 0 \text{ as } y \to x, |y| > |x|$$

and

$$\mathbb{P}(d(\varphi_{0,t}(x), \varphi_{0,t}(y)) \ge \varepsilon) \le \mathbb{P}(\tau_{0,|y|} < t < T_{|x|,|y|}) \to 0 \text{ as } y \to x, |y| < |x|.$$

The case $x \in G^-$ holds similarly. □

Proposition 3.3. $\forall s < t < u, x \in G$:

$$\varphi_{s,u}(x) = \varphi_{t,u}(\varphi_{s,t}(x)) \qquad a.s.$$

Proof. Set $y = \varphi_{s,t}(x)$. Then, with probability 1, $\forall r \geq t$ $Y_{s,r}(\varepsilon(x)|x|) = Y_{t,r}(Y_{s,t}(\varepsilon(x)|x|))$ and so, a.s. $\forall r \geq t$ $Z_{s,r}(x) = Z_{t,r}(y)$. All the equalities below hold a.s.

• 1st case: $u \leq \tau_{s,x}$. We have $\tau_{t,y} = \inf\{r \geq t, Z_{t,r}(y) = 0\} = \inf\{r \geq t, Z_{s,r}(x) = 0\} = \tau_{s,x}$. Consequently $u \leq \tau_{t,y}$ and $\varphi_{s,u}(x) = \vec{e}(x)|Z_{s,u}(x)| = \vec{e}(y)|Z_{t,u}(y)| = \varphi_{t,u}(y) = \varphi_{t,u}(\varphi_{s,t}(x))$.

• 2nd case: $t \leq \tau_{s,x} < u$. We still have $\tau_{t,y} = \tau_{s,x}$ and so $g_{t,u}(y) = g_{s,u}(x)$. It is clear by construction that: $\varphi_{s,u}(x) = \varphi_{t,u}(y) = \varphi_{t,u}(\varphi_{s,t}(x))$.

• 3rd case: $\tau_{s,x} < t, \tau_{t,y} \leq u$. Since $\tau_{t,y}$ is a common zero of $(Z_{s,r}(x))_{r\geq s}$ and $(Z_{t,r}(y))_{r\geq t}$ before u, it comes that $g_{t,u}(y) = g_{s,u}(x)$ and therefore $\varphi_{s,u}(x) = \varphi_{t,u}(y) = \varphi_{t,u}(\varphi_{s,t}(x))$.

• 4th case: $\tau_{s,x} < t, u < \tau_{t,y}$. In such a case, we have $\varphi_{t,u}(y) = \vec{e}(y)|Z_{t,u}(y)| = \vec{e}(y)|Z_{s,u}(x)|$. Since $r \longmapsto Z_{s,r}(x)$ does not touch 0 in the interval $[t, u]$ and $\varphi_{s,t}(x) = y$, we easily see that $\varphi_{s,u}(x) = \vec{e}(y)|Z_{s,u}(x)| = \varphi_{t,u}(y)$. □

Proposition 3.4. *φ is a coalescing solution of (E).*

Proof. We use these notations: $Y_u := Y_{0,u}(0)$, $\varphi_u := \varphi_{0,u}(0)$. We first show that φ is an $W(\alpha_1, ..., \alpha_N)$ on G. Define for all $n \geq 1$: $T_0^n(Y) = 0$,

$$\begin{aligned} T_{k+1}^n(Y) &= \inf\{r \geq T_k^n(Y), |\varphi_r - \varphi_{T_k^n}| = \frac{1}{2^n}\} = \inf\{r \geq T_k^n(Y), |Y_r - Y_{T_k^n}| = \frac{1}{2^n}\} \\ &= \inf\{r \geq T_k^n(Y), ||Y_r| - |Y_{T_k^n}|| = \frac{1}{2^n}\}, k \geq 0. \end{aligned}$$

Remark that $|Y|$ is a reflected Brownian motion and denote $T_k^n(Y)$ simply by T_k^n. From the proof of Proposition 2.5, $\lim_{n\to+\infty} \sup_{t\leq K} |T_{\lfloor 2^{2n}t \rfloor}^n - t| = 0$ $a.s$ for all $K > 0$. Set $\varphi_k^n = 2^n \varphi_{T_k^n}$. Then, since almost surely $t \longrightarrow \varphi_t$ is continuous, a.s $\forall t \geq 0$, $\lim_{n\to+\infty} \frac{1}{2^n}\varphi_{\lfloor 2^{2n}t \rfloor}^n = \varphi_t$. By Proposition 2.5, it remains to show that for all $n \geq 0$, $(\varphi_k^n, k \geq 0)$ is a Markov chain (started at 0) whose transition mechanism is described by (2.1). If $Y_k^n = 2^n Y_{T_k^n}$, then, by the proof of Proposition 2.5 (since SBM is a special case of $W(\alpha_1, ..., \alpha_N)$), for all $n \geq 0$, $(Y_k^n)_{k\geq 1}$ is a Markov chain on $\mathbb{Z}$ started at 0 whose law is described by

$$Q(0, 1) = 1 - Q(0, -1) = \alpha^+, \quad Q(m, m + 1) = Q(m, m - 1) = \frac{1}{2} \ \forall m \neq 0.$$

Let $k \geq 1$ and $x_0, .., x_k \in G$ such that $x_0 = x_k = 0$ and $|x_{h+1} - x_h| = 1$ if $h \in [0, k - 1]$. We write

$$\{x_h, x_h = 0, h \in [1, k]\} = \{x_{i_0}, .., x_{i_q}\}, \ i_0 = 0 < i_1 < ... < i_q = k$$

and

$$\{x_h, x_h \neq 0, h \in [1, k]\} = \{x_h\}_{h\in[i_0+1,i_1-1]} \cup ... \cup \{x_h\}_{h\in[i_{q-1}+1,i_{k-1}]}.$$

Assume that

$$\{x_h\}_{h\in[i_0+1,i_1-1]} \subset D_{j_0}, ..., \{x_h\}_{h\in[i_{q-1}+1,i_{k-1}]} \subset D_{j_{q-1}}$$

and define

$$A_h^n = (Y_h^n = \varepsilon(x_h)|x_h|), \quad E = (\vec{e}(\varphi_{i_0+1}^n) = \vec{e}_{j_0}, ..., \vec{e}(\varphi_{i_{q-1}+1}^n) = \vec{e}_{j_{q-1}}).$$

If $i \in [1,p]$, we have

$$(\varphi^n_{k+1} = \vec{e}_i, \varphi^n_k = x_k, ..., \varphi^n_0 = x_0) = \bigcap_{h=0}^{k} A^n_h \bigcap (Y^n_{k+1} - Y^n_k = 1) \bigcap E \bigcap (\vec{e}(\varphi^n_{k+1}) = \vec{e}_i)$$

and $(\varphi^n_k = x_k, ..., \varphi^n_0 = x_0) = \bigcap_{h=0}^{k} A^n_h \bigcap E$. Now

$$\mathbb{P}(\varphi^n_{k+1} = \vec{e}_i | \varphi^n_0 = x_0, ..., \varphi^n_k = 0) = \frac{\alpha_i}{\alpha^+} \mathbb{P}(Y^n_{k+1} - Y^n_k = 1 | Y^n_k = 0) = \alpha_i.$$

Obviously, the previous argument can be applied to show that the transition probabilities of $(\varphi^n_k, k \geq 0)$ are given by (2.1) and so φ is an $W(\alpha_1, ..., \alpha_N)$ on G started at 0. Using (2.2) for φ, it follows that $\forall f \in D(\alpha_1, ...\alpha_N)$,

$$f(\varphi_t) = f(0) + \int_0^t f'(\varphi_s) dB_s + \frac{1}{2} \int_0^t f''(\varphi_s) ds$$

where

$$B_t = |\varphi_t| - \tilde{L}_t(|\varphi|) = |Y_t| - \tilde{L}_t(|Y|) = \int_0^t \widetilde{sgn}(Y_s) dY_s$$

by Tanaka's formula for symmetric local time. But Y solves (1.3) and therefore $\int_0^t \widetilde{sgn}(Y_s) dY_s = \int_0^t \widetilde{sgn}(Y_s) W(ds)$. Since a.s $\widetilde{sgn}(Y_s) = \varepsilon(\varphi_s)$ for all $s \geq 0$, it comes that $\forall f \in D(\alpha_1, ...\alpha_N)$,

$$f(\varphi_{0,t}(x)) = f(x) + \int_0^t f'(\varphi_{0,s}(x)) \varepsilon(\varphi_{0,s}(x)) W(ds) + \frac{1}{2} \int_0^t f''(\varphi_{0,s}(x)) ds$$

when $x = 0$. Finally, by distinguishing the cases $t \leqslant \tau_{0,x}$ and $t > \tau_{0,x}$, we see that the previous equation is also satisfied for $x \neq 0$. $\square$

Corollary 3.5. *K^{m^+,m^-} is a stochastic flow of kernels solution of (E).*

Proof. By Proposition 3.2 (i) and Jensen inequality, K^{m^+,m^-} is a stochastic flow of kernels. The fact that K^{m^+,m^-} is a solution of (E) is a consequence of the previous proposition and is similar to Lemma 4.6 [11]. $\square$

Remarks 3.6. • *If $(m^+, m^-) = (\delta_{(\frac{\alpha_1}{\alpha^+}, ..., \frac{\alpha_p}{\alpha^+})}, \delta_{(\frac{\alpha_{p+1}}{\alpha^-}, ..., \frac{\alpha_N}{\alpha^-})})$, then*

$$\begin{aligned} K^W_{s,t}(x) &= \delta_{\vec{e}(x)|Z_{s,t}(x)|} 1\!\!1_{t \leq \tau_{s,x}} \\ &+ \Big(\sum_{i=1}^{p} \frac{\alpha_i}{\alpha^+} \delta_{\vec{e}_i |Z_{s,t}(x)|} 1\!\!1_{Z_{s,t}(x) > 0} + \sum_{i=p+1}^{N} \frac{\alpha_i}{\alpha^-} \delta_{\vec{e}_i |Z_{s,t}(x)|} 1\!\!1_{Z_{s,t}(x) \leq 0}\Big) 1\!\!1_{t > \tau_{s,x}} \end{aligned} \tag{3.9}$$

is a Wiener solution of (E).

• *If $(m^+, m^-) = (\sum_{i=1}^{p} \frac{\alpha_i}{\alpha^+} \delta_{0,..,0,1,0,..,0}, \sum_{i=p+1}^{N} \frac{\alpha_i}{\alpha^-} \delta_{0,..,0,1,0,..,0})$, then $K^{m^+,m^-} = \delta_\varphi$.*

4 Unicity of flows associated to (E).

Lemma 4.1. *Let (K, W) be a solution of (E). Then*
(i) $\forall s \in \mathbb{R}, x \in G$, a.s $t \longmapsto K_{s,t}(x,.)$ is continuous from $[s, +\infty[$ into $\mathcal{P}(G)$.
(ii)$\forall x \in G, s \in \mathbb{R}$, a.s

$$K_{s,t}(x) = \delta_{x+\vec{e}(x)\varepsilon(x)W_{s,t}}, \text{ if } s \leq t \leq \tau_{s,x} \text{ where } \tau_{s,x} = \inf\{r \geq s,\ \varepsilon(x)|x| + W_{s,r} = 0\}.$$

Proof. (i) We take $s = 0$. Recall the definition of $D'(\alpha_1, ..., \alpha_N)$ from (2.3). Then, $D'(\alpha_1, ..., \alpha_N)$ is dense in $C_0(G)$ as shown in Proposition 2.8. Since (K, W) satisfies (E),

$$\lim_{u \to t} K_{0,u} f(x) = K_{0,t} f(x) \text{ for all } t \geq 0, f \in D'(\alpha_1, ..., \alpha_N).$$

The result extends for all $f \in C_0(G)$ by an approximation argument.
ii) We follow [11] (Lemma 3.1). Assume that $x \neq 0, x \in D_i$, $1 \leq i \leq p$ and take $s = 0$. Let $\beta_i = 1$ and consider a set of numbers $(\beta_j)_{1 \leq j \leq N, j \neq i}$ such that $\sum_{j=1}^{N} \beta_j \alpha_j = 0$. If $f(h\vec{e}_j) = \beta_j h$ for all $1 \leq j \leq N$, then $f \in D(\alpha_1, ..., \alpha_N)$. Set $\tilde{\tau}_x = \inf\{r; K_{0,r}(x)(\cup_{j \neq i} D_j) > 0\}$ and apply f in (E) to get

$$\int_{\mathring{D}_i} |y| K_{0,t}(x, dy) = |x| + W_t \text{ for all } t \leq \tilde{\tau}_x. \tag{4.1}$$

By applying $f_k(y) = |y|^2 e^{\frac{-|y|}{k}}$, $k \geq 1$ in (E), we have for all $t \geq 0$

$$K_{0,t \wedge \tilde{\tau}_x} f_k(x) = f_k(x) + \int_0^t 1\!\!1_{[0,\tilde{\tau}_x]}(u) K_{0,u}(\varepsilon f_k')(x) W(du) + \frac{1}{2} \int_0^{t \wedge \tilde{\tau}_x} K_{0,u} f_k''(x) du.$$

As $k \to \infty$, $K_{0,t \wedge \tilde{\tau}_x} f_k(x)$ tends to $\int_0^t |y|^2 K_{0,t \wedge \tilde{\tau}_x}(x, dy)$ by monotone convergence. Let $A > 0, xe^{-x} \leq A$ for all $x \geq 0$. Since $|f_k'(y) - 2|y|| \leq (4 + A)|y|$,

$$\int_0^t 1\!\!1_{[0,\tilde{\tau}_x]}(u) K_{0,u}(\varepsilon f_k')(x) W(du) \longrightarrow \int_0^t 1\!\!1_{[0,\tilde{\tau}_x]}(u) \int_G 2|y| K_{0,u}(x, dy) W(du)$$

as $k \to \infty$ using (4.1) and dominated convergence for stochastic integrals ([16] page 142). From $|f_k''(y)| \leq 2e^{\frac{-1}{k}|y|} + \frac{4+A}{k}|y|$, we get $\int_0^{t \wedge \tilde{\tau}_x} K_{0,u} f_k''(x) du \longrightarrow 0$ as $k \to \infty$. By identifying the limits, we have

$$\int_{\mathring{D}_i} (|y| - |x| - W_t)^2 K_{0,t}(x, dy) = 0 \ \ \forall\ t \leq \tilde{\tau}_x.$$

This proves that for $t \leq \tilde{\tau}_x$, $K_{0,t}(x) = \delta_{x+\vec{e}(x)W_t}$. The fact that $\tau_{0,x} = \tilde{\tau}_x$ easily follows. □

The previous lemma entails the following

Corollary 4.1. *If (K, W) is a solution of (E), then $\sigma(W) \subset \sigma(K)$.*

Proof. For all $x \in D_1$, we have $K_{0,t}(x) = \delta_{\vec{e_1}(|x|+W_t)}$ if $t \leq \tau_{0,x}$. If f is a positive function on G such that $f_1(h) = h$, then $W_t = K_{0,t} f(x) - |x|$ for all $t \leq \tau_{0,x}, x \in D_1$. By considering a sequence $(x_k)_{k \geq 0}$ converging to ∞, this shows that $\sigma(W_t) \subset \sigma(K_{0,t}(y), y \in D_1)$. □

4.1 Unicity of the Wiener solution.

In order to complete the proof of Theorem 1.4, we will prove the following

Proposition 4.2. *Equation (E) has at most one Wiener solution. That is, if K and K' are two Wiener solutions, then for all $s \le t, x \in G$,* $\quad K_{s,t}(x) = K'_{s,t}(x)$ *a.s.*

Proof. Denote by P the semigroup of $W(\alpha_1, ..., \alpha_N)$, A and $D(A)$ being respectively its generator and its domain on $C_0(G)$. Recall the definition of $D'(\alpha_1, ..., \alpha_N)$ from (2.3) and that

$$\forall t > 0 \quad P_t(C_0(G)) \subset D'(\alpha_1, ..., \alpha_N) \subset D(A)$$

(see Proposition 2.8). Define
$\mathcal{S} = \{f : G \longrightarrow \mathbb{R} : f, f', f'' \in C_b(G^*)$ and are extendable by continuity at 0 on each ray, $\lim_{x\to\infty} f(x) = 0\}$.
For $t > 0, h$ a measurable bounded function on G^*, let $\lambda_t h(x) = 2p_t h_j(|x|)$, if $x \in \Delta_j$, where h_j is the extension of h_j that equals 0 on $]-\infty, 0]$. Then, the following identity can be easily checked using the explicit expression of P:

$$(P_t f)' = -P_t f' + \lambda_t f' \quad \text{on } G^* \quad \text{for all } f \in \mathcal{S}. \tag{4.2}$$

Fix $f \in \mathcal{S}$. We will verify that $(P_t f)' \in \mathcal{S}$. For $x = h\vec{e}_j \in G^*$, we have

$$(P_t f)'(x) = -2\sum_{i=1}^{N} \alpha_i \int_{\mathbb{R}} f_i'(y-h)p_t(0,y)dy + \int_{\mathbb{R}} f_j'(y+h)p_t(0,y)dy + \int_{\mathbb{R}} f_j'(y-h)p_t(0,y)dy$$

Clearly $(P_t f)' \in C_b(G^*)$ and is extendable by continuity at 0 on each ray. Furthermore, a simple integration by parts yields

$$\int_{\mathbb{R}} f_j'(y+h)p_t(0,y)dy = C\int_{\mathbb{R}} f_j(y+h)yp_t(0,y)dy \text{ for some } C \in \mathbb{R}$$

and since $\lim_{x\to\infty} f(x) = 0$, we get $\lim_{x\to\infty}(P_t f)'(x) = 0$. It is also easy to check that $(P_t f)'', (P_t f)''' \in C_b(G^*)$ and are extendable by continuity at 0 on each ray which shows that $(P_t f)' \in \mathcal{S}$.
Let (K, W) be a stochastic flow that solves (E) (not necessarily a Wiener flow) and fix $x = h\vec{e}_j \in G^*$. Our aim now is to establish the following identity

$$K_{0,t}f(x) = P_t f(x) + \int_0^t K_{0,u}(D(P_{t-u}f))(x)W(du) \tag{4.3}$$

where $Dg(x) = \varepsilon(x).g'(x)$. Note that $\int_0^t K_{0,u}(D(P_{t-u}f))(x)W(du)$ is well defined. In fact

$$\int_0^t E[K_{0,u}(D(P_{t-u}f))(x)]^2 du \le \int_0^t P_u((D(P_{t-u}f))^2)(x)du \le \int_0^t ||(P_{t-u}f)'||_\infty^2 du$$

and the right-hand side is bounded since (4.2) is satisfied and f' is bounded. Set $g = P_\varepsilon f = P_{\frac{\varepsilon}{2}} P_{\frac{\varepsilon}{2}} f$. Then, since $P_{\frac{\varepsilon}{2}} f \in C_0(G)$ ($\lim_{x\to\infty} P_{\frac{\varepsilon}{2}} f(x) = 0$ comes from $\lim_{x\to\infty} f(x) = 0$), we have $g \in D'(\alpha_1, ..., \alpha_N)$. Now

$$K_{0,t}g(x) - P_t g(x) - \int_0^t K_{0,u}(D(P_{t-u}g))(x)W(du) = \sum_{p=0}^{n-1}(K_{0,\frac{(p+1)t}{n}} P_{t-\frac{(p+1)t}{n}} g - K_{0,\frac{pt}{n}} P_{t-\frac{pt}{n}} g)(x)$$

$$-\sum_{p=0}^{n-1}\int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} K_{0,u}D((P_{t-u}-P_{t-\frac{(p+1)t}{n}})g)(x)W(du)-\sum_{p=0}^{n-1}\int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} K_{0,u}D(P_{t-\frac{(p+1)t}{n}}g)(x)W(du).$$

For all $p \in \{0,..,n-1\}$, $g_{p,n} = P_{t-\frac{(p+1)t}{n}}g \in D'(\alpha_1,...,\alpha_N)$ and so by replacing in (E), we get

$$\int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} K_{0,u}Dg_{p,n}(x)W(du) = K_{0,\frac{(p+1)t}{n}}g_{p,n}(x) - K_{0,\frac{pt}{n}}g_{p,n}(x) - \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} K_{0,u}Ag_{p,n}(x)du$$

$$= K_{0,\frac{(p+1)t}{n}}g_{p,n}(x) - K_{0,\frac{pt}{n}}g_{p,n}(x) - \frac{t}{n}K_{0,\frac{pt}{n}}Ag_{p,n}(x) - \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} (K_{0,u} - K_{0,\frac{pt}{n}})Ag_{p,n}(x)du$$

Then we can write

$$K_{0,t}g(x) - P_tg(x) - \int_0^t K_{0,u}(D(P_{t-u}g))(x)W(du) = A_1(n) + A_2(n) + A_3(n),$$

where

$$A_1(n) = -\sum_{p=0}^{n-1} K_{0,\frac{pt}{n}}[P_{t-\frac{pt}{n}}g - P_{t-\frac{(p+1)t}{n}}g - \frac{t}{n}.AP_{t-\frac{(p+1)t}{n}}g](x),$$

$$A_2(n) = -\sum_{p=0}^{n-1}\int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} K_{0,u}D((P_{t-u} - P_{t-\frac{(p+1)t}{n}})g)(x)W(du),$$

$$A_3(n) = \sum_{p=0}^{n-1}\int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} (K_{0,u} - K_{0,\frac{pt}{n}})AP_{t-\frac{(p+1)t}{n}}g(x)du.$$

Using $||K_{0,u}f||_\infty \le ||f||_\infty$ if f is a bounded measurable function, we obtain

$$|A_1(n)| \le \sum_{p=0}^{n-1} ||P_{t-\frac{(p+1)t}{n}}[P_{\frac{t}{n}}g - g - \frac{t}{n}.Ag]||_\infty \le n||P_{\frac{t}{n}}g - g - \frac{t}{n}.Ag||_\infty \xrightarrow[n\to+\infty]{} 0.$$

Note that $A_2(n)$ is the sum of orthogonal terms in $L^2(\Omega)$. Consequently

$$||A_2(n)||^2_{L^2(\Omega)} = \sum_{p=0}^{n-1} ||\int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} K_{0,u}D((P_{t-u} - P_{t-\frac{(p+1)t}{n}})g)(x)W(du)||^2_{L^2(\Omega)}.$$

By applying Jensen inequality, we arrive at

$$||A_2(n)||^2_{L^2(\Omega)} \le \sum_{p=0}^{n-1}\int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} P_uV_u^2(x)du$$

where $V_u = (P_{t-u}g)' - (P_{t-\frac{(p+1)t}{n}}g)'$. By (4.2) one can decompose V_u as follows:

$$V_u = X_u + Y_u; \quad X_u = -P_{t-u}g' + P_{t-\frac{(p+1)t}{n}}g', \quad Y_u = \lambda_{t-u}g' - \lambda_{t-\frac{(p+1)t}{n}}g'$$

Using the trivial inequality $(a+b)^2 \le 2a^2 + 2b^2$, we obtain: $P_uV_u^2(x) \le 2P_uX_u^2(x) + 2P_uY_u^2(x)$ and so

$$||A_2(n)||^2_{L^2(\Omega)} \le 2B_1(n) + 2B_2(n)$$

where $B_1(n)=\sum_{p=0}^{n-1}\int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} P_uX_u^2(x)du,\quad B_2(n)=\sum_{p=0}^{n-1}\int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} P_uY_u^2(x)du.$

If $p\in[0,n-1]$ and $u\in[\frac{pt}{n},\frac{(p+1)t}{n}]$, then $P_uX_u^2(x)\le P_{u+t-\frac{p+1}{n}t}(g'-P_{\frac{p+1}{n}t-u}g')^2(x)$. The change of variable $v=(p+1)t-nu$ yields

$$\begin{aligned} B_1(n) &\le \int_0^t P_{t-\frac{v}{n}}(P_{\frac{v}{n}}g'-g')^2(x)dv \\ &\le \int_0^t (P_tg'^2(x)-2P_{t-\frac{v}{n}}(g'P_{\frac{v}{n}}g')(x)+P_{t-\frac{v}{n}}g'^2(x))dv. \end{aligned}$$

By writing $P_{t-\frac{v}{n}}(g'P_{\frac{v}{n}}g')(x)$ as a function of p, we prove that $\lim_{n\to\infty}P_{t-\frac{v}{n}}(g'P_{\frac{v}{n}}g')(x)=P_tg'^2(x)$. Since g' is bounded, by dominated convergence this shows that $B_1(n)$ tends to 0 as $n\to+\infty$. For $B_2(n)$, we write

$$P_uY_u^2(x)=2\sum_{i=1}^{N}\alpha_ip_u((Y_u^2)_i)(-|x|)+p_u((Y_u^2)_j)(|x|)-p_u((Y_u^2)_j)(-|x|)$$

where $(Y_u)_i=2p_{t-u}g_i'-2p_{t-\frac{(p+1)t}{n}}g_i'$, defined on $\mathbb{R}_+^*$. It was shown before that this quantity tends to 0 as $n\to+\infty$ when (p,g_i') is replaced by (P,g') in general and consequently $B_2(n)$ tends to 0 as $n\to+\infty$. Now

$$||A_3(n)||_{L^2(\Omega)}\le\sum_{p=0}^{n-1}||\int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}}(K_{0,u}-K_{0,\frac{pt}{n}})AP_{t-\frac{(p+1)t}{n}}g(x)du||_{L^2(\Omega)}.$$

Set $h_{p,n}=AP_{t-\frac{(p+1)t}{n}}g$. Then $h_{p,n}\in D'(\alpha_1,...,\alpha_N)$ for all $p\in[0,n-1]$ (if $p=n-1$ remark that $h_{p,n}=P_{\frac{\varepsilon}{2}}AP_{\frac{\varepsilon}{2}}f$). By the Cauchy-Schwarz inequality

$$||A_3(n)||_{L^2(\Omega)}\le\sqrt{t}\left\{\sum_{p=0}^{n-1}\int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}}E[((K_{0,u}-K_{0,\frac{pt}{n}})h_{p,n}(x))^2]du\right\}^{\frac{1}{2}}.$$

If $u\in[\frac{pt}{n},\frac{(p+1)t}{n}]$:

$$\begin{aligned} E[((K_{0,u}-K_{0,\frac{pt}{n}})h_{p,n}(x))^2] &\le E[K_{0,\frac{pt}{n}}(K_{\frac{pt}{n},u}h_{p,n}-h_{p,n})^2(x)] \\ &\le E[K_{0,\frac{pt}{n}}(K_{\frac{pt}{n},u}h_{p,n}^2-2h_{p,n}K_{\frac{pt}{n},u}h_{p,n}+h_{p,n}^2)(x)] \\ &\le ||P_{u-\frac{pt}{n}}h_{p,n}^2-2h_{p,n}P_{u-\frac{pt}{n}}h_{p,n}+h_{p,n}^2||_\infty \\ &\le 2||h_{p,n}||_\infty||P_{u-\frac{pt}{n}}h_{p,n}-h_{p,n}||_\infty+||P_{u-\frac{pt}{n}}h_{p,n}^2-h_{p,n}^2||_\infty \end{aligned}$$

Therefore $||A_3(n)||_{L^2(\Omega)}\le\sqrt{t}(2C_1(n)+C_2(n))^{\frac{1}{2}}$, where

$$C_1(n)=\sum_{p=0}^{n-1}||h_{p,n}||_\infty\int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}}||P_{u-\frac{pt}{n}}h_{p,n}-h_{p,n}||_\infty du,\ C_2(n)=\sum_{p=0}^{n-1}\int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}}||P_{u-\frac{pt}{n}}h_{p,n}^2-h_{p,n}^2||_\infty du.$$

From $||h_{p,n}||_\infty\le||Ag||_\infty$ and $||P_{u-\frac{pt}{n}}h_{p,n}-h_{p,n}||_\infty\le||P_{u-\frac{pt}{n}}Ag-Ag||_\infty$, we get

$$C_1(n)\le||Ag||_\infty\sum_{p=0}^{n-1}\int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}}||P_{u-\frac{pt}{n}}Ag-Ag||_\infty du\le||Ag||_\infty\int_0^t||P_{\frac{z}{n}}Ag-Ag||_\infty dz.$$

As $Ag \in C_0(G)$, $C_1(n)$ tends to 0 obviously. On the other hand, $h_{p,n}^2 \in D(\alpha_1, ..., \alpha_N)$ (this can be easily verified since $h_{p,n}$ is continuous and $\sum_{i=0}^{N} \alpha_i (h_{p,n})'_i(0+) = 0$). We may apply (2.2) to get

$C_2(n) = \frac{1}{n} \sum_{p=0}^{n-1} \int_0^t ||P_{\frac{z}{n}} h_{p,n}^2 - h_{p,n}^2||_\infty dz \le \frac{1}{2n} \sum_{p=0}^{n-1} \int_0^t \int_0^{\frac{z}{n}} ||(h_{p,n}^2)''||_\infty du dz$

Now we verify that $h'_{p,n}, h''_{p,n}$ are uniformly bounded with respect to n and $0 \le p \le n-1$. In fact $||h''_{p,n}||_\infty = ||2Ah_{p,n}||_\infty \le 2||AP_{\frac{\varepsilon}{2}} f||_\infty$. Write $h_{p,n} = P_{t-\frac{p+1}{n}t+\frac{\varepsilon}{2}} P_{\frac{\varepsilon}{4}} AP_{\frac{\varepsilon}{4}} f$ where $P_{\frac{\varepsilon}{4}} AP_{\frac{\varepsilon}{4}} f \in D'(\alpha_1, ..., \alpha_N)$. Then, by (4.2), $||h'_{p,n}||_\infty$ is uniformly bounded with respect to $n, p \in [0, n-1]$ and so the same holds for $||(h_{p,n}^2)''||_\infty$. As a result $C_2(n)$ tends to 0 as $n \to \infty$. Finally

$$K_{0,t}g(x) = P_t g(x) + \int_0^t K_{0,u}(D(P_{t-u}g))(x) W(du).$$

Now, let ε go to 0, then $K_{0,t}g(x)$ tends to $K_{0,t}f(x)$ in $L^2(\Omega)$. Furthermore

$|| \int_0^t K_{0,u}(D(P_{t-u}g))(x) W(du) - \int_0^t K_{0,u}(D(P_{t-u}f))(x) W(du)||_{\mathbb{L}^2(\Omega)}^2$

$\le \int_0^t P_u((P_{t-u}g)' - (P_{t-u}f)')^2(x) du.$

Using the derivation formula (4.2), the right side may be decomposed as $I_\varepsilon + J_\varepsilon$, where

$$I_\varepsilon = \int_0^t P_u(P_{t-u}g' - P_{t-u}f')^2(x) du, \quad J_\varepsilon = \int_0^t P_u(\lambda_{t-u}g' - \lambda_{t-u}f')^2(x) du.$$

By Jensen inequality, $I_\varepsilon \le tP_t(g' - f')^2(x)$. Since $g'(y) = -P_\varepsilon f'(y) + 2\lambda_\varepsilon f'(y) \longrightarrow f'(y)$ as $\varepsilon \to 0, P_t(x, dy)$ *a.s*, we get $I_\varepsilon \longrightarrow 0$ as $\varepsilon \to 0$ by dominated convergence. Similarly J_ε tends to 0 as $\varepsilon \to 0$. This establishes (4.3). Now assume that (K, W) is a Wiener solution of (E) and let $f \in \mathcal{S}$. Since $K_{0,t}f(x) \in L^2(\mathcal{F}_\infty^{W_{0,.}})$, let $K_{0,t}f(x) = P_t f(x) + \sum_{n=1}^\infty J_t^n f(x)$ be the decomposition in Wiener chaos of $K_{0,t}f(x)$ in L^2 sens ([16] page 202). By iterating (4.3) (recall that $(P_t f)' \in \mathcal{S}$), we see that for all $n \ge 1$

$$J_t^n f(x) = \int_{0<s_1<...<s_n<t} P_{s_1}(D(P_{s_2-s_1}...D(P_{t-s_n}f)))(x) dW_{0,s_1}...dW_{0,s_n}.$$

If K' is another Wiener flow satisfying (4.3), then $K_{0,t}f(x)$ and $K'_{0,t}f(x)$ must have the same Wiener chaos decomposition for all $f \in \mathcal{S}$, that is $K_{0,t}f(x) = K'_{0,t}f(x)$ a.s. Consequently $K_{0,t}f(x) = K'_{0,t}f(x)$ a.s for all $f \in D'(\alpha_1, ..., \alpha_N)$ since this last set is included in $\mathcal{S}$ and the result extends for all $f \in C_0(G)$ by a density argument. This completes the proof when $x \ne 0$. The case $x = 0$ can be deduced from property (4) in the Definition 2.1. □

Consequence: We already know that K^W given by (3.9) is a Wiener solution of (E). Since $\sigma(W) \subset \sigma(K)$, we can define K^* the stochastic flow obtained by filtering K with respect to $\sigma(W)$ (Lemma 3-2 (ii) in [10]). Then $\forall s \le t, x \in G, \quad K_{s,t}^*(x) = E[K_{s,t}(x)|\sigma(W)]$ *a.s.* As a result, (K^*, W) solves also (E) and we may apply the last theorem so that

$$\forall s \le t, x \in G, \quad E[K_{s,t}(x)|\sigma(W)] = K_{s,t}^W(x) \ a.s. \tag{4.4}$$

From now on, (K, W) **is a solution of** (E). We retain the notations introduced in Section 3 for all functions of W $(Y_{s,t}(x), Z_{s,t}(x), g_{s,t}(x)...)$. In the next section, starting from K, we construct a flow of mappings φ^c which is a solution of (E). This flow will play an important role to characterize the law of K.

4.2 Construction of a stochastic flow of mappings solution of (E) from K.

For $n \in \mathbb{N}^*$, define $P^n = E[K_{0,t}^{\otimes n}]$. Let $(P^{(n),c})_{n\in\mathbb{N}^*}$ be the family of Markovian semigroups given by Theorem 4.1 of [10].

Proposition 4.3. *$(P^{(n),c})_{n\in\mathbb{N}^*}$ is a compatible family of Feller semigroups.*

Proof. By (4.4), $\forall x \in G$ with probability 1, for all $t \geq \tau_{0,x}$, $K_{0,t}(x)$ is supported on

$$\{|Z_{0,t}(x)|\vec{e}_i, \quad 1 \leq i \leq p\} \text{ if } Z_{0,t}(x) > 0$$

and is supported on

$$\{|Z_{0,t}(x)|\vec{e}_i, \quad p+1 \leq i \leq N\} \text{ if } Z_{0,t}(x) \leq 0.$$

For each $(x, y) \in G^2$, let $(X_t^x, X_t^y)_{t\geq 0}$ be the two point motion started at (x, y) associated with P^2 constructed as in Section 2.6 [10]. Then $T^{x,y} := \inf\{r \geq 0, X_r^x = X_r^y\} < +\infty$ a.s (if $Z_{0,t}(x) = Z_{0,t}(y) = 0$ then necessarily $X_t^x = X_t^y = 0$). Furthermore, by Lemma 3.2, $\forall t > 0, \varepsilon > 0, x \in G$,

$$\lim_{y\to x} \mathbb{P}(\{T^{x,y} > t\} \bigcap \{|X_t^x - X_t^y| > \varepsilon\}) \leq \lim_{y\to x} \mathbb{P}(\tau_{x,y} < t < T_{\varepsilon(x)|x|,\varepsilon(y)|y|}) = 0.$$

Now the result follows from Theorem 4.1 [10]. ☐

Consequence: Let φ^c be the coalescing flow of mappings associated with $(P^{(n),c})_{n\in\mathbb{N}^*}$ (Theorem 1.1 in [10]). Let ν (respectively ν^c) be the Feller convolution semigroup associated with K (respectively φ^c), then ν^c weakly dominates ν (Theorem 4.2 [10]). This essentially says ([11] page 23) that there exists a joint realization (K^1, K^2) such that K^1 (resp. K^2) is a stochastic flow of kernels associated to ν^c (resp.ν) (so $K^1 \stackrel{law}{=} \delta_{\varphi^c}$, $K^2 \stackrel{law}{=} K$) satisfying $E[K^1|K^2] = K^2$.
To simplify notations, we shall assume that K and φ^c are defined both on the original space $(\Omega, \mathcal{A}, \mathbb{P})$ and

$$E[\delta_{\varphi^c}|K] = K \tag{4.5}$$

Conditioning by $\sigma(W)$, by Proposition 4.2, we obtain

$$E[\delta_{\varphi^c}|\sigma(W)] = K^W \tag{4.6}$$

with K^W being the flow given by (3.9).

Proposition 4.4. *For all $s_1 \leq s_2 \in \mathbb{R}, x_1, x_2 \in G$, define*

$$S_{s_1,s_2}^{x_1,x_2} := \inf\{r \geq sup(s_1, s_2) : \varphi_{s_1,r}^c(x_1) = \varphi_{s_2,r}^c(x_2)\}$$

Then

$$S_{s_1,s_2}^{x_1,x_2} = T_{s_1,s_2}^{x_1,x_2}, \quad \varphi_{s_1,r}^c(x_1) = \varphi_{s_2,r}^c(x_2), \text{ and } K_{s_1,r}(x_1) = K_{s_2,r}(x_2) \quad \forall r \geq T_{s_1,s_2}^{x_1,x_2} \quad a.s.$$

Proof. By (4.6), for all s, y with probability 1,

$$\delta_{\varphi^c_{s,t}(y)} \text{ is supported on } \{|Z_{s,t}(y)|\vec{e}_j, 1 \le j \le N\} \text{ for all } t \ge s.$$

This shows that $|\varphi^c_{s,t}(y)| = |Z_{s,t}(y)|$. Similarly, using (4.6), we have

$$\varphi^c_{s,t}(y) \in G^+ \Leftrightarrow Z_{s,t}(y) \ge 0 \text{ and } \quad \varphi^c_{s,t}(y) \in G^- \Leftrightarrow Z_{s,t}(y) \le 0. \tag{4.7}$$

This establishes the first identity. By the flow property of Y

$$T^{x_1,x_2}_{s_1,s_2} = \inf\{r \ge sup(s_1, s_2) : Z_{s_2,r}(x_1) = Z_{s_2,r}(\varphi^c_{s_1,s_2}(x_1))\} \ \ a.s.$$

From the first assertion and the independence of increments, it immediately follows that $S^{z,x_2}_{s_2,s_2} = T^{x_1,x_2}_{s_1,s_2} \ \ (z := \varphi^c_{s_1,s_2}(x_1)) \ \ a.s.$ Since φ^c is a coalescing flow, with probability 1

$$\varphi^c_{s_2,r}(\varphi^c_{s_1,s_2}(x_1)) = \varphi^c_{s_2,r}(x_2) \ \ \forall r \ge S^{z,x_2}_{s_2,s_2}.$$

and so $\varphi^c_{s_2,r}(\varphi^c_{s_1,s_2}(x_1)) = \varphi^c_{s_2,r}(x_2) \ \ \forall r \ge T^{x_1,x_2}_{s_1,s_2}$. Fix $r \in \mathbb{R}$, then conditionally to $\{r > T^{x_1,x_2}_{s_1,s_2}\}$, we have by the flow property of φ^c

$$\varphi^c_{s_1,r}(x_1) = \varphi^c_{s_2,r}(\varphi^c_{s_1,s_2}(x_1)) = \varphi^c_{s_2,r}(x_2).$$

Now, the result easily extends by continuity for φ^c and using (4.5) for K. □

Proposition 4.5. *φ^c is a stochastic flow of mappings solution of (E).*

Proof. For all $x \in G$, $\varphi^c_{0,.}(x)$ is an $W(\alpha_1, ..., \alpha_N)$ started at x. By Theorem 2.6, $\forall f \in D(\alpha_1, ..., \alpha_N)$,

$$f(\varphi^c_{0,t}(x)) = f(x) + \int_0^t f'(\varphi^c_{0,u}(x))dB_u + \frac{1}{2}\int_0^t f''(\varphi^c_{0,u}(x))du \ \ a.s.$$

with $B_t = |\varphi_{0,t}(x)| - \tilde{L}_t(|\varphi_{0,.}(x)|) - |x| = |Z_{0,t}(x)| - \tilde{L}_t(|Z_{0,.}(x)|) - |x|$. Tanaka's formula and (4.7) yield

$$B_t = \int_0^t \widetilde{sgn}(Z_{0,u}(x))dZ_{0,u}(x) = \int_0^t \widetilde{sgn}(Z_{0,u}(x))W(du) = \int_0^t \varepsilon(\varphi^c_{0,u}(x))W(du).$$

Likewise for all $s \le t, x \in G, f \in D(\alpha_1, ..., \alpha_N)$,

$$f(\varphi^c_{s,t}(x)) = f(x) + \int_s^t f'(\varphi^c_{s,u}(x))\varepsilon(\varphi^c_{s,u}(x))W(du) + \frac{1}{2}\int_s^t f''(\varphi^c_{s,u}(x))du \ \ a.s.$$

□

4.3 Two probability measures associated to K.

For all $t \ge \tau_{s,x}$, set

$$V^{+,i}_{s,t}(x) = K_{s,t}(x)(\mathring{D}_i) \ \forall 1 \le i \le p$$

and

$$V^{-,N}_{s,t}(x) = K_{s,t}(x)(D_N), \quad V^{-,i}_{s,t}(x) = K_{s,t}(x)(\mathring{D}_i) \ \ \forall p+1 \le i \le N-1$$

$$V^+_{s,t}(x) = (V^{+,i}_{s,t}(x))_{1 \le i \le p}, V^-_{s,t}(x) = (V^{-,i}_{s,t}(x))_{p+1 \le i \le N}, \ V_{s,t}(x) = (V^+_{s,t}(x), V^-_{s,t}(x)).$$

For $s = 0$, we use these abbreviated notations

$$Z_t(x) = Z_{0,t}(x), \quad V_t^+(x) = V_{0,t}^+(x), \quad V_t^-(x) = V_{0,t}^-(x), \; V_t(x) = (V_t^+(x), V_t^-(x))$$

and if $x = 0$,

$$Z_t = Z_{0,t}(0), \quad V_t^+ = V_{0,t}^+(0), \quad V_t^- = V_{0,t}^-(0), \; V_t = (V_t^+, V_t^-).$$

By (4.4), $\forall x \in G, s \leq t$, with probability 1

$$\begin{aligned} K_{s,t}(x) &= \delta_{\vec{e}(x)|Z_{s,t}(x)|} 1\!\!1_{t \leq \tau_{s,x}} \\ &+ \Big(\sum_{i=1}^{p} V_{s,t}^{+,i}(x) \delta_{\vec{e}_i|Z_{s,t}(x)|} 1\!\!1_{Z_{s,t}(x)>0} + \sum_{i=p+1}^{N} V_{s,t}^{-,i}(x) \delta_{\vec{e}_i|Z_{s,t}(x)|} 1\!\!1_{Z_{s,t}(x)\leq 0}\Big) 1\!\!1_{t>\tau_{s,x}}. \end{aligned}$$

Similarly to Lemma 4.1, $\forall s \in \mathbb{R}, \mu \in \mathcal{P}(G)$, the mapping $t \longmapsto \mu K_{s,t}$ is continuous from $[s, +\infty[$ into $\mathcal{P}(G)$ and therefore $(\mu K_{0,t})_{t\geq 0}$ is a Feller process taking values in $\mathcal{P}(G)$ for all $\mu \in \mathcal{P}(G)$. Denote by $\mathbb{P}_\mu$ the law of this process and define

$$\mathcal{F}_{s,t}^K = \sigma(K_{v,u}, s \leq v \leq u \leq t), \quad \mathcal{F}_{s,t}^W = \sigma(W_{v,u}, s \leq v \leq u \leq t)$$

Assume that these σ-fields include all $\mathbb{P}$-negligible sets. Then, the following strong Markov property holds

Lemma 4.2. *Let $\mu \in \mathcal{P}(G)$, and let T be a finite $(\mathcal{F}_{0,t}^K)_{t\geq 0}$ stopping time. Then, the law of $(\mu K_{0,T+t})_{t\geq 0}$ knowing of $\mathcal{F}_{0,T}^K$ is $\mathbb{P}_{\mu K_{0,T}}$.*

As a consequence of the preceding lemma, for each $x \in G$, $(K_{0,r+\tau_{0,x}}(x))_{r\geq 0}$ is independent of $\mathcal{F}_{0,\tau_{0,x}}^K$ and is equal in law to $(K_{0,r}(0))_{r\geq 0}$.
Consider the following random times:

$$T = \inf\{r \geq 0 : Z_r = 1\}, \quad L = \sup\{r \in [0, T] : Z_r = 0\}$$

and the following σ-fields:

$$\mathcal{F}_{L-} = \sigma(X_L, X \text{ is bounded } \; (\mathcal{F}_{0,t}^W)_{t\geq 0} - \text{previsible process})$$

$$\mathcal{F}_{L+} = \sigma(X_L, X \text{ is bounded } \; (\mathcal{F}_{0,t}^W)_{t\geq 0} - \text{progressive process})$$

Then $\mathcal{F}_{L+} = \mathcal{F}_{L-}$ (Lemma 4.11 in [11]). Let $f : \mathbb{R}^N \longrightarrow \mathbb{R}$ be a bounded continuous function and set $X_t = E[f(V_t)|\sigma(W)]$. Thanks to (4.5), the process $r \longrightarrow V_r$ is constant on the excursions of $r \longrightarrow Z_r$. By following the same steps as in Section 4.2 [11], we show that there is an $\mathcal{F}^W$-progressive version of X that is constant on the excursions of Z out of 0 (Lemma 4.12 [11]). We take for X this version. Then X_T is $\mathcal{F}_{L+}$ measurable and $E[X_T|\mathcal{F}_{L-}] = E[f(V_T)]$ (Lemma 4.13 [11]). This implies that V_T is independent of $\sigma(W)$ (Lemma 4.14 [11]) and the same holds if we replace T by $\inf\{t \geq 0 : Z_t = a\}$ where $a \geq 0$. Define by induction $T_{0,n}^+ = 0$ and for $k \geq 1$:

$$S_{k,n}^+ = \inf\{t \geq T_{k-1,n}^+ : Z_t = 2^{-n}\}, \quad T_{k,n}^+ = \inf\{t \geq S_{k,n}^+ : Z_t = 0\}.$$

Set $V_{k,n}^+ = V_{S_{k,n}^+}^+$. Then, we have the following

Lemma 4.3. *For all n, $(V_{k,n}^+)_{k\geq 1}$ is a sequence of i.i.d random variables. Moreover, this sequence is independent of W.*

Proof. For all $k \geq 2$, $V_{k,n}^{+}$ is $\sigma(\delta_0 K_{0,T_{k-1,n}^{+}+t}, t \geq 0)$ measurable and $V_{k-1,n}^{+}$ is $\mathcal{F}_{0,T_{k-1,n}^{+}}^{K}$ measurable which proves the first claim by Lemma 4.2. Now, we show by induction on q that $(V_{1,n}^{+}, ..., V_{q,n}^{+})$ is independent of $\sigma(W)$. For $q = 1$, this has been justified. Suppose $(V_{1,n}^{+}, ..., V_{q-1,n}^{+})$ is independent of $\sigma(W)$ and write

$$\sigma(W) = \sigma(Z_{u \wedge T_{q-1,n}^{+}}, u \geq 0) \vee \sigma(Z_{u+T_{q-1,n}^{+}}, u > 0).$$

Since $(V_{1,n}^{+}, ..., V_{q-1,n}^{+})$ is $\mathcal{F}_{0,T_{q-1,n}^{+}}^{K}$ measurable and

$$\sigma(Z_{u+T_{q-1,n}^{+}}, u > 0) \vee \sigma(V_{q,n}^{+}) \subset \sigma(\delta_0 K_{0,T_{q-1,n}^{+}+t}, t \geq 0)$$

we conclude that $(V_{1,n}^{+}, ..., V_{q,n}^{+})$ and $\sigma(W)$ are independent. □

Let m_n^{+} be the common law of $(V_{k,n}^{+})_{k \geq 1}$ for each $n \geq 1$ and define m^{+} as the law of V_1^{+} under $\mathbb{P}(.|Z_1 > 0)$. Then, we have the

Lemma 4.4. *The sequence $(m_n^{+})_{n \geq 1}$ converges weakly towards m^{+}. For all $t > 0$, the law of V_t^{+} under $\mathbb{P}(.|Z_t > 0)$ is given by m^{+}.*

Proof. For each bounded continuous function $f : \mathbb{R}^p \longrightarrow \mathbb{R}$,

$$\begin{aligned} E[f(V_t^{+})|W]1\!\!1_{\{Z_t>0\}} &= \lim_{n\to\infty} \sum_k E[1\!\!1_{t \in [S_{k,n}^{+}, T_{k,n}^{+}[} f(V_{k,n}^{+})|W] \\ &= \lim_{n\to\infty} \sum_k 1\!\!1_{t \in [S_{k,n}^{+}, T_{k,n}^{+}[} \left(\int f dm_n^{+} \right) \\ &= [1\!\!1_{\{Z_t>0\}} \lim_{n\to\infty} \int f dm_n^{+} + \varepsilon_n(t)] \end{aligned}$$

with $\lim_{n\to\infty} \varepsilon_n(t) = 0$ a.s. Consequently

$$\lim_{n\to\infty} \int f dm_n^{+} = \frac{1}{\mathbb{P}(Z_t > 0)} E[f(V_t^{+}) 1\!\!1_{Z_t>0}].$$

The left hand side does not depend on t, which completes the proof. □

Remark 4.6. *As a consequence of the lemma, we get*

$$E[f(V_t^{+})|W]1\!\!1_{\{Z_t>0\}} = 1\!\!1_{\{Z_t>0\}} \int f dm^{+}$$

for each measurable bounded $f : \mathbb{R}^p \longrightarrow \mathbb{R}$. We define analogously the measure m^{-} by considering the following stopping times: $T_{0,n}^{-} = 0$ and for $k \geq 1$:

$$S_{k,n}^{-} = \inf\{t \geq T_{k-1,n}^{-} : Z_t = -2^{-n}\}, \quad T_{k,n}^{-} = \inf\{t \geq S_{k,n}^{-} : Z_t = 0\}.$$

Set $V_{k,n}^{-} = V_{S_{k,n}^{-}}^{-}$ and let m_n^{-} be the common law of $(V_{k,n}^{-})_{k \geq 1}$. Denote by m^{-} the law of V_1^{-} under $\mathbb{P}(.|Z_1 < 0)$. Then, the sequence $(m_n^{-})_{n \geq 1}$ converges weakly towards m^{-}. Moreover, for all $t > 0$, the law of V_t^{-} under $\mathbb{P}(.|Z_t < 0)$ is given by m^{-}. As a result, we have

$$E[f(V_t^{-})|W]1\!\!1_{\{Z_t<0\}} = 1\!\!1_{\{Z_t<0\}} \int f dm^{-}$$

for each measurable bounded $f : \mathbb{R}^{N-p} \longrightarrow \mathbb{R}$*. If we follow the same steps as before but consider* $(Z_{u+\tau_{0,x}}(x), u \geq 0)$ *for all* x*, we show that the law of* $V_{0,t}^{+}(x)$ *under* $\mathbb{P}(.|Z_{0,t}(x) > 0, t > \tau_{0,x})$ *does not depend on* $t > 0$*. Denote by* m_x^+ *such a law. Then, thanks to Lemma 4.2,* m_x^+ *does not depend on* $x \in G$*. Thus* $m_x^+ = m^+$ *for all* x *and*

$$E[f(V_t^+(x))|W]1\!\!1_{\{Z_t(x)>0,t>\tau_{0,x}\}} = 1\!\!1_{\{Z_t(x)>0,t>\tau_{0,x}\}} \int f dm^+ \tag{4.8}$$

for each measurable bounded $f : \mathbb{R}^p \longrightarrow \mathbb{R}$*. Similarly*

$$E[h(V_t^-(x))|W]1\!\!1_{\{Z_t(x)<0,t>\tau_{0,x}\}} = 1\!\!1_{\{Z_t(x)<0,t>\tau_{0,x}\}} \int h dm^- \tag{4.9}$$

for each measurable bounded $h : \mathbb{R}^{N-p} \longrightarrow \mathbb{R}$*.*

4.4 Unicity in law of K.

To prove part (b) of Theorem 1.5, we begin by the following

Lemma 4.5. *For all* $x \in G, t > s$

$$\mathbb{P}(\exists y \in G_{\mathbb{Q}} : K_{s,r}(y) = K_{0,r}(x)\ \forall r \geq g_{0,t}(x) | \tau_{0,x} < s < g_{0,t}(x)) = 1.$$

Proof. By (4.5), it will be sufficient to prove the lemma for the kernel δ_{φ^c}. Using Proposition 4.4, this can be reduced to showing that, for all $t > s, z \in \mathbb{R}$

$$\mathbb{P}(\exists y \in \mathbb{Q} : Y_{s,r}(y) = Y_{0,r}(z)\ \forall r \geq g_{0,t}(z) | \tau_{0,z} < s < g_{0,t}(z)) = 1. \tag{4.10}$$

By the proof of Lemma 3.3, for all $a \in \mathbb{R}$

$$\mathbb{P}(\exists \varepsilon > 0 : Y_{s,t}(a - \varepsilon) = Y_{s,t}(a + \varepsilon) | t > \tau_{s,a}) = 1.$$

Fix $z \in \mathbb{R}$. From the flow property of Y, we have

$$\mathbb{P}(\exists \varepsilon > 0 : Y_{s,t}(Y_{0,s}(z) - \varepsilon) = Y_{s,t}(Y_{0,s}(z) + \varepsilon) | t > \tau_{s,Y_{0,s}(z)}) = 1.$$

Suppose $t > \tau_{s,Y_{0,s}(z)}$. Let $\varepsilon > 0, Y_{s,t}(Y_{0,s}(z) - \varepsilon) = Y_{s,t}(Y_{0,s}(z) + \varepsilon)$ and $b \in]Y_{0,s}(z) - \varepsilon, Y_{0,s}(z)[\cap \mathbb{Q}$. Then, by the comparison principle (3.2)

$$Y_{s,r}(Y_{0,s}(z) - \varepsilon) \leq Y_{s,r}(b) \leq Y_{s,r}(Y_{0,s}(z)) \leq Y_{s,r}(Y_{0,s}(z) + \varepsilon), \quad \forall r \geq s.$$

Consequently $Y_{s,t}(b) = Y_{0,t}(z)$ and so $Y_{s,r}(b) = Y_{0,r}(z), \forall r \geq g_{0,t}(z)$. Since $\{\tau_{0,z} < s < g_{0,t}(z)\} \subset \{t > \tau_{s,Y_{0,s}(z)}\}$, this gives (4.10) and consequently the lemma. □

Now, we will show by induction on n the following fact: $\forall n \geq 1, t > 0, x_1, ..., x_n \in G$:

$$(K_{0,t}(x_1), ..., K_{0,t}(x_n), W) \overset{law}{=} (K_{0,t}^{m^+,m^-}(x_1), ..., K_{0,t}^{m^+,m^-}(x_n), W') \tag{4.11}$$

with (K^{m^+,m^-}, W') being the solution constructed in Section 3 associated with (m^+, m^-). For $n = 1$, the result follows from (4.8) and (4.9). Notice that if $t < \tau_{0,z}$, then $K_{0,t}(z)$ is a function of W and if $t > T_{0,0}^{z_1,z_2}$, then $(K_{0,t}(z_1), K_{0,t}(z_2)) = (K_{0,t}(z_1), K_{0,t}(z_1))$. Assume that (4.11) holds for $n \geq 1$. Then, by the induction hypothesis and the preceding observation, it will be enough to show, for all $x_{n+1} \in G$ that

$$(K_{0,t}(x_1), ..., K_{0,t}(x_{n+1}), W) \overset{law}{=} (K_{0,t}^{m^+,m^-}(x_1), ..., K_{0,t}^{m^+,m^-}(x_{n+1}), W') \tag{4.12}$$

conditionally to $A = \{\sup_{1\leq i\leq n+1} \tau_{0,x_i} < t < T^{x_1,\dots,x_{n+1}}_{0,\dots,0}\}$ (resp. its analogous set for W') at the left (resp. right). Notice that on A, $g_{0,t}(x_1),\dots,g_{0,t}(x_{n+1})$ are distinct. By summing over all possible cases, we need only to check (4.12) conditionally to

$$E = \{\sup_{1\leq i\leq n+1} \tau_{0,x_i} < t < T^{x_1,\dots,x_{n+1}}_{0,\dots,0}, g_{0,t}(x_1) < \dots < g_{0,t}(x_n) < g_{0,t}(x_{n+1})\}$$

(resp. its analogous set for W') at the left (resp.right). Recall the definition of f from Section 3.2 and set $S = f(g_{0,t}(x_n), g_{0,t}(x_{n+1}))$. Then

$$E = \bigcup_{s\in\mathbb{D}} E_s, E_s = A \bigcap \{g_{0,t}(x_1) < \dots < g_{0,t}(x_n) < s < g_{0,t}(x_{n+1}), S = s\}.$$

From Lemma 4.5, conditionally to E_s, there exists $y \in G_{\mathbb{Q}}$ such that $K_{0,t}(x_{n+1}) = K_{s,t}(y)$. On E_s, we also have $V_t(x_1) = V_s(x_1),\dots,V_t(x_n) = V_s(x_n)$ since $(V_r(x_i), r \geq \tau_{0,x_i})$ is constant on the excursions of $(Z_r(x_i), r \geq \tau_{0,x_i})$. Consequently $(K_{0,t}(x_1),\dots,K_{0,t}(x_{n+1}), W)$ will be a measurable function of $(V_s(x_1),\dots.,V_s(x_n), K_{s,t}(y), W)$. Now since $\sigma(W) = \mathcal{F}^W_{0,s} \vee \sigma(W_{r+s} - W_s, r \geq 0)$, by the independence of increments of the flow and the induction hypothesis, (4.12) holds conditionally to E_s (resp. its analogous set for W') at the left (resp. right). This completes the proof.

Remark 4.7. *Suppose that K is a stochastic flow of mappings then by the definition of (m^+, m^-), we necessarily have $(m^+, m^-) = (\sum_{i=1}^{p} \frac{\alpha_i}{\alpha^+}\delta_{0,..,0,1,0,..,0}, \sum_{i=p+1}^{N} \frac{\alpha_i}{\alpha^-}\delta_{0,..,0,1,0,..,0})$. This shows that there is only one flow of mappings solving (E).*

4.5 The case $\alpha^+ = \frac{1}{2}, N > 2$.

Let K^W be the flow given by (3.9), where $Z_{s,t}(x) = \varepsilon(x)|x| + W_t - W_s$. It is easy to verify that K^W is a Wiener flow. **Fix** $s \in \mathbb{R}, x \in G$. Then, by following ideas of Section 3.2, one can construct a real white noise W and a process $(X^x_{s,t}, t \geq s)$ which is an $W(\alpha_1,\dots,\alpha_N)$ started at x such that

- (i) for all $t \geq s, f \in D(\alpha_1,\dots,\alpha_N)$,

$$f(X^x_{s,t}) = f(x) + \int_s^t (\varepsilon f')(X^x_{s,u})W(du) + \frac{1}{2}\int_s^t f''(X^x_{s,u})du \quad a.s.$$

- (ii) for all $t \geq s$, $K^W_{s,t}(x) = E[\delta_{X^x_{s,t}}|\sigma(W)]$ a.s.

By conditioning with respect to $\sigma(W)$ in (i), this shows that K^W solves (E).
Now, let (K, W) be any other solution of (E) and set $P^n_t = E[K^{\otimes n}_{0,t}]$. From the hypothesis $\alpha^+ = \frac{1}{2}$, we see that $h(x) = \varepsilon(x)|x|$ belongs to $D(\alpha_1,\dots,\alpha_N)$ and by applying h in (E), we get $K_{0,t}h(x) = h(x)+W_t$. Denote by (X^{x_1}, X^{x_2}) the two-point motion started at $(x_1, x_2) \in G^2$ associated to P^2. Since $|X^{x_i}|$ is a reflected Brownian motion started at $|x_i|$ (Theorem 2.6), we have $E[|X^{x_i}_t|^2] = t + |x_i|^2$. From the preceding observation $E[h(X^{x_1}_t)h(X^{x_2}_t)] = E[K_{0,t}h(x_1)K_{0,t}h(x_2)] = h(x_1)h(x_2) + t$ and therefore

$$E[(h(X^{x_1}_t) - h(X^{x_2}_t) - h(x_1) + h(x_2))^2] = 0.$$

This shows that $h(X^{x_1}_t) - h(X^{x_2}_t) = h(x_1) - h(x_2)$. Now we will check by induction on n that P^n does not depend on K. For $n = 1$, this follows from Proposition 2.8. Suppose the result holds for n and let $(x_1, ..., x_{n+1}) \in G^{n+1}$ such that $h(x_i) \neq h(x_j), i \neq j$. Let $\tau_{x_i} = \inf\{r \geq 0 : X_r^{x_i} = 0\} = \inf\{r \geq 0 : h(X_r^{x_i}) = 0\}$ and $(x_i, x_j) \in G^+ \times G^-$ such that $h(x_i) < h(x_k), h(x_h) < h(x_j)$ for all $(x_k, x_h) \in G^+ \times G^-$ (when (x_i, x_j) does not exist the proof is simpler). Clearly τ_{x_k} is a function of X^{x_h} for all $h, k \in [1, n+1]$ and so for all measurable bounded $f : G^{n+1} \longrightarrow \mathbb{R}$,

$$f(X_t^{x_1}, ..., X_t^{x_{n+1}})1\!\!1_{\{t<\tau_{x_i}, \inf_{1\leq k\leq n+1} \tau_{x_k}=\tau_{x_i}\}} \text{ is a function of } X^{x_i}$$

and

$$f(X_t^{x_1}, ..., X_t^{x_{n+1}})1\!\!1_{\{t<\tau_{x_j}, \inf_{1\leq k\leq n+1} \tau_{x_k}=\tau_{x_j}\}} \text{ is a function of } X^{x_j}.$$

where $t > 0$ is fixed. This shows that $E[f(X_t^{x_1}, ..., X_t^{x_{n+1}})1\!\!1_{\{t<\inf_{1\leq k\leq n+1} \tau_{x_k}\}}]$ only depends on P^1. Consider the following stopping times

$$S_0 = \inf_{1\leq i\leq n+1} \tau_{x_i}, S_{k+1} = \inf\{r \geq S_k : \exists j \in [1, n+1], X_r^{x_j} = 0, X_{S_k}^{x_j} \neq 0\}, k \geq 0.$$

Remark that $(S_k)_{k\geq 0}$ is a function of X^{x_h} for all $h \in [1, n+1]$. By summing over all possible cases we need only check the unicity in law of $(X_t^{x_1}, ..., X_t^{x_{n+1}})$ conditionally to $A = \{S_k < t < S_{k+1}, X_{S_k}^{x_h} = 0\}$ where $k \geq 0, h \in [1, n+1]$ are fixed. Write $A = B \cap \{t - S_k < T\}$ where $B = \{S_k < t, X_{S_k}^{x_h} = 0\} = \{S_k < t, X_{S_k}^{x_i} \neq 0 \text{ if } i \neq h\}$ and $T = \inf\{r \geq 0, \exists j \neq h : X_{r+S_k}^{x_j} = 0\}$. On A, $X_t^{x_i}$ is a function of $(X_{S_k}^{x_i}, X_t^{x_h})$ and therefore for all measurable bounded $f : G^{n+1} \longrightarrow \mathbb{R}$,

$$f(X_t^{x_1}, ..., X_t^{x_{n+1}})1\!\!1_A \text{ may be written as } g((X_{S_k}^{x_i})_{i\neq h}, X_t^{x_h})1\!\!1_A$$

where g is measurable bounded from G^{n+1} into $\mathbb{R}$. By the strong Markov property for $X = (X^{x_1}, ..., X^{x_{n+1}})$, we have

$$1\!\!1_B E[1\!\!1_{\{t-S_k<T\}} g((X_{S_k}^{x_i})_{i\neq h}, X_t^{x_h})|\mathcal{F}_{S_k}^X] = 1\!\!1_B \psi(t - S_k, (X_{S_k}^{x_i})_{i\neq h})$$

where

$$\psi(u, y_1, ..., y_n) = E[1\!\!1_{\{u<\inf\{r\geq 0:\exists j\in[1,n], X_r^{y_j}=0\}\}} g(y_1, ..., y_n, X_u^0)].$$

This shows that $E[f(X_t^{x_1}, ..., X_t^{x_{n+1}})1\!\!1_A]$ only depends on the law of $(X^{x_i})_{i\neq h}$. As a result, $P_t^{n+1}((x_1, ..., x_{n+1}), dy)$ is unique whenever $h(x_i) \neq h(x_j), i \neq j$ and by an approximation argument for all $(x_1, ..., x_{n+1}) \in G^{n+1}$. Since a stochastic flow of kernels is uniquely determined by the compatible system of its n-point motions, this proves (2) of Theorem 1.5.

5 Appendix: Freidlin-Sheu formula.

In this section, we shall prove Theorem 2.6. We begin by
Preliminary remarks. We recall that if Y is a semimartingale satisfying $\langle Y \rangle = \langle |Y| \rangle$ then $\tilde{L}_t(Y) = \tilde{L}_t(|Y|)$. Let $L_t(Y)$ be the (non symmetric) local time at 0 of Y and $\alpha \in [0, 1]$. If Y is a $SBM(\alpha)$, then $L_t(Y) = 2\alpha\tilde{L}_t(Y)$ by identifying Tanaka's formulas for symmetric and non symmetric local time for Y.

Let Q be the semigroup of the reflecting Brownian motion on $\mathbb{R}$ and define $\Phi(x) = |x|$. Then $X_t = \Phi(Z_t)$ and it can be easily checked that $P_t(f \circ \Phi) = Q_t f \circ \Phi$ for all bounded measurable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which proves (i). (ii) is an easy consequence of Tanaka's formula for local time.

(iii) Set $\tau_z = \inf\{r \geq 0, Z_r = 0\}$. For $t \leq \tau_z$, (2.2) holds from Itô's formula applied to the semimartingale X. By discussing the cases $t \leq \tau_z$ and $t > \tau_z$, one can assume that $z = 0$ and so in the sequel we take $z = 0$.
For all $i \in [1, N]$, define $Z_t^i = |Z_t|\mathbb{1}_{Z_t \in D_i} - |Z_t|\mathbb{1}_{Z_t \notin D_i}$. Then $Z_t^i = \Phi^i(Z_t)$ where $\Phi^i(x) = |x|\mathbb{1}_{x \in D_i} - |x|\mathbb{1}_{x \notin D_i}$. Let Q^i be the semigroup of the $SBM(\alpha_i)$. Then the following relation is easy to check: $P_t(f \circ \Phi^i) = Q_t^i f \circ \Phi^i$ for all bounded measurable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which shows that Z^i is a $SBM(\alpha_i)$ started at 0. We use the notation $(\mathbb{P})$ to denote the convergence in probability.
Let $\delta > 0$. Define $\tau_0^\delta = \theta_0^\delta = 0$ and for $n \geqslant 1$

$$\theta_n^\delta = \inf\{r \geq \tau_{n-1}^\delta, |Z_r| = \delta\}, \quad \tau_n^\delta = \inf\{r \geq \theta_n^\delta, Z_r = 0\}.$$

Let $f \in C_b^2(G^*)$ and $t > 0$. Then

$$f(Z_t) - f(0) = \sum_{n=0}^{\infty} f(Z_{\theta_{n+1}^\delta \wedge t}) - f(Z_{\theta_n^\delta \wedge t}) = Q_1^\delta + Q_2^\delta + Q_3^\delta$$

where

$$Q_1^\delta = \sum_{n=0}^{\infty}(f(Z_{\theta_{n+1}^\delta \wedge t}) - f(Z_{\tau_n^\delta \wedge t})) - \sum_{i=1}^{N}\sum_{n=0}^{\infty} \delta f_i'(0+)\mathbb{1}_{\{\theta_{n+1}^\delta \leq t, Z_{\theta_{n+1}^\delta} \in D_i\}},$$

$$Q_2^\delta = \sum_{i=1}^{N}\sum_{n=0}^{\infty} \delta f_i'(0+)\mathbb{1}_{\{\theta_{n+1}^\delta \leq t, Z_{\theta_{n+1}^\delta} \in D_i\}}, \quad Q_3^\delta = \sum_{n=0}^{\infty} f(Z_{\tau_n^\delta \wedge t}) - f(Z_{\theta_n^\delta \wedge t}).$$

We first show that $Q_1^\delta \xrightarrow[\delta \to 0]{} 0$ $(\mathbb{P})$ and for this write $Q_1^\delta = Q_{(1,1)}^\delta + Q_{(1,2)}^\delta$ with

$$Q_{(1,1)}^\delta = \sum_{n=0}^{\infty}\sum_{i=1}^{N}(f(Z_{\theta_{n+1}^\delta}) - f(Z_{\tau_n^\delta}) - \delta f_i'(0+))\mathbb{1}_{\{\theta_{n+1}^\delta \leq t, Z_{\theta_{n+1}^\delta} \in D_i\}},$$

$$Q_{(1,2)}^\delta = \sum_{n=0}^{\infty}\sum_{i=1}^{N}(f(Z_t) - f(Z_{\tau_n^\delta \wedge t}))\mathbb{1}_{\{\theta_{n+1}^\delta > t, Z_{\theta_{n+1}^\delta} \in D_i\}}.$$

Since $f \in C_b^2(G^*)$, we have

(i) $\forall i \in [1, N]; \quad \int_0^\delta (f_i'(u) - f_i'(0+))du = f_i(\delta) - f_i(0) - \delta f_i'(0+).$

(ii) There exists $M > 0$ such that $\forall i \in [1, N], \ u \geq 0 : |f_i'(u) - f_i'(0+)| \leq Mu.$

Consequently

$$|Q_{(1,1)}^\delta| = |\sum_{n=0}^{\infty}\sum_{i=1}^{N}(f_i(\delta) - f_i(0) - \delta f_i'(0+))\mathbb{1}_{\{\theta_{n+1}^\delta \leq t, Z_{\theta_{n+1}^\delta} \in D_i\}}| \leq \frac{NM\delta^2}{2}\sum_{n=0}^{\infty}\mathbb{1}_{\theta_{n+1}^\delta \leq t}.$$

It is known that $\delta \sum_{n=0}^{\infty} \mathbb{1}_{\theta_{n+1}^\delta \leq t} \xrightarrow[\delta \to 0]{} \frac{1}{2}L_t(X)$ $(\mathbb{P})$ ([16]) and therefore $Q_{(1,1)}^\delta \xrightarrow[\delta \to 0]{} 0$ $(\mathbb{P})$.
Let $C > 0$ such that $\forall i \in [1, N], \ u \geq 0 : |f_i(u) - f_i(0)| \leq Cu$. Then

$$\begin{aligned}
|Q_{(1,2)}^\delta| &= |\sum_{n=0}^{\infty}\sum_{i=1}^{N}(f(Z_t) - f(Z_{\tau_n^\delta \wedge t}))\mathbb{1}_{\{\theta_{n+1}^\delta > t, Z_{\theta_{n+1}^\delta} \in D_i\}}| \\
&\leqslant \sum_{n=0}^{\infty}\sum_{i=1}^{N}|f_i(X_t) - f_i(0)|\mathbb{1}_{\{\tau_n^\delta < t < \theta_{n+1}^\delta, Z_{\theta_{n+1}^\delta} \in D_i\}} \\
&\leqslant CX_t \sum_{n=0}^{\infty}\mathbb{1}_{\{\tau_n^\delta < t < \theta_{n+1}^\delta\}} \leq C\delta
\end{aligned}$$

which shows that $Q^\delta_{(1,2)} \xrightarrow[\delta\to 0]{} 0$ a.s and so $Q^\delta_1 \xrightarrow[\delta\to 0]{} 0$ $(\mathbb{P})$.

Now define $Q^\delta_{(2,i)} = \delta \sum_{n=0}^{\infty} 1\!\!1_{\{\theta^\delta_{n+1}\le t, Z_{\theta^\delta_{n+1}}\in D_i\}}$. Since $\sum_{n=0}^{\infty} 1\!\!1_{\{\theta^\delta_{n+1}\le t, Z_{\theta^\delta_{n+1}}\in D_i\}}$ is the number of upcrossings of Z^i from 0 to δ before time t, we have $Q^\delta_{(2,i)} \xrightarrow[\delta\to 0]{} \frac{1}{2}L_t(Z^i)$ $(\mathbb{P})$. Using our preliminary remarks, we see that $Q^\delta_2 \xrightarrow[\delta\to 0]{} (\sum_{i=1}^{N} \alpha_i f_i'(0+))\tilde{L}_t(X)$ $(\mathbb{P})$.

We now establish that $Q^\delta_3 \xrightarrow[\delta\to 0]{} \int_0^t f'(Z_s)dB_s + \frac{1}{2}\int_0^t f''(Z_s)ds$ $(\mathbb{P})$. For this write $Q^\delta_3 = Q^\delta_{(3,1)} + Q^\delta_{(3,2)}$ with

$$Q^\delta_{(3,1)} = \sum_{n=0}^{\infty} (f(Z_{\tau^\delta_n}) - f(Z_{\theta^\delta_n}))1\!\!1_{\{\tau^\delta_n \le t\}} = \sum_{n=0}^{\infty}\sum_{i=1}^{N} (f(0) - f_i(\delta))1\!\!1_{\{\tau^\delta_n\le t, Z_{\theta^\delta_n}\in D_i\}},$$

$$Q^\delta_{(3,2)} = \sum_{i=1}^{N} (f(Z_t) - f_i(\delta))\sharp\{n\in\mathbb{N} : \theta^\delta_n < t < \tau^\delta_n, Z_{\theta^\delta_n}\in D_i\}.$$

It is clear that $\sharp\{n\in\mathbb{N} : \theta^\delta_n < t < \tau^\delta_n, Z_{\theta^\delta_n}\in D_i\} \xrightarrow[\delta\to 0]{} 1\!\!1_{\{Z_t\in \mathring{D}_i\}}$ a.s and so $Q^\delta_{(3,2)}$ converges to $f(Z_t) - f(0)$ as $\delta\to 0$ a.s. Define $\tau^{\delta,i}_0 = \theta^{\delta,i}_0 = 0$ and

$$\theta^{\delta,i}_n = \inf\{r \ge \tau^{\delta,i}_{n-1}, Z_r = \delta\vec{e_i}\}; \quad \tau^{\delta,i}_n = \inf\{r\ge \theta^{\delta,i}_n, Z_r = 0\}, n\geqslant 1.$$

Using $\sum_{n=0}^{\infty} 1\!\!1_{\{\tau^\delta_n\le t, Z_{\theta^\delta_n}\in D_i\}} = \sum_{n=0}^{\infty} 1\!\!1_{\{\tau^{\delta,i}_n\le t\}}$, we get $Q^\delta_{(3,1)} = \sum_{n=0}^{\infty}\sum_{i=1}^{N}(f(0)-f_i(\delta))1\!\!1_{\{\tau^{\delta,i}_n\le t\}}$.
On the other hand

$$\sum_{n=0}^{\infty}(f_i(X_{\tau^{\delta,i}_n\wedge t}) - f_i(X_{\theta^{\delta,i}_n\wedge t})) = \sum_{n=0}^{\infty}(f_i(X_{\tau^{\delta,i}_n}) - f_i(X_{\theta^{\delta,i}_n}))1\!\!1_{\{\tau^{\delta,i}_n\le t\}} + \sum_{n=0}^{\infty}(f_i(X_t) - f_i(0))1\!\!1_{\{\theta^{\delta,i}_n < t < \tau^{\delta,i}_n\}}$$

and therefore

$$Q^\delta_{(3,1)} = \sum_{i=1}^{N}\sum_{n=0}^{\infty}(f_i(X_{\tau^{\delta,i}_n\wedge t}) - f_i(X_{\theta^{\delta,i}_n\wedge t})) - \sum_{i=1}^{N}(f_i(X_t) - f_i(0))\times\sharp\{n\in\mathbb{N}, \theta^{\delta,i}_n < t < \tau^{\delta,i}_n\}.$$

Since $\sharp\{n\in\mathbb{N}, \theta^{\delta,i}_n < t < \tau^{\delta,i}_n\} \xrightarrow[\delta\to 0]{} 1\!\!1_{\{Z_t\in\mathring{D}_i\}}$ a.s, we deduce that

$$Q^\delta_3 \overset{\delta\to 0}{=} \sum_{i=1}^{N}\sum_{n=0}^{\infty}(f_i(X_{\tau^{\delta,i}_n\wedge t}) - f_i(X_{\theta^{\delta,i}_n\wedge t})) + o(1) \ \ a.s.$$

For all $i\in[1,N]$, let $\tilde{f}_i$ be C^2 on $\mathbb{R}$ such that $\tilde{f}_i = f_i$ on $\mathbb{R}_+$, $\tilde{f}_i' = f_i'$, $\tilde{f}_i'' = f_i''$ on $\mathbb{R}^*_+$
Now a.s:

$$\forall s\in[0,t], i\in[1,N] \quad \sum_{n=0}^{\infty} 1\!\!1_{[\theta^{\delta,i}_n\wedge t, \tau^{\delta,i}_n\wedge t[}(s) \xrightarrow[\delta\to 0]{} 1\!\!1_{\{Z_s\in\mathring{D}_i\}}.$$

By dominated convergence for stochastic integrals, we have:

$$\sum_{n=0}^{\infty}\tilde{f}_i(X_{\tau^{\delta,i}_n\wedge t}) - \tilde{f}_i(X_{\theta^{\delta,i}_n\wedge t}) = \int_0^t \sum_{n=0}^{\infty} 1\!\!1_{[\theta^{\delta,i}_n\wedge t, \tau^{\delta,i}_n\wedge t[}(s) d\tilde{f}_i(X_s) \xrightarrow[\delta\to 0]{} \int_0^t 1\!\!1_{\{Z_s\in\mathring{D}_i\}} d\tilde{f}_i(X_s) \ (\mathbb{P}).$$

Finally

$$\begin{aligned}\int_0^t f'(Z_s)dB_s + \frac{1}{2}\int_0^t f''(Z_s)ds &= \sum_{i=1}^{N}\int_0^t 1\!\!1_{\{Z_s\in\mathring{D}_i\}}(f_i'(X_s)dB_s + \frac{1}{2}f_i''(X_s)ds)\\ &= \sum_{i=1}^{N}\int_0^t 1\!\!1_{\{Z_s\in\mathring{D}_i\}}d\tilde{f}_i(X_s).\end{aligned}$$

by Itô's formula and using the fact that $d\tilde{L}_s(X)$ is carried by $\{s; Z_s = 0\}$. Now the proof of Theorem 2.6 is complete.

Acknowledgements. I sincerely thank my supervisor Yves Le Jan for his guidance through my Ph.D. thesis and for his careful reading of this paper. I'am also grateful to Olivier Raimond for very useful discussions, for his careful reading of this paper and for his assistance in proving Lemma 3.3.

References

[1] Martin Barlow, Krzysztof Burdzy, Haya Kaspi, and Avi Mandelbaum. Coalescence of skew Brownian motions. In *Séminaire de Probabilités, XXXV*, volume 1755 of *Lecture Notes in Math.*, pages 202–205. Springer, Berlin, 2001.

[2] Martin Barlow, Jim Pitman, and Marc Yor. On Walsh's Brownian motions. In *Séminaire de Probabilités, XXIII*, volume 1372 of *Lecture Notes in Math.*, pages 275–293. Springer, Berlin, 1989.

[3] Krzysztof Burdzy and Zhen-Qing Chen. Local time flow related to skew Brownian motion. *Ann. Probab.*, 29(4):1693–1715, 2001.

[4] Krzysztof Burdzy and Haya Kaspi. Lenses in skew Brownian flow. *Ann. Probab.*, 32(4):3085–3115, 2004.

[5] Mark Freidlin and Shuenn-Jyi Sheu. Diffusion processes on graphs: stochastic differential equations, large deviation principle. *Probab. Theory Related Fields*, 116(2):181–220, 2000.

[6] J. M. Harrison and L. A. Shepp. On skew Brownian motion. *Ann. Probab.*, 9(2):309–313, 1981.

[7] T. Jeulin. Grossissement d'une filtration et applications. In *Séminaire de Probabilités, XIII (Univ. Strasbourg, Strasbourg, 1977/78)*, volume 721 of *Lecture Notes in Math.*, pages 574–609. Springer, Berlin, 1979.

[8] Hiroshi Kunita. *Stochastic flows and stochastic differential equations*, volume 24 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997. Reprint of the 1990 original.

[9] Yves Le Jan and Olivier Raimond. Integration of Brownian vector fields. *Ann. Probab.*, 30(2):826–873, 2002.

[10] Yves Le Jan and Olivier Raimond. Flows, coalescence and noise. *Ann. Probab.*, 32(2):1247–1315, 2004.

[11] Yves Le Jan and Olivier Raimond. Flows associated to Tanaka's SDE. *ALEA Lat. Am. J. Probab. Math. Stat.*, 1:21–34, 2006.

[12] LeGall.J.F. *Mouvement brownien Processus de Branchement et Superprocessus*. 1994. Notes de cours de DEA paris 6.

[13] LeGall.J.F. *Calcul stochastique et processus de Markov*. 2010. Notes de cours de Master 2 paris sud.

[14] Antoine Lejay. On the constructions of the skew Brownian motion. *Probab. Surv.*, 3:413–466 (electronic), 2006.

[15] Bertrand Micaux. *Flots stochastiques d'opérateurs dirigés par des bruits Gaussiens et Poissonniens*. 2007. Thèse présentée pour obtenir le grade de docteur en sciences de l'université Paris XI, spécialité mathématiques.

[16] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

[17] B. Tsirelson. Triple points: from non-Brownian filtrations to harmonic measures. *Geom. Funct. Anal.*, 7(6):1096–1142, 1997.

[18] J.B Walsh. *A diffusion with discontinuous local time*, volume 52 of *Temps locaux Astérisque*. Société Mathématique de France, Paris, 1978.

[19] S. Watanabe. The stochastic flow and the noise associated to Tanaka's stochastic differential equation. *Ukraïn. Mat. Zh.*, 52(9):1176–1193, 2000.

[20] Ludmila L. Zaitseva. On stochastic continuity of generalized diffusion processes constructed as the strong solution to an SDE. *Theory Stoch. Process.*, 11(1-2):125–135, 2005.