The relativistic mean-field equations of the atomic nucleus

Simona Rota Nodari

▶ **To cite this version:**

Simona Rota Nodari. The relativistic mean-field equations of the atomic nucleus. 2011. hal-00553265v1

HAL Id: hal-00553265

https://hal.science/hal-00553265v1

Preprint submitted on 6 Jan 2011 (v1), last revised 18 Nov 2011 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

THE RELATIVISTIC MEAN-FIELD EQUATIONS OF THE ATOMIC NUCLEUS

SIMONA ROTA NODARI

ABSTRACT. In nuclear physics, the relativistic mean-field theory describes the nucleus as a system of Dirac nucleons which interact via meson fields. In a static case and without nonlinear self-coupling of the σ meson, the relativistic mean-field equations become a system of Dirac equations where the potential is given by the meson and photon fields. The aim of this work is to prove the existence of solutions of these equations. We consider a minimization problem with constraints that involve negative spectral projectors and we apply the concentration-compactness lemma to find a minimizer of this problem. We show that this minimizer is a solution of the relativistic mean-field equations considered.

1. INTRODUCTION

In this paper, we present the first mathematically rigorous result concerning the existence of solutions of the relativistic mean-field equations of the atomic nucleus in a static case and without nonlinear self-coupling of the σ meson.

In nuclear physics, the relativistic mean-field RMF theory describes the nucleus as a system of Dirac nucleons which interact in a relativistic covariant manner via meson fields. During the last years, the relativistic mean-field theory has received wide attention due to its successful description of lots of nuclear phenomena. The relativistic mean-field model is considered to be the relativistic generalization of the nonrelativistic models such as the Skyrme force or the Gogny force Hartree-Fock theory, using effective mesonic degrees of freedom rather than instantaneous forces. The relativistic model describes successfully the single-particle structure of nuclei as the nonrelativistic ones and provides a natural explanation of some relativistic effects as the spin-orbit force (see [1],[2],[3],[4]).

The model is formulated on the basis of two approximations, the mean-field and the no-sea approximation. Thanks to the mean-field approximation, the fields for the mesons and the photons are treated as classical fields and the nucleons behave as noninteracting particles moving in these mean fields. This implies that the nucleon field operator can be expanded in single-particles states $\psi_\alpha(x^\mu)$,

$$\psi = \sum_\alpha \psi_\alpha(x^\mu)\,\hat{a}_\alpha \tag{1.1}$$

where $\hat{a}_\alpha$ is the annihilation operator for a nucleon in the state α, while the densities become simple bilinear sums over the ψ_α. The no-sea approximation corresponds to neglecting the vacuum polarization, that means that we have a number of occupied single-particle orbitals ψ_α, $\alpha = 1, \ldots, \Omega$, which determines the densities.

Date: January 6, 2011.

The Lagrangian density of the RMF theory can be written as

$$\mathcal{L} = \mathcal{L}_{nucleons} + \mathcal{L}_{mesons} + \mathcal{L}_{coupling}. \tag{1.2}$$

The free Lagrangian for the nucleons is

$$\mathcal{L}_{nucleons} = \sum_{\alpha=1}^{\Omega} w_\alpha \bar{\psi}_\alpha (i\gamma^\mu \partial_\mu - m_b)\psi_\alpha \tag{1.3}$$

where m_b denotes the nucleon mass, γ^μ are the Dirac matrices and w_α are occupation weights, $0 \leq w_\alpha \leq 1$. The Lagrangian for the free meson fields is

$$\begin{aligned} \mathcal{L}_{mesons} &= \frac{1}{2}(\partial^\mu \sigma \partial_\mu \sigma - m_\sigma^2 \sigma^2) \\ &\quad -\frac{1}{2}(\overline{\partial^\mu \omega^\nu}\partial_\mu \omega_\nu - m_\omega^2 \omega^\mu \omega_\mu) \\ &\quad -\frac{1}{2}(\overline{\partial^\mu \boldsymbol{R}^\nu} \cdot \partial_\mu \boldsymbol{R}_\nu - m_\rho^2 \boldsymbol{R}^\mu \cdot \boldsymbol{R}_\mu) \\ &\quad -\frac{1}{2}\overline{\partial^\mu A^\nu}\partial_\mu A_\nu \end{aligned} \tag{1.4}$$

where σ, ω^μ and $\boldsymbol{R}^\mu$ describe respectively the σ, ω and ρ meson field, and A^μ stands for the photon field. Moreover, an antisymmetrized derivative is defined via

$$\overline{\partial^\mu A^\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu.$$

We remind that the σ meson is an isoscalar scalar meson which provides a medium range attractive interaction, the ω meson is an isoscalar vector meson leading to a short range repulsive interaction, the ρ meson is an isovector vector meson needed for a better description of isospin-dependent effects in the nuclei, and the photon describes the electromagnetic interaction.
Finally, the Lagrangian for the coupling is

$$\mathcal{L}_{coupling} = -g_\sigma \sigma \rho_s - g_\omega \omega^\mu \rho_\mu - g_\rho \boldsymbol{R}^\mu \cdot \boldsymbol{\rho}_\mu - eA^\mu \rho_\mu^c - U(\sigma) \tag{1.5}$$

where $U(\sigma) = \frac{1}{3}b_2\sigma^3 + \frac{1}{4}b_3\sigma^4$ represents a nonlinear self-coupling of the σ meson. Note that in Reinhard's paper [1] the coupling constant for the ρ meson is $2g_\rho$.
The densities are

$$\rho_s = \sum_{\alpha=1}^{\Omega} w_\alpha \bar{\psi}_\alpha \psi_\alpha, \tag{1.6}$$

$$\rho_\mu = \sum_{\alpha=1}^{\Omega} w_\alpha \bar{\psi}_\alpha \gamma_\mu \psi_\alpha, \tag{1.7}$$

$$\boldsymbol{\rho}_\mu = \sum_{\alpha=1}^{\Omega} w_\alpha \bar{\psi}_\alpha \boldsymbol{\tau} \gamma_\mu \psi_\alpha, \tag{1.8}$$

$$\rho_\mu^c = \sum_{\alpha=1}^{\Omega} w_\alpha \bar{\psi}_\alpha \frac{1}{2}(1+\tau_0)\gamma_\mu \psi_\alpha. \tag{1.9}$$

We remind that $\boldsymbol{R}$, $\boldsymbol{\rho}$ and $\boldsymbol{\tau}$ are vectors in isospin space and $\cdot$ denotes the vector product therein.
The model contains as free parameters the meson masses m_σ, m_ω and m_ρ, as well as the coupling constants g_σ, g_ω, g_ρ, b_2 and b_3. For the nucleon mass m_b the free value is usually employed.

Most applications of the relativistic mean-field model are concerned with stationary states; then, like in [1], we want to derive the field equations for the static case. Moreover, we remark that it is generally true that proton and neutron states do not mix, that means that the single-particle states are eingenstates of τ_0. As a consequence, only the components with isospin projection 0 appear, i.e. $R_{0\mu}$ and $\rho_{0\mu}$.
Stationarity implies that all time derivatives and also the spatial components of densities and fields vanish; only the fields σ, ω_0, R_{00} and A_0 remain and they are independent of time. Furthermore, the single-particle wave functions may be separated as

$$\psi_\alpha(\boldsymbol{x},t) = e^{-i\varepsilon_\alpha t}\psi_\alpha(\boldsymbol{x}) \tag{1.10}$$

where the ε_α are the single-particle energies and $\varepsilon_\alpha > 0$.
Varying the action integral $S = \int \mathcal{L}\, dx^4$ with respect to the wave functions and to the fields with all the above simplifications inserted yields

$$\begin{aligned}
\varepsilon_\alpha\gamma_0\psi_\alpha &= [-i\boldsymbol{\gamma}\cdot\nabla + m_b + g_\sigma\sigma + g_\omega\omega_0\gamma_0 \\
&\quad + g_\rho R_{00}\gamma_0\tau_0 + \frac{1}{2}eA_0\gamma_0(1+\tau_0)\Big]\psi_\alpha, && (1.11)\\
(-\Delta + m_\sigma^2)\sigma + U'(\sigma) &= -g_\sigma\rho_s, && (1.12)\\
(-\Delta + m_\omega^2)\omega_0 &= g_\omega\rho_0, && (1.13)\\
(-\Delta + m_\rho^2)R_{00} &= g_\rho\rho_{00}, && (1.14)\\
-\Delta A_0 &= e\rho_0^c. && (1.15)
\end{aligned}$$

This set of equations, together with the definition of the densities, constitutes a self-consistent field problem that can be solved numerically using an iterative scheme (see [1], [5]).

In this paper, we consider the case without nonlinear self-coupling of the σ meson, i.e. $b_2 = b_3 = 0$, and we choose a fixed occupation, that means that the occupation weights w_α are defined as

$$w_\alpha = \begin{cases} 1 & \alpha = 1,\ldots,A \\ 0 & \text{otherwise} \end{cases} \tag{1.16}$$

where A is the nucleon number.
In this case, the equations (1.12-1.15) can be solved explicitly and we obtain

$$\begin{aligned}
\sigma &= -\frac{g_\sigma}{4\pi}\left(\frac{e^{-m_\sigma|\cdot|}}{|\cdot|}\star\rho_s\right), && (1.17)\\
\omega_0 &= \frac{g_\omega}{4\pi}\left(\frac{e^{-m_\omega|\cdot|}}{|\cdot|}\star\rho_0\right), && (1.18)\\
R_{00} &= \frac{g_\rho}{4\pi}\left(\frac{e^{-m_\rho|\cdot|}}{|\cdot|}\star\rho_{00}\right), && (1.19)\\
A_0 &= \frac{e}{4\pi}\left(\frac{1}{|\cdot|}\star\rho_0^c\right). && (1.20)
\end{aligned}$$

Hence, the equation (1.11) becomes

$$\begin{aligned}\varepsilon_\alpha\psi_\alpha &= \left[H_0 - \beta\frac{g_\sigma^2}{4\pi}\left(\frac{e^{-m_\sigma|\cdot|}}{|\cdot|}\star\rho_s\right) + \frac{g_\omega^2}{4\pi}\left(\frac{e^{-m_\omega|\cdot|}}{|\cdot|}\star\rho_0\right)\right. \\ &\left. +\tau_0\frac{g_\rho^2}{4\pi}\left(\frac{e^{-m_\rho|\cdot|}}{|\cdot|}\star\rho_{00}\right) + \frac{1}{2}(1+\tau_0)\frac{e^2}{4\pi}\left(\frac{1}{|\cdot|}\star\rho_0^c\right)\right]\psi_\alpha \end{aligned} \tag{1.21}$$

where $H_0 = -i\boldsymbol{\alpha}\cdot\nabla + \beta m_b$ is the free Dirac operator,

$$\beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}$$

for $k = 1, 2, 3$, with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The operator H_0 acts on 4-spinors, i.e. functions $\psi \in \mathcal{H} := L^2(\mathbb{R}^3, \mathbb{C}^4)$. It is self-adjoint on $\mathcal{H}$, with domain $H^1(\mathbb{R}^3, \mathbb{C}^4)$ and form-domain $E := H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$. Moreover, it is defined to ensure

$$H_0^2 = -\Delta + m_b^2.$$

The spectrum of H_0 is $(-\infty, -m_b] \cup [m_b, +\infty)$, and the projector associated with the negative (resp. positive) part of the spectrum of H_0 will be denoted by Λ^- (resp. Λ^+). Finally, we endow the space E with the norm $\|\psi\|_E^2 := (\psi, |H_0|\,\psi)_{L^2}$.

Using the convention $\tau_0 = 1$ for the protons and $\tau_0 = -1$ for the neutrons, the densities can be written as

$$\rho_s = \sum_{k=1}^{A} \bar{\psi}_k\psi_k, \tag{1.22}$$

$$\rho_0 = \sum_{k=1}^{A} \psi_k^*\psi_k, \tag{1.23}$$

$$\rho_{00} = \sum_{k=1}^{Z} \psi_k^*\psi_k - \sum_{k=Z+1}^{A} \psi_k^*\psi_k, \tag{1.24}$$

$$\rho_0^c = \sum_{k=1}^{Z} \psi_k^*\psi_k \tag{1.25}$$

with Z the number of protons, $N = A - Z$ the number of neutrons and $\bar{\psi}_i = \beta\psi_i^*$; furthermore, the nonlinear Dirac equations are given by

$$\begin{aligned} H_{p,\Psi}\psi_i &:= \left[H_0 - \beta\frac{g_\sigma^2}{4\pi}\left(\frac{e^{-m_\sigma|\cdot|}}{|\cdot|}\star\rho_s\right) + \frac{g_\omega^2}{4\pi}\left(\frac{e^{-m_\omega|\cdot|}}{|\cdot|}\star\rho_0\right)\right. \\ &\left. +\frac{g_\rho^2}{4\pi}\left(\frac{e^{-m_\rho|\cdot|}}{|\cdot|}\star\rho_{00}\right) + \frac{e^2}{4\pi}\left(\frac{1}{|\cdot|}\star\rho_0^c\right)\right]\psi_i = \varepsilon_i\psi_i \end{aligned} \tag{1.26}$$

if $1 \leq i \leq Z$, and

$$H_{n,\Psi}\psi_i := \left[H_0 - \beta\frac{g_\sigma^2}{4\pi}\left(\frac{e^{-m_\sigma|\cdot|}}{|\cdot|}\star\rho_s\right) + \frac{g_\omega^2}{4\pi}\left(\frac{e^{-m_\omega|\cdot|}}{|\cdot|}\star\rho_0\right) - \frac{g_\rho^2}{4\pi}\left(\frac{e^{-m_\rho|\cdot|}}{|\cdot|}\star\rho_{00}\right)\right]\psi_i = \varepsilon_i\psi_i \tag{1.27}$$

if $Z+1 \leq i \leq A$, with $\Psi = (\psi_1, \ldots, \psi_Z, \psi_{Z+1}, \ldots, \psi_A)$ and under the constraints $\int_{\mathbb{R}^3}\psi_i^*\psi_j = \delta_{ij}$ for $1 \leq i,j \leq Z$ and for $Z+1 \leq i,j \leq A$.
In what follows, $V_{p,\Psi}$ and $V_{n,\Psi}$ denote the potentials of the nonlinear Dirac equations, namely $V_{\mu,\Psi} = H_{\mu,\Psi} - H_0$ for $\mu = p, n$.

Note that the scalars ε_i can be seen as Lagrange multipliers; indeed, the nonlinear Dirac equations are the Euler-Lagrange equations of the energy functional

$$\begin{aligned}\mathcal{E}(\Psi) &= \sum_{j=1}^{A}\int_{\mathbb{R}^3}\psi_j^* H_0\psi_j - \frac{g_\sigma^2}{8\pi}\int\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\rho_s(x)\rho_s(y)}{|x-y|}e^{-m_\sigma|x-y|}\,dxdy \\ &+\frac{g_\omega^2}{8\pi}\int\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\rho_0(x)\rho_0(y)}{|x-y|}e^{-m_\omega|x-y|}\,dxdy \\ &+\frac{g_\rho^2}{8\pi}\int\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\rho_{00}(x)\rho_{00}(y)}{|x-y|}e^{-m_\rho|x-y|}\,dxdy \\ &+\frac{e^2}{8\pi}\int\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\rho_0^c(x)\rho_0^c(y)}{|x-y|}\,dxdy\end{aligned} \tag{1.28}$$

under the constraints $\int_{\mathbb{R}^3}\psi_i^*\psi_j = \delta_{ij}$ for $1 \leq i,j \leq Z$ and for $Z+1 \leq i,j \leq A$.
Since this functional is not bounded from below under the constraints $\int_{\mathbb{R}^3}\psi_i^*\psi_j = \delta_{ij}$, as in [6] (see also [7]), we introduce the following minimization problem

$$\begin{aligned}I &= \inf\Big\{\mathcal{E}(\Psi); \Psi \in (H^{1/2})^A, \int_{\mathbb{R}^3}\psi_i^*\psi_j = \delta_{ij}, 1 \leq i,j \leq Z, Z+1 \leq i,j \leq A, \\ &\Lambda_{p,\Psi}^-(\psi_1,\ldots,\psi_Z) = 0,\, \Lambda_{n,\Psi}^-(\psi_{Z+1},\ldots,\psi_A) = 0\Big\}\end{aligned} \tag{1.29}$$

together with its extension

$$\begin{aligned}I(\lambda_1,\ldots,\lambda_A) &= \inf\Big\{\mathcal{E}(\Psi); \Psi \in (H^{1/2})^A, \int_{\mathbb{R}^3}\psi_i^*\psi_j = \lambda_i\delta_{ij}, 1 \leq i,j \leq Z, \\ &Z+1 \leq i,j \leq A,\, \Lambda_{p,\Psi}^-(\psi_1,\ldots,\psi_Z) = 0, \\ &\Lambda_{n,\Psi}^-(\psi_{Z+1},\ldots,\psi_A) = 0\Big\}\end{aligned} \tag{1.30}$$

where, for $\mu = p, n$, $\Lambda_{\mu,\Psi}^- = \chi_{(-\infty,0)}(H_{\mu,\Psi})$ is the negative spectral projector of the operator $H_{\mu,\Psi}$,

$$\Lambda_{p,\Psi}^-(\psi_1,\ldots,\psi_Z) = (\Lambda_{p,\Psi}^-\psi_1,\ldots,\Lambda_{p,\Psi}^-\psi_Z)$$

and

$$\Lambda_{n,\Psi}^-(\psi_{Z+1},\ldots,\psi_A) = (\Lambda_{n,\Psi}^-\psi_{Z+1},\ldots,\Lambda_{n,\Psi}^-\psi_A).$$

In this paper, we prove that, for $g_\sigma, g_\omega, g_\rho$ and e sufficiently small, a solution of the equations (1.26) and (1.27) can be obtained as a solution of the minimization problem (1.29).

Theorem 1.1. *If $g_\sigma, g_\omega, g_\rho$ and e are sufficiently small, a minimizer of (1.29) is a solution of the equations (1.26) and (1.27).*

Moreover, the application of the concentration-compactness method ([8],[9]) to the minimization problem (1.29) yields the following theorem that is our main result.

Theorem 1.2. *If $g_\sigma, g_\omega, g_\rho$ and e are sufficiently small, any minimizing sequence of (1.29) is relatively compact up to a translation if and only if the following condition holds*

$$I < I(\lambda_1, \dots, \lambda_A) + I(1-\lambda_1, \dots, 1-\lambda_A) \tag{1.31}$$

for all $\lambda_k \in [0,1]$, $k = 1, \dots, A$, such that $\sum_{k=1}^{A} \lambda_k \in (0, A)$.
In particular, if (1.31) holds, there exists a minimum of (1.29).

This result is relevant both from mathematical and physical point of view since it provides a condition that ensures the existence of a ground state solution of the equations (1.26) and (1.27). Furthermore, this is the first result relating the existence of critical points of a strongly indefinite energy functional to strict concentration-compactness inequalities.

The condition $g_\sigma, g_\omega, g_\rho$ and e sufficiently small means that we are in a weakly relativistic regime. In our proof of the theorems 1.1 and 1.2, this condition is required for several reasons. First of all, if $g_\sigma, g_\omega, g_\rho$ and e are sufficiently small, we can show that $H_{\mu,\Psi}$ is a self-adjoint isomorphism between $H^{1/2}$ and its dual $H^{-1/2}$, whose inverse is bounded independently of Ψ. Moreover, we need this condition to prove that a minimizing sequence of (1.29) is bounded in $\left(H^{1/2}(\mathbb{R}^3)\right)^A$. Finally, in both theorems, we have to apply the implicit function theorem with $g_\sigma, g_\omega, g_\rho$ and e as parameters.

In section 2, we introduce some useful properties of the potential $V_{\mu,\Psi}$ and of the operator $H_{\mu,\Psi}$ for $\mu = p, n$. In section 3, we show how we can apply the concentration-compactness argument to the minimization problem (1.29). Finally, in the last section, we prove the theorem 1.1.

2. Properties of the potential $V_{\mu,\Psi}$

In this section, we describe some useful properties of the potential $V_{\mu,\Psi}$ and we give a condition on the parameters $(g_\sigma, g_\omega, g_\rho, e, N, Z)$ which implies that $H_{\mu,\Psi}$ is a self-adjoint isomorphism and its inverse is bounded independently of Ψ.

Lemma 2.1. *For any $\Psi \in \left(H^{1/2}(\mathbb{R}^3)\right)^A$,*

$$\begin{aligned} V_{p,\Psi} &\in L^r(\mathbb{R}^3), \quad 3 < r < \infty \\ V_{n,\Psi} &\in L^r(\mathbb{R}^3), \quad 1 \le r < \infty. \end{aligned}$$

Proof. The proof of this lemma is an application of Young's inequality : if $f \in L^p(\mathbb{R}^3)$, $g \in L^q(\mathbb{R}^3)$, then

$$\|f \star g\|_{L^r} \le \|f\|_{L^p} \|g\|_{L^q}$$

with $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, $1 \le p, q, r \le \infty$.
We remark that if $\Psi \in \left(H^{1/2}(\mathbb{R}^3)\right)^A$, then ρ_s, ρ_0, ρ_{00} and ρ_0^c are in $L^p(\mathbb{R}^3)$ for $1 \le p \le \frac{3}{2}$.
Furthermore, using the definition of the Gamma function, we can show that, for any $\lambda > 0$, $\frac{e^{-\lambda|x|}}{|x|} \in L^q(\mathbb{R}^3)$ for $1 \le q < 3$.
Finally, we observe that $\frac{1}{|x|}$ can be written as $\frac{1}{|x|} = h_1(x) + h_2(x)$ with $h_1 \in L^\alpha(\mathbb{R}^3)$

for $1 \leq \alpha < 3$ and $h_2 \in L^\beta(\mathbb{R}^3)$ for $3 < \beta \leq \infty$, where $h_1(x) = \frac{1}{|x|}$ for $|x| \leq 1$, $h_1(x) = 0$ otherwise.
Hence,

$$V_{p,\Psi} = \underbrace{-\beta\frac{g_\sigma^2}{4\pi}\left(\frac{e^{-m_\sigma|\cdot|}}{|\cdot|}\star\rho_s\right)+\frac{g_\omega^2}{4\pi}\left(\frac{e^{-m_\omega|\cdot|}}{|\cdot|}\star\rho_0\right)+\frac{g_\rho^2}{4\pi}\left(\frac{e^{-m_\rho|\cdot|}}{|\cdot|}\star\rho_{00}\right)}_{\in L^1(\mathbb{R}^3)\bigcap L^{r_c}(\mathbb{R}^3)}$$
$$+\underbrace{\frac{e^2}{4\pi}\left(h_1(x)\star\rho_0^c\right)}_{\in L^1(\mathbb{R}^3)\bigcap L^{r_c}(\mathbb{R}^3)}+\underbrace{\frac{e^2}{4\pi}\left(h_2(x)\star\rho_0^c\right)}_{\in L^r(\mathbb{R}^3),\ 3<r\leq r_c}\in L^r(\mathbb{R}^3)$$

for $3 < r \leq r_c$ with $r_c = \frac{9-3\varepsilon}{\varepsilon}$ for any $\varepsilon > 0$, and

$$V_{n,\Psi} = -\beta\frac{g_\sigma^2}{4\pi}\left(\frac{e^{-m_\sigma|\cdot|}}{|\cdot|}\star\rho_s\right)+\frac{g_\omega^2}{4\pi}\left(\frac{e^{-m_\omega|\cdot|}}{|\cdot|}\star\rho_0\right)-\frac{g_\rho^2}{4\pi}\left(\frac{e^{-m_\rho|\cdot|}}{|\cdot|}\star\rho_{00}\right)\in L^r(\mathbb{R}^3)$$

for $1 \leq r \leq r_c$.

□

Lemma 2.2 ([10])**.** *The coulombic potential $V(x) = \frac{1}{|x|}$ satisfies the following Hardy-type inequalities:*

$$(\varphi,(\mu\star V)\varphi)_{L^2} \leq \frac{1}{2}\left(\frac{\pi}{2}+\frac{2}{\pi}\right)(\varphi,|H_0|\varphi)_{L^2}\,, \tag{2.1}$$

for all $\varphi \in \Lambda^+(H^{1/2}) \cup \Lambda^-(H^{1/2})$ and for all probability measures μ on $\mathbb{R}^3$. Moreover,

$$(\varphi,(\mu\star V)\varphi)_{L^2} \leq \frac{\pi}{2}(\varphi,|H_0|\varphi)_{L^2}\,,\ \forall\varphi\in H^{1/2}, \tag{2.2}$$
$$\|(\mu\star V)\varphi\|_{L^2} \leq 2\,\|\nabla\varphi\|_{L^2}\,,\ \forall\varphi\in H^1. \tag{2.3}$$

Then, using this lemma and proceeding like in [10] (Lemma 3.1) , we obtain the following estimates.

Lemma 2.3. *Assume that*

$$\frac{g_\sigma^2 A + g_\rho^2\max(Z,N)}{4\pi} < \frac{2}{\pi/2+2/\pi}, \tag{2.4}$$
$$\frac{g_\sigma^2 A + g_\omega^2 A + g_\rho^2 Z + e^2 Z}{4\pi} < \frac{2}{\pi/2+2/\pi}, \tag{2.5}$$
$$\frac{g_\sigma^2 A + g_\omega^2 A + g_\rho^2 N}{4\pi} < \frac{2}{\pi/2+2/\pi}. \tag{2.6}$$

There is a constant $h_\mu > 0$, such that for any $\Psi \in \left(H^{1/2}(\mathbb{R}^3)\right)^A$ such that

$$\mathrm{Gram}_{L^2}(\Psi) \leq \mathbb{1},$$

and $\psi \in H^{1/2}(\mathbb{R}^3)$,

$$h_\mu\|\psi\|_{H^{1/2}} \leq \|H_{\mu,\Psi}\psi\|_{H^{-1/2}} \tag{2.7}$$

with $\mu = p, n$. In other words, $H_{\mu,\Psi}$ is a self-adjoint isomorphism between $H^{1/2}$ and its dual $H^{-1/2}$, whose inverse is bounded independently of Ψ.

Finally, a straightforward application of the inequality (2.3) yields the following lemma.

Lemma 2.4. *Assume that*

$$d_p = \frac{(g_\sigma^2+g_\omega^2+g_\rho^2)A+e^2Z}{2\pi} < 1, \tag{2.8}$$

$$d_n = \frac{(g_\sigma^2+g_\omega^2+g_\rho^2)A}{2\pi} < 1. \tag{2.9}$$

For any $\Psi \in \left(H^{1/2}(\mathbb{R}^3)\right)^A$ *such that* $\mathrm{Gram}_{L^2}(\Psi) \leq \mathbb{1}$,

$$V_{\mu,\Psi} \leq d_\mu^{1/2}|H_0| \tag{2.10}$$

$$(1-d_\mu)^{1/2}|H_0| \leq |H_{\mu,\Psi}| \tag{2.11}$$

for $\mu = p, n$.

Remark. Our estimates are far from optimal. In particular, we do not give any condition on m_σ, m_ω and m_ρ. We can expect that taking into account the meson masses, one can obtain better estimates.

3. Proof of the theorem 1.2

This theorem is an application of the concentration-compactness argument (see [8],[9]). Like in [11], if $(\psi_1^k, \dots, \psi_A^k)$ is a minimizing sequence of (1.29), then we apply the lemma below (proved in [8]) with the probability P_k in $\mathbb{R}^3$ whose density is $\frac{1}{A}\rho^k$ and $\rho^k = \sum_{i=1}^{A} |\psi_i^k|^2$.

Lemma 3.1. *Let* $(P_k)_k$ *be a sequence of probability measures on* $\mathbb{R}^N$. *Then there exists a subsequence that we still denote by* P_k *such that one of the following properties holds:*

i. *(compactness up to a translation)* $\exists y^k \in \mathbb{R}^N$, $\forall \varepsilon > 0$, $\exists R < \infty$

$$P_k\left(B\left(y^k, R\right)\right) \geq 1-\varepsilon;$$

ii. *(vanishing)* $\forall R < \infty$

$$\sup_{y\in\mathbb{R}^N} P_k\left(B\left(y,R\right)\right) \underset{k}{\to} 0;$$

iii. *(dichotomy)* $\exists \alpha \in (0,1)$, $\forall \varepsilon > 0$, $\forall M < \infty$, $\exists R_0 \geq M$, $\exists y^k \in \mathbb{R}^N$, $\exists R_k \underset{k}{\to} +\infty$ *such that*

$$\left|P_k\left(B\left(y^k, R_0\right)\right) - \alpha\right| \leq \varepsilon, \quad \left|P_k\left(B\left(y^k, R_k\right)^c\right) - (1-\alpha)\right| \leq \varepsilon.$$

In the following subsections, we prove that if the condition (1.31) holds, then we can rule out dichotomy and vanishing.

First, we make a few preliminary observations; let $\Psi^k = (\psi_1^k, \dots, \psi_A^k)$ be a minimizing sequence and $g_\sigma, g_\omega, g_\rho$ and e such that $d_\mu < \frac{4}{5}$ for $\mu = p, n$, then Ψ^k is bounded in $\left(H^{1/2}(\mathbb{R}^3)\right)^A$. Indeed, since Ψ^k is a minimizing sequence, there exists

a constant C such that

$$\begin{aligned} C \geq \mathcal{E}(\Psi^k) &= \sum_{j=1}^{A} \left(\psi_j^k, H_0 \psi_j^k\right)_{L^2} + \frac{1}{2} \sum_{j=1}^{Z} \left(\psi_j^k, V_{p,\Psi^k} \psi_j^k\right)_{L^2} \\ &\quad + \frac{1}{2} \sum_{j=Z+1}^{A} \left(\psi_j^k, V_{n,\Psi^k} \psi_j^k\right)_{L^2} \\ &= \sum_{j=1}^{Z} \left(\psi_j^k, H_{p,\Psi^k} \psi_j^k\right)_{L^2} + \sum_{j=Z+1}^{A} \left(\psi_j^k, H_{n,\Psi^k} \psi_j^k\right)_{L^2} \\ &\quad - \frac{1}{2} \sum_{j=1}^{Z} \left(\psi_j^k, V_{p,\Psi^k} \psi_j^k\right)_{L^2} - \frac{1}{2} \sum_{j=Z+1}^{A} \left(\psi_j^k, V_{n,\Psi^k} \psi_j^k\right)_{L^2} \end{aligned}$$

Then, using the fact that, for any $k \in \mathbb{N}$, $\psi_j^k = \Lambda_{p,\Psi^k}^+ \psi_j^k$ for $1 \leq j \leq Z$, $\psi_j^k = \Lambda_{n,\Psi^k}^+ \psi_j^k$ for $Z+1 \leq j \leq A$ and the inequalities (2.10) and (2.11), we obtain

$$\begin{aligned} C \geq & \sum_{j=1}^{Z} \left(\psi_j^k, |H_{p,\Psi^k}| \psi_j^k\right)_{L^2} + \sum_{j=Z+1}^{A} \left(\psi_j^k, |H_{n,\Psi^k}| \psi_j^k\right)_{L^2} \\ & - \frac{1}{2} \sum_{j=1}^{Z} d_p^{1/2} \left(\psi_j^k, |H_0| \psi_j^k\right)_{L^2} - \frac{1}{2} \sum_{j=Z+1}^{A} d_n^{1/2} \left(\psi_j^k, |H_0| \psi_j^k\right)_{L^2} \\ \geq & \sum_{j=1}^{Z} \left[(1-d_p)^{1/2} - \frac{d_p^{1/2}}{2} \right] \|\psi_j^k\|_{H^{1/2}}^2 + \sum_{j=Z+1}^{A} \left[(1-d_n)^{1/2} - \frac{d_n^{1/2}}{2} \right] \|\psi_j^k\|_{H^{1/2}}^2. \end{aligned}$$

As a conclusion, if $2(1-d_\mu)^{1/2} - d_\mu^{1/2} > 0$, that means $d_\mu < \frac{4}{5}$, then $\|\Psi^k\|_{(H^{1/2})^A}^2$ is bounded independently of k and I is bounded from below.

3.1. **Dichotomy does not occur.** If dichotomy occurs (case iii.), then, roughly speaking, $\Psi^k = (\psi_1^k, \ldots, \psi_A^k)$ can be split into two parts that we denote by $\Psi_1^k = (\psi_{1,1}^k, \ldots, \psi_{A,1}^k)$ and $\Psi_2^k = (\psi_{1,2}^k, \ldots, \psi_{A,2}^k)$. More precisely, let ξ, ζ be cut-off functions: $0 \leq \xi \leq 1$, $0 \leq \zeta \leq 1$, $\xi(x) = 1$ if $|x| \leq 1$, $\xi(x) = 0$ if $|x| \geq 2$, $\zeta(x) = 0$ if $|x| \leq 1$, $\zeta(x) = 1$ if $|x| \geq 2$, $\xi, \zeta \in \mathcal{D}(\mathbb{R}^3)$ and let ξ_μ, ζ_μ denote $\xi\left(\frac{\cdot}{\mu}\right)$, $\zeta\left(\frac{\cdot}{\mu}\right)$. We set

$$\begin{aligned} \psi_{i,1}^k &= \xi_{R_0}(\cdot - y^k) \psi_i^k \\ \psi_{i,2}^k &= \zeta_{R_k}(\cdot - y^k) \psi_i^k \end{aligned}$$

with $R_k \underset{k}{\to} +\infty$. We remind that $\operatorname{dist}\left(\operatorname{supp} \psi_{i,1}^k, \operatorname{supp} \psi_{i,2}^k\right) \underset{k}{\to} +\infty$ and

$$\left\| \psi_i^k - \left(\psi_{i,1}^k + \psi_{i,2}^k\right) \right\|_{L^p} \underset{k}{\to} 0 \tag{3.1}$$

for $2 \leq p < 3$ (see [8]).

Next, we may assume that

$$\int_{\mathbb{R}^3} \psi_{i,1}^{k^*} \psi_{j,1}^k = \lambda_i \delta_{ij}, \quad \int_{\mathbb{R}^3} \psi_{i,2}^{k^*} \psi_{j,2}^k = (1-\lambda_i) \delta_{ij} \tag{3.2}$$

for $1 \leq i, j \leq Z$, $Z+1 \leq i, j \leq A$ and $0 \leq \lambda_i \leq 1$. In fact, suppose that $\tilde{\Psi}^k$ is a minimizing sequence for which the dichotomy case occurs. We remind that $\mathcal{E}(\Psi)$ is

invariant under the transformations of $(\psi_1, \dots, \psi_A)$ of the form

$$U = \begin{pmatrix} U_p & 0 \\ 0 & U_n \end{pmatrix}$$

where U_p (resp. U_n) is a $Z \times Z$ (resp. $N \times N$) unitary matrix; then, using this kind of transformations, it is clear that we may diagonalize $\mathrm{Gram}_{L^2}(\tilde\Psi_1^k)$ into $\mathrm{diag}(\lambda_1^k, \dots, \lambda_A^k)$ and $\mathrm{Gram}_{L^2}(\tilde\Psi_2^k)$ into $\mathrm{diag}(1-\lambda_1^k, \dots, 1-\lambda_A^k)$. In particular, writing $\tilde\Psi_1^k = (\tilde\Psi_{p,1}^k, \tilde\Psi_{n,1}^k)$ and $\tilde\Psi_2^k = (\tilde\Psi_{p,2}^k, \tilde\Psi_{n,2}^k)$, we have

$$\begin{aligned}
&\mathrm{Gram}_{L^2}(\tilde\Psi_{p,1}^k) = \mathrm{diag}(\lambda_1^k, \dots, \lambda_Z^k),\\
&\mathrm{Gram}_{L^2}(\tilde\Psi_{n,1}^k) = \mathrm{diag}(\lambda_{Z+1}^k, \dots, \lambda_A^k),\\
&\mathrm{Gram}_{L^2}(\tilde\Psi_{p,2}^k) = \mathrm{diag}(1-\lambda_1^k, \dots, 1-\lambda_Z^k),\\
&\mathrm{Gram}_{L^2}(\tilde\Psi_{n,2}^k) = \mathrm{diag}(1-\lambda_{Z+1}^k, \dots, 1-\lambda_A^k)
\end{aligned}$$

with $0 \le \lambda_i^k \le 1$ for all $k \in \mathbb{N}$. Since $\{\lambda_i^k\}_k$ is a bounded sequence in $\mathbb{R}$, up to a subsequence $\lambda_i^k \underset{k}{\to} \lambda_i$ with $0 \le \lambda_i \le 1$; so, to obtain (3.2), it is enough to consider a small perturbation of $\tilde\Psi_1^k$ and $\tilde\Psi_2^k$. More precisely, if $0 < \lambda_i < 1$, we take

$$\psi_{i,1}^k = \sqrt{\tfrac{\lambda_i}{\lambda_i^k}}\tilde\psi_{i,1}^k\,, \quad \psi_{i,2}^k = \sqrt{\tfrac{1-\lambda_i}{1-\lambda_i^k}}\tilde\psi_{i,2}^k\,;$$

if $\lambda_i = 0$,

$$\psi_{i,1}^k = 0\,, \quad \psi_{i,2}^k = \sqrt{\tfrac{1}{1-\lambda_i^k}}\tilde\psi_{i,2}^k\,;$$

if $\lambda_i = 1$,

$$\psi_{i,1}^k = \sqrt{\tfrac{1}{\lambda_i^k}}\tilde\psi_{i,1}^k\,, \quad \psi_{i,2}^k = 0\,.$$

We remark that $\Psi_1^k = (\psi_{1,1}^k, \dots, \psi_{A,1}^k)$ and $\Psi_2^k = (\psi_{1,2}^k, \dots, \psi_{A,2}^k)$ do not necessarily satisfy the constraints of $I(\lambda_1, \dots, \lambda_A)$ and $I(1-\lambda_1, \dots, 1-\lambda_A)$ respectively, then we proceed as follows.
First, we show that

$$\begin{aligned}
&\Lambda^-_{p,\Psi_1^k}(\psi_{1,1}^k, \dots, \psi_{Z,1}^k) \underset{k}{\to} 0 \text{ in } \left(H^{1/2}(\mathbb{R}^3)\right)^Z,\\
&\Lambda^-_{n,\Psi_1^k}(\psi_{Z+1,1}^k, \dots, \psi_{A,1}^k) \underset{k}{\to} 0 \text{ in } \left(H^{1/2}(\mathbb{R}^3)\right)^N,
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
&\Lambda^-_{p,\Psi_2^k}(\psi_{1,2}^k, \dots, \psi_{Z,2}^k) \underset{k}{\to} 0 \text{ in } \left(H^{1/2}(\mathbb{R}^3)\right)^Z,\\
&\Lambda^-_{n,\Psi_2^k}(\psi_{Z+1,2}^k, \dots, \psi_{A,2}^k) \underset{k}{\to} 0 \text{ in } \left(H^{1/2}(\mathbb{R}^3)\right)^N.
\end{aligned} \tag{3.4}$$

Second, using the implicit function theorem, we construct $\Phi_1^k = (\Phi_{p,1}^k, \Phi_{n,1}^k), \Phi_2^k = (\Phi_{p,2}^k, \Phi_{n,2}^k) \in \left(H^{1/2}(\mathbb{R}^3)\right)^Z \times \left(H^{1/2}(\mathbb{R}^3)\right)^N$, small perturbations of Ψ_1^k, Ψ_2^k in $\left(H^{1/2}(\mathbb{R}^3)\right)^A$, such that

$$\Lambda^-_{p,\Phi_1^k}\Phi_{p,1}^k = 0, \quad \Lambda^-_{n,\Phi_1^k}\Phi_{n,1}^k = 0, \tag{3.5}$$

$$\Lambda^-_{p,\Phi_2^k}\Phi_{p,2}^k = 0, \quad \Lambda^-_{n,\Phi_2^k}\Phi_{n,2}^k = 0 \tag{3.6}$$

and

$$\mathrm{Gram}_{L^2}(\Phi_{\mu,i}^k) = \mathrm{Gram}_{L^2}(\Psi_{\mu,i}^k) \tag{3.7}$$

for $\mu = p, n$ and $i = 1, 2$.
In conclusion, thanks to the continuity of $\mathcal{E}$ in $H^{1/2}(\mathbb{R}^3)$, we obtain

$$\begin{aligned} I &= \lim_{k\to\infty} \mathcal{E}(\Psi^k) \geq \underline{\lim}_{k\to\infty} \mathcal{E}(\Psi_1^k) + \underline{\lim}_{k\to\infty} \mathcal{E}(\Psi_2^k) \\ &= \underline{\lim}_{k\to\infty} \mathcal{E}(\Phi_1^k) + \underline{\lim}_{k\to\infty} \mathcal{E}(\Phi_2^k) \\ &\geq I(\lambda_1, \ldots, \lambda_A) + I(1-\lambda_1, \ldots, 1-\lambda_A) \end{aligned}$$

that clearly contradicts (1.31). We remind that the first inequality is obtained by using the properties of localization of $\Psi_1^k, \Psi_2^k, \nabla\Psi_1^k$ and $\nabla\Psi_2^k$.

We start by showing that

$$\Lambda^-_{p,\Psi_1^k}(\psi_{1,1}^k, \ldots, \psi_{Z,1}^k) \underset{k}{\to} 0 \text{ in } \left(H^{1/2}(\mathbb{R}^3)\right)^Z.$$

Using the formula (see [12])

$$\begin{aligned} \Lambda_B^- - \Lambda_A^- &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left[(A-i\eta)^{-1} - (B-i\eta)^{-1}\right] d\eta \qquad (3.8) \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} (A-i\eta)^{-1}(B-A)(B-i\eta)^{-1}\, d\eta, \end{aligned}$$

we can write

$$\begin{aligned} &\Lambda^-_{p,\Psi_1^k}\psi_{i,1}^k - \Lambda^-_{p,\Psi^k}\psi_{i,1}^k = \qquad (3.9) \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} (H_{p,\Psi^k} - i\eta)^{-1}(H_{p,\Psi_1^k} - H_{p,\Psi^k})(H_{p,\Psi_1^k} - i\eta)^{-1}\psi_{i,1}^k\, d\eta. \end{aligned}$$

for $i = 1, \ldots, Z$. Hence, if we prove that

$$\left\|\Lambda^-_{p,\Psi^k}\psi_{i,1}^k\right\|_{H^{1/2}} \underset{k}{\to} 0$$

and

$$\int_{-\infty}^{+\infty} \left\|(H_{p,\Psi^k} - i\eta)^{-1}(H_{p,\Psi_1^k} - H_{p,\Psi^k})(H_{p,\Psi_1^k} - i\eta)^{-1}\psi_{i,1}^k\right\|_{H^{1/2}} d\eta \underset{k}{\to} 0,$$

we can conclude that

$$\left\|\Lambda^-_{p,\Psi_1^k}\psi_{i,1}^k\right\|_{H^{1/2}} \underset{k}{\to} 0.$$

First of all, we consider

$$f^k(\eta) = \left\|(H_{p,\Psi^k} - i\eta)^{-1}(H_{p,\Psi_1^k} - H_{p,\Psi^k})(H_{p,\Psi_1^k} - i\eta)^{-1}\psi_{i,1}^k\right\|_{H^{1/2}}$$

and we prove that, $\forall\eta \in \mathbb{R}$,

$$f^k(\eta) \underset{k}{\to} 0.$$

In particular, we decompose the proof of this fact into two lemmas.

Lemma 3.2. *Assume that $g_\sigma, g_\omega, g_\rho$ and e are sufficiently small and let Ψ^k be a sequence in $\left(H^{1/2}(\mathbb{R}^3)\right)^A$ such that $\mathrm{Gram}_{L^2}(\Psi^k) \leq \mathbb{1}$, $\forall k \in \mathbb{N}$, and $\left\|\Psi^k\right\|_{(L^p)^A}$ is bounded independently of k for $2 \leq p \leq 3$. Then for any $\varphi \in L^2(\mathbb{R}^3)$ and for any $\eta \in \mathbb{R}$, there exists a constant $\hat{h}_p$ such that*

$$\left\|(H_{p,\Psi^k} - i\eta)^{-1}\varphi\right\|_{H^{1/2}} \leq \frac{1}{(m_b^2+\eta^2)^{1/4}}\left(\|\varphi\|_{L^2} + \frac{C}{(\hat{h}_p^2+\eta^2)^{1/3}}\|\varphi\|_{L^2}\right) \qquad (3.10)$$

with C a constant that does not depend on k.

Proof. First of all, we write

$$\begin{aligned} (H_{p,\Psi^k} - i\eta)^{-1}\varphi &= \chi \\ \varphi &= (H_{p,\Psi^k} - i\eta)\chi \\ \varphi &= (H_0 - i\eta)\chi + V_{p,\Psi^k}\chi \\ (H_0 - i\eta)^{-1}(\varphi - V_{p,\Psi^k}\chi) &= \chi \end{aligned}$$

where $V_{p,\Psi^k} = H_{p,\Psi^k} - H_0$. It is easy to show that if $\varphi \in L^2(\mathbb{R}^3)$, then $\chi \in H^{1/2}(\mathbb{R}^3)$; indeed, there exists a constant $h_p > 0$ such that

$$\begin{aligned} \|\varphi\|^2_{L^2} &= \big((H_{p,\Psi^k} - i\eta)\chi, (H_{p,\Psi^k} - i\eta)\chi\big)_{L^2} = \big\|H_{p,\Psi^k}\chi\big\|^2_{L^2} + \eta^2 \|\chi\|^2_{L^2} \\ &\geq m_b h_p^2 \|\chi\|^2_{H^{1/2}} + \eta^2 \|\chi\|^2_{L^2} \geq m_b^2 h_p^2 \|\chi\|^2_{L^2} + \eta^2 \|\chi\|^2_{L^2} \end{aligned}$$

thanks to Sobolev embeddings and lemma 2.3.
Next, to have a good estimate of the $H^{1/2}$-norm, we use its definition and we obtain

$$\begin{aligned} \|\chi\|^2_{H^{1/2}} &= \big\|(H_0 - i\eta)^{-1}(\varphi - V_{p,\Psi^k}\chi)\big\|^2_{H^{1/2}} \\ &= \int_{\mathbb{R}^3} (m_b^2 + |p|^2)^{1/2} \left| \frac{\hat{\varphi}(p) - \widehat{V_{p,\Psi^k}\chi}(p)}{\hat{H}_0(p) - i\eta} \right|^2 dp \\ &\leq \int_{\mathbb{R}^3} \frac{(m_b^2 + |p|^2)^{1/2}}{\hat{H}_0(p)^2 + \eta^2} \left(|\hat{\varphi}(p)|^2 + \left|\widehat{V_{p,\Psi^k}\chi}(p)\right|^2 \right) dp \\ &= \int_{\mathbb{R}^3} \frac{(m_b^2 + |p|^2)^{1/2}}{m_b^2 + |p|^2 + \eta^2} \left(|\hat{\varphi}(p)|^2 + \left|\widehat{V_{p,\Psi^k}\chi}(p)\right|^2 \right) dp \\ &\leq \int_{\mathbb{R}^3} \frac{(m_b^2 + |p|^2)^{1/2}}{(m_b^2 + |p|^2)^{1/2}(m_b^2 + \eta^2)^{1/2}} \left(|\hat{\varphi}(p)|^2 + \left|\widehat{V_{p,\Psi^k}\chi}(p)\right|^2 \right) dp \\ &\leq \frac{1}{(m_b^2 + \eta^2)^{1/2}} \left(\|\varphi\|^2_{L^2} + \big\|V_{p,\Psi^k}\chi\big\|^2_{L^2} \right). \end{aligned}$$

To conclude, we have to find an estimate for $\big\|V_{p,\Psi^k}\chi\big\|_{L^2}$. In particular, we have

$$\begin{aligned} \big\|V_{p,\Psi^k}\chi\big\|_{L^2} &\leq \big\|V_{p,\Psi^k}\big\|_{L^{18}} \|\chi\|_{L^{9/4}} \leq \big\|V_{p,\Psi^k}\big\|_{L^{18}} \|\chi\|^{2/3}_{L^2} \|\chi\|^{1/3}_{L^3} \\ &\leq C \|\varphi\|^{1/3}_{L^2} \big\|(H_{p,\Psi^k} - i\eta)^{-1}\varphi\big\|^{2/3}_{L^2} \leq \frac{C}{(\hat{h}_p^2 + \eta^2)^{1/3}} \|\varphi\|_{L^2} \end{aligned}$$

where $\hat{h}_p = m_b h_p$ and C is a constant that does not depend on k. Hence,

$$\big\|(H_{p,\Psi^k} - i\eta)^{-1}\varphi\big\|_{H^{1/2}} \leq \frac{1}{(m_b^2 + \eta^2)^{1/4}} \left(\|\varphi\|_{L^2} + \frac{C}{(\hat{h}_p^2 + \eta^2)^{1/3}} \|\varphi\|_{L^2} \right)$$

$\forall \eta \in \mathbb{R}$.

□

Lemma 3.3. *Assume that $g_\sigma, g_\omega, g_\rho$ and e are sufficiently small; let Ψ^k be a minimizing sequence of (1.29) and Ψ_1^k, Ψ_2^k defined as above. Then, for $i = 1, \dots, Z$,*

$$(H_{p,\Psi_1^k} - H_{p,\Psi^k})(H_{p,\Psi_1^k} - i\eta)^{-1}\psi_{i,1}^k \xrightarrow[k]{L^2} 0 \quad \forall \eta \in \mathbb{R}. \tag{3.11}$$

Proof. First of all, we study the behavior of

$$(H_{p,\Psi_1^k} - i\eta)^{-1}\psi_{i,1}^k$$

far from y^k. In particular, we consider a translation by x^k such that $|x^k - y^k| \underset{k}{\to} \infty$, and we analyze

$$(H_{p,\Psi_1^k(\bar{x}+x^k)} - i\eta)^{-1}\psi_{i,1}^k(\bar{x}+x^k) := \tau^k(\bar{x}+x^k);$$

hence τ^k is a solution of

$$(H_{p,\Psi_1^k(\bar{x}+x^k)} - i\eta)\tau^k(\bar{x}+x^k) = \psi_{i,1}^k(\bar{x}+x^k).$$

We remark that $\forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^3)$

$$\int_{\mathbb{R}^3} \psi_{i,1}^k(\bar{x}+x^k)\varphi(\bar{x})\,d\bar{x} = \int_{\mathbb{R}^3} \xi_{R_0}(\bar{x}+(x^k-y^k))\psi_i^k(\bar{x}+x^k)\varphi(\bar{x})\,d\bar{x} \underset{k}{\to} 0,$$

then $\psi_{i,1}^k(\cdot + x^k)$ converges weakly to zero in $H^{1/2}(\mathbb{R}^3)$ and, thanks to compact Sobolev embeddings, there exists a subsequence that converges in L^p_{loc} for $2 \le p < 3$. We still denote by $\psi_{i,1}^k(\cdot + x^k)$ this subsequence.

In the same way, we find that $(H_{p,\Psi_1^k(\cdot+x^k)} - i\eta)\tau^k(\cdot + x^k) \xrightarrow[k]{L^p_{loc}} (H_0 - i\eta)\tau^\infty(\cdot)$. Then $(H_0 - i\eta)\tau^\infty = 0$, that implies $\tau^\infty = 0$.
Hence, we have that $\forall \varepsilon > 0$, $\exists \Delta > 0$ such that $\forall R > 0$

$$\sup_{x\in\mathbb{R}^3} \|\tau^k\|_{L^p(B(x,R)\setminus B(y^k,\Delta))} \le \varepsilon \tag{3.12}$$

for k sufficiently large and $2 \le p < 3$.
Now, using (3.12), we can estimate the norm of τ^k in $L^p\left(\mathbb{R}^3 \setminus B(y^k,\Delta)\right)$ for $2 < p < 3$. If we denote by Q_l the cube in $\mathbb{R}^3$, then

$$\begin{aligned}
\|\tau^k\|^p_{L^p(\mathbb{R}^3\setminus B(y^k,\Delta))} &= \sum_{l\in\mathbb{Z}^3} \|\tau^k\|^p_{L^p(Q_l\setminus B(y^k,\Delta))} \\
&\le \sum_{l\in\mathbb{Z}^3} \|\tau^k\|^{p\theta}_{L^2(Q_l\setminus B(y^k,\Delta))} \|\tau^k\|^{p(1-\theta)}_{L^3(Q_l\setminus B(y^k,\Delta))} \\
&\le \varepsilon^{p\theta} \sum_{l\in\mathbb{Z}^3} \|\tau^k\|^{p(1-\theta)}_{L^3(Q_l)} \le \varepsilon^{p\theta} C^{p(1-\theta)} \|\tau^k\|^{p(1-\theta)}_{H^{1/2}(\mathbb{R}^3)} \\
&\le \varepsilon^{p\theta} \tilde{C}^{p(1-\theta)}
\end{aligned}$$

for $0 < \theta < 1$ such that $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{3}$ thanks to the interpolation inequality and Sobolev embeddings.
Hence, $\forall \varepsilon > 0$, $\exists \Delta > 0$, such that

$$\|\tau^k\|_{L^p(\mathbb{R}^3\setminus B(y^k,\Delta))} \le \varepsilon \tag{3.13}$$

for k sufficiently large and $2 < p < 3$.
Second, we consider the potential

$$\begin{aligned} W^k := H_{p,\Psi_1^k} - H_{p,\Psi^k} \quad = \quad & \Bigg[-\beta \frac{g_\sigma^2}{4\pi} \sum_{j=1}^{A} \left(\frac{e^{-m_\sigma |\cdot|}}{|\cdot|} \star \left(\bar\psi_{j,1}^k \psi_{j,1}^k - \bar\psi_j^k \psi_j^k \right) \right) \\ & + \frac{g_\omega^2}{4\pi} \sum_{j=1}^{A} \left(\frac{e^{-m_\omega |\cdot|}}{|\cdot|} \star \left(\left|\psi_{j,1}^k\right|^2 - \left|\psi_j^k\right|^2 \right) \right) \\ & + \frac{g_\rho^2}{4\pi} \sum_{j=1}^{Z} \left(\frac{e^{-m_\rho |\cdot|}}{|\cdot|} \star \left(\left|\psi_{j,1}^k\right|^2 - \left|\psi_j^k\right|^2 \right) \right) \\ & - \frac{g_\rho^2}{4\pi} \sum_{j=Z+1}^{A} \left(\frac{e^{-m_\rho |\cdot|}}{|\cdot|} \star \left(\left|\psi_{j,1}^k\right|^2 - \left|\psi_j^k\right|^2 \right) \right) \\ & + \frac{e^2}{4\pi} \sum_{j=1}^{Z} \left(\frac{1}{|\cdot|} \star \left(\left|\psi_{j,1}^k\right|^2 - \left|\psi_j^k\right|^2 \right) \right) \Bigg] \end{aligned} \tag{3.14}$$

and we estimate the L^7-norm of W^k in $B_\Delta^k := B(y^k, \Delta)$. Using (3.1) and the definitions of $\psi_{j,1}^k$ and $\psi_{j,2}^k$, we obtain, for $1 \le p < \frac{3}{2}$,

$$\begin{aligned} \left\| \bar\psi_{j,1}^k \psi_{j,1}^k - \bar\psi_j^k \psi_j^k \right\|_{L^p(B_\Delta^k)} \quad \le \quad & \left\| \bar\psi_{j,1}^k \psi_{j,1}^k - (\bar\psi_{j,1}^k + \bar\psi_{j,2}^k)(\psi_{j,1}^k + \psi_{j,2}^k) \right\|_{L^p(B_\Delta^k)} \\ & + C \left\| \psi_{j,1}^k + \psi_{j,2}^k - \psi_j^k \right\|_{L^{2p}(\mathbb{R}^3)} \underset{k}{\to} 0 \end{aligned}$$

and, in the same way,

$$\begin{aligned} \left\| \left|\psi_{j,1}^k\right|^2 - \left|\psi_j^k\right|^2 \right\|_{L^p(B_\Delta^k)} \quad \le \quad & \left\| \left|\psi_{j,1}^k\right|^2 - \left|(\psi_{j,1}^k + \psi_{j,2}^k)\right|^2 \right\|_{L^p(B_\Delta^k)} \\ & + C \left\| \psi_{j,1}^k + \psi_{j,2}^k - \psi_j^k \right\|_{L^{2p}(\mathbb{R}^3)} \underset{k}{\to} 0. \end{aligned}$$

Next, we remark that this potential contains three types of terms; for the first one, we have

$$\begin{aligned} & \left\| \frac{e^{-m_\sigma |\cdot|}}{|\cdot|} \star \left(\bar\psi_{j,1}^k \psi_{j,1}^k - \bar\psi_j^k \psi_j^k \right) \right\|_{L^7(B_\Delta^k)} \\ \le \quad & \left\| \frac{e^{-m_\sigma |\cdot|}}{|\cdot|} \right\|_{L^{35/12}(B_\Delta^k)} \left\| \left(\bar\psi_{j,1}^k \psi_{j,1}^k - \bar\psi_j^k \psi_j^k \right) \right\|_{L^{5/4}(B_\Delta^k)} \underset{k}{\to} 0. \end{aligned}$$

Similarly, for the second type of terms, we obtain

$$\begin{aligned} & \left\| \frac{e^{-m_\omega |\cdot|}}{|\cdot|} \star \left(\left|\psi_{j,1}^k\right|^2 - \left|\psi_j^k\right|^2 \right) \right\|_{L^7(B_\Delta^k)} \\ \le \quad & \left\| \frac{e^{-m_\omega |\cdot|}}{|\cdot|} \right\|_{L^{35/12}(B_\Delta^k)} \left\| \left(\left|\psi_{j,1}^k\right|^2 - \left|\psi_j^k\right|^2 \right) \right\|_{L^{5/4}(B_\Delta^k)} \underset{k}{\to} 0. \end{aligned}$$

For the last term, we remind that $\frac{1}{|x|}$ can be written as $\frac{1}{|x|} = h_1(x) + h_2(x)$ with $h_1 \in L^{35/12}(\mathbb{R}^3)$ and $h_2 \in L^7(\mathbb{R}^3)$, where $h_1(x) = \frac{1}{|x|}$ for $|x| \le 1$, $h_1(x) = 0$

otherwise. Then

$$\begin{aligned} & \left\| \frac{1}{|\cdot|} \star \left(|\psi_{j,1}^k|^2 - |\psi_j^k|^2 \right) \right\|_{L^7(B_\Delta^k)} \\ \leq \; & \|h_1\|_{L^{35/12}(B_\Delta^k)} \left\| \left(|\psi_{j,1}^k|^2 - |\psi_j^k|^2 \right) \right\|_{L^{5/4}(B_\Delta^k)} \\ & + \|h_2\|_{L^7(B_\Delta^k)} \left\| \left(|\psi_{j,1}^k|^2 - |\psi_j^k|^2 \right) \right\|_{L^1(B_\Delta^k)} \underset{k}{\to} 0. \end{aligned}$$

Finally,

$$\|W^k\|_{L^7(B(y^k,\Delta))} \underset{k}{\to} 0. \tag{3.15}$$

In conclusion, using (3.13) and (3.15), we obtain

$$\begin{aligned} & \left\| (H_{p,\Psi_1^k} - H_{p,\Psi^k})(H_{p,\Psi_1^k} - i\eta)^{-1} \psi_{i,1}^k \right\|_{L^2(\mathbb{R}^3)}^2 = \left\| W^k \tau^k \right\|_{L^2(\mathbb{R}^3)}^2 \\ \leq \; & \left\| W^k \tau^k \right\|_{L^2(B(y^k,\Delta))}^2 + \left\| W^k \tau^k \right\|_{L^2(\mathbb{R}^3 \setminus B(y^k,\Delta))}^2 \\ \leq \; & \left\| W^k \right\|_{L^7(B(y^k,\Delta))}^2 \left\| \tau^k \right\|_{L^{14/5}(\mathbb{R}^3)}^2 + \left\| W^k \right\|_{L^7(\mathbb{R}^3)}^2 \left\| \tau^k \right\|_{L^{14/5}(\mathbb{R}^3 \setminus B(y^k,\Delta))}^2 \underset{k}{\to} 0. \end{aligned}$$

□

Hence, if we apply the lemma 3.2 to $\varphi = (H_{p,\Psi_1^k} - H_{p,\Psi^k})(H_{p,\Psi_1^k} - i\eta)^{-1}\psi_{i,1}^k$ and we use the result of lemma 3.3, we can conclude that

$$\left\| (H_{p,\Psi^k} - i\eta)^{-1} (H_{p,\Psi_1^k} - H_{p,\Psi^k})(H_{p,\Psi_1^k} - i\eta)^{-1} \psi_{i,1}^k \right\|_{H^{1/2}} \underset{k}{\to} 0 \tag{3.16}$$

for all $\eta \in \mathbb{R}$.
Finally, to prove that $\int_{-\infty}^{+\infty} f^k(\eta)\, d\eta \to 0$ as $k \to \infty$, we use the Lebesgue's dominated convergence theorem.
Indeed, the sequence f^k converges to $f = 0$ for all $\eta \in \mathbb{R}$ and is dominated by an integrable function g. In particular, using lemma 3.2 and its proof, we remark that, $\forall k \in \mathbb{N}$,

$$\begin{aligned} |f^k(\eta)| \; & \leq \; \frac{1}{(m_b^2 + \eta^2)^{1/4}} \left(1 + \frac{C}{(\hat{h}_p^2 + \eta^2)^{1/3}} \right) \left\| W^k (H_{p,\Psi_1^k} - i\eta)^{-1} \psi_{i,1}^k \right\|_{L^2} \\ & \leq \; \frac{\tilde{C}}{(m_b^2 + \eta^2)^{1/4} (\hat{h}_p^2 + \eta^2)^{1/3}} \left(1 + \frac{C}{(\hat{h}_p^2 + \eta^2)^{1/3}} \right) \left\| \psi_{i,1}^k \right\|_{L^2} := g(\eta) \\ & \leq \; \frac{\tilde{C}}{(m_b^2 + \eta^2)^{1/4} (\hat{h}_p^2 + \eta^2)^{1/3}} \left(1 + \frac{C}{(\hat{h}_p^2 + \eta^2)^{1/3}} \right) := g(\eta) \end{aligned}$$

and $g(\eta) \in L^1(\mathbb{R})$. Then

$$\int_{-\infty}^{+\infty} f^k(\eta)\, d\eta \underset{k}{\to} 0.$$

Now, to prove that $\Lambda_{p,\Psi^k}^- \psi_{i,1}^k = \Lambda_{p,\Psi^k}^- \xi_{R_0}(\cdot - y^k)\psi_i^k$ converges to 0 in $H^{1/2}(\mathbb{R}^3)$, we give an estimate on the commutator $\left[\Lambda_{p,\Psi^k}^-, \xi_{R_0}(\cdot - y^k) \right]$. Writing $\xi_{R_0,y^k}(\cdot) =$

$\xi_{R_0}(\cdot - y^k)$ and using Cauchy's formula, we infer

$$\begin{aligned}\left[\Lambda^-_{p,\Psi^k}, \xi_{R_0,y^k}\right] &= -\frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{H_{p,\Psi^k}+i\eta}\xi_{R_0,y^k} - \xi_{R_0,y^k}\frac{1}{H_{p,\Psi^k}+i\eta}\,d\eta \\ &= -\frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{H_{p,\Psi^k}+i\eta}\left[\xi_{R_0,y^k}, H_{p,\Psi^k}+i\eta\right]\frac{1}{H_{p,\Psi^k}+i\eta}\,d\eta \\ &= -\frac{i}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{H_{p,\Psi^k}+i\eta}\alpha\cdot\nabla\xi_{R_0,y^k}\frac{1}{H_{p,\Psi^k}+i\eta}\,d\eta.\end{aligned}$$

Hence,

$$\begin{aligned}\left\|\left[\Lambda^-_{p,\Psi^k}, \xi_{R_0,y^k}\right]\psi^k_i\right\|_{H^{1/2}} &\leq C_1\int_{-\infty}^{+\infty}\frac{\left\|\nabla\xi_{R_0,y^k}\right\|_{L^{18}}}{(m_b^2+\eta^2)^{1/4}(\hat{h}_p^2+\eta^2)^{1/3}}\,d\eta \\ &\quad + C_2\int_{-\infty}^{+\infty}\frac{\left\|\nabla\xi_{R_0,y^k}\right\|_{L^{18}}}{(m_b^2+\eta^2)^{1/4}(\hat{h}_p^2+\eta^2)^{1/3}(\hat{h}_p^2+\eta^2)^{1/3}}\,d\eta \\ &\leq \frac{C}{R_0^\delta}\end{aligned}$$

with $\delta > 0$. Then, choosing R_0 large enough, we obtain

$$\left\|\left[\Lambda^-_{p,\Psi^k}, \xi_{R_0,y^k}\right]\psi^k_i\right\|_{H^{1/2}} \leq \varepsilon$$

and, since $\Lambda^-_{p,\Psi^k}\psi^k_{i,1} = \left[\Lambda^-_{p,\Psi^k}, \xi_{R_0,y^k}\right]\psi^k_i + \xi_{R_0,y^k}\Lambda^-_{p,\Psi^k}\psi^k_i$ and $\Lambda^-_{p,\Psi^k}\psi^k_i = 0$, we conclude that

$$\Lambda^-_{p,\Psi^k_1}(\psi^k_{1,1},\dots,\psi^k_{Z,1}) \underset{k}{\to} 0$$

in $\left(H^{1/2}(\mathbb{R}^3)\right)^Z$.
Moreover, with the same arguments used above, we prove that

$$\Lambda^-_{n,\Psi^k_1}(\psi^k_{Z+1,1},\dots,\psi^k_{A,1}) \underset{k}{\to} 0$$

in $\left(H^{1/2}(\mathbb{R}^3)\right)^N$.
Furthermore, to show that

$$\Lambda^-_{p,\Psi^k_2}(\psi^k_{1,2},\dots,\psi^k_{Z,2}) \underset{k}{\to} 0 \quad \text{and} \quad \Lambda^-_{n,\Psi^k_2}(\psi^k_{Z+1,2},\dots,\psi^k_{A,2}) \underset{k}{\to} 0$$

in $\left(H^{1/2}(\mathbb{R}^3)\right)^Z$ and $\left(H^{1/2}(\mathbb{R}^3)\right)^N$ respectively, we can proceed as before; only the proof of

$$(H_{p,\Psi^k_2} - H_{p,\Psi^k})(H_{p,\Psi^k_2} - i\eta)^{-1}\psi^k_{i,1} \xrightarrow[k]{L^2} 0, \quad \forall\eta\in\mathbb{R}, \tag{3.17}$$

is slightly different. In this case,

$$\|H_{p,\Psi^k_2} - H_{p,\Psi^k}\|_{L^7(\mathbb{R}^3\setminus B(y^k,2R_0))} \underset{k}{\to} 0$$

thanks to the localization property of $\psi^k_{i,1}$ and $(H_{p,\Psi^k_2}-i\eta)^{-1}\psi^k_{i,2}$ converges strongly to zero in $L^p(B(y^k,2R_0))$ for $2<p<3$. In conclusion,

$$\begin{aligned}
&\left\|(H_{p,\Psi^k_2}-H_{p,\Psi^k})(H_{p,\Psi^k_2}-i\eta)^{-1}\psi^k_{i,2}\right\|^2_{L^2(\mathbb{R}^3)}\\
\leq\ & C_1\left\|(H_{p,\Psi^k_2}-i\eta)^{-1}\psi^k_{i,2}\right\|^2_{L^{14/5}(B(y^k,2R_0))}\\
&+C_2\left\|(H_{p,\Psi^k_2}-H_{p,\Psi^k})\right\|^2_{L^7(\mathbb{R}^3\setminus B(y^k,2R_0))}\underset{k}{\to}0.
\end{aligned}$$

Now, we want to construct $\Phi^k_1=(\Phi^k_{p,1},\Phi^k_{n,1}),\Phi^k_2=(\Phi^k_{p,2},\Phi^k_{n,2})\in\left(H^{1/2}(\mathbb{R}^3)\right)^Z\times\left(H^{1/2}(\mathbb{R}^3)\right)^N$, small perturbations of Ψ^k_1,Ψ^k_2 in $\left(H^{1/2}(\mathbb{R}^3)\right)^A$, that satisfy the constraints of $I(\lambda_1,\dots,\lambda_A)$ and $I(1-\lambda_1,\dots,1-\lambda_A)$ respectively. For this purpose, we use the following lemma and his corollary.

Lemma 3.4. *Take $\Psi=(\Psi_p,\Psi_n)\in\left(H^{1/2}(\mathbb{R}^3)\right)^Z\times\left(H^{1/2}(\mathbb{R}^3)\right)^N$ such that*

i. $\mathrm{Gram}_{L^2}(\Psi_p):=G_p\leq\mathbb{1}_Z$ *and* $\mathrm{Gram}_{L^2}(\Psi_n):=G_n\leq\mathbb{1}_N$ *are invertible matrices;*

ii.
$$\begin{aligned}
\left\|\Lambda^-_{p,\Psi}\Psi_p\right\|_{(H^{1/2})^Z}&\leq\tilde{\delta}\\
\left\|\Lambda^-_{n,\Psi}\Psi_n\right\|_{(H^{1/2})^N}&\leq\tilde{\delta}
\end{aligned}$$

for $\tilde{\delta}>0$ small enough.

If g_σ,g_ω,g_ρ and e are sufficiently small, there exists $\Phi=(\Phi_p,\Phi_n)\in\left(H^{1/2}(\mathbb{R}^3)\right)^Z\times\left(H^{1/2}(\mathbb{R}^3)\right)^N$ such that

$$\Lambda^-_{p,\Phi}\Phi_p=0, \tag{3.18}$$
$$\Lambda^-_{n,\Phi}\Phi_n=0. \tag{3.19}$$

Moreover,

$$\mathrm{Gram}_{L^2}(\Phi_p)=G_p, \tag{3.20}$$
$$\mathrm{Gram}_{L^2}(\Phi_n)=G_n. \tag{3.21}$$

Corollary 3.5. *Take $\Psi^k=(\Psi^k_p,\Psi^k_n)\in\left(H^{1/2}(\mathbb{R}^3)\right)^Z\times\left(H^{1/2}(\mathbb{R}^3)\right)^N$ a sequence of functions bounded in $\left(H^{1/2}(\mathbb{R}^3)\right)^A$ such that*

i. $\mathrm{Gram}_{L^2}\left(\Psi^k_p\right):=G_p\leq\mathbb{1}_Z$ *and* $\mathrm{Gram}_{L^2}\left(\Psi^k_n\right):=G_n\leq\mathbb{1}_N$ *are invertible matrices that do not depend on k for any $k\in\mathbb{N}$;*

ii.
$$\begin{aligned}
\left\|\Lambda^-_{p,\Psi^k}\Psi^k_p\right\|_{(H^{1/2})^Z}&\underset{k}{\to}0\\
\left\|\Lambda^-_{n,\Psi^k}\Psi^k_n\right\|_{(H^{1/2})^N}&\underset{k}{\to}0.
\end{aligned}$$

If g_σ,g_ω,g_ρ and e are sufficiently small, there is a constant $k_0\in\mathbb{N}$ such that, for any $k\geq k_0$, there exists $\Phi^k=(\Phi^k_p,\Phi^k_n)\in\left(H^{1/2}(\mathbb{R}^3)\right)^Z\times\left(H^{1/2}(\mathbb{R}^3)\right)^N$ with the following properties:

a.

$$\Lambda^-_{p,\Phi^k}\Phi^k_p = 0, \tag{3.22}$$
$$\Lambda^-_{n,\Phi^k}\Phi^k_n = 0. \tag{3.23}$$

b.

$$\left\|\Phi^k_p - \Psi^k_p\right\|_{(H^{1/2})^Z} \underset{k}{\to} 0, \tag{3.24}$$
$$\left\|\Phi^k_n - \Psi^k_n\right\|_{(H^{1/2})^N} \underset{k}{\to} 0. \tag{3.25}$$

c.

$$\mathrm{Gram}_{L^2}\left(\Phi^k_p\right) = G_p, \tag{3.26}$$
$$\mathrm{Gram}_{L^2}\left(\Phi^k_n\right) = G_n. \tag{3.27}$$

Proof of lemma 3.4. Given an $M \times M$ matrix $B = (b_{ij})$, we denote $\Phi \bullet B$ the right action of B on $\Phi = (\varphi_1, \ldots, \varphi_M) \in \left(L^2(\mathbb{R}^3)\right)^M$. More precisely,

$$(\Phi \bullet B) := \left(\sum_{i=1}^M b_{i1}\varphi_i, \ldots, \sum_{i=1}^M b_{iM}\varphi_i\right)$$

and, by straightforward calculation, we obtain

$$\mathrm{Gram}_{L^2}(\Phi \bullet B) = B^* \mathrm{Gram}_{L^2}(\Phi) B$$

where B^* denotes the conjugate transpose of B.
First of all, for $\mu = p, n$, we consider

$$\tilde{\Psi}_\mu = \Psi_\mu \bullet G_\mu^{-1/2} \tag{3.28}$$

and we observe that

$$\mathrm{Gram}_{L^2}\left(\tilde{\Psi}_p\right) = \mathbb{1}_Z,$$
$$\mathrm{Gram}_{L^2}\left(\tilde{\Psi}_n\right) = \mathbb{1}_N.$$

Second, we define

$$\tilde{\Phi}^+_p = \Lambda^+_{p,\Psi}\tilde{\Psi}_p \bullet \left[\mathrm{Gram}_{L^2}\left(\Lambda^+_{p,\Psi}\tilde{\Psi}_p\right)\right]^{-1/2} \in \left(\Lambda^+_{p,\Psi}H^{1/2}\right)^Z, \tag{3.29}$$
$$\tilde{\Phi}^+_n = \Lambda^+_{n,\Psi}\tilde{\Psi}_n \bullet \left[\mathrm{Gram}_{L^2}\left(\Lambda^+_{n,\Psi}\tilde{\Psi}_n\right)\right]^{-1/2} \in \left(\Lambda^+_{n,\Psi}H^{1/2}\right)^N. \tag{3.30}$$

Remark that $\mathrm{Gram}_{L^2}\left(\Lambda^+_{p,\Psi}\tilde{\Psi}_p\right)$ and $\mathrm{Gram}_{L^2}\left(\Lambda^+_{n,\Psi}\tilde{\Psi}_n\right)$ are invertible matrices thanks to the hypothesis ii. of the lemma.
Next, we look for $\left(\tilde{\Phi}^-_p, \tilde{\Phi}^-_n\right) \in \left(\Lambda^-_{p,\Psi}H^{1/2}\right)^Z \times \left(\Lambda^-_{n,\Psi}H^{1/2}\right)^N$ such that, taking

$$\Phi_p = l_{\tilde{\Phi}^+_p}(\tilde{\Phi}^-_p) \bullet G_p^{1/2}$$
$$\Phi_n = l_{\tilde{\Phi}^+_n}(\tilde{\Phi}^-_n) \bullet G_n^{1/2},$$

we have

$$\Lambda^-_{p,\Psi}\Lambda^-_{p,\Phi}\Phi_p = 0, \tag{3.31}$$
$$\Lambda^-_{n,\Psi}\Lambda^-_{n,\Phi}\Phi_n = 0, \tag{3.32}$$

with $\Phi = (\Phi_p, \Phi_n)$ and $l_{\tilde{\Phi}_p^+}$, $l_{\tilde{\Phi}_n^+}$ defined by

$$l_{\tilde{\Phi}_\mu^+}(\tilde{\Phi}_\mu^-) := \left(\tilde{\Phi}_\mu^+ + \tilde{\Phi}_\mu^-\right) \bullet \left[\mathrm{Gram}_{L^2}\left(\tilde{\Phi}_\mu^+ + \tilde{\Phi}_\mu^-\right)\right]^{-1/2}$$

for $\mu = p, n$.

We observe that $l_{\tilde{\Phi}_p^+}$ and $l_{\tilde{\Phi}_n^+}$ are smooth maps from $\left(\Lambda_{p,\Psi}^- H^{1/2}\right)^Z$ to $\left(H^{1/2}\right)^Z$ and from $\left(\Lambda_{n,\Psi}^- H^{1/2}\right)^N$ to $\left(H^{1/2}\right)^N$ respectively; furthermore,

$$\mathrm{Gram}_{L^2}(\Phi_p) = G_p,$$
$$\mathrm{Gram}_{L^2}(\Phi_n) = G_n.$$

Now, to prove the existence of $\tilde{\Phi}_p^-$ and $\tilde{\Phi}_n^-$, we apply the implicit function theorem. We remark that the equations (3.31) and (3.32) can be written as $F(g, \tilde{\Phi}_p^-, \tilde{\Phi}_n^-) = 0$ where F is a nonlinear $\mathcal{C}^1$ operator and $g = (g_\sigma, g_\omega, g_\rho, e)$. In particular,

$$\Lambda_{\mu,\Psi}^- \Lambda_{\mu,\Phi}^- \Phi_\mu =$$
$$\Lambda_{\mu,\Psi}^- \Phi_\mu + \Lambda_{\mu,\Psi}^- \left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} (H_{\mu,\Psi} - i\eta)^{-1}(H_{\mu,\Phi} - H_{\mu,\Psi})(H_{\mu,\Phi} - i\eta)^{-1}\Phi_\mu \, d\eta\right)$$

and

$$\begin{aligned}\Lambda_{\mu,\Psi}^- \Phi_\mu &= \Lambda_{\mu,\Psi}^- \left(\tilde{\Phi}_\mu^+ + \tilde{\Phi}_\mu^-\right) \bullet \left[\mathbb{1} + \mathrm{Gram}_{L^2}\left(\tilde{\Phi}_\mu^-\right)\right]^{-1/2} \bullet G_\mu^{1/2} \\ &= \tilde{\Phi}_\mu^- \bullet \left[\mathbb{1} + \mathrm{Gram}_{L^2}\left(\tilde{\Phi}_\mu^-\right)\right]^{-1/2} \bullet G_\mu^{1/2}.\end{aligned}$$

Hence, we define

$$F(g, \tilde{\Phi}_p^-, \tilde{\Phi}_n^-) = \begin{pmatrix} F_p(g, \tilde{\Phi}_p^-, \tilde{\Phi}_n^-) \\ F_n(g, \tilde{\Phi}_p^-, \tilde{\Phi}_n^-) \end{pmatrix}$$

where

$$F_p(g, \tilde{\Phi}_p^-, \tilde{\Phi}_n^-) = \tilde{\Phi}_p^- \bullet \left[\mathbb{1} + \mathrm{Gram}_{L^2}\left(\tilde{\Phi}_p^-\right)\right]^{-1/2} \bullet G_p^{1/2} + K_p(g, \tilde{\Phi}_p^-, \tilde{\Phi}_n^-), \quad (3.33)$$

$$F_n(g, \tilde{\Phi}_p^-, \tilde{\Phi}_n^-) = \tilde{\Phi}_n^- \bullet \left[\mathbb{1} + \mathrm{Gram}_{L^2}\left(\tilde{\Phi}_n^-\right)\right]^{-1/2} \bullet G_n^{1/2} + K_n(g, \tilde{\Phi}_p^-, \tilde{\Phi}_n^-) \quad (3.34)$$

and

$$K_\mu(g, \tilde{\Phi}_p^-, \tilde{\Phi}_n^-) = \Lambda_{\mu,\Psi}^- \frac{1}{2\pi}\int_{-\infty}^{+\infty} (H_{\mu,\Psi} - i\eta)^{-1}(H_{\mu,\Phi} - H_{\mu,\Psi})(H_{\mu,\Phi} - i\eta)^{-1}\Phi_\mu \, d\eta$$

for $\mu = p, n$.

Using the definitions (1.26) and (1.27), we obtain

$$K_p(0, \tilde{\Phi}_p^-, \tilde{\Phi}_n^-) = K_n(0, \tilde{\Phi}_p^-, \tilde{\Phi}_n^-) = 0,$$

and then $F(0, 0, 0) = 0$.

Now, to apply the implicit function theorem, we have to check that

$$F : \mathbb{R}^4 \times \left(\Lambda_{p,\Psi}^- H^{1/2}\right)^Z \times \left(\Lambda_{n,\Psi}^- H^{1/2}\right)^N \to \left(\Lambda_{p,\Psi}^- H^{1/2}\right)^Z \times \left(\Lambda_{n,\Psi}^- H^{1/2}\right)^N$$

is a $\mathcal{C}^1$ operator and $D_2F(0,0,0) := F_{\tilde{\Phi}_p^-,\tilde{\Phi}_n^-}(0,0,0)$ is an isomorphism. We remark that

$$D_2F(0,0,0)(\chi,\tau) = \begin{pmatrix} \chi \bullet G_p^{1/2} \\ \tau \bullet G_n^{1/2} \end{pmatrix}, \tag{3.35}$$

and then it is an isomorphism, since $G_p^{1/2}$ and $G_n^{1/2}$ are invertible matrices.
Proceeding as above, we can easily show that F is well defined in $\left(\Lambda_{p,\Psi}^- H^{1/2}\right)^Z \times \left(\Lambda_{n,\Psi}^- H^{1/2}\right)^N$.
Next, we have to prove that $F(g,\tilde{\Phi}_p^-,\tilde{\Phi}_n^-)$ is $\mathcal{C}^1$; by classical arguments, it is enough to show that for $(\chi,\tau) \in \left(\Lambda_{p,\Psi}^- H^{1/2}\right)^Z \times \left(\Lambda_{n,\Psi}^- H^{1/2}\right)^N$

$$\begin{aligned}
&\frac{\partial F_p(g,\tilde{\Phi}_p^-,\tilde{\Phi}_n^-)}{\partial \tilde{\Phi}_p^-}\chi \in \left(\Lambda_{p,\Psi}^- H^{1/2}\right)^Z, \\
&\frac{\partial F_p(g,\tilde{\Phi}_p^-,\tilde{\Phi}_n^-)}{\partial \tilde{\Phi}_n^-}\tau \in \left(\Lambda_{p,\Psi}^- H^{1/2}\right)^Z, \\
&\frac{\partial F_n(g,\tilde{\Phi}_p^-,\tilde{\Phi}_n^-)}{\partial \tilde{\Phi}_p^-}\chi \in \left(\Lambda_{n,\Psi}^- H^{1/2}\right)^N, \\
&\frac{\partial F_n(g,\tilde{\Phi}_p^-,\tilde{\Phi}_n^-)}{\partial \tilde{\Phi}_n^-}\tau \in \left(\Lambda_{n,\Psi}^- H^{1/2}\right)^N,
\end{aligned}$$

and we leave the details of this part to the reader.
Then, applying the implicit function theorem, we conclude that there exist $U \subset \mathbb{R}^4$, $V_p \subset \left(\Lambda_{p,\Psi}^- H^{1/2}\right)^Z$ and $V_n \subset \left(\Lambda_{n,\Psi}^- H^{1/2}\right)^N$ neighborhoods of 0, and a unique continuously differentiable function $f : U \to V_p \times V_n$ such that $F(g,f(g)) = 0$; that means that for $g_\sigma, g_\omega, g_\rho, e$ sufficiently small, there exists $(\tilde{\Phi}_p^-,\tilde{\Phi}_n^-) \in V_p \times V_n$ such that

$$\begin{aligned}
\Lambda_{p,\Psi}^- \Lambda_{p,\Phi}^- \Phi_p &= 0, \\
\Lambda_{n,\Psi}^- \Lambda_{n,\Phi}^- \Phi_n &= 0.
\end{aligned}$$

In particular, $U = \bar{B}(0,\gamma)$, $V_p = \bar{B}(0,\eta)$ and $V_n = \bar{B}(0,\eta)$ with $\gamma,\eta > 0$ and from the proof of the implicit function theorem, we know that, fixed η, we can choose γ such that $f : U \to V_p \times V_n$. Then we take η and γ such that $D_2F(g,\chi,\tau)$ is invertible $\forall (g,\chi,\tau) \in U \times V_p \times V_n$.
Now, we denote $B_p := \left[\mathrm{Gram}_{L^2}\left(\Lambda_{p,\Psi}^+ \tilde{\Psi}_p\right)\right]^{1/2}$ and we remark that

$$\begin{aligned}
\tilde{\Psi}_p &= \Lambda_{p,\Psi}^+ \tilde{\Psi}_p + \Lambda_{p,\Psi}^- \tilde{\Psi}_p \\
&= \tilde{\Phi}_p^+ \bullet B_p + \Lambda_{p,\Psi}^- \tilde{\Psi}_p.
\end{aligned}$$

So we may write

$$\tilde{\Psi}_p \bullet B_p^{-1} = \tilde{\Phi}_p^+ + \Lambda_{p,\Psi}^- \tilde{\Psi}_p \bullet B_p^{-1}.$$

As a consequence,

$$l_{\tilde{\Phi}_p^+}(\Lambda_{p,\Psi}^- \tilde{\Psi}_p \bullet B_p^{-1}) = (\tilde{\Psi}_p \bullet B_p^{-1}) \bullet \left[\mathrm{Gram}_{L^2}\left(\tilde{\Psi}_p \bullet B_p^{-1}\right)\right]^{-1/2}.$$

We can easily compute

$$\mathrm{Gram}_{L^2}(\tilde{\Psi}_p \bullet B_p^{-1}) = (B_p^*)^{-1}\mathrm{Gram}_{L^2}(\tilde{\Psi}_p)B_p^{-1} = (B_p B_p^*)^{-1}$$

where B_p^* denotes the conjugate transpose of B_p. Since B_p is hermitian,

$$\mathrm{Gram}_{L^2}(\tilde{\Psi}_p \bullet B_p^{-1}) = (B_p^2)^{-1} = (B_p^{-1})^2,$$

$$l_{\tilde{\Phi}_p^+}(\Lambda_{p,\Psi}^-\tilde{\Psi}_p \bullet B_p^{-1}) = (\tilde{\Psi}_p \bullet B_p^{-1}) \bullet (B_p^2)^{1/2} = \tilde{\Psi}_p$$

and

$$l_{\tilde{\Phi}_p^+}(\Lambda_{p,\Psi}^-\tilde{\Psi}_p \bullet B_p^{-1}) \bullet G_p^{1/2} = \Psi_p.$$

Hence

$$\|\Phi_p - \Psi_p\|_{(H^{1/2})^Z} = \left\| \left[l_{\tilde{\Phi}_p^+}(\tilde{\Phi}_p^-) - l_{\tilde{\Phi}_p^+}(\tilde{\Psi}_p^- \bullet B_p^{-1}) \right] \bullet G_p^{1/2} \right\|_{(H^{1/2})^Z} \tag{3.36}$$

with $\tilde{\Psi}_p^- = \Lambda_{p,\Psi}^-\tilde{\Psi}_p$. In the same way,

$$\|\Phi_n - \Psi_n\|_{(H^{1/2})^N} = \left\| \left[l_{\tilde{\Phi}_n^+}(\tilde{\Phi}_n^-) - l_{\tilde{\Phi}_n^+}(\tilde{\Psi}_n^- \bullet B_n^{-1}) \right] \bullet G_n^{1/2} \right\|_{(H^{1/2})^N} \tag{3.37}$$

with $\tilde{\Psi}_n^- = \Lambda_{n,\Psi}^-\tilde{\Psi}_n$ and $B_n := \left[\mathrm{Gram}_{L^2}\left(\Lambda_{n,\Psi}^+\tilde{\Psi}_n\right)\right]^{1/2}$.
We remind that the maps $l_{\tilde{\Phi}_p^+}$ and $l_{\tilde{\Phi}_n^+}$ are smooth; then, to have an estimation of the norms (3.36) and (3.37), it is enough to estimate

$$\left\|\tilde{\Phi}_p^- - \tilde{\Psi}_p^- \bullet B_p^{-1}\right\|_{(H^{1/2})^Z} \text{ and } \left\|\tilde{\Phi}_n^- - \tilde{\Psi}_n^- \bullet B_n^{-1}\right\|_{(H^{1/2})^N}.$$

Indeed, $\forall \varepsilon > 0$, $\exists \delta_p, \delta_n > 0$ such that

$$\left\|\tilde{\Phi}_p^- - \tilde{\Psi}_p^- \bullet B_p^{-1}\right\|_{(H^{1/2})^Z} \leq \delta_p \Rightarrow \left\|l_{\tilde{\Phi}_p^+}(\tilde{\Phi}_p^-) - l_{\tilde{\Phi}_p^+}(\tilde{\Psi}_p^- \bullet B_p^{-1})\right\|_{(H^{1/2})^Z} \leq \varepsilon$$

and

$$\left\|\tilde{\Phi}_n^- - \tilde{\Psi}_n^- \bullet B_n^{-1}\right\|_{(H^{1/2})^N} \leq \delta_n \Rightarrow \left\|l_{\tilde{\Phi}_n^+}(\tilde{\Phi}_n^-) - l_{\tilde{\Phi}_n^+}(\tilde{\Psi}_n^- \bullet B_n^{-1})\right\|_{(H^{1/2})^N} \leq \varepsilon.$$

Now, for $\tilde{\delta}$ small enough, $(\tilde{\Psi}_p^- \bullet B_p^{-1}, \tilde{\Psi}_n^- \bullet B_n^{-1}) \in V_p \times V_n$; then $F(g, \tilde{\Psi}_p^- \bullet B_p^{-1}, \tilde{\Psi}_n^- \bullet B_n^{-1})$ is differentiable and $D_2F(g, \tilde{\Psi}_p^- \bullet B_p^{-1}, \tilde{\Psi}_n^- \bullet B_n^{-1}) := Q$ is invertible $\forall g \in U$. Using this fact, we can write

$$F(g, \tilde{\Phi}_p^-, \tilde{\Phi}_n^-) = F(g, \tilde{\Psi}_p^- \bullet B_p^{-1}, \tilde{\Psi}_n^- \bullet B_n^{-1}) + Q(\tilde{\Phi}^- - \tilde{\Psi}^- \bullet B^{-1}) + u(g, \tilde{\Phi}_p^-, \tilde{\Phi}_n^-)$$

with

$$\tilde{\Phi}^- = (\tilde{\Phi}_p^-, \tilde{\Phi}_n^-) = f(g),\ \tilde{\Psi}^- = (\tilde{\Psi}_p^-, \tilde{\Psi}_n^-),\ B = \begin{pmatrix} B_p & 0 \\ 0 & B_n \end{pmatrix}$$

and

$$\lim_{y \to \tilde{\Psi}^- \bullet B^{-1}} \frac{\|u(g,y)\|_{(H^{1/2})^A}}{\left\|y - \tilde{\Psi}^- \bullet B^{-1}\right\|_{(H^{1/2})^A}} = 0, \tag{3.38}$$

and this implies

$$(\tilde{\Phi}^- - \tilde{\Psi}^- \bullet B^{-1}) = -Q^{-1}F(g, \tilde{\Psi}_p^- \bullet B_p^{-1}, \tilde{\Psi}_n^- \bullet B_n^{-1}) - Q^{-1}u(g, \tilde{\Phi}_p^-, \tilde{\Phi}_n^-).$$

Moreover, thanks to (3.38), we know that there exists $\bar{\delta} > 0$, such that

$$\|u(g,y)\|_{(H^{1/2})^A} \leq \frac{1}{2\|Q^{-1}\|}\left\|y - \tilde{\Psi}^- \bullet B^{-1}\right\|_{(H^{1/2})^A}$$

if $\left\| y - \tilde{\Psi}^- \bullet B^{-1} \right\|_{(H^{1/2})^A} \leq \bar{\delta}$.
Then, choosing $\eta \leq \frac{\bar{\delta}}{2}$, we have

$$\begin{aligned} \left\| \tilde{\Phi}^- - \tilde{\Psi}^- \bullet B^{-1} \right\|_{(H^{1/2})^A} &\leq \|Q^{-1}\| \left\| F(g, \tilde{\Psi}_p^- \bullet B_p^{-1}, \tilde{\Psi}_n^- \bullet B_n^{-1}) \right\|_{(H^{1/2})^A} \\ &\quad + \frac{1}{2} \left\| \tilde{\Phi}^- - \tilde{\Psi}^- \bullet B^{-1} \right\|_{(H^{1/2})^A} \end{aligned}$$

and

$$\left\| \tilde{\Phi}^- - \tilde{\Psi}^- \bullet B^{-1} \right\|_{(H^{1/2})^A} \leq C \left\| \begin{pmatrix} \Lambda_{p,\Psi}^- \Psi_p \\ \Lambda_{n,\Psi}^- \Psi_n \end{pmatrix} \right\|_{(H^{1/2})^A} \leq C\tilde{\delta}. \tag{3.39}$$

Finally, choosing $\tilde{\delta} \leq \frac{\min(\delta_p, \delta_n)}{C}$, we obtain

$$\|\Phi_p - \Psi_p\|_{(H^{1/2})^Z} \leq \varepsilon \tag{3.40}$$

and

$$\|\Phi_n - \Psi_n\|_{(H^{1/2})^N} \leq \varepsilon. \tag{3.41}$$

To conclude the proof of the lemma, we have to show that

$$\Lambda_{p,\Psi}^- : \mathrm{Im}\Lambda_{p,\Phi}^- \to \mathrm{Im}\Lambda_{p,\Psi}^-$$
$$\Lambda_{n,\Psi}^- : \mathrm{Im}\Lambda_{n,\Phi}^- \to \mathrm{Im}\Lambda_{n,\Psi}^-$$

are one-to-one operators. We remark that

$$\begin{aligned} \left\| \Lambda_{p,\Phi}^- \Lambda_{p,\Psi}^- - I_{\mathrm{Im}\Lambda_{p,\Phi}^-} \right\| &= \left\| \Lambda_{p,\Phi}^- \Lambda_{p,\Psi}^- - \Lambda_{p,\Phi}^- \right\| = \left\| \Lambda_{p,\Phi}^- \left(\Lambda_{p,\Psi}^- - \Lambda_{p,\Phi}^- \right) \right\| \\ &\leq \left\| \Lambda_{p,\Phi}^- \right\| \left\| \Lambda_{p,\Psi}^- - \Lambda_{p,\Phi}^- \right\| \leq \left\| \Lambda_{p,\Psi}^- - \Lambda_{p,\Phi}^- \right\| < 1. \end{aligned}$$

As a consequence, $\Lambda_{p,\Phi}^- \Lambda_{p,\Psi}^-$ is an invertible operator and $\Lambda_{p,\Psi}^-$ is one-to-one from $\mathrm{Im}\Lambda_{p,\Phi}^-$ into $\mathrm{Im}\Lambda_{p,\Psi}^-$. In the same way, we can prove that $\Lambda_{n,\Psi}^-$ is one-to-one from $\mathrm{Im}\Lambda_{n,\Phi}^-$ into $\mathrm{Im}\Lambda_{n,\Psi}^-$.
In conclusion,

$$\begin{aligned} \Lambda_{p,\Phi}^- \Phi_p &= 0, \\ \Lambda_{n,\Phi}^- \Phi_n &= 0. \end{aligned}$$

□

Proof of corollary 3.5. To prove this corollary, we apply the lemma 3.4 to Ψ^k for any $k \in \mathbb{N}$ and, to obtain (3.24) and (3.25), we use the inequalities (3.36), (3.37) and (3.39).

□

Now, using the corollary 3.5, we can conclude that if $g_\sigma, g_\omega, g_\rho$ and e are sufficiently small, there is a constant $k_0 \in \mathbb{N}$ such that, for any $k \geq k_0$, there exists $\Phi_1^k = (\Phi_{p,1}^k, \Phi_{n,1}^k) \in \left(H^{1/2}(\mathbb{R}^3)\right)^Z \times \left(H^{1/2}(\mathbb{R}^3)\right)^N$ with the following properties:

i.

$$\Lambda_{p,\Phi_1^k}^- \Phi_{p,1}^k = 0, \tag{3.42}$$
$$\Lambda_{n,\Phi_1^k}^- \Phi_{n,1}^k = 0. \tag{3.43}$$

ii.

$$\left\|\Phi_{p,1}^k - \Psi_{p,1}^k\right\|_{(H^{1/2})^Z} \underset{k}{\to} 0, \tag{3.44}$$

$$\left\|\Phi_{n,1}^k - \Psi_{n,1}^k\right\|_{(H^{1/2})^N} \underset{k}{\to} 0. \tag{3.45}$$

iii. For $1 \leq i, j \leq Z$, $Z+1 \leq i, j \leq A$,

$$\int_{\mathbb{R}^3} \phi_{i,1}^{k^*} \phi_{j,1}^k = \lambda_i \delta_{ij}. \tag{3.46}$$

In particular, if

$$\mathrm{Gram}_{L^2}\left(\Psi_{p,1}^k\right) = \mathrm{diag}(\lambda_1, \ldots, \lambda_Z) \text{ and } \mathrm{Gram}_{L^2}\left(\Psi_{n,1}^k\right) = \mathrm{diag}(\lambda_{Z+1}, \ldots, \lambda_A)$$

are invertible matrices, we apply the corollary 3.5 to Ψ_1^k.
On the other hand, if $\mathrm{Gram}_{L^2}\left(\Psi_{p,1}^k\right)$ or $\mathrm{Gram}_{L^2}\left(\Psi_{n,1}^k\right)$ is not an invertible matrix; then there exists $i \in \{1, \ldots, A\}$ such that $\lambda_i = 0$. As a consequence, $\psi_{i,1}^k = 0$ for any $k \in \mathbb{N}$.
We assume, without loss of generality, that $\lambda_i = 0$ for $1 \leq i < r_p$, $Z+1 \leq i < r_n$ and $\lambda_i \neq 0$ for $r_p \leq i \leq Z$, $r_n \leq i \leq A$, and we denote $\hat{\Psi}_{p,1}^k = (\psi_{r_p,1}^k, \ldots, \psi_{Z,1}^k)$ and $\hat{\Psi}_{n,1}^k = (\psi_{r_n,1}^k, \ldots, \psi_{A,1}^k)$. Since

$$\mathrm{Gram}_{L^2}\left(\hat{\Psi}_{p,1}^k\right) = \mathrm{diag}(\lambda_{r_p}, \ldots, \lambda_Z) \text{ and } \mathrm{Gram}_{L^2}\left(\hat{\Psi}_{n,1}^k\right) = \mathrm{diag}(\lambda_{r_n}, \ldots, \lambda_A)$$

are invertible matrices, we can apply the corollary 3.5 to $\hat{\Psi}_1^k = (\hat{\Psi}_{p,1}^k, \hat{\Psi}_{n,1}^k)$ to obtain, for any $k \geq k_0$, $\hat{\Phi}_1^k = (\hat{\Phi}_{p,1}^k, \hat{\Phi}_{n,1}^k)$ such that

i.

$$\Lambda^-_{p,\hat{\Phi}_1^k} \hat{\Phi}_{p,1}^k = 0,$$
$$\Lambda^-_{n,\hat{\Phi}_1^k} \hat{\Phi}_{n,1}^k = 0,$$

ii.

$$\left\|\hat{\Phi}_{p,1}^k - \hat{\Psi}_{p,1}^k\right\|_{(H^{1/2})^{Z-r_p+1}} \underset{k}{\to} 0,$$
$$\left\|\hat{\Phi}_{n,1}^k - \hat{\Psi}_{n,1}^k\right\|_{(H^{1/2})^{N-r_n+1}} \underset{k}{\to} 0,$$

iii. $\mathrm{Gram}_{L^2}\left(\hat{\Psi}_{p,1}^k\right) = \mathrm{Gram}_{L^2}\left(\hat{\Phi}_{p,1}^k\right)$ and $\mathrm{Gram}_{L^2}\left(\hat{\Psi}_{n,1}^k\right) = \mathrm{Gram}_{L^2}\left(\hat{\Phi}_{n,1}^k\right)$.

To conclude, it is enough to take $\Phi_{p,1}^k = (0, \ldots, 0, \hat{\phi}_{r_p,1}^k, \ldots, \hat{\phi}_{Z,1}^k)$ and $\Phi_{n,1}^k = (0, \ldots, 0, \hat{\phi}_{r_n,1}^k, \ldots, \hat{\phi}_{A,1}^k)$, and remark that $H_{\mu,\Phi_1^k} = H_{\mu,\hat{\Phi}_1^k}$ for $\mu = p, n$.

In the same way, if $g_\sigma, g_\omega, g_\rho$ and e are sufficiently small, there is a constant $k_0 \in \mathbb{N}$ such that, for any $k \geq k_0$, there exists $\Phi_2^k = (\Phi_{p,2}^k, \Phi_{n,2}^k) \in \left(H^{1/2}(\mathbb{R}^3)\right)^Z \times \left(H^{1/2}(\mathbb{R}^3)\right)^N$ with the following properties:

i.

$$\Lambda^-_{p,\Phi_2^k} \Phi_{p,2}^k = 0, \tag{3.47}$$

$$\Lambda^-_{n,\Phi_2^k} \Phi_{n,2}^k = 0. \tag{3.48}$$

ii.

$$\left\|\Phi_{p,2}^k - \Psi_{p,2}^k\right\|_{(H^{1/2})^Z} \underset{k}{\to} 0, \tag{3.49}$$

$$\left\|\Phi_{n,2}^k - \Psi_{n,2}^k\right\|_{(H^{1/2})^N} \underset{k}{\to} 0. \tag{3.50}$$

iii. For $1 \leq i,j \leq Z$, $Z+1 \leq i,j \leq A$,

$$\int_{\mathbb{R}^3} \phi_{i,2}^{k^*} \phi_{j,2}^k = (1-\lambda_i)\delta_{ij}. \tag{3.51}$$

Using (3.44), (3.45), (3.49), (3.50) and the continuity of $\mathcal{E}$, we remark that

$$\underline{\lim}_{k\to\infty} \mathcal{E}(\Psi_1^k) = \underline{\lim}_{k\to\infty} \mathcal{E}(\Phi_1^k),$$
$$\underline{\lim}_{k\to\infty} \mathcal{E}(\Psi_2^k) = \underline{\lim}_{k\to\infty} \mathcal{E}(\Phi_2^k),$$

and then, if dichotomy occurs, we have

$$\begin{aligned} I &= \lim_{k\to\infty} \mathcal{E}(\Psi^k) \geq \underline{\lim}_{k\to\infty} \mathcal{E}(\Psi_1^k) + \underline{\lim}_{k\to\infty} \mathcal{E}(\Psi_2^k) \\ &= \underline{\lim}_{k\to\infty} \mathcal{E}(\Phi_1^k) + \underline{\lim}_{k\to\infty} \mathcal{E}(\Phi_2^k) \\ &\geq I(\lambda_1,\dots,\lambda_A) + I(1-\lambda_1,\dots,1-\lambda_A). \end{aligned} \tag{3.52}$$

It is now clear that (3.52) contradicts (1.31).

3.2. **Vanishing does not occur.** If vanishing occurs (case ii.), then $\forall R < \infty$

$$\sup_{y\in\mathbb{R}^3} \int_{B(y,R)} \left|\psi_j^k\right|^2 \underset{k}{\to} 0$$

for $j = 1,\dots,A$ and $\psi_1^k,\dots,\psi_A^k$ converge strongly in $L^p(\mathbb{R}^3)$ to 0 for $2 < p < 3$ (see lemma 7.2 of [13]). As a consequence,

$$\lim_{k\to\infty} \mathcal{E}(\Psi^k) = \sum_{j=1}^A \lim_{k\to\infty} \int_{\mathbb{R}^3} \psi_j^{k^*} H_0 \psi_j^k,$$

and

$$I(\lambda_1,\dots,\lambda_A) = m_b \sum_{j=1}^A \lambda_j$$

thanks to the constraints of the problem.
This contradicts (1.31) because we have

$$I = m_b A = m_b \sum_{j=1}^A \lambda_j + m_b \sum_{j=1}^A (1-\lambda_j) = I(\lambda_1,\dots,\lambda_A) + I(1-\lambda_1,\dots,1-\lambda_A).$$

At this point, we have shown that any minimizing sequence satisfies the following compactness criterion: $\exists y^k \in \mathbb{R}^3$, $\forall \varepsilon > 0$, $\exists R < \infty$

$$\frac{1}{A} \sum_{j=1}^A \int_{B(y^k,R)} \left|\psi_j^k\right|^2 \geq 1 - \varepsilon.$$

We denote $\tilde{\Psi}^k = \Psi^k(\cdot + y^k)$ and we remark that the energy functional $\mathcal{E}$ is invariant by translations and $\tilde{\Psi}^k$ is in the minimizing set; then $\tilde{\Psi}^k$ is a minimizing sequence of (1.29). Since $\tilde{\Psi}^k$ is bounded in $\left(H^{1/2}(\mathbb{R}^3)\right)^A$, $\tilde{\Psi}^k$ converges weakly in $\left(H^{1/2}(\mathbb{R}^3)\right)^A$,

almost everywhere on $\mathbb{R}^3$ and in $\left(L^p_{loc}(\mathbb{R}^3)\right)^A$ for $2 \leq p < 3$ to some $\tilde{\Psi}$; moreover, thanks to the concentration-compactness argument, $\tilde{\Psi}^k$ converges strongly to $\tilde{\Psi}$ in $\left(L^2(\mathbb{R}^3)\right)^A$ and in $\left(L^p(\mathbb{R}^3)\right)^A$ for $2 \leq p < 3$.
As $\|\tilde{\psi}_j - \tilde{\psi}_j^k\|_{L^2} \to 0$ for $k \to +\infty$, it is clear that

$$\int_{\mathbb{R}^3} \tilde{\psi}_i^* \tilde{\psi}_j = \lim_{k\to+\infty} \int_{\mathbb{R}^3} \tilde{\psi}_i^{k^*} \tilde{\psi}_j^k = \delta_{ij}$$

for $1 \leq i, j \leq Z$ and $Z+1 \leq i, j \leq A$. Furthermore, $\Lambda^-_{\mu,\tilde{\Psi}} \tilde{\Psi}_\mu = 0$ for $\mu = p, n$. Indeed, as before,

$$\Lambda^-_{\mu,\tilde{\Psi}} \tilde{\psi}_j - \Lambda^-_{\mu,\tilde{\Psi}^k} \tilde{\psi}_j = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (H_{\mu,\tilde{\Psi}^k} - i\eta)^{-1} (V_{\mu,\tilde{\Psi}} - V_{\mu,\tilde{\Psi}^k}) (H_{\mu,\tilde{\Psi}} - i\eta)^{-1} \tilde{\psi}_j \, d\eta$$

and

$$\left\| (H_{\mu,\tilde{\Psi}^k} - i\eta)^{-1} (V_{\mu,\tilde{\Psi}} - V_{\mu,\tilde{\Psi}^k}) (H_{\mu,\tilde{\Psi}} - i\eta)^{-1} \tilde{\psi}_j \right\|_{H^{1/2}} \underset{k}{\to} 0$$

since $\|\tilde{\psi}_j - \tilde{\psi}_j^k\|_{L^p} \underset{k}{\to} 0$ for $2 \leq p < 3$. Then, applying the Lebesgue's dominated convergence theorem as above, we obtain

$$\left\| \Lambda^-_{\mu,\tilde{\Psi}} \tilde{\psi}_j - \Lambda^-_{\mu,\tilde{\Psi}^k} \tilde{\psi}_j \right\|_{H^{1/2}} \underset{k}{\to} 0$$

for $\mu = p$ if $1 \leq j \leq Z$ and $\mu = n$ if $Z+1 \leq j \leq A$. As a consequence,

$$\left\| \Lambda^-_{\mu,\tilde{\Psi}} \tilde{\psi}_j - \Lambda^-_{\mu,\tilde{\Psi}^k} \tilde{\psi}_j \right\|_{L^2} \underset{k}{\to} 0$$

and

$$\left\| \Lambda^-_{\mu,\tilde{\Psi}} \tilde{\psi}_j \right\|_{L^2} = \lim_{k\to+\infty} \left\| \Lambda^-_{\mu,\tilde{\Psi}^k} \tilde{\psi}_j \right\|_{L^2} = \lim_{k\to+\infty} \left\| \Lambda^-_{\mu,\tilde{\Psi}^k} \tilde{\psi}_j^k \right\|_{L^2} = 0,$$

thanks to the properties of the spectral projection $\Lambda^-_{\mu,\tilde{\Psi}^k}$ and using the fact that $\|\tilde{\psi}_j - \tilde{\psi}_j^k\|_{L^2} \underset{k}{\to} 0$. So we can conclude that $\tilde{\Psi}$ satisfies the constraints of the minimization problem (1.29).

Finally, we have to prove that

$$\mathcal{E}(\tilde{\Psi}) \leq \liminf_{k\to+\infty} \mathcal{E}(\tilde{\Psi}^k).$$

It is clear that if $\|\tilde{\psi}_j - \tilde{\psi}_j^k\|_{L^p} \underset{k}{\to} 0$ for $2 \leq p < 3$, then

$$\left(\tilde{\psi}_j, V_{\mu,\tilde{\Psi}} \tilde{\psi}_j \right) = \lim_{k\to+\infty} \left(\tilde{\psi}_j^k, V_{\mu,\tilde{\Psi}^k} \tilde{\psi}_j^k \right) \tag{3.53}$$

for $\mu = p$ if $1 \leq j \leq Z$ and $\mu = n$ if $Z+1 \leq j \leq A$. Moreover, we observe that

$$\left\| \Lambda^- \tilde{\psi}_j - \Lambda^- \tilde{\psi}_j^k \right\|_{H^{1/2}} \leq \left\| (\Lambda^- - \Lambda^-_{\mu,\tilde{\Psi}^k})(\tilde{\psi}_j - \tilde{\psi}_j^k) \right\|_{H^{1/2}} + \left\| (\Lambda^-_{\mu,\tilde{\Psi}^k} - \Lambda^-_{\mu,\tilde{\Psi}}) \tilde{\psi}_j \right\|_{H^{1/2}}$$

and, with the same arguments used above, we obtain

$$\left\| \Lambda^- \tilde{\psi}_j \right\|_{H^{1/2}} = \lim_{k\to+\infty} \left\| \Lambda^- \tilde{\psi}_j^k \right\|_{H^{1/2}}. \tag{3.54}$$

Then, using (3.53), (3.54) and the weak lower semicontinuity of the $H^{1/2}$-norm, we get

$$\begin{aligned}\mathcal{E}(\tilde{\Psi}) &= \sum_{j=1}^{A}\left(\tilde{\psi}_j, |H_0|\tilde{\psi}_j\right)_{L^2} - 2\sum_{j=1}^{A}\left(\Lambda^-\tilde{\psi}_j, |H_0|\Lambda^-\tilde{\psi}_j\right)_{L^2} \\ &\quad + \frac{1}{2}\sum_{j=1}^{Z}\left(\tilde{\psi}_j, V_{p,\tilde{\Psi}}\tilde{\psi}_j\right)_{L^2} + \frac{1}{2}\sum_{j=Z+1}^{A}\left(\tilde{\psi}_j, V_{n,\tilde{\Psi}}\tilde{\psi}_j\right)_{L^2} \\ &\leq \liminf_{k\to+\infty}\mathcal{E}(\tilde{\Psi}^k) \leq \mathcal{E}(\tilde{\Psi}).\end{aligned}$$

As a conclusion, $\tilde{\Psi}$ is a minimizer of (1.29) and the minimizing sequence Ψ^k is relatively compact in $(H^{1/2})^A$ up to a translation.

4. Solutions of the relativistic mean-field equations

In this section, we prove that, in a weakly relativistic regime, a minimizer of (1.29) is a solution of the equations (1.26) and (1.27).

Let

$$X = \left\{\gamma \in \mathcal{B}(\mathcal{H}); \gamma = \gamma^*, (m_b^2 - \Delta)^{1/4}\gamma(m_b^2 - \Delta)^{1/4} \in \sigma_1(\mathcal{H})\right\} \tag{4.1}$$

where $\mathcal{B}(\mathcal{H})$ is the space of bounded linear maps from $\mathcal{H}$ to $\mathcal{H}$ and $\sigma_1(\mathcal{H})$ is the space of trace-class operators on $\mathcal{H}$.

Now, to each $P \in \mathbb{N}$, we associate

$$\Gamma_P = \left\{\gamma \in X; \gamma^2 = \gamma, \mathrm{tr}(\gamma) = P\right\}. \tag{4.2}$$

Given $\gamma = (\gamma_p, \gamma_n) \in X \times X$, we define

$$\begin{aligned}H_{p,\gamma}\gamma_p &:= \left[H_0 - \beta\frac{g_\sigma^2}{4\pi}\left(\frac{e^{-m_\sigma|\cdot|}}{|\cdot|}\star\rho_s\right) + \frac{g_\omega^2}{4\pi}\left(\frac{e^{-m_\omega|\cdot|}}{|\cdot|}\star\rho_0\right)\right. \\ &\quad \left. + \frac{g_\rho^2}{4\pi}\left(\frac{e^{-m_\rho|\cdot|}}{|\cdot|}\star\rho_{00}\right) + \frac{e^2}{4\pi}\left(\frac{1}{|\cdot|}\star\rho_p\right)\right]\gamma_p\end{aligned} \tag{4.3}$$

$$\begin{aligned}H_{n,\gamma}\gamma_n &:= \left[H_0 - \beta\frac{g_\sigma^2}{4\pi}\left(\frac{e^{-m_\sigma|\cdot|}}{|\cdot|}\star\rho_s\right) + \frac{g_\omega^2}{4\pi}\left(\frac{e^{-m_\omega|\cdot|}}{|\cdot|}\star\rho_0\right)\right. \\ &\quad \left. - \frac{g_\rho^2}{4\pi}\left(\frac{e^{-m_\rho|\cdot|}}{|\cdot|}\star\rho_{00}\right)\right]\gamma_n\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}\rho_s(x) &= \bar{\rho}_p(x) + \bar{\rho}_n(x) \\ \rho_0(x) &= \rho_p(x) + \rho_n(x) \\ \rho_{00}(x) &= \rho_p(x) - \rho_n(x)\end{aligned}$$

with $\bar{\rho}_p(x) = \mathrm{tr}(\beta\gamma_p(x,x))$, $\bar{\rho}_n(x) = \mathrm{tr}(\beta\gamma_n(x,x))$, $\rho_p(x) = \mathrm{tr}(\gamma_p(x,x))$ and $\rho_n(x) = \mathrm{tr}(\gamma_n(x,x))$.

Finally, we define

$$\begin{aligned}\Lambda_{p,\gamma}^{\pm} &= \chi_{\mathbb{R}^\pm}(H_{p,\gamma}), \\ \Lambda_{n,\gamma}^{\pm} &= \chi_{\mathbb{R}^\pm}(H_{n,\gamma}).\end{aligned}$$

Let $\tilde{\Psi} = (\tilde{\Psi}_p, \tilde{\Psi}_n)$ be a minimizer of the problem (1.29); to prove that $\tilde{\psi}_i$ is a solution of (1.26) for $1 \leq i \leq Z$ and of (1.27) for $Z+1 \leq i \leq A$, we proceed as follow: first, we consider $\tilde{\gamma}_p$ and $\tilde{\gamma}_n$ the orthogonal projectors defined by

$$\tilde{\gamma}_p = \sum_{i=1}^{Z} |\tilde{\psi}_i\rangle \langle\tilde{\psi}_i| \tag{4.5}$$

and

$$\tilde{\gamma}_n = \sum_{i=Z+1}^{A} |\tilde{\psi}_i\rangle \langle\tilde{\psi}_i| , \tag{4.6}$$

and we denote $\tilde{\gamma} = (\tilde{\gamma}_p, \tilde{\gamma}_n)$; then, we show that the commutator of the operator $H_{\mu,\tilde{\gamma}}$ and of the projector $\tilde{\gamma}_\mu$ is zero for $\mu = p, n$. This implies

$$\begin{aligned} H_{p,\tilde{\Psi}}\tilde{\psi}_i &= \varepsilon_i\tilde{\psi}_i \quad \text{for } 1 \leq i \leq Z, \\ H_{n,\tilde{\Psi}}\tilde{\psi}_i &= \varepsilon_i\tilde{\psi}_i \quad \text{for } Z+1 \leq i \leq A. \end{aligned}$$

First of all, we observe that if $\tilde{\Psi}$ is a minimizer of (1.29), then the vector $\tilde{\gamma} = (\tilde{\gamma}_p, \tilde{\gamma}_n)$ is a minimizer of the energy

$$\begin{aligned} \mathcal{E}(\gamma_p, \gamma_n) &= \operatorname{tr}(H_0\gamma_p) + \operatorname{tr}(H_0\gamma_n) - \frac{g_\sigma^2}{8\pi}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_s(x)\rho_s(y)}{|x-y|} e^{-m_\sigma|x-y|}\, dxdy \\ &\quad + \frac{g_\omega^2}{8\pi}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_0(x)\rho_0(y)}{|x-y|} e^{-m_\omega|x-y|}\, dxdy \\ &\quad + \frac{g_\rho^2}{8\pi}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_{00}(x)\rho_{00}(y)}{|x-y|} e^{-m_\rho|x-y|}\, dxdy \\ &\quad + \frac{e^2}{8\pi}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_p(x)\rho_p(y)}{|x-y|}\, dxdy \end{aligned} \tag{4.7}$$

on $\Gamma_{Z,N}^+ = \Gamma_Z^+ \times \Gamma_N^+$ where

$$\begin{aligned} \Gamma_Z^+ &= \left\{\gamma_p \in \Gamma_Z; \gamma_p = \Lambda_{p,\gamma}^+\gamma_p\Lambda_{p,\gamma}^+\right\}, \\ \Gamma_N^+ &= \left\{\gamma_n \in \Gamma_N; \gamma_n = \Lambda_{n,\gamma}^+\gamma_n\Lambda_{n,\gamma}^+\right\}. \end{aligned}$$

Next, we remind that

$$H_{\mu,\gamma} = H_{\mu,\gamma}^+ + H_{\mu,\gamma}^-$$

with $H_{\mu,\gamma}^+ = \Lambda_{\mu,\gamma}^+ H_{\mu,\gamma}\Lambda_{\mu,\gamma}^+$ and $H_{\mu,\gamma}^- = \Lambda_{\mu,\gamma}^- H_{\mu,\gamma}\Lambda_{\mu,\gamma}^-$ for $\mu = p, n$. Then

$$[H_{\mu,\gamma}, \gamma_\mu] = \left[H_{\mu,\gamma}^+, \gamma_\mu\right] + \left[H_{\mu,\gamma}^-, \gamma_\mu\right].$$

It is clear that $\Gamma_{Z,N}^+$ is a subset of

$$\bar{\Gamma}_{Z,N} = \left\{\gamma = (\gamma_p, \gamma_n) \in \Gamma_Z \times \Gamma_N; \left[H_{p,\gamma}^-, \gamma_p\right] = 0, \left[H_{n,\gamma}^-, \gamma_n\right] = 0\right\}$$

and, since $\tilde{\gamma} \in \Gamma_{Z,N}^+$, we obtain $\left[H_{\mu,\tilde{\gamma}}^-, \tilde{\gamma}_\mu\right] = 0$ for $\mu = p, n$. Thus, to conclude, we have to prove that $\left[H_{\mu,\tilde{\gamma}}^+, \tilde{\gamma}_\mu\right] = 0$ for $\mu = p, n$. We proceed by contradiction. We suppose that $\left[H_{p,\tilde{\gamma}}^+, \tilde{\gamma}_p\right]$ and $\left[H_{n,\tilde{\gamma}}^+, \tilde{\gamma}_n\right]$ are different from zero and we define

$$\tilde{\gamma}_p^\varepsilon = \mathcal{U}_p^\varepsilon\tilde{\gamma}_p\left(\mathcal{U}_p^\varepsilon\right)^{-1} := \exp\left(-\varepsilon\left[H_{p,\tilde{\gamma}}^+, \tilde{\gamma}_p\right]\right)\tilde{\gamma}_p\exp\left(\varepsilon\left[H_{p,\tilde{\gamma}}^+, \tilde{\gamma}_p\right]\right), \tag{4.8}$$

$$\tilde{\gamma}_n^\varepsilon = \mathcal{U}_n^\varepsilon\tilde{\gamma}_n\left(\mathcal{U}_n^\varepsilon\right)^{-1} := \exp\left(-\varepsilon\left[H_{n,\tilde{\gamma}}^+, \tilde{\gamma}_n\right]\right)\tilde{\gamma}_n\exp\left(\varepsilon\left[H_{n,\tilde{\gamma}}^+, \tilde{\gamma}_p\right]\right). \tag{4.9}$$

In particular,

$$\tilde{\gamma}_p^\varepsilon = \sum_{i=1}^{Z} |\tilde{\psi}_i^\varepsilon\rangle \langle\tilde{\psi}_i^\varepsilon|, \tag{4.10}$$

$$\tilde{\gamma}_n^\varepsilon = \sum_{i=Z+1}^{A} |\tilde{\psi}_i^\varepsilon\rangle \langle\tilde{\psi}_i^\varepsilon| \tag{4.11}$$

with $\tilde{\psi}_i^\varepsilon = \mathcal{U}_p^\varepsilon \tilde{\psi}_i$ for $1 \leq i \leq Z$ and $\tilde{\psi}_i^\varepsilon = \mathcal{U}_n^\varepsilon \tilde{\psi}_i$ for $Z+1 \leq i \leq A$.
Using the lemma 3.4, we construct $\gamma^\varepsilon = (\gamma_p^\varepsilon, \gamma_n^\varepsilon)$, small perturbation of $\tilde{\gamma}^\varepsilon = (\tilde{\gamma}_p^\varepsilon, \tilde{\gamma}_n^\varepsilon)$ such that

$$\gamma_p^\varepsilon = \Lambda^+_{p,\gamma^\varepsilon} \gamma_p^\varepsilon \Lambda^+_{p,\gamma^\varepsilon},$$
$$\gamma_n^\varepsilon = \Lambda^+_{n,\gamma^\varepsilon} \gamma_n^\varepsilon \Lambda^+_{n,\gamma^\varepsilon}.$$

We remark that $\gamma^\varepsilon \in \Gamma^+_{Z,N}$ and

$$\gamma_p^\varepsilon = \sum_{i=1}^{Z} |\phi_i^\varepsilon\rangle \langle\phi_i^\varepsilon|, \tag{4.12}$$

$$\gamma_n^\varepsilon = \sum_{i=Z+1}^{A} |\phi_i^\varepsilon\rangle \langle\phi_i^\varepsilon| \tag{4.13}$$

where

$$(\phi_1^\varepsilon, \ldots, \phi_Z^\varepsilon) = \Phi_p^\varepsilon = \Phi_p^{\varepsilon\,+} + \Phi_p^{\varepsilon\,-} + O(\varepsilon^2) \tag{4.14}$$
$$(\phi_{Z+1}^\varepsilon, \ldots, \phi_A^\varepsilon) = \Phi_n^\varepsilon = \Phi_n^{\varepsilon\,+} + \Phi_n^{\varepsilon\,-} + O(\varepsilon^2) \tag{4.15}$$

with $\Phi_\mu^{\varepsilon\,+} = \Lambda^+_{\mu,\tilde{\Psi}^\varepsilon} \tilde{\Psi}_\mu^\varepsilon \bullet \left[\mathrm{Gram}_{L^2}(\Lambda^+_{\mu,\tilde{\Psi}^\varepsilon} \tilde{\Psi}_\mu^\varepsilon)\right]^{-1/2}$ for $\mu = p, n$. Finally, we remind that

$$\tilde{\Psi}_\mu^\varepsilon = \Phi_\mu^{\varepsilon\,+} + \Lambda^-_{p,\tilde{\Psi}^\varepsilon} \tilde{\Psi}_\mu^\varepsilon \bullet B_\mu^{-1} + O(\varepsilon^2) \tag{4.16}$$

with $B_\mu = \left[\mathrm{Gram}_{L^2}(\Lambda^+_{\mu,\tilde{\Psi}^\varepsilon} \tilde{\Psi}_\mu^\varepsilon)\right]^{1/2}$ for $\mu = p, n$ (see the proof of lemma 3.4).
Then, to show that we have a contradiction, we want to prove that

$$\mathcal{E}(\gamma_p^\varepsilon, \gamma_n^\varepsilon) < \mathcal{E}(\tilde{\gamma}_p, \tilde{\gamma}_n).$$

For this purpose, we calculate $\mathcal{E}(\gamma_p^\varepsilon, \gamma_n^\varepsilon) - \mathcal{E}(\tilde{\gamma}_p, \tilde{\gamma}_n)$; since $(\gamma_p^\varepsilon, \gamma_n^\varepsilon)$ is a small perturbation of $(\tilde{\gamma}_p, \tilde{\gamma}_n)$, we can write

$$\mathcal{E}(\gamma_p^\varepsilon, \gamma_n^\varepsilon) - \mathcal{E}(\tilde{\gamma}_p, \tilde{\gamma}_n) = \mathrm{tr}\left(H_{p,\tilde{\gamma}}(\gamma_p^\varepsilon - \tilde{\gamma}_p)\right) + \mathrm{tr}\left(H_{n,\tilde{\gamma}}(\gamma_n^\varepsilon - \tilde{\gamma}_n)\right) + o(\varepsilon). \tag{4.17}$$

To study the sign of (4.17), we remind that given an operator T and an orthogonal projector P, we can consider the block decomposition of T defined by

$$T = \left(\begin{array}{c|c} PTP & PT(1-P) \\ \hline (1-P)TP & (1-P)T(1-P) \end{array}\right) := \left(\begin{array}{c|c} T_{++} & T_{+-} \\ \hline T_{-+} & T_{--} \end{array}\right). \tag{4.18}$$

Moreover, let R be another orthogonal projector and consider $Q = R - P$; then $P + Q$ is a projector and

$$\begin{aligned} P+Q &= (P+Q)^2 \\ P+Q &= P + PQ + QP + Q^2 \\ Q^2 &= (1-P)Q - QP \\ Q^2 &= (1-P)Q(1-P) - QP + (1-P)QP \\ Q^2 &= Q_{--} - Q_{++}. \end{aligned} \tag{4.19}$$

As a consequence, if $Q = O(\varepsilon)$, then $Q_{++} = O(\varepsilon^2)$ and $Q_{--} = O(\varepsilon^2)$.
Since

$$H_{\mu,\tilde{\gamma}} = \left(\begin{array}{c|c} H^+_{\mu,\tilde{\gamma}} & 0 \\ \hline 0 & H^-_{\mu,\tilde{\gamma}} \end{array}\right)$$

for $\mu = p, n$, then we have

$$\begin{aligned} \mathcal{E}(\gamma^\varepsilon_p, \gamma^\varepsilon_n) \;-\; & \mathcal{E}(\tilde{\gamma}_p, \tilde{\gamma}_n) = \mathrm{tr}\left(H^+_{p,\tilde{\gamma}}\Lambda^+_{p,\tilde{\gamma}}(\gamma^\varepsilon_p - \tilde{\gamma}_p)\Lambda^+_{p,\tilde{\gamma}}\right) \\ & + \mathrm{tr}\left(H^-_{p,\tilde{\gamma}}\Lambda^-_{p,\tilde{\gamma}}(\gamma^\varepsilon_p - \tilde{\gamma}_p)\Lambda^-_{p,\tilde{\gamma}}\right) + \mathrm{tr}\left(H^+_{n,\tilde{\gamma}}\Lambda^+_{n,\tilde{\gamma}}(\gamma^\varepsilon_n - \tilde{\gamma}_n)\Lambda^+_{n,\tilde{\gamma}}\right) \\ & + \mathrm{tr}\left(H^-_{n,\tilde{\gamma}}\Lambda^-_{n,\tilde{\gamma}}(\gamma^\varepsilon_n - \tilde{\gamma}_n)\Lambda^-_{n,\tilde{\gamma}}\right) + o(\varepsilon) \\ & := T^+_p + T^-_p + T^+_n + T^-_n + o(\varepsilon). \end{aligned} \tag{4.20}$$

First of all, we analyze the relation between $\Lambda^\pm_{\mu,\tilde{\gamma}}$, $\Lambda^\pm_{\mu,\tilde{\gamma}^\varepsilon}$ and $\Lambda^\pm_{\mu,\gamma^\varepsilon}$ for $\mu = p, n$. Using (3.8), we obtain

$$\begin{aligned} \Lambda^\pm_{\mu,\tilde{\gamma}} &= \Lambda^\pm_{\mu,\tilde{\gamma}^\varepsilon} + O(\varepsilon), \\ \Lambda^\pm_{\mu,\tilde{\gamma}} &= \Lambda^\pm_{\mu,\gamma^\varepsilon} + O(\varepsilon), \\ \Lambda^\pm_{\mu,\tilde{\gamma}^\varepsilon} &= \Lambda^\pm_{\mu,\gamma^\varepsilon} + O(\varepsilon). \end{aligned}$$

Then, if we take $P = \Lambda^+_{\mu,\gamma^\varepsilon}$, $Q = \Lambda^+_{\mu,\tilde{\gamma}} - \Lambda^+_{\mu,\gamma^\varepsilon}$ and we apply (4.18)-(4.19), we obtain

$$\Lambda^-_{\mu,\tilde{\gamma}} = \Lambda^-_{\mu,\gamma^\varepsilon} + \left(\begin{array}{c|c} O(\varepsilon^2) & O(\varepsilon) \\ \hline O(\varepsilon) & O(\varepsilon^2) \end{array}\right) = \left(\begin{array}{c|c} O(\varepsilon^2) & O(\varepsilon) \\ \hline O(\varepsilon) & 1 + O(\varepsilon^2) \end{array}\right)$$

for $\mu = p, n$.
Moreover, since $\gamma^\varepsilon \in \Gamma^+_{Z,N}$, we can write

$$\gamma^\varepsilon_\mu = \left(\begin{array}{c|c} \gamma^\varepsilon_{\mu++} & 0 \\ \hline 0 & 0 \end{array}\right)$$

and

$$\Lambda^-_{\mu,\tilde{\gamma}}\gamma^\varepsilon_\mu\Lambda^-_{\mu,\tilde{\gamma}} = \left(\begin{array}{c|c} O(\varepsilon^4) & O(\varepsilon^3) \\ \hline O(\varepsilon^3) & O(\varepsilon^2) \end{array}\right).$$

So we can conclude that $T^-_p = o(\varepsilon)$ and $T^-_n = o(\varepsilon)$.
Next, we remark that

$$\begin{aligned} T^+_\mu &= \mathrm{tr}\left(H^+_{\mu,\tilde{\gamma}}\Lambda^+_{\mu,\tilde{\gamma}}(\gamma^\varepsilon_\mu - \tilde{\gamma}^\varepsilon_\mu + \tilde{\gamma}^\varepsilon_\mu - \tilde{\gamma}_\mu)\Lambda^+_{\mu,\tilde{\gamma}}\right) \\ &= \mathrm{tr}\left(H^+_{\mu,\tilde{\gamma}}\Lambda^+_{\mu,\tilde{\gamma}}(\gamma^\varepsilon_\mu - \tilde{\gamma}^\varepsilon_\mu)\Lambda^+_{\mu,\tilde{\gamma}}\right) + \mathrm{tr}\left(H^+_{\mu,\tilde{\gamma}}\Lambda^+_{\mu,\tilde{\gamma}}(\tilde{\gamma}^\varepsilon_\mu - \tilde{\gamma}_\mu)\Lambda^+_{\mu,\tilde{\gamma}}\right). \end{aligned}$$

To calculate $\mathrm{tr}\left(H^+_{\mu,\tilde\gamma}\Lambda^+_{\mu,\tilde\gamma}(\gamma^\varepsilon_\mu-\tilde\gamma^\varepsilon_\mu)\Lambda^+_{\mu,\tilde\gamma}\right)$, we consider the block decomposition of $\Lambda^+_{\mu,\tilde\gamma}-\Lambda^+_{\mu,\tilde\gamma^\varepsilon}$ for $P=\Lambda^+_{\mu,\tilde\gamma^\varepsilon}$. As before, we have

$$\Lambda^+_{\mu,\tilde\gamma}=\Lambda^+_{\mu,\tilde\gamma^\varepsilon}+\left(\begin{array}{c|c}O(\varepsilon^2)&O(\varepsilon)\\\hline O(\varepsilon)&O(\varepsilon^2)\end{array}\right)=\left(\begin{array}{c|c}1+O(\varepsilon^2)&O(\varepsilon)\\\hline O(\varepsilon)&O(\varepsilon^2)\end{array}\right)$$

for $\mu=p,n$.
Now, we observe that, in general, $\gamma^\varepsilon_\mu-\tilde\gamma^\varepsilon_\mu=O(\varepsilon)$ and, more precisely, $\Lambda^+_{\mu,\tilde\gamma^\varepsilon}(\gamma^\varepsilon_\mu-\tilde\gamma^\varepsilon_\mu)\Lambda^+_{\mu,\tilde\gamma^\varepsilon}=O(\varepsilon^2)$. Indeed, using the definitions from (4.10) to (4.16), we have

$$\begin{aligned}\Lambda^+_{\mu,\tilde\gamma^\varepsilon}(\gamma^\varepsilon_\mu-\tilde\gamma^\varepsilon_\mu)\Lambda^+_{\mu,\tilde\gamma^\varepsilon}&=\sum_i\Lambda^+_{\mu,\tilde\gamma^\varepsilon}\left(|\phi^\varepsilon_i\rangle\langle\phi^\varepsilon_i|-|\tilde\psi^\varepsilon_i\rangle\langle\tilde\psi^\varepsilon_i|\right)\Lambda^+_{\mu,\tilde\gamma^\varepsilon}\\&=\sum_i\left(|\phi^{\varepsilon\,+}_i\rangle\langle\phi^{\varepsilon\,+}_i|-|\phi^{\varepsilon\,+}_i\rangle\langle\phi^{\varepsilon\,+}_i|\right)+O(\varepsilon^2)=O(\varepsilon^2).\end{aligned}$$

Then

$$\Lambda^+_{\mu,\tilde\gamma}(\gamma^\varepsilon_\mu-\tilde\gamma^\varepsilon_\mu)\Lambda^+_{\mu,\tilde\gamma}=\left(\begin{array}{c|c}O(\varepsilon^2)&O(\varepsilon^3)\\\hline O(\varepsilon^3)&O(\varepsilon^4)\end{array}\right)$$

and

$$T^+_\mu=\mathrm{tr}\left(H^+_{\mu,\tilde\gamma}\Lambda^+_{\mu,\tilde\gamma}(\tilde\gamma^\varepsilon_\mu-\tilde\gamma_\mu)\Lambda^+_{\mu,\tilde\gamma}\right)+o(\varepsilon).$$

Next , we consider $\tilde\gamma^\varepsilon_\mu-\tilde\gamma_\mu$. By definition,

$$\begin{aligned}\tilde\gamma^\varepsilon_\mu-\tilde\gamma_\mu&=\mathcal{U}^\varepsilon_\mu\tilde\gamma_\mu(\mathcal{U}^\varepsilon_\mu)^{-1}-\tilde\gamma_\mu\\&=\left(1-\varepsilon\left[H^+_{\mu,\tilde\gamma},\tilde\gamma_\mu\right]\right)\tilde\gamma_\mu\left(1+\varepsilon\left[H^+_{\mu,\tilde\gamma},\tilde\gamma_\mu\right]\right)-\tilde\gamma_\mu+o(\varepsilon)\\&=-\varepsilon\left[\left[H^+_{\mu,\tilde\gamma},\tilde\gamma_\mu\right],\tilde\gamma_\mu\right]+o(\varepsilon).\end{aligned}$$

Then

$$T^+_\mu=-\varepsilon\,\mathrm{tr}\left(H^+_{\mu,\tilde\gamma}\left[\left[H^+_{\mu,\tilde\gamma},\tilde\gamma_\mu\right],\tilde\gamma_\mu\right]\right)+o(\varepsilon)$$

for $\mu=p,n$ and

$$\begin{aligned}\mathcal{E}(\gamma^\varepsilon_p,\gamma^\varepsilon_n)-\mathcal{E}(\tilde\gamma_p,\tilde\gamma_n)&=-\varepsilon\sum_{\mu=p,n}\mathrm{tr}\left(H^+_{\mu,\tilde\gamma}\left[\left[H^+_{\mu,\tilde\gamma},\tilde\gamma_\mu\right],\tilde\gamma_\mu\right]\right)+o(\varepsilon)\\&=2\varepsilon\sum_{\mu=p,n}\mathrm{tr}\left((H^+_{\mu,\tilde\gamma}\tilde\gamma_\mu)^2-(H^+_{\mu,\tilde\gamma})^2\tilde\gamma_\mu^2\right)+o(\varepsilon)\\&=2\varepsilon\sum_{\mu=p,n}\langle(H^+_{\mu,\tilde\gamma}\tilde\gamma_\mu)^*,H^+_{\mu,\tilde\gamma}\tilde\gamma_\mu\rangle-\langle H^+_{\mu,\tilde\gamma}\tilde\gamma_\mu,H^+_{\mu,\tilde\gamma}\tilde\gamma_\mu\rangle\\&\quad+o(\varepsilon)\end{aligned}\tag{4.21}$$

where $\langle A,B\rangle=\mathrm{tr}(A^*B)$ is the Hilbert–Schmidt inner product.
Then, using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\left|\langle(H^+_{\mu,\tilde\gamma}\tilde\gamma_\mu)^*,H^+_{\mu,\tilde\gamma}\tilde\gamma_\mu\rangle\right|&\le\langle(H^+_{\mu,\tilde\gamma}\tilde\gamma_\mu)^*,(H^+_{\mu,\tilde\gamma}\tilde\gamma_\mu)^*\rangle^{1/2}\langle H^+_{\mu,\tilde\gamma}\tilde\gamma_\mu,H^+_{\mu,\tilde\gamma}\tilde\gamma_\mu\rangle^{1/2}\\&=\langle H^+_{\mu,\tilde\gamma}\tilde\gamma_\mu,H^+_{\mu,\tilde\gamma}\tilde\gamma_\mu\rangle\end{aligned}$$

and

$$\mathcal{E}(\gamma^\varepsilon_p,\gamma^\varepsilon_n)-\mathcal{E}(\tilde\gamma_p,\tilde\gamma_n)\le0\,;$$

furthermore, the equality holds if and only if $(H^+_{\mu,\tilde{\gamma}}\tilde{\gamma}_\mu)^* = \pm H^+_{\mu,\tilde{\gamma}}\tilde{\gamma}_\mu$.
First, we consider the case $(H^+_{\mu,\tilde{\gamma}}\tilde{\gamma}_\mu)^* = H^+_{\mu,\tilde{\gamma}}\tilde{\gamma}_\mu$; this implies $\tilde{\gamma}_\mu H^+_{\mu,\tilde{\gamma}} = H^+_{\mu,\tilde{\gamma}}\tilde{\gamma}_\mu$ that means $\left[H^+_{\mu,\tilde{\gamma}}, \tilde{\gamma}_\mu\right] = 0$. Then we have a contradiction.
Second, if $(H^+_{\mu,\tilde{\gamma}}\tilde{\gamma}_\mu)^* = -H^+_{\mu,\tilde{\gamma}}\tilde{\gamma}_\mu$, then

$$\begin{aligned}\tilde{\gamma}_\mu H^+_{\mu,\tilde{\gamma}} + H^+_{\mu,\tilde{\gamma}}\tilde{\gamma}_\mu &= 0\\ \tilde{\gamma}_\mu H^+_{\mu,\tilde{\gamma}} + \tilde{\gamma}_\mu H^+_{\mu,\tilde{\gamma}}\tilde{\gamma}_\mu &= 0\\ \tilde{\gamma}_\mu H^+_{\mu,\tilde{\gamma}} - H^+_{\mu,\tilde{\gamma}}\tilde{\gamma}_\mu &= 0\end{aligned}$$

that contradicts the hypothesis $\left[H^+_{\mu,\tilde{\gamma}}, \tilde{\gamma}_\mu\right] \neq 0$ for $\mu = p, n$.
Finally, we can conclude that if $\left[H^+_{\mu,\tilde{\gamma}}, \tilde{\gamma}_\mu\right] \neq 0$ for $\mu = p, n$, then we can construct $\gamma^\varepsilon \in \Gamma^+_{Z,N}$ such that

$$\mathcal{E}(\gamma^\varepsilon_p, \gamma^\varepsilon_n) - \mathcal{E}(\tilde{\gamma}_p, \tilde{\gamma}_n) < 0,$$

and thus we have a contradiction with the fact that $\tilde{\gamma}$ minimize the energy on $\Gamma^+_{Z,N}$. This implies that $\left[H^+_{\mu,\tilde{\gamma}}, \tilde{\gamma}_\mu\right]$ must be equal to zero and, as a consequence,

$$[H_{\mu,\tilde{\gamma}}, \tilde{\gamma}_\mu] = 0$$

for $\mu = p, n$.
As a conclusion, if $g_\sigma, g_\omega, g_\rho$ and e are sufficiently small, $\tilde{\Psi}$ is a solution of the equations (1.26) and (1.27).

Acknowledgment. The author would like to thank Professor Eric Séré and Professor Bernhard Ruf for helpful discussions.
This work was partially supported by the Grant ANR-10-BLAN 0101 of the French Ministry of Research.

References

[1] Reinhard, P.G. The relativistic mean-field description of nuclei and nuclear dynamics. *Rep. Prog. Phys.*, vol. 52, pp. 439–514 (1989).
[2] Greiner, W., Maruhn, J. *Nuclear Models*. Springer-Verlag, Berlin (1996).
[3] Ring, P. Relativistic Mean Field Theory in Finite Nuclei. *Prog. Part. Nucl. Phys.*, vol. 37, pp. 193–236 (1996).
[4] Meng, J., Toki, H., Zhou, S., Zhang, S., Long, W., Geng, L. Relativistic continuum Hartree Bogoliubov theory for ground-state properties of exotic nuclei. *Prog. Part. Nucl. Phys.*, vol. 57, pp. 470–563 (2006).
[5] Bottcher, C., Strayer, M., McGrory, J., eds. *Nuclear and atomic physics at one gigaflop : proceedings of the Nuclear and Atomic Physics Conference at One Gigaflop, held at Oak Ridge, Tennessee, April 13–16, 1988*. Harwood Academic Publishers (1989).
[6] Esteban, M.J., Séré, E. Nonrelativistic Limit of the Dirac-Fock Equations. *Ann. Henri Poincaré*, vol. 2, pp. 941–961 (2001).
[7] Esteban, M.J., Lewin, M., Séré, E. Variational methods in relativistic quantum mechanics. *Bull. Amer. Math. Soc. (N.S.)*, vol. 45 (4), pp. 535–593 (2008).
[8] Lions, P.L. The concentration-compactness principle in the calculus of variations. The locally compact case. I. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, vol. 1 (2), pp. 109–145 (1984).
[9] Lions, P.L. The concentration-compactness principle in the calculus of variations. The locally compact case. II. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, vol. 1 (4), pp. 223–283 (1984).
[10] Esteban, M.J., Séré, E. Solutions of the Dirac-Fock Equations for Atoms and Molecules. *Comm. Math. Phys.*, vol. 203, pp. 499–530 (1999).
[11] Gogny, D., Lions, P. Hartree-Fock theory in nuclear physics. *RAIRO Modél. Math. Anal. Numér.*, vol. 20, pp. 571–637 (1986).

[12] Kato, T. *Perturbation Theory for Linear Operators*. Berlin : Springer-Verlag (1980).
[13] Lenzmann, E., Lewin, M. Minimizers for the Hartree-Fock-Bogoliubov theory of neutron stars and white dwarfs. *Duke Math. J.*, vol. 152 (2), pp. 257–315 (2010).

Ceremade (UMR CNRS 7534) Université Paris-Dauphine
Place du Maréchal de Lattre de Tassigny
75775 Paris Cedex 16 France

Dipartimento di Matematica, Università di Milano
Via Saldini 50
20133 Milano, Italy
E-mail address: rotanodari@ceremade.dauphine.fr