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Chaotic quasi-collision trajectories in the 3-centre problem

Linda Dimare

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Abstract We study a particular kind of chaotic dynamics for the planar 3-centre problem on small negative energy level sets. We know that chaotic motions exist, if we make the assumption that one of the centres is far away from the other two (see Bolotin and Negrini, J Differ Equ 190:539–558, 2003): this result has been obtained by the use of the Poincaré-Melnikov theory. Here we change the assumption on the third centre: we do not make any hypothesis on its position, and we obtain a perturbation of the 2-centre problem by assuming its intensity to be very small. Then, for a dense subset of possible positions of the perturbing centre in \mathbb{R}^2 , we prove the existence of uniformly hyperbolic invariant sets of periodic and chaotic almost collision orbits by the use of a general result of Bolotin and MacKay (Celest Mech Dyn Astron 77:49–75, 2000; Celest Mech Dyn Astron 94(4):433–449, 2006). To apply it, we must preliminarily construct *chains of collision arcs* in a proper way. We succeed in doing that by the classical regularisation of the 2-centre problem and the use of the periodic orbits of the regularised problem passing through the third centre.

Keywords 3-centre problem · Regularisation · Collisions · Chaotic motion

1 Introduction

We consider the motion of a particle in the plane, under the gravitational action of three point masses at fixed positions (the planar restricted 3-centre problem). We fix a Cartesian reference system Oxy on the plane and choose suitable dimensionless coordinates such that the two centres with greater masses occupy the positions $C_1 = (1, 0)$, $C_2 = (-1, 0)$. Following the common terminology, we refer to these as primaries. We suppose for simplicity that the primaries have equal intensities $a_1 = a_2 = a > 0$ (symmetric problem), and we always consider all the three centres having positive intensities.

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Let $C = (x_0, y_0) \in \mathbb{R}^2 \setminus \{C_1, C_2\}$ be the position of the third centre and $\varepsilon > 0$ be its intensity. We assume ε to be very small and consider the limit $\varepsilon \rightarrow 0$: this means that ε is a perturbation parameter. In other words, we make the hypothesis that the conditions are such that we can consider the problem as a one-parameter perturbation of an integrable one: the 2-centre problem.

Let M be the smooth Riemannian manifold $M = \mathbb{R}^2 \setminus \{C_1, C_2\}$, with the induced Euclidean metric on the tangent bundle TM . The configuration space for the motion of a particle in the potential field generated by the centres C_1, C_2, C , is $M \setminus \{C\}$. Let $P = (x, y)$ denote the position of the particle on $M \setminus \{C\}$. Then the system has smooth Lagrangian function on $T(M \setminus \{C\})$ given by

$$L_\varepsilon = L_0 + \frac{\varepsilon}{\sqrt{(x - x_0)^2 + (y - y_0)^2}},$$

with L_0 the Lagrangian of the 2-centre problem

$$L_0 = \frac{\dot{x}^2 + \dot{y}^2}{2} + \frac{a}{\sqrt{(x + 1)^2 + y^2}} + \frac{a}{\sqrt{(x - 1)^2 + y^2}}.$$

Here we use the Newtonian notation for derivatives with respect to time: $\dot{x} = dx/dt, \dot{y} = dy/dt$. Note that L_0 is a smooth function on M , while L_ε has a Newtonian singularity at the point C . To simplify notation, let us denote by $W(x, y)$ and $\varepsilon V(x, y)$ the potentials due respectively to the primaries and the third centre C , so that

$$W(x, y) = -\frac{a}{\sqrt{(x + 1)^2 + y^2}} - \frac{a}{\sqrt{(x - 1)^2 + y^2}},$$

$$V(x, y) = -\frac{1}{\sqrt{(x - x_0)^2 + (y - y_0)^2}}.$$

The Lagrangian and the Hamiltonian of the problem have the form

$$L_\varepsilon = L_0 - \varepsilon V, \quad H_\varepsilon = H_0 + \varepsilon V,$$

with L_0, H_0 the Lagrangian and Hamiltonian of the 2-centre problem

$$L_0 = \frac{\dot{x}^2 + \dot{y}^2}{2} - W, \quad H_0 = \frac{p_x^2 + p_y^2}{2} + W.$$

For negative values of the energy $H_\varepsilon = E < 0$, with $E \rightarrow 0$, we prove the existence of chaotic motions passing arbitrarily close to the perturbing centre C . This is made by the use of the shadowing result proved in [Bolotin and MacKay \(2000\)](#).

For $E > 0$ non-integrability has been established in [Bolotin \(1984\)](#): he proves that for the n -centre problem on the plane \mathbb{R}^2 , with $n > 2$, it does not exist an analytic integral of motion which is non-constant on the energy shell $H^{-1}(E)$, with $E > 0$. In [Bolotin \(1985\)](#) the same author extends this result to a wider class of Lagrangian systems defined on any 2-dimensional manifold M , with n Newtonian singularities on M : he shows non-integrability when n is greater than two times the Euler characteristic of the manifold, $n > 2\chi(M)$, and the energy is over a suitably defined threshold, $E > E_{th}$. In particular, we have analytic non-integrability for the restricted circular many-body problem, in which a particle moves in a rotating plane, under the action of the gravitational attraction of n centres fixed on this plane, when $n > 2$. Nevertheless, this generalisation does not add any additional information about the n -centre problem on a fixed plane, in which case it reduces to the result given in [Bolotin \(1984\)](#).

More recent outcomes exist in literature, valid for positive energy values $E > 0$: see Bolotin and Negrini (2001), Knauf and Taimanov (2005). In Bolotin and Negrini (2001) the authors study the n -centre problem in \mathbb{R}^3 , showing that if $n \geq 3$ and $E \geq 0$, then the topological entropy is positive. In Knauf and Taimanov (2005), it is proved that no analytic independent integral exist for the n -centre problem in the space, if $n \geq 3$ and the energy is greater than some threshold $E > E_{th}$. Moreover, for both the planar and the spatial problem, smooth integrals are found on energy levels with $E > E_{th}$, where $E_{th} = 0$ in the planar case.

The case $E < 0$ has been investigated in Bolotin and Negrini (2003). The authors study the restricted 3-centre problem on the plane, when the third centre is very far from the other two and consider small negative energies E , in the limit $E \rightarrow 0$. Then they have a two-parameter perturbation of the 2-centre problem on the zero-energy level. They succeed in applying the Poincaré-Melnikov theory, thus proving the existence of chaotic motions.

We too study the case of small negative energies, but our point of view is different, in that the position of the third centre does not go to infinity. This allows us to study the orbits which undergo close encounters with the perturbing centre C .

As in Bolotin and Negrini (2003) the unperturbed system is the 2-centre problem with Lagrangian L_0 . After the classical regularisation of singularities, obtained by the use of elliptic coordinates $(\xi, \phi) \in \mathbb{R} \times S^1$ and a reparametrisation of time, the system separates: we have the motion of the standard pendulum for the variable ϕ and a two-well potential for ξ , with a hyperbolic equilibrium for $\xi = 0$. We are interested in the motions which go very close to the centre C , then we consider trajectories of the unperturbed problem that pass through C . The perturbing term of the Hamiltonian is not ε -small in the vicinity of a collision with C and this results in a lost of control on the stable and unstable manifolds, so that a perturbative study of their intersection is not possible.

Anyway, we will see that for ε small enough a hyperbolic invariant set with chaotic dynamics exists, for a dense set of possible positions of the third centre in \mathbb{R}^2 . Moreover, the obtained trajectories shadow chains of collision orbits through the third centre C .

We will apply a general result of Bolotin and MacKay (see Bolotin and MacKay 2000), valid for Lagrangian systems with Newtonian singularities on a d -dimensional Riemannian manifold, with $d = 2, 3$: they show that if the singular part of the Lagrangian is a perturbing term, then for each chain formed by arcs of the unperturbed problem which start and end at a singular point (*collision arcs*), there is a unique orbit with the same energy which shadows it. Indeed, after local regularisation of the perturbing singularities, the corresponding equilibria are hyperbolic and, by the use of the strong λ -lemma, shadowing arcs are obtained locally in small neighbourhoods of the singularities. It is then proved that these arcs can be joined to gain the entire shadowing orbits. In fact, the latter correspond to nondegenerate critical points of a formal functional defined on small spherical neighbourhoods (circular in the 2-dimensional case) of the singularities. This functional can be seen as the energy functional of a Frenkel-Kontorova model. There is a correspondence between the equilibrium states of Frenkel-Kontorova models and the orbits of symplectic twist maps, and in this correspondence nondegeneracy of the equilibrium states of the energy functional corresponds to hyperbolicity, as shown in Aubry et al. (1992). With this approach the existence and hyperbolicity of the shadowing orbits is proved and, by proper estimates on the mixed second derivatives of the formal functional, it is also derived that the Lyapunov exponents of the corresponding Poincaré map are of order $\log \varepsilon^{-1}$ (see Bolotin and MacKay 2006).

The result of Bolotin and MacKay (2000) has been applied by the two authors to the planar circular restricted problem of three bodies, in which a massless particle moves under the gravitational attraction of the other two bodies, the primaries, and the latter are supposed on a circular orbit about their centre of mass. The second primary is supposed to have small

mass ε . The result is the existence of periodic and chaotic orbits which undergo consecutive close encounters with the smaller primary and which approach, as $\varepsilon \rightarrow 0$, arcs of Kepler ellipses around the first primary, starting and ending at collision with the second.

A similar existence result has been obtained also in [Font et al. \(2002\)](#) by a completely different method. A direct study of the orbits which have close encounters with the small primary is carried out and the proof in this case is constructive. Indeed, an approximation of the *first return map*, defined on a region of the phase space whose projection is a small circle around the second primary, is explicitly computed. It is shown that the first return map is *horseshoe* like and it allows to conclude about the existence of orbits with consecutive infinite close approaches with the small primary. A complete numerical study of these orbits is carried out in [Font et al. \(2009\)](#).

As previously outlined, in this paper we follow the variational approach of Bolotin and MacKay applied to the 3-centre problem: in this manner we don't get an horseshoe map, but we still have a symbolic dynamics on the set of quasi-collision orbits found.

In our case the unperturbed system is simply the 2-centre problem with Lagrangian L_0 and the perturbing term is given by the potential due to the third centre C . Fixed a small negative value of the energy $E < 0$, by the classical regularisation of the 2-centre problem the system separates and we can easily find periodic orbits with energy E passing through C . We prove by the implicit function theorem that for almost all the possible positions of the perturbing centre C on the plane, these orbits do not collide with the primaries C_1, C_2 , if $|E|$ is chosen sufficiently small. Then they are periodic orbits of the not-regularised 2-centre problem and we can use them to define a finite set of collision arcs, starting and ending at C , and take as collision chains the infinite sequences formed with the chosen arcs. We can verify that the considered orbits of the unperturbed problem are nondegenerate (see Sect. 2 for the definition) and then the theorem of Bolotin and MacKay can be applied to obtain a hyperbolic invariant set formed by the orbits that shadow the collision chains. Finally, taking as alphabet the finite set of collision arcs chosen, a symbolic dynamics is naturally defined on the set of shadowing orbits by considering the shift on the collision chains. We can also get easily the positivity of entropy.

In Sect. 2, we will recall the results of [Bolotin and MacKay \(2000, 2006\)](#) for the case at hand, starting from some basic definitions, and state our main result. In Sect. 3 we will perform the classical regularisation of the 2-centre problem and compute the periods of the resulting separated problem. This will allow us to get infinite classes of periodic orbits with fixed energy E , passing through the centre C (Sect. 4): this is the core of the paper. The delicate point here is to show that the periodic orbits of the regularised problem previously found do not meet the primaries for any small enough value of the energy. In Sect. 5 we will verify that these trajectories satisfy the conditions which allow to apply the results of [Bolotin and MacKay \(2000, 2006\)](#) and conclude our proof.

2 Basic definitions and theorems

Consider the unperturbed system with Lagrangian L_0 . A solution $\gamma : [0, T] \rightarrow M$ of fixed energy E for this system will be called a *collision arc* if $\gamma(0) = \gamma(T) = C$, and $\gamma(t) \neq C$ for any $t \in (0, T)$. In particular, the latter condition means that there are no *early* collisions.

Fix an energy value $E < 0$, such that C belongs to the set $D = \{(x, y) \in M \mid W(x, y) < E\}$. Denote by Ω the set of $W^{1,2}$ curves lying in D , starting and ending at collision in C . A collision arc γ of energy E is a critical point of the Maupertuis-Jacobi functional J_E defined on Ω by

$$J_E(\gamma) = \int_0^T g_E(\gamma(t), \dot{\gamma}(t))dt,$$

with

$$g_E(x, y, \dot{x}, \dot{y}) = \sqrt{2(E - W(x, y))(\dot{x}^2 + \dot{y}^2)}.$$

The arc γ is said to be *nondegenerate* if it is a nondegenerate critical point of the functional J_E on Ω .

This definition of nondegeneracy is the most natural, but it is quite complicated to be used for verifications in concrete examples. There are at least four equivalent definitions of nondegeneracy, for which we refer the reader to Bolotin and MacKay (2000). In particular, a sufficient condition for nondegeneracy is the following. Denote the general solution of the system (L_0) by $q(t) = f(q_0, v_0, t)$, where $q_0 \in M, v_0 \in TM_{q_0}$ are the initial position and velocity. Then a collision arc γ with energy E corresponds to a solution (v_0, T) of the system of three equations

$$f(C, v_0, T) = C, \quad H_0(C, v_0) = E.$$

If the Jacobian of the system at this solution is nonzero, then γ is nondegenerate. Actually this is the characterisation of nondegeneracy which will be used in Sect. 5.

Suppose that the system (L_0) has a finite set of nondegenerate collision arcs through the centre $C, \gamma_k : [0, T_k] \rightarrow D, k \in K$, with the same energy E , where K denote a finite set of labels. A sequence $(\gamma_{k_i})_{i \in \mathbb{Z}}, k_i \in K$, is called a *collision chain* if it satisfies the condition of *direction change*:

$$\dot{\gamma}_{k_i}(T_{k_i}) \neq \pm \dot{\gamma}_{k_{i+1}}(0), \text{ for any } i \in \mathbb{Z}.$$

Collision chains correspond to paths in the graph Γ with the set of vertices K and the set of edges

$$\Gamma = \{(k, k') \in K^2 \mid \dot{\gamma}_k(T_k) \neq \pm \dot{\gamma}_{k'}(0)\}. \tag{1}$$

We are going to use the following results:

Theorem 1 (Bolotin-MacKay, 2000) *Given a finite set K of nondegenerate collision arcs with the same energy E , there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ and any collision chain $(\gamma_{k_i})_{i \in \mathbb{Z}}, k_i \in K$, there exists a unique (up to a time shift) trajectory $\gamma : \mathbb{R} \rightarrow D \setminus \{C\}$ of energy E of system (L_ε) , which shadows the chain $(\gamma_{k_i})_{i \in \mathbb{Z}}$ within order ε . More precisely, there exist constants $B, B' > 0$, independent of ε and the collision chain, and a sequence of times $(t_i)_{i \in \mathbb{Z}}$, such that $|t_{i+1} - t_i - T_{k_i}| \leq B\varepsilon, \text{dist}(\gamma(t), \gamma_{k_i}([0, T_{k_i}])) \leq B\varepsilon$, for $t_i \leq t \leq t_{i+1}$, and $\text{dist}(\gamma(t), C) \geq B'\varepsilon$.*

From Theorem 1 it follows that there is an invariant subset Λ_ε on the energy shell $\{H_\varepsilon = E\}$ on which the system (L_ε) is a suspension of a subshift of finite type. The subshift is given by the shift on the set of paths of the graph Γ , that is on the set of all the possible collision chains, and the set Λ_ε is formed by the orbits which shadow them.

The important fact about the invariant set Λ_ε is that it is uniformly hyperbolic.

Theorem 2 (Bolotin-MacKay, 2006) *There exists a cross-section $N \subset \{H_\varepsilon = E\}$, such that the corresponding invariant set $M_\varepsilon = \Lambda_\varepsilon \cap N$ of the Poincaré map is uniformly hyperbolic with Lyapunov exponents of order $\log \varepsilon^{-1}$.*

In particular, the set Λ_ε is uniformly hyperbolic as a suspension of a hyperbolic invariant set with bounded transition times.

The reader can see [Bolotin and MacKay \(2006\)](#) for the construction of the cross-section N and the definition of the Poincaré map: we omit these technical details here, because it would require some machinery, which is not necessary for the sequel.

We will show that the assumptions of [Theorem 1](#) are all satisfied for the case at hand, for almost all the possible positions of the third centre C in M . More precisely, we will prove that, fixed any of these admissible positions of C , for any small enough negative energy value, there exist non-degenerate collision arcs, with which we can construct collision chains. In particular, our collision arcs are pieces of periodic orbits of the 2-centre problem (L_0) , which pass through the centre C . We will then derive the following

Theorem 3 *Let $I \subset \mathbb{Q}^+$ be a finite set of positive rationals. There exists a dense open subset $X_I \subset M$ of possible positions for the third centre C , such that, fixed $C \in X_I$, the following is true. There is a small value $E_0 > 0$, depending on C and I , such that, fixed an energy value $E \in (-E_0, 0)$, we have that there is $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$ and any sequence $(q_k)_{k \in \mathbb{Z}}$, $q_k \in I$, there exists a trajectory of the planar restricted 3-centre problem (L_ε) on the energy shell $\{H_\varepsilon = E\}$, which avoids collision with the third centre C by order ε and is within order ε a concatenation of pieces of periodic orbits for the planar restricted 2-centre problem (L_0) , passing through C and of classes q_k , $k \in \mathbb{Z}$ (see [Sect. 3.3](#) for notation).*

The resulting invariant set formed by these orbits is uniformly hyperbolic.

In particular, as it follows from [Theorem 2](#), fixed a small enough energy value $E < 0$ and $\varepsilon > 0$, there is a cross section in the energy shell $\{H_\varepsilon = E\}$, such that the associated Poincaré map has a chaotic invariant set with Lyapunov exponents of order $\log \varepsilon^{-1}$ and it contains infinitely many periodic orbits, corresponding to periodic collision chains. The topological entropy of the Poincaré map is $O(\varepsilon)$ -close to that of the topological Markov chain associated to the graph Γ , defined by [\(1\)](#). It's easy to verify that the finite set of collision arcs that we construct in the proof of [Theorem 3](#) determine positive topological entropy (see “Appendix”).

[Theorem 3](#) is still true if we substitute in the statement the set X_I with a set X , which is dense in M and is independent of the set of rationals I . In this case, we do not know if the set X is open or not, the only thing that we can say about it is that it is dense (see [Remark 6](#)).

3 Periodic orbits for the regularised problem

In order to apply [Theorem 1](#), we must find collision arcs with fixed energy $E < 0$ for the unperturbed system (L_0) . A natural choice is to look for periodic orbits through the third centre C . As a first step, we recall the classical regularisation of singularities of Euler. A deep analysis of the 2-centre problem was made in the first volume of [Charlier \(1902\)](#). Following his ideas we get the existence of infinite classes of periodic orbits for the separated problem obtained after regularisation.¹

3.1 Regularisation

We introduce the elliptic coordinates, defined by the map $x + iy = \cosh(\xi + i\phi)$ from the cylinder $\mathbb{R} \times S^1$ to \mathbb{R}^2 . This transformation has two ramification points at the two primaries $C_1 = (0, 0)$, $C_2 = (0, \pi)$. In elliptic coordinates the Lagrangian L_ε becomes

¹ A classification of the periodic orbits of the 2-centre problem based on the variation of the energy parameters can be found in [Duan and Yuan \(1999\)](#) and [Duan et al. \(1995\)](#). We will not need this classification: according to it our orbits are all of the same kind, because of the constraints that we impose on the parameters.

$$L_\varepsilon = \frac{\dot{\xi}^2 + \dot{\phi}^2}{2} (\cosh^2 \xi - \cos^2 \phi) - W(\xi, \phi) - \varepsilon V(\xi, \phi),$$

where the potentials W, V have the form

$$W(\xi, \phi) = -\frac{2a \cosh \xi}{\cosh^2 \xi - \cos^2 \phi},$$

$$V(\xi, \phi) = -\frac{1}{\sqrt{\cosh^2 \xi - \sin^2 \phi + (x_0^2 + y_0^2) - 2(x_0 \cosh \xi \cos \phi + y_0 \sinh \xi \sin \phi)}}.$$

Consider the problem on the energy level set $\{H_\varepsilon = E\}$. The Hamiltonian in elliptic coordinates is

$$H_\varepsilon - E = \frac{1}{\cosh^2 \xi - \cos^2 \phi} \mathcal{H}_\varepsilon,$$

with

$$\mathcal{H}_\varepsilon = \frac{p_\xi^2 + p_\phi^2}{2} - 2a \cosh \xi - [E - \varepsilon V(\xi, \phi)] (\cosh^2 \xi - \cos^2 \phi),$$

where the symbols p_ξ, p_ϕ denote the conjugate momenta.

The orbits of the problem with Hamiltonian H_ε on the energy level $\{H_\varepsilon = E\}$ are, up to time parametrisation, orbits of the system with Hamiltonian \mathcal{H}_ε on the energy level $\{\mathcal{H}_\varepsilon = 0\}$. The regularised Hamiltonian \mathcal{H}_ε has no singularities at C_1, C_2 : this means that an orbit for \mathcal{H}_ε is an orbit for H_ε only if it does not pass through the primaries. The new time parameter τ is given by

$$\tau = \int_0^t \frac{1}{\cosh^2 \xi(s) - \cos^2 \phi(s)} ds, \tag{2}$$

and this formula allows us to pass from a solution for the system $(\mathcal{H}_\varepsilon)$, on the zero energy level, to a solution for (H_ε) with energy E .

The Lagrangian corresponding to \mathcal{H}_ε is

$$\mathcal{L}_\varepsilon = \frac{(\xi')^2 + (\phi')^2}{2} + 2a \cosh \xi + [E - \varepsilon V(\xi, \phi)] (\cosh^2 \xi - \cos^2 \phi),$$

where the prime sign denote derivation with respect to the new time parameter: $\xi' = d\xi/d\tau, \phi' = d\phi/d\tau$.

3.2 The separated problem

We have to find orbits of the 2-centre problem (L_0) with fixed energy $E < 0$, then we put $\varepsilon = 0$ and study the regularised system on the energy level $\{\mathcal{H}_0 = 0\}$. The Lagrangian is

$$\mathcal{L}_0 = \frac{(\xi')^2 + (\phi')^2}{2} + 2a \cosh \xi + E(\cosh^2 \xi - \cos^2 \phi).$$

The system separates and we have the two one-dimensional problems

$$\begin{cases} \frac{(\xi')^2}{2} - 2a \cosh \xi + |E| \cosh^2 \xi = -E_1 \\ \frac{(\phi')^2}{2} - |E| \cos^2 \phi = E_1 \end{cases} \tag{3}$$

Fig. 1 Potential energy for the variable ξ when $a = 1$ and $E = -0.5$.

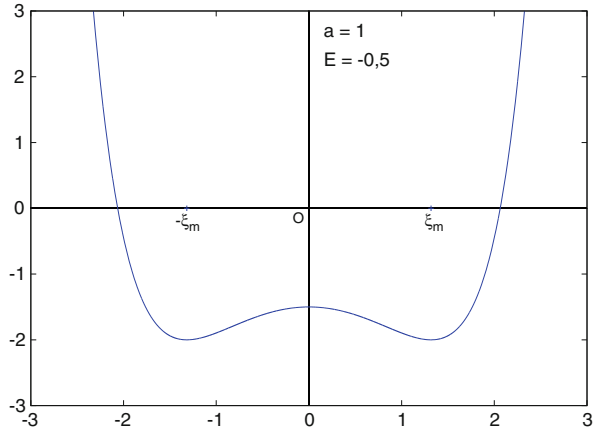
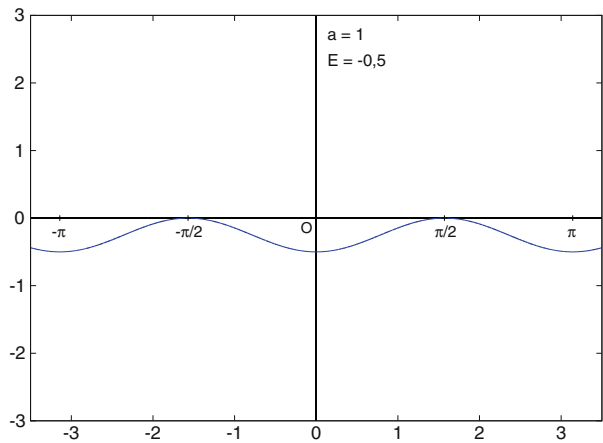


Fig. 2 Potential energy for the variable ϕ when $a = 1$ and $E = -0.5$.



where we have taken into account that $E < 0$. If we assume $|E| < a$, then the potential in the variable ξ has a local maximum when $\xi = 0$, with negative maximum value $|E| - 2a$, and two minima $\pm \xi_m$, defined by $\cosh(\pm \xi_m) = a/|E|$, where the potential takes the value $-a^2/|E|$. Furthermore, it goes to infinity for $\xi \rightarrow \pm\infty$. Finally, if we choose $-E_1 > |E| - 2a$, i.e. we consider a level over the separatrix, the motion in ξ is periodic with two inversion points at ξ_- and ξ_+ , defined by the relation

$$\cosh(\xi_{\pm}) = -\frac{a}{E} \left(1 + \sqrt{1 + \frac{EE_1}{a^2}} \right).$$

The motion in ϕ is simply the motion of the standard pendulum. Then, in the domain $E_1 > 0$, we have rotational closed orbits for ϕ and there exist action-angle variables. The graphs of the potential energies for ξ and ϕ separately are plotted in Figs. 1 and 2, when $a = 1$ and $E = -0.5$.

After these considerations, we make the following assumptions:

$$|E| < a, \quad E_1 > 0, \quad |E| + E_1 < 2a. \tag{4}$$

For future convenience, we scale the energy parameter E_1 and define

$$A_1 := E_1/2a, \quad \beta := |E|/E_1.$$

In this manner the conditions (4) becomes

$$A_1, \beta > 0, \quad 2\beta A_1 < 1, \quad A_1 < \frac{1}{1 + \beta}, \tag{5}$$

and the regularised system (3) takes the form

$$\begin{cases} \frac{(\xi')^2}{4a} = \cosh \xi - \beta A_1 \cosh^2 \xi - A_1 \\ \frac{(\phi')^2}{4a} = \beta A_1 \cos^2 \phi + A_1 \end{cases} \tag{6}$$

With the assumptions (5) on β, A_1 , both the one-dimensional motions are periodic, with periods T_1, T_2 respectively for ξ, ϕ , which depend only on the values of β, A_1 . If the periods T_1, T_2 have rational ratio $T_1/T_2 \in \mathbb{Q}$, then the corresponding orbit on the cylinder $\mathbb{R} \times S^1$ is periodic. To find a periodic orbit for the system (\mathcal{L}_0) , corresponding to a fixed value of the energy parameter β , we have to show that there is at least a value of A_1 for which T_1, T_2 have rational ratio. Before doing that we must compute the analytical expressions of the periods.

Lemma 1 *Consider the system (6), with the assumptions (5). Then the one-dimensional motions are both periodic with periods T_1, T_2 , for the coordinates ξ, ϕ respectively, given by*

$$T_1 = \frac{2\sqrt{2a^{-1}}}{\sqrt[4]{1 - 4\beta A_1^2}} K(\kappa_1), \quad T_2 = \frac{2\sqrt{a^{-1}}}{\sqrt{A_1(1 + \beta)}} K(\kappa_2), \tag{7}$$

where

$$\kappa_1^2 = \frac{A_1(1 - \beta) + \sqrt{1 - 4\beta A_1^2}}{2\sqrt{1 - 4\beta A_1^2}}, \quad \kappa_2^2 = \frac{\beta}{1 + \beta}, \quad \kappa_1, \kappa_2 > 0,$$

and $K(\kappa)$ is the elliptic integral of the first type:

$$K(\kappa) = \text{cn}^{-1}(0, \kappa) = \int_0^1 \frac{dv}{\sqrt{(1 - v^2)(1 - \kappa^2 v^2)}}.$$

Proof It is a straightforward computation, starting from the integral expressions of $\tau(\xi), \tau(\phi)$. □

Remark 1 Observe that $\lim_{\kappa \rightarrow 1^-} K(\kappa) = +\infty$ and

$$\kappa_1 = 1 \iff \begin{cases} \beta < 1 \\ A_1 = \frac{1}{1 + \beta} \end{cases}.$$

In particular, the second condition means that the motion of ξ takes place on the separatrix energy level. Then, if $\beta < 1$ the period T_1 goes to infinity as $A_1 \rightarrow \frac{1}{1+\beta}$:

$$\lim_{A_1 \rightarrow \frac{1}{1+\beta}} T_1 = +\infty.$$

Correspondingly, for T_2 we have:

$$\lim_{A_1 \rightarrow 0} T_2 = +\infty.$$

We are interested to the limit $E \rightarrow 0$, then we can suppose that the parameter E_1 is greater than $|E|$. It corresponds to make the hypothesis that $\beta < 1$. Our definitive assumptions are:

$$\beta \in (0, 1), \quad 0 < A_1 < \frac{1}{1+\beta}.$$

3.3 Periodic orbits

The expressions of the periods T_1, T_2 given in Lemma 1 and the Remark 1 give the limits

$$\begin{aligned} \lim_{A_1 \rightarrow \frac{1}{1+\beta}} T_1 &= +\infty, & \lim_{A_1 \rightarrow 0} T_2 &= +\infty, \\ \lim_{A_1 \rightarrow 0} T_1 &= \frac{4}{\sqrt{2a}} K\left(\frac{1}{\sqrt{2}}\right), & \lim_{A_1 \rightarrow \frac{1}{1+\beta}} T_2 &= \frac{2}{\sqrt{a}} K\left(\sqrt{\frac{\beta}{1+\beta}}\right). \end{aligned} \tag{8}$$

Moreover, from the definition of $K(\kappa)$, we easily conclude that, fixed $\beta \in (0, 1)$, T_2 is a strictly decreasing function of $A_1 \in (0, \frac{1}{1+\beta})$, while T_1 is strictly increasing. Indeed, $K(\kappa)$ is a strictly increasing function of κ^2 , κ_2 does not depend on A_1 and κ_1^2 has positive derivative with respect to A_1 given by

$$\frac{\partial \kappa_1^2}{\partial A_1} = \frac{1-\beta}{2} (1-4\beta A_1^2)^{-\frac{3}{2}} > 0.$$

By these observations, the following is shown.

Proposition 1 *Let $\beta \in (0, 1)$ be fixed. For any positive rational $q \in \mathbb{Q}^+$, there exists a unique value $\hat{A}_1(\beta, q) \in (0, \frac{1}{1+\beta})$ for A_1 , such that*

$$qT_1(\beta, \hat{A}_1) = T_2(\beta, \hat{A}_1). \tag{9}$$

In particular, the system (6) has a periodic solution in correspondence of the value \hat{A}_1 , with energy $E = -2a\beta\hat{A}_1$.

We have seen that for any $\beta \in (0, 1)$ and any positive rational $q \in \mathbb{Q}^+$, there is a periodic orbit for the regularised system (\mathcal{L}_0) with energy given by the relation $E = -2a\beta\hat{A}_1$. We can classify these periodic orbits, identifying each class with the rational number q . Then for any fixed value of the parameter $\beta \in (0, 1)$, we have exactly one value of \hat{A}_1 for each class $q \in \mathbb{Q}^+$. Orbits of different classes do not have the same energy E ; more precisely we have

Proposition 2 *Let $\beta \in (0, 1)$ be fixed. The function $\hat{A}_1(\beta, \cdot) : \mathbb{Q}^+ \rightarrow (0, \frac{1}{1+\beta})$, defined by the equality (9), is strictly decreasing. Moreover, there exist the limits*

$$\lim_{q \rightarrow 0^+} \hat{A}_1(\beta, q) = \frac{1}{1+\beta}, \quad \lim_{q \rightarrow +\infty} \hat{A}_1(\beta, q) = 0.$$

Proof If $q > q'$ then $qT_1(\beta, A_1) > q'T_1(\beta, A_1)$ and we have

$$T_2(\beta, \hat{A}_1(\beta, q)) = qT_1(\beta, \hat{A}_1(\beta, q)) > q'T_1(\beta, \hat{A}_1(\beta, q)).$$

T_2 is strictly decreasing with respect to A_1 , while T_1 is strictly increasing, then $\hat{A}_1(\beta, q) < \hat{A}_1(\beta, q')$.

The monotony of $\hat{A}_1(\beta, \cdot)$ assures the existence of the limits

$$L_1 = \lim_{q \rightarrow 0^+} \hat{A}_1(\beta, q), \quad L_2 = \lim_{q \rightarrow +\infty} \hat{A}_1(\beta, q).$$

Clearly $L_1, L_2 \in \left[0, \frac{1}{1+\beta}\right]$ and from the knowledge of the limits (8), we easily obtain the desired values for L_1, L_2 . □

It follows that periodic orbits with many “loops” in ξ and few in the variable ϕ tend to the separatrix level for ϕ ; viceversa, orbits with many loops in ϕ tend to the separatrix level for ξ . In other words, if we increase only for a single variable the number of loops before the orbit closes, we will obtain a limit orbit which does not close anymore in finite time. For example, take $q = m/n$, with $m, n \in \mathbb{N}$. Increasing the number of loops in ξ corresponds to make m larger and consequently \hat{A}_1 smaller. The periodic orbit increases the number of “oscillations” in ξ , while making the same number of revolutions in the variable ϕ . As m goes to infinity we have that the corresponding orbit makes an infinite number of times the same trajectory in ξ , tending to close, but without being able to reach the limit values $\phi = \pm\pi/2$ in a finite time interval, in the future and in the past respectively: in particular, it does not complete even a single revolution for ϕ . Furthermore, the energy $E = -2a\beta\hat{A}_1$ tends to zero.

We conclude that to form easily a finite set of collision arcs with the same energy, we should fix $q \in \mathbb{Q}^+$ and look for collision arcs only in the set of periodic orbits of the same class q .

4 Construction of collision arcs

In the previous section we have found infinite classes of periodic orbits for the regularised problem \mathcal{L}_0 . We would like to use them to construct collision arcs.

The next step is then to show that, among the periodic orbits of the regularised system (\mathcal{L}_0), there is at least one which passes through the third centre C . Actually this is true if the parameter β is sufficiently small. We must also verify that the obtained orbits are solutions of the not regularised problem (L_0), that is they do not pass through the primaries: this is the most delicate point of the proof of Theorem 3. Moreover, in Sect. 4.3 we will face the problem of *early collisions*.

4.1 Periodic orbits through the third centre

Let $(\xi_0, \phi_0) \in \mathbb{R} \times S^1 \setminus \{(0, 0), (0, \pi)\}$ be fixed elliptic coordinates for the position of the third centre C . Then, among the orbits corresponding to the value $\hat{A}_1(\beta, q)$, surely there is one which pass through the centre C , if $\xi_0 \in (\xi_-(\beta, \hat{A}_1), \xi_+(\beta, \hat{A}_1))$, where ξ_{\pm} are the inversion points. Note that in Cartesian coordinates this corresponds to say that the centre C lies in the region internal to the ellipse defined by the equation $\xi = \xi_+$. By construction, for each $\beta \in (0, 1)$, we have $\hat{A}_1 \in (0, \frac{1}{1+\beta})$, then $\lim_{\beta \rightarrow 0} \beta \hat{A}_1 = 0$ and

$$\lim_{\beta \rightarrow 0} \cosh(\xi_{\pm}) = \lim_{\beta \rightarrow 0} \frac{1 + \sqrt{1 - 4\beta \hat{A}_1^2}}{2\beta \hat{A}_1} = +\infty.$$

Thus we conclude that

Proposition 3 *Fixed a class $q \in \mathbb{Q}^+$, there exists $\beta_0 > 0$, such that for any $\beta \in (0, \beta_0)$ there is a periodic orbit of system (\mathcal{L}_0) , associated with the value $\hat{A}_1(\beta, q)$, which passes through C , and the coordinate ξ_0 is not an inversion point of the corresponding one-dimensional motion in ξ . □*

Note that the periodic orbits associated with the same value of \hat{A}_1 differ only for the sign of the velocities ξ'_0, ϕ'_0 at the centre C . Thus we have exactly two orbits on the configuration space $\mathbb{R} \times S^1$: if one has velocity (ξ'_0, ϕ'_0) at C , the other has velocity $(-\xi'_0, \phi'_0)$. The remaining two possibilities give the same orbits, but with the opposite direction of motion. Moreover $\xi'_0 \neq 0$, because ξ_0 is not an inversion point. Then the two trajectories corresponding to \hat{A}_1 meet transversely at C on the cylinder $\mathbb{R} \times S^1$.

We remark that there is the possibility that the two orbits coincide: it can happen when the trajectory has an autointersection at (ξ_0, ϕ_0) before closing. This is a case of *early collision* and it will be treated in Proposition 6.

The obtained solutions are not yet the collision arcs that we desire. In fact, they are orbits for the regularised system (\mathcal{L}_0) : to be orbits of the 2-centre problem with Lagrangian L_0 , it's enough they do not pass through the primaries C_1, C_2 . This is a delicate problem and to face it we will need a general result about the regularity of \hat{A}_1 as function of β , in a neighbourhood of $\beta = 0$.

4.2 Avoiding collision with the primaries

In this subsection we will obtain the central result of the paper: we will show that for almost all the possible positions of the third centre in $\mathbb{R} \times S^1$, the periodic orbits through C corresponding to \hat{A}_1 don't collide with the primaries for any sufficiently small value of β . It means that they are solutions of the not-regularised system (L_0) and allows us to proceed with the final verifications, in order to apply Theorem 1.

First of all we would like to recall a useful property which characterises the periodic orbits of the regularised 2-centre problem passing through one of the primaries. Note that periodic orbits through the primaries always exist because a primary has $\xi = 0$.

Proposition 4 *Let $\beta \in (0, 1)$ arbitrarily fixed. Given $q \in \mathbb{Q}^+$, let $m, n \in \mathbb{N}$ such that $q = m/n$ and $(m, n) = 1$. Let γ be a periodic orbit of the system (\mathcal{L}_0) associated with $\hat{A}_1(\beta, q)$ and suppose that it passes through one of the centres C_1, C_2 , which have elliptic coordinates $(0, 0), (0, \pi)$ respectively.*

If n is odd, then the orbit goes through both the primaries in a period, and the collisions happens at a time distance of half the period from each other.

If n is even, then the orbit passes through only one of the primaries and it happens two times in a period, at a time distance of half the period. In the configuration space $\mathbb{R} \times S^1$ the orbit has a transverse self-intersection at the position of the centre.

Proof The system (\mathcal{L}_0) has the form (6). Without loss of generality we can suppose that the orbit $\gamma(\tau) = (\xi(\tau), \phi(\tau))$ passes through one of the centres C_1, C_2 at time $\tau = 0$. The orbit γ collides with a primary at time $\tau \neq 0$ if and only if $\xi(\tau) = 0$ and $\phi(\tau) \in \{0, \pi\}$. Then we must have $\tau = k \frac{T_1}{2} = j \frac{T_2}{2}$, with $k, j \in \mathbb{Z} \setminus \{0\}$. Then $\frac{k}{j} = \frac{m}{n} = q$ and this implies that there

is $i \in \mathbb{Z}$ such that $k = im$ and $j = in$, because $(m, n) = 1$. It follows that $\gamma(\tau)$ is a primary if and only if $\tau = im \frac{T_1}{2} = in \frac{T_2}{2} = i \frac{T}{2}$, where $T = mT_1 = nT_2$ is the period of the orbit. This concludes the proof. \square

If we assume $q = 1$, then, using the above Proposition, we can exclude at once the collision with the primaries for some simple cases. To simplify notation we place $\hat{A}_1(\beta) = \hat{A}_1(\beta, 1)$.

Proposition 5 *Let γ be a periodic orbit through the third centre C , corresponding to the value $\hat{A}_1(\beta)$, with $\beta < \beta_0$, as in Proposition 3. Let (ξ_0, ϕ_0) be fixed elliptic coordinates for the centre C . If $\phi_0 = k \frac{\pi}{2}$, $k \in \mathbb{Z}$, or $\xi_0 = 0$, then the orbit γ cannot pass through the primaries.*

Remark 2 Note that the positions with $\phi_0 = k \frac{\pi}{2}$, $k \in \mathbb{Z}$, and the ones with $\xi_0 = 0$ correspond in Cartesian coordinates (x, y) to points on the coordinate axes. In particular, if $\xi_0 = 0$ then the three centres are collinear on the x -axis, and C lies between the primaries. If $\phi_0 = k\pi$ then the three centres are still aligned on the x -axis, but C is external. Finally, if $\phi_0 = (2k + 1) \frac{\pi}{2}$, then the centre C lies on the y -axis and the configuration of the centres is symmetric with respect to this axis.

In particular, Proposition 5 says that, for $q = 1$, the periodic orbits through the primaries intersect the coordinate axes only at the primaries and for $\xi = \xi_{\pm}$.

Proof We are in the case $q = 1$, then from Proposition 4 we know that the orbit γ passes through a primary if and only if for any time τ such that $\xi(\tau) = 0$, we have $\phi(\tau) = k\pi$, $k \in \mathbb{Z}$, and viceversa. The centre C does not coincide with a primary, then it cannot happen that $\phi_0 = k\pi$, $k \in \mathbb{Z}$, and $\xi_0 = 0$ at the same time and in these cases the statement is obvious. Now suppose $\phi_0 = (2k + 1) \frac{\pi}{2}$. The time intervals from C to a position with $\phi = i\pi$, $i \in \mathbb{Z}$, are $(2j + 1) \frac{T}{4}$, $j \in \mathbb{Z}$, where $T = T_1 = T_2$ is the period of γ . Look at the variable ξ : the only positions which have a time distance of $(2j + 1) \frac{T}{4}$ from $\xi = 0$ are the inversion points ξ_{\pm} , but $\xi_0 \neq \xi_{\pm}$. \square

Remark 3 Note that the proof of Proposition 5 cannot be generalised to arbitrary fixed values of $q \in \mathbb{Q}^+$. For example, if $q = 1/2$, then, if a periodic orbit passes through C_1 , it certainly passes through a point with elliptic coordinates $(\xi, \pi/2)$, with $\xi \in (0, \xi_+)$: in fact the time passed from the last passage through C_1 to this point is $T_2/4 = T_1/8$. Nevertheless, it's easy to see that even for this case the passage through the primaries is excluded when $\xi_0 = 0$ or $\phi_0 = k\pi$: indeed, the time between two positions with $\xi = 0$ is $T_1/2 = T_2$, and the points with $\phi_0 = k\pi$ along an orbit which passes through a primary must have $\xi_0 = 0, \xi_{\pm}$.

Our next step is to study the regularity of the function $\hat{A}_1(\beta, q)$ with respect to the real parameter β , to be used for proving the central result of the subsection: in particular, we are interested in the behaviour near $\beta = 0$.

Lemma 2 *Let $q \in \mathbb{Q}^+$ be fixed. Then \hat{A}_1 is a smooth function of $\beta \in (0, 1)$ and it can be smoothly extended to $\beta = 0$. In particular, there exists the limit*

$$\hat{A}_1(0, q) := \lim_{\beta \rightarrow 0} \hat{A}_1(\beta, q) \in (0, 1) ,$$

the periods T_1 and T_2 are smooth for $\beta = 0$ and

$$qT_1(0, \hat{A}_1(0, q)) = T_2(0, \hat{A}_1(0, q)).$$

Proof Denote by F the function

$$F(\beta, A_1) := [qT_1 - T_2](\beta, A_1),$$

defined on the domain $\mathcal{D} = \left\{ (\beta, A_1) \in \mathbb{R}^2 \mid \beta \in (0, 1), 0 < A_1 < \frac{1}{1+\beta} \right\}$.

We observe that κ_1, κ_2 are C^∞ functions of $(\beta, A_1) \in \mathcal{D}$, and $\kappa_1, \kappa_2 \in (0, 1)$; then, by derivation under the integral sign, we conclude that F is C^∞ on the same domain. Furthermore $\frac{\partial F}{\partial A_1} \geq 0$, then from the implicit function theorem we can assert the regularity of \hat{A}_1 on $(0, 1)$.

We want to extend the definition of the function \hat{A}_1 to $\beta = 0$. First of all we prove that the function F can be smoothly extended to $\beta = 0$. We study T_1 and T_2 separately. We start from T_1 . If $\beta \in [0, 1)$ and $A_1 < \frac{1}{1+\beta}$, then $(1 - 4\beta A_1^2) \geq 0$. It follows that k_1^2 is well defined and C^∞ for $\beta = 0$ and that $k_1^2 \in (\frac{1}{2}, 1)$. Then T_1 is C^∞ for $\beta = 0$. Now see T_2 . For $\beta \in [0, 1)$, we have $\kappa_2^2 \in [0, \frac{1}{2})$ and it is a C^∞ function of β . To assert the regularity of T_2 , it was enough to show $\kappa_2^2 < 1$, then we surely have T_2 defined and smooth for $\beta = 0$.

The next step is to verify that for $\beta = 0$ there is a unique value $\hat{A}_1(0, q) \in (0, 1)$, such that $F(0, \hat{A}_1(0, q)) = 0$ and that for this value $\frac{\partial F}{\partial A_1}(0, \hat{A}_1(0, q)) \geq 0$. From the definition, we easily see that $\lim_{\kappa \rightarrow 0} K(\kappa) = \frac{\pi}{2}$, while $\lim_{\kappa \rightarrow 1^-} K(\kappa) = +\infty$. Moreover

$$qT_1(0, A_1) = \frac{4q}{\sqrt{2a}} K\left(\sqrt{\frac{A_1 + 1}{2}}\right), \quad T_2(0, A_1) = \frac{\pi}{\sqrt{aA_1}}.$$

$T_1(0, A_1)$ is a strictly increasing function of A_1 , while $T_2(0, A_1)$ is strictly decreasing, and

$$\begin{aligned} \lim_{A_1 \rightarrow 0} F(0, A_1) &= -\infty, \\ \lim_{A_1 \rightarrow 1^-} F(0, A_1) &= \frac{1}{\sqrt{2a}} \left[\lim_{A_1 \rightarrow 1^-} 4qK\left(\sqrt{\frac{A_1 + 1}{2}}\right) - \sqrt{2}\pi \right] = +\infty. \end{aligned}$$

We conclude that $\hat{A}_1(0, q)$ is uniquely determined from the equality $F(0, \hat{A}_1(0, q)) = 0$, and $\hat{A}_1(0, q) \in (0, 1)$. Moreover $\frac{\partial F}{\partial A_1}(0, \hat{A}_1(0, q)) \geq 0$, then the regularity of the function \hat{A}_1 in $\beta = 0$ follows from the implicit function theorem. □

Now we are ready to state and show the main result.

Theorem 4 *Let $q \in \mathbb{Q}^+$ a fixed positive rational number. There is a dense open subset $X'_q \subset \mathbb{R} \times S^1$, such that fixed $(\xi_0, \phi_0) \in X'_q$, there is $\beta_0 > 0$, such that for each $\beta \in (0, \beta_0)$, the periodic orbits through (ξ_0, ϕ_0) , associated with $\hat{A}_1(\beta, q)$, do not pass through the primaries C_1, C_2 . In particular, after scaling time, they are orbits of the not-regularised 2-centre problem with Lagrangian L_0 , and they have energy $E = -2a\beta\hat{A}_1$.*

Proof Let $(\xi_0, \phi_0) \in \mathbb{R} \times S^1$ be the position of the centre C in elliptic coordinates and let $\gamma(\tau) = (\xi(\tau), \phi(\tau))$ be a periodic orbit associated with $\hat{A}_1(\beta, q)$, which pass through (ξ_0, ϕ_0) with velocity (ξ'_0, ϕ'_0) . Without loss of generality we can assume $\phi'_0 > 0$ (see Sect. 4.1). Suppose that the orbit γ passes through a primary C_1 or C_2 : at that instant we have $\xi = 0$ and $\phi \in \{0, \pi\}$. Let $q = \frac{m}{n}$, with $m, n \in \mathbb{Z}$ positive integers and $(m, n) = 1$. Let $\Delta\tau$ be the shortest time interval to go from a primary to the centre C along the orbit γ . Thanks to Proposition 4, we must have $\Delta\tau < \frac{T}{2}$, where $T = mT_1 = nT_2$ is the period of γ .

We have two possibilities, corresponding to start from C_1 or from C_2 . The orbit solves the separated system (6). Then, in the first case

$$\Delta\tau = \frac{1}{2\sqrt{a}} \int_0^{\phi_0} \frac{1}{\sqrt{\beta\hat{A}_1 \cos^2 \phi + \hat{A}_1}} d\phi + iT_2,$$

while in the second case

$$\Delta\tau = \frac{1}{2\sqrt{a}} \int_0^{\phi_0} \frac{1}{\sqrt{\beta\hat{A}_1 \cos^2 \phi + \hat{A}_1}} d\phi \pm \frac{T_2}{2} + i'T_2,$$

with $i, i' \in \mathbb{Z}, i, i' \geq 0$, and in the second case with sign \pm respectively when $\phi_0 < \pi, \phi_0 \geq \pi$.

The condition $\Delta\tau < \frac{T}{2} = \frac{nT_2}{2}$ implies that $i < \frac{n}{2}$ and $i' < \frac{n+1}{2}$. Look now at the variable ξ . In both cases we must have

$$\Delta\tau = \pm \frac{1}{2\sqrt{a}} \int_0^{\xi_0} \frac{1}{\sqrt{\cosh \xi - \beta\hat{A}_1 \cosh^2 \xi - \hat{A}_1}} d\xi + j\frac{T_1}{2}.$$

with $j \in \mathbb{Z}, j \geq 0$, and the condition $\Delta\tau < \frac{T}{2} = \frac{mT_1}{2}$ implies that $j \leq m$.

We denote by $P(\phi_0, \beta)$ and $Q(\xi_0, \beta)$ the functions

$$P(\phi_0, \beta) = \frac{1}{2\sqrt{a}} \int_0^{\phi_0} \frac{1}{\sqrt{\beta\hat{A}_1 \cos^2 \phi + \hat{A}_1}} d\phi,$$

$$Q(\xi_0, \beta) = \frac{1}{2\sqrt{a}} \int_0^{\xi_0} \frac{1}{\sqrt{\cosh \xi - \beta\hat{A}_1 \cosh^2 \xi - \hat{A}_1}} d\xi.$$

From Lemma 2 and the fact that ξ_0 is not an inversion point, we deduce that these functions are smooth in a neighbourhood of $\beta = 0$. The condition to pass through a primary for the first case is

$$\Delta\tau = P(\phi_0, \beta) + iT_2 = \pm Q(\xi_0, \beta) + j\frac{T_1}{2},$$

while for the second case it is

$$\Delta\tau = P(\phi_0, \beta) \pm \frac{T_2}{2} + i'T_2 = \pm Q(\xi_0, \beta) + j\frac{T_1}{2}.$$

We have a finite set of possible values for the integers i, i', j and the equality $qT_1 = \frac{m}{n}T_1 = T_2$ holds, then we can summarise all the conditions with the following one: if the orbit γ passes through a primary, then $G^+(\xi_0, \phi_0, \beta) \in S$ or $G^-(\xi_0, \phi_0, \beta) \in S$, where $S \subset \mathbb{Q}$ is a finite subset of rationals, depending only on the fixed parameter $q \in \mathbb{Q}^+$, and the functions G^+, G^- are defined by

$$G^\pm(\xi_0, \phi_0, \beta) = \frac{P(\phi_0, \beta) \pm Q(\xi_0, \beta)}{T_1(\beta, \hat{A}_1(\beta, q))}.$$

The functions G^\pm are smooth in all the variables: in particular, if $G^\pm(\xi_0, \phi_0, 0) \notin S$, there is $\beta_0 > 0$, such that $G^\pm(\xi_0, \phi_0, \beta) \notin S$ for any $\beta \in (0, \beta_0)$. Then, to exclude the collision with the primaries for small values of β , it is enough that the centre C belongs to the set

$$X'_q = \{(\xi_0, \phi_0) \in \mathbb{R} \times S^1 \mid G^\pm(\xi_0, \phi_0, 0) \notin S\}.$$

The set S is finite, then it is clear that X'_q is open and dense in $\mathbb{R} \times S^1$. Indeed, we easily see from the definitions of the functions G^\pm that their partial derivatives with respect to ϕ_0 are different from zero; then X'_q is a finite intersection of open dense subset of $\mathbb{R} \times S^1$. Moreover, the complement of X'_q has zero Lebesgue measure. □

Remark 4 When $q = 1$, it follows easily from Proposition 4 that at least one of the two periodic orbits through C corresponding to the same $\hat{A}_1(\beta)$ does not collide with the primaries. Indeed, without loss of generality we can assume $\phi'_0 > 0$ (see the end of Sect. 4.1). Suppose $\phi_0 \in (0, \frac{\pi}{2})$: then the time to pass from the centre C_1 to the centre C must be less than $\frac{T}{4}$, where $T = T_1 = T_2$ is the period. If $\xi_0 > 0$ the only possibility to collide with the primaries is that $\xi'_0 > 0$, while if $\xi_0 < 0$ we must have $\xi'_0 < 0$. For the other possible intervals of values of ϕ_0 a similar reasoning works.

For our purposes, the fact of having at least one orbit through C that does not collide with the primaries is not enough: indeed, after we have obtained the collision arcs, we want to construct *collision chains* with them, and to do it we need at least two orbits of the system (L_0) with the same energy E , which pass through C with not-parallel tangent fields. The latter condition will be investigated in the next section.

Remark 5 Theorem 4 would be improved if we showed one of the following:

- (i) the partial derivatives $\frac{\partial G^\pm}{\partial \beta}(\xi_0, \phi_0, 0)$ are zero only for isolated values of (ξ_0, ϕ_0) ;
- (ii) the partial derivatives $\frac{\partial G^\pm}{\partial \beta}(\xi_0, \phi_0, 0)$ do not vanish for every $\xi_0 \in \mathbb{R}, \phi_0 \in S^1$, with (ξ_0, ϕ_0) not a primary.

For case (i) we would have that the collision with the primaries is possible only for isolated positions $(\xi_0, \phi_0) \in \mathbb{R} \times S^1$, while for case (ii) the collision would be excluded for any position of the centre C .

At present, we don't have any of these improvements.

Corollary 1 *Let $I \subset \mathbb{Q}^+$ be a finite set of positive rationals. There is a dense open subset $X'_I \subset \mathbb{R} \times S^1$, such that for each $(\xi_0, \phi_0) \in X'_I$, there is $\beta_0 > 0$, such that for any $\beta \in (0, \beta_0)$, the periodic orbits through (ξ_0, ϕ_0) corresponding to $\hat{A}_1(\beta, q)$, with $q \in I$, do not pass through the primaries. In particular, they are periodic orbits for the system (L_0) .*

Proof It is an immediate consequence of the construction of the subset $X'_q \subset \mathbb{R} \times S^1$ in the proof of Theorem 4: we can apply this theorem for any $q \in I$, then take the intersection of the sets X'_q thus obtained, and the resulting set maintains the same properties of the sets X'_q . □

Remark 6 Denote by $\psi: \mathbb{R} \times S^1 \rightarrow \mathbb{R}^2$ the map to pass from elliptic to Cartesian coordinates, $\psi(\xi, \phi) = \cosh(\xi + i\phi) = x + iy$. The map ψ is open and surjective then, fixed a finite subset $I \subset \mathbb{Q}^+$, the image $X_I = \psi(X'_I)$ is an open dense subset of \mathbb{R}^2 . Then we can say that the thesis of Corollary 1 holds for any position of the centre C in an open dense subset $X_I \subset M$.

Consider now the set $X' = \bigcap_{q \in \mathbb{Q}^+} X'_q$; by the classical Baire’s Category Theorem the set X' is dense in $\mathbb{R} \times S^1$. Then, the set $X = \psi(X')$ is dense in \mathbb{R}^2 and for any finite subset $I \subset \mathbb{Q}^+$, it is contained in X_I . Then we can say that the thesis of Corollary 1 holds for any position $C \in X \subset M$. In this manner the only information that we have lost is that we don’t know if the set X is open, while we have gained the independence of the dense set X of positions of C from the set of rationals I . Note that the choice of a sufficiently small β cannot be independent of I , instead.

4.3 Early collisions

We defined a collision arc in Sect. 2 to be a critical point of the Maupertuis-Jacobi functional which starts and ends at collision with the centre C and does not meet this centre at intermediate times. If we take one of the periodic orbits through C found in the preceding section, and we pass to Cartesian coordinates, then we cannot be sure that the orbit does not pass newly through C before a period has passed. This can happen in two ways: when the orbit, considered in elliptic coordinates, meets the point (ξ_0, ϕ_0) and when it meets $(-\xi_0, -\phi_0)$.

When a periodic orbit starting from the centre C passes newly through C in a time shorter than its period, then we talk of *early collision*. To understand when an early collision occurs we need to study the behaviour of periodic orbits a little deeper.

An easy example is given by the case $q = 1$: in this case any periodic orbit crosses the y -axis, is symmetric with respect to this axis and, if it does not collide with the primaries, it has a transverse autointersection at the point of crossing. The symmetry comes from the fact that in a time interval of half the orbit’s period we have the passage from a point (ξ, ϕ) to a point $(-\xi, \phi + \pi)$, that is from a point $(x, y) = (\cosh(\xi) \cos(\phi), \sinh(\xi) \sin(\phi))$, to its symmetric $(-x, y)$. The intersections with the y -axis occur when $\phi = \pm\pi/2$ and from one intersection to the next there is a time interval of half the period. In elliptic coordinates the orbit passes through a point $(\xi, \pi/2)$ and after half a period it arrives at $(-\xi, -\pi/2)$, but these two points coincide in Cartesian coordinates and are on the y -axis. Then each orbit autointersects at a point on the y -axis. The transformation of the velocities when we pass from elliptic to Cartesian coordinates is given by:

$$v = U(\xi, \phi) \cdot \begin{pmatrix} \xi' \\ \phi' \end{pmatrix}, \quad U(\xi, \phi) = \begin{pmatrix} \sinh \xi \cos \phi & -\cosh \xi \sin \phi \\ \cosh \xi \sin \phi & \sinh \xi \cos \phi \end{pmatrix}, \tag{10}$$

where $U(\xi, \phi)$ is an invertible matrix, except when $(\xi, \phi) \in \{(0, 0), (0, \pi)\}$. We observe that $U(-\xi, -\phi) = -U(\xi, \phi)$. If at the point of intersection with the y -axis $\xi = \pm\xi_+$, then $\xi' = 0$ and the velocity has zero y -component: then the two crossings are not transverse. But this is the case in which the orbit collides with the primaries, reversing its direction at the collisions. If $\xi \neq \pm\xi_+$ instead, the velocities at the point of intersection with the y -axis have both the components different from zero, then they are transverse, because by symmetry they differ only in the sign of their x -component.

An illustration of this situation is given in Fig. 3, where we have drawn 19 periodic orbits corresponding to the value of $\beta = 1/7$, included the two orbits that collide with the primaries: these latter orbits are in green, while the primaries are marked with red asterisks. We have also drawn the periodic orbit $\xi = \xi_+$: we cannot use it to construct collision arcs, because it has different energy, but this orbit is interesting in itself because it encloses all the trajectories with the same values of the parameters (and then the same energy), and it is tangent to all. The behaviour of a couple of periodic orbits through the same point can be seen in Fig. 4, where the starting point is marked with a black asterisk.

Fig. 3 Periodic orbits in Cartesian coordinates when $a = 1, q = 1, \beta = 1/7$ and the orbits through the primaries

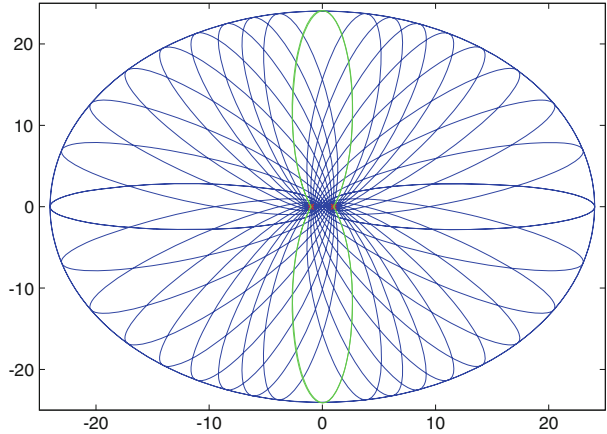
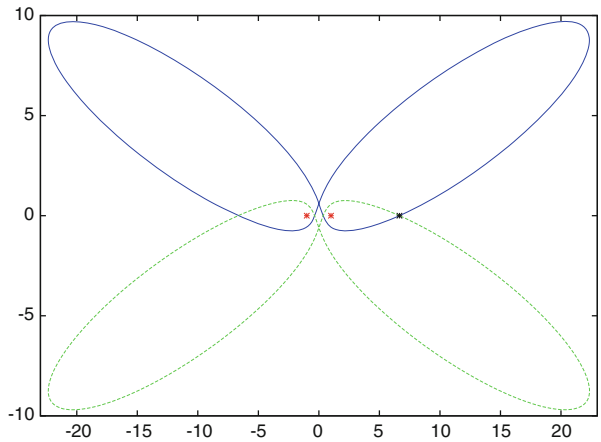


Fig. 4 Periodic orbits through $(\xi_0, \phi_0) = (\frac{2}{3}\xi_+, 0)$ in Cartesian coordinates, when $a = 1, q = 1, \beta = 1/7$



Then, if $q = 1$, we can say that there is an early collision at C if and only if C lies on the y -axis: in this case there is a unique, up to reversing the direction of motion, periodic orbit through C .

In particular, we see that the period does not change in the passage from elliptic to Cartesian coordinates. Actually, this holds for any value of the parameter $q \in \mathbb{Q}^+$. In fact, if we take a periodic orbit that starts from a point (ξ_0, ϕ_0) , which is not a primary, we see that the matrix $U(\xi_0, \phi_0)$ given by (10) is invertible and $U(-\xi_0, -\phi_0) = -U(\xi_0, \phi_0)$. Then the only possibility to have a shorter period when we pass to Cartesian coordinates is that the orbit passes through $(-\xi_0, -\phi_0)$, with velocity $(-\xi'_0, -\phi'_0)$. This is clearly impossible because the sign of ϕ' never changes along the orbit.

Anyway, the situation complicates if $q \neq 1$ and we cannot get a global view easily: as an example, in Fig. 5 we have drawn the orbits through the point $(\xi_0, \phi_0) = (\frac{2}{3}\xi_+, 0)$, when $a = 1, \beta = 1/7$ and $q = 2$. To better see the autointersections we have put an enlargement in Fig. 6.

We note the following remarkable fact:

Proposition 6 *Let $(\xi_0, \phi_0) \in \mathbb{R} \times S^1$ be fixed and consider a periodic orbit $(\xi(\tau), \phi(\tau))$ through this point, corresponding to some fixed values of β, q . If $(\xi(\tau), \phi(\tau))$ passes newly*

Fig. 5 Periodic orbits through $(\xi_0, \phi_0) = (\frac{2}{3}\xi_+, 0)$ in Cartesian coordinates, when $a = 1, q = 2, \beta = 1/7$

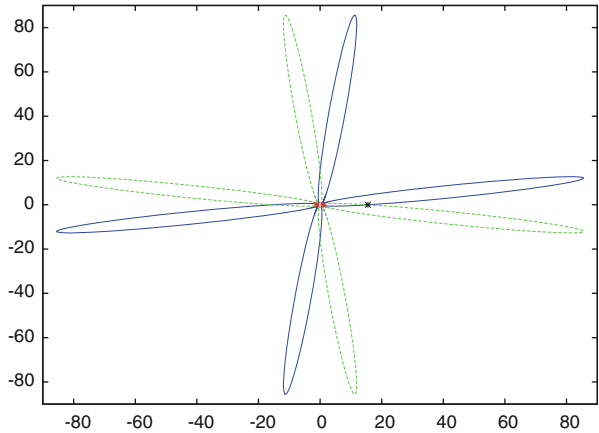
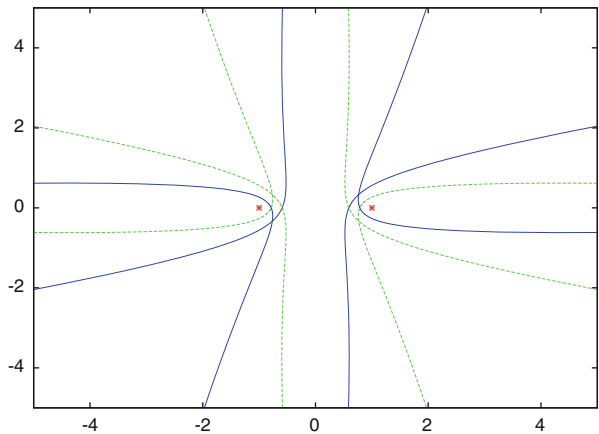


Fig. 6 An enlargement of Fig. 5, where the autointersections of the two orbits are visible



through (ξ_0, ϕ_0) before closing, then it is the unique periodic orbit through (ξ_0, ϕ_0) corresponding to the same fixed values of β, q , up to reverse the direction of motion. Moreover, if ξ_0 is not an inversion point, the orbit has a transverse autointersection at (ξ_0, ϕ_0) .

Proof If the initial velocity at (ξ_0, ϕ_0) is (ξ'_0, ϕ'_0) , then the only other possible velocity through the same point along the same orbit is $(-\xi'_0, \phi'_0)$. If the orbit arrives at (ξ_0, ϕ_0) before that a period is passed, then it must happen with velocity $(-\xi'_0, \phi'_0)$, and this implies that the two possible periodic orbits through (ξ_0, ϕ_0) coincide. \square

At this stage, we know completely early collisions only for the case $q = 1$. In general, we can only say that early collisions cannot be excluded: we don't have any knowledge about the conditions that determine them. Anyway, it is not a problem for the proof of Theorem 3. When an early collision occurs, we will take as collision arc the partial arc of the periodic orbit through C , which starts from C and ends at the first next passage through C . The final time T of the collision arc in this case will not be the period, but it will be the time of the first return to C .

Finally, we want to stress that a collision arc must end at C with one of the four possible velocities determined by the choice of the parameters, which in elliptic coordinates are $\{\pm(\xi'_0, \phi'_0), \pm(-\xi'_0, \phi'_0)\}$, where (ξ'_0, ϕ'_0) is the initial velocity of the arc.

5 Nondegeneracy and direction change

In this section we will verify that the collision arcs obtained from Corollary 1 satisfy the non-degeneracy condition and moreover they meet transversely at the centre C : this will allow us to apply Theorem 1 and derive our final result, Theorem 3.

5.1 Nondegeneracy

Let $q \in \mathbb{Q}^+$ be fixed and $m, n \in \mathbb{Z}$ be positive coprime integers such that $q = m/n$. Denote by $(\xi_0, \phi_0) \in \mathbb{R} \times S^1$ the position of the centre C in elliptic coordinates. Suppose that $(\xi_0, \phi_0) \in X'_q$, where the set $X'_q \subset \mathbb{R} \times S^1$ is the one given by Theorem 4. Take β small enough, so that the periodic orbits of system (\mathcal{L}_0) corresponding to $\hat{A}_1(\beta, q)$ and passing through C do not pass through the primaries. Let $\gamma(t)$ be one of the resulting collision arcs for the system (L_0) , which starts from the centre C with velocity v_0 and whose energy is $E = -2a\beta\hat{A}_1(\beta, q)$. Then $\gamma(t)$ solves the system

$$f(C, v_0, T) = C, \quad H_0(C, v_0) = E. \tag{11}$$

As reminded in Sect. 2, to show the nondegeneracy of γ it is sufficient to verify that the Jacobian of system (11) is nonzero. Actually we will verify a slight variant of this condition.

It is convenient to consider as variables the parameters (β, A_1) instead of the coordinates of the initial velocity v_0 . This procedure is right only if there is a local diffeomorphism which allows to pass from v_0 to (β, A_1) . The transformation of the velocities in the passage from elliptic to Cartesian coordinates is given by the invertible matrix $U(\xi_0, \phi_0)$ defined by (10). We know that the orbit γ corresponds to a solution of the separated system (6), and in elliptic coordinates we have $(\xi_0, \phi_0) = \frac{d\tau}{dt}(0)(\xi'_0, \phi'_0)$, with $\frac{d\tau}{dt}(0) = (\cosh^2(\xi_0) - \cos^2(\phi_0))^{-1}$. Then v_0 is a C^∞ function of (ξ'_0, ϕ'_0) . From the system (6) we see that (ξ'_0, ϕ'_0) is a C^∞ function of (β, A_1) and then v_0 is. By the same reasoning we see that (β, A_1) is locally a C^∞ function of the velocity (ξ'_0, ϕ'_0) and then of v_0 .

For all nearby trajectories from the same initial point C , we evaluate the m -th positive instant of time at which they meet the ellipse $\xi = \xi_0$, with the velocity $\dot{\xi}$ equal to the initial velocity $\dot{\xi}_0 \neq 0$; then we consider the time distance from the n -th passage through ϕ_0 , which is given by $nF(\beta, A_1)$, where

$$F(\beta, A_1) = [qT_1 - T_2](\beta, A_1).$$

This is equivalent to consider the value of the angle coordinate ϕ at the instant at which we have the m -th oriented crossing of the line $\xi = \xi_0$.

Note that in this manner the time variable T is fixed as function of (β, A_1) , $T = mT_1(\beta, A_1)$, so that we have reduced the order of system (11) of a unit. The orbit γ satisfies

$$F(\beta, A_1) = 0, \quad -2a\beta A_1 = E,$$

and it is nondegenerate if the Jacobian determinant of this system at the solution $(\beta, \hat{A}_1(\beta, q))$, corresponding to the orbit γ , is different from zero.

The Jacobian matrix is

$$J = \begin{pmatrix} \frac{\partial F}{\partial \beta} & \frac{\partial F}{\partial A_1} \\ -2aA_1 & -2a\beta \end{pmatrix}.$$

As observed in Sect. 3.3, we have $\partial F/\partial A_1 \gtrsim 0$. Furthermore, we have that $\hat{A}_1(\beta, q) \in (0, \frac{1}{1+\beta})$. These properties remain true for $\beta = 0$, as proved in Lemma 2, and moreover T_1, T_2 are C^∞ functions of (β, A_1) . It follows that when $\beta = 0$ and $A_1 = \hat{A}_1(0, q)$, the determinant is well defined and different from zero, and by regularity the same is true also for values $\beta, \hat{A}_1(\beta, q)$, with $\beta > 0$ sufficiently small.

We conclude that the collision arc γ is nondegenerate for β small enough.

5.2 Direction change

Let $q \in \mathbb{Q}^+$ be fixed and $m, n \in \mathbb{Z}$ be positive coprime integers such that $q = m/n$. In order to construct *collision chains* to which apply Theorem 1, we should have at least two collision arcs which start and arrive at the centre C with transverse tangent fields.

From the results of Sect. 4, if in elliptic coordinates our centre (ξ_0, ϕ_0) belongs to the dense open subset $X'_q \subset \mathbb{R} \times S^1$, then for small enough values of β , after reparametrisation of time and change of coordinates, the orbits associated to $\hat{A}_1(\beta, q)$ do not collide with the primaries and they are periodic orbits for the not-regularised two-centre problem (L_0) . As observed in Sect. 4.1, there are exactly two orbits corresponding to $\hat{A}_1(\beta, q)$, which pass through the centre (ξ_0, ϕ_0) with velocities $(\xi'_0, \phi'_0), (-\xi'_0, \phi'_0)$ respectively. Changing sign to ϕ'_0 , we obtain simply the same orbits, with reversed direction of motion. Moreover, the two transverse orbits coincide in the case of an autointersection at (ξ_0, ϕ_0) , as we have seen in Proposition 6.

Now consider the two orbits associated with $\hat{A}_1(\beta, q)$: suppose they pass through the centre (ξ_0, ϕ_0) at time $\tau = 0$ and that their velocities are $(\xi'_0, \phi'_0), (-\xi'_0, \phi'_0)$. They are obviously transverse at the point (ξ_0, ϕ_0) in the cylinder $\mathbb{R} \times S^1$, because we have chosen β small enough to have $\xi'_0 \neq 0$ (see Proposition 3). We must verify that this transversality is conserved after time reparametrisation and changing from elliptic to Cartesian coordinates. The reparametrisation of time is given by formula (2): it maintains the directions, because the centre (ξ_0, ϕ_0) is not a primary. As seen in Sect. 4.3, the passage to Cartesian coordinates is given by an invertible matrix U (see (10)), then the transversality is conserved.

Look now at the other possible elliptic coordinates for C : they are $(-\xi_0, -\phi_0)$, then the possible velocities at this point are the same as the ones at (ξ_0, ϕ_0) . It follows that there are no more orbits through C that we can consider, for the same values of β, q . We conclude that there are two transverse directions for the collision arcs starting from C . In correspondence of the same values of the parameters β, q we obtain four collision arcs, divided in pairs of arcs with transverse initial velocities.

Finally, we observe that, in correspondence of some fixed values of β, q , if an early collision occurs at C , then there is a unique (up to reverse the direction of motion) periodic orbit through C with transverse autointersection at C . Indeed, since the passage through the primaries is excluded, the matrix U is invertible and we cannot have inversion points along the orbit, because $\phi' \neq 0$.

We summarise the results of this and the preceding subsection in the following

Proposition 7 *Let $I \subset \mathbb{Q}^+$ be a finite set of positive rationals. Suppose that the centre C has elliptic coordinates $(\xi_0, \phi_0) \in X'_I$, where $X'_I \subset \mathbb{R} \times S^1$ is the dense subset given by Corollary 1. Then, there is $\beta_0 > 0$ such that for each $\beta \in (0, \beta_0)$ and for each $q \in I$, there are exactly four collision arcs at C for the system (L_0) , associated with the value $\hat{A}_1(\beta, q)$, which are nondegenerate. Moreover, the four collision arcs divide into two pairs, according to their initial velocities at C : any couple is formed by two arcs with opposite initial velocities at C , and each arc in one pair has transverse initial velocity to each arc in the other.*

5.3 Proof of Theorem 3

We're going to conclude the proof of Theorem 3.

Let $I \subset \mathbb{Q}^+$ be a finite set of positive rationals and $X_I = \psi(X'_I) \subset M$ as in Remark 6. The energy of the collision arcs at C is given as function of the parameters $(\beta, q), q \in I$, by the relation

$$E = -2a\beta\hat{A}_1(\beta, q).$$

Thanks to the regularity of the function $\hat{A}_1(\beta, q)$ for $\beta \in [0, 1)$ and to the fact that $\hat{A}_1(0, q) \in (0, 1)$ (Lemma 2), to require β small enough is equivalent to ask for the absolute value of the energy E to be small enough. In particular, if $\beta < \beta_0$, with β_0 sufficiently small, then the energy $E(\beta, q)$ is a strictly decreasing function of β :

$$E(\cdot, q): (0, \beta_0) \rightarrow (-E_0, 0),$$

with $E(0, q) = 0, E(\beta_0, q) = -E_0 < 0$. It follows that all the previous results can be stated using the energy parameter E instead of β .

If we choose the energy $E < 0$ sufficiently close to zero, then for each $q \in I$, there are four nondegenerate collision arcs of energy E through the centre C : they are pieces of periodic trajectories on the configuration space $M = \mathbb{R}^2 \setminus \{C_1, C_2\}$.

For β fixed, the energy increase with the class q (see Proposition 2). This means that we cannot state a general result valid for a fixed energy E and any $q \in \mathbb{Q}^+$. On the other hand, we don't need such a result, because to apply Theorem 1 we only want a finite number of collision arcs. What is certainly true is that for any finite set of classes $I \subset \mathbb{Q}^+$ of cardinality i , we can choose a small enough energy value $E < 0$ to form a set of $4i$ collision arcs of energy E , by taking for each $q \in I$ the four arcs obtained by our procedure.

By the monotony of the function $\hat{A}_1(q, \beta)$ with respect to q and its relation with the energy E , we are sure that the arcs with the same energy E , but different classes, cannot have the same value of \hat{A}_1 . Then the velocities ξ'_0, ϕ'_0 cannot coincide (see system (6)). However, we can't state that collision arcs of different classes determine a different set of directions at the point C . What we can certainly assure is that if we fix an arc of class $q \in I$, then for each class $q' \in I$, not necessarily different from q , we can always choose a pair of arcs of class q' , which start at C with directions transverse to the velocity with which the arc of class q has arrived at C .

Thus, for any fixed sequence $\{q_k\}_{k \in \mathbb{Z}}, q_k \in I$, we can construct infinite collision chains $\{\gamma_k\}_{k \in \mathbb{Z}}$, such that any γ_k is a piece of a periodic orbit of the 2-centre problem (L_0) of class q_k .

The assumptions of Theorem 1 are all satisfied and we can apply this theorem and then Theorem 2 to get our main result Theorem 3, which is now definitely proved.

Acknowledgments I wish to thank P. Negrini for having introduced me to the subject.

Appendix: Positivity of entropy

In this appendix we give a proof of the positivity of the topological entropy of the Poincaré map for the case at hand. We do it by computing the exponential growth rate of the number of periodic orbits.

By local uniqueness, if $(\gamma_{k_i})_{i \in \mathbb{Z}}$ is a periodic collision chain, then the shadowing orbit is also periodic. We compute for each positive integer n the number P_n of periodic collision chains with period n and take the limit

$$p = \limsup_{n \rightarrow \infty} \frac{\log(P_n)}{n}.$$

Let q be fixed and consider only the collision arcs of class q . They are exactly four, corresponding to the two possible transverse directions at the centre C . We distinguish two cases:

- (i) there are exactly two transverse periodic orbits of the unperturbed problem through C (up to reverse the direction of motion), and the arcs are given by the entire periodic orbits;
- (ii) there is only one periodic orbit of the unperturbed problem through C (up to reverse the direction of motion), with a transverse autointersection at C , and the arcs correspond to parts of the periodic orbit between two successive passages through C .

In case (i), for a sequence $(\gamma_{k_i})_{i \in \mathbb{Z}}$, to be a collision chain we must have that $\dot{\gamma}_{k_i}(0) = \dot{\gamma}_{k_i}(T_{k_i}) \neq \pm \dot{\gamma}_{k_{i+1}}(0)$. Then we cannot have periodic orbits with odd period $n = 2m + 1$, $m \in \mathbb{Z}^+$ and $P_{2m+1} = 0$ for each $m \in \mathbb{Z}^+$. The number of periodic orbits with even period is $P_{2m} = 2^{2m+1}$ instead, then $p = \log 2$.

In case (ii), for $(\gamma_{k_i})_{i \in \mathbb{Z}}$ to be a collision chain, we must have that $\dot{\gamma}_{k_i}(0) = \pm \dot{\gamma}_{k_{i+1}}(0)$, because there are only four possible velocities at C , divided in two pairs each containing the parallel ones, and any arc γ_{k_i} arrives at C with velocity $\dot{\gamma}_{k_i}(T_{k_i})$ transverse to the initial one $\dot{\gamma}_{k_i}(0)$. Then $P_n = 2^{n+1}$, for any $n \in \mathbb{Z}^+$, and again $p = \log 2$.

We conclude that the entropy corresponding to any choice of $I \subset \mathbb{Q}^+$ is $h_{\text{top}} \geq \log 2$.

References

Aubry, S., MacKay, R.S., Baesens, C.: Equivalence of uniform hyperbolicity for symplectic twist maps and phonon gap for Frenkel-Kontorova models. *Physica D* **56**, 123–134 (1992)

Bolotin S.V.: Nonintegrability of the n -center problem for $n > 2$, *Mosc. Univ. Mech. Bull.* **39**, No. 3, 24–28 (1984); translation from *Vestnik Mosk. Gos. Univ., Ser. I, math. mekh.* **3** (1984), 65–68.

Bolotin, S.V.: Influence of singularities of the potential energy on the integrability of dynamical systems. *J. Appl. Math. Mech.* **48**(3), 255–260 (1985)

Bolotin, S.V., MacKay, R.S.: Periodic and chaotic trajectories of the second species for the n -Centre problem. *Celest. Mech. Dyn. Astron.* **77**, 49–75 (2000)

Bolotin, S.V., MacKay, R.S.: Non-planar second species periodic and chaotic trajectories for the circular restricted three-body problem. *Celest. Mech. Dyn. Astron.* **94**(4), 433–449 (2006)

Bolotin, S.V., Negrini, P.: Regularization and topological entropy for the spatial n -center problem. *Ergod. Theory Dynam. Syst.* **21**, 383–399 (2001)

Bolotin, S.V., Negrini, P.: Chaotic behavior in the 3-center problem. *J. Differ. Equ.* **190**, 539–558 (2003)

Charlier C.L.: *Die Mechanik des Himmels*. Bd. I, II, Verlag Von Veit & Comp, Leipzig, (1902)

Duan, Y., Yuan, J.: Periodic orbits of the hydrogen molecular ion. *Eur. Phys. J D* **6**(3), 319–326 (1999)

Duan, Y., Yuan, J., Bao, C.: Periodic orbits of the hydrogen molecular ion and their quantization. *Phys. Rev. A* **52**(5), 3497–3502 (1995)

Font, J., Nunes, A., Simó, C.: Consecutive quasi collisions in the planar circular RTBP. *Nonlinearity* **15**, 115–142 (2002)

Font, J., Nunes, A., Simó, C.: A numerical study of the orbits of second species of the planar circular RTBP. *Celest. Mech. Dyn. Astron.* **103**, 143–162 (2009)

Knauf, A., Taimanov, I.A.: On the integrability of the n -centre problem. *Math. Ann.* **331**, 631–649 (2005)