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# Some ring-shaped potentials as a generalized 4-D isotropic oscillator. Periodic orbits

Eva Tresaco · Sebastián Ferrer

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**Abstract** A generalized integrable biparametric family of 4-D isotropic oscillators is proposed. It allows to treat in a unified way, Pöschl-Teller, Hartmann and other ring-shaped systems. This approach, based in the use of two canonical extensions, helps to simplify the studies of classical aspects of those systems. As an illustration, an analysis of the periodic solutions of those system is presented.

**Keywords** 4-D isotropic oscillators · Generalized Hartmann potential · Ring-shaped systems · Pöschl-Teller potential · Superintegrability

## 1 Introduction

This paper deals with with a 4-D integrable dynamical system defined by the parametric Hamiltonian function

$$\mathcal{H}_O = \frac{1}{2} \left( Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2 + \omega (q_1^2 + q_2^2 + q_3^2 + q_4^2) + \frac{a}{q_1^2 + q_2^2} + \frac{b}{q_3^2 + q_4^2} \right), \quad (1)$$

(where  $\omega$ ,  $a$  and  $b$  are parameters), and its relation with two families of 3-D integrable Hamiltonian systems  $\mathcal{H} = \frac{1}{2} \|X\|^2 + V_i$  with axial symmetry, namely systems with potentials given by

$$V_1 = -\frac{\mu}{\sqrt{x_1^2 + x_2^2 + x_3^2}} + \frac{P}{x_1^2 + x_2^2} + \frac{Q x_3}{(x_1^2 + x_2^2) \sqrt{x_1^2 + x_2^2 + x_3^2}}, \quad (2)$$

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dubbed as Smorodinsky-Winternitz potential (see [Mardoyan 2003](#)), and

$$V_2 = \frac{\Omega^2}{2} (x_1^2 + x_2^2 + x_3^2) + \frac{P}{2x_3^2} + \frac{Q}{2(x_1^2 + x_2^2)}, \quad (3)$$

(where  $\mu$ ,  $\Omega$ ,  $P$  and  $Q$  are parameters). Note that, written in spherical variables, potentials  $V_1$  also appear in the literature under the Pöschl-Teller form

$$V_1 = -\frac{\mu}{r} + \frac{P+Q}{4r^2 \sin^2 \frac{\phi}{2}} + \frac{P-Q}{4r^2 \cos^2 \frac{\phi}{2}}.$$

The particular case of the system (1) when  $a = b$  was considered by [Kibler and Negali \(1984a\)](#) when they studied the Hartmann potential using the Kustaanheimo-Stiefel transformation. In this sense, the proposed Hamiltonian (1) represents a generalization of theirs.

Potentials  $V_i$  belong to a larger family of integrable systems which are known to be separable ([Makarov et al. 1967](#)). These potentials have received special attention since the pioneer work of Hartmann et al., due to its relation to the benzene molecule, as well as other models in quantum chemistry and nuclear physics. When we take  $Q = 0$  in potential  $V_1$  we have the [Hartmann \(1972\)](#) model. Continuing the study done by [Kibler and Negali \(1984a\)](#), the solution was detailed by [Kibler and Winternitz \(1987\)](#) in parabolic coordinates. With respect to potential  $V_2$ , the case  $P = 0$  has been studied by [Quesne \(1988\)](#). Finally, [Kibler et al. \(1992\)](#) developed a complete study of both potential models.

The ring-shaped features come from the fact that coefficients have to be taken within specific ranges. Systems defined by these potentials are super-integrable but not maximally super-integrable, having four globally defined single-valued integrals of motion. They admit two maximally super-integrable systems as limiting cases, viz, the Coulomb-Kepler system and the isotropic harmonic oscillator system in three dimensions. This is related with the fact that Schrödinger equation is separable, among others, in spherical, parabolic and spheroidal coordinates. All finite trajectories are quasi-periodic; they become truly periodical if a commensurability condition is imposed on an angular momentum component. [Kibler et al. \(1994\)](#) studied in detail the coefficients of the interbasis expansions between three bases (spherical, parabolic and spheroidal) in the case of potential  $V_1$ . For the path integral approach applied to these and related systems we mention the review paper of [Grosche \(1992\)](#) and references therein. Recently the normalized wavefunctions and explicit expressions for their radial average values have been presented by [Chen et al. \(2002\)](#), where an updated list of references on these problems is given. Similar studies for potential  $V_2$  were done by [Kibler and Winternitz \(1990\)](#), and [Kibler et al. in 1996](#).

This paper deals with the relation of these 3-D systems with a 4-D integrable dynamical system defined by the Hamiltonian function (1), focused on aspects related to classical dynamics. With respect to quantum mechanics approach, we refer to the classic paper of [Calogero \(1969\)](#) (Sect. 2) which, with minor changes, can be applied to the two coupled one degree of freedom systems defining our model.

Also, it is presented a description of the periodic solutions found. The role of periodic orbits in Hamiltonian systems was first recognized by [Poincaré \(1892\)](#), stating that periodic orbits are the only opening through which we may try to penetrate the stronghold when studying complex dynamical systems. [Deprit and Henrard \(1967\)](#) also considered periodic solutions as the skeleton around which the dynamics is organized. Accordingly, we may find many studies in the literature focused on the analysis of periodic trajectories in order to get information of the behavior of the dynamical model. Today periodic orbits are at the basis

of both classical and quantum mechanics. See, for example, recent works by [Bhaduri et al. \(2005\)](#), [Palacián \(2007\)](#) and [Hadjidemetriou \(2008\)](#).

It is clear that, like any other superintegrable system, the system defined by the Hamiltonian function (1) opens different lines for analysis: analytical, numerical, geometric and topological. For a general view on this the reader may consult [Fassò \(2005\)](#) and references therein. Here, having established the connection among the 3-D and 4-D models [Ferrer \(2006\)](#), we focus on conditions for periodic orbits. Geometric properties connected to invariant manifolds as well as symmetries of the system, which complement the work presented here, make the content of another paper (see [Balsas et al. 2009](#)).

This paper is organized as follows. In Sect. 2, assuming  $\omega > 0$ , we carry out the integration of the system defined by Hamiltonian (1), which is given by means of elementary functions. Section 3 presents an analysis of some families of periodic orbits. In Sect. 4 we establish the relation between the oscillator and the systems defined by potentials (2) and (3) respectively, making use of well known point transformations in 4-D and their canonical extensions; for each case there is a linear system which relates parameters  $P$  and  $Q$  of the potentials with integrals and parameters of the 4-D oscillator.

## 2 The biparametric oscillator and its integration

Prior to show the relation of both families of ring-shaped systems with the system defined by the Hamiltonian function (1), we focus first on the integration of our oscillator. The Hamiltonian function (1) defines an integrable system in  $\Delta = \mathbb{R}^4 - \{(0, 0) \times \mathbb{R}^2\} \cup \{\mathbb{R}^2 \times (0, 0)\}$ .

### 2.1 Still another canonical extension. Superintegrability

There is a large literature on the issue of integrability and superintegrability which we do not consider necessary to treat here. We wish only mention that Liouville-Arnold conditions for integrability are satisfied for our system.

Let us consider a canonical extension, the polar-polar transformation  $(q_1, q_2, q_3, q_4) \rightarrow (\rho_1, \rho_2, \alpha_1, \alpha_2)$ , considered previously by [Kibler and Winternitz \(1987\)](#), and given by

$$q_1 = \rho_1 \cos \alpha_1, \quad q_2 = \rho_1 \sin \alpha_1, \quad q_3 = \rho_2 \cos \alpha_2, \quad q_4 = \rho_2 \sin \alpha_2 \tag{4}$$

and its canonical extension,  $(Q_1, Q_2, Q_3, Q_4) \rightarrow (P_1, P_2, A_1, A_2)$

$$P_1 = \frac{q_1 Q_1 + q_2 Q_2}{\sqrt{q_1^2 + q_2^2}}, \quad A_1 = q_1 Q_2 - q_2 Q_1, \quad P_2 = \frac{q_3 Q_3 + q_4 Q_4}{\sqrt{q_3^2 + q_4^2}}, \quad A_2 = q_3 Q_4 - q_4 Q_3 \tag{5}$$

Then, the Hamiltonian (1) in the new variables reads

$$\mathcal{H}_O = \frac{1}{2} \left( P_1^2 + P_2^2 + \frac{A_1^2}{\rho_1^2} + \frac{A_2^2}{\rho_2^2} + \omega (\rho_1^2 + \rho_2^2) + \frac{a}{\rho_1^2} + \frac{b}{\rho_2^2} \right) \tag{6}$$

Note that  $\alpha_1$  and  $\alpha_2$  are cyclic, thus  $A_1$  and  $A_2$  are first integrals. Take into account that we will have to deal with the quadratures of

$$\dot{\alpha}_1 = \frac{\partial \mathcal{H}_O}{\partial A_1} = \frac{A_1}{\rho_1^2}, \quad \dot{\alpha}_2 = \frac{\partial \mathcal{H}_O}{\partial A_2} = \frac{A_2}{\rho_2^2} \tag{7}$$

after having integrated the rest of the system.

Introducing the quantities

$$k_1 = A_1^2 + a, \quad k_2 = A_2^2 + b, \tag{8}$$

our Hamiltonian may be written as

$$\mathcal{H}_O = \frac{1}{2} \left( P_1^2 + P_2^2 + \omega (\rho_1^2 + \rho_2^2) + \frac{k_1}{\rho_1^2} + \frac{k_2}{\rho_2^2} \right), \tag{9}$$

in other words, the two dimensional Smorodinsky-Winternitz system (see [Evans 1990](#)) in the variables  $(\rho_1, \rho_2, P_1, P_2)$ . Other authors call this Hamiltonian (9) a set of two non-interacting isotonic systems with the same frequency.

Apart from the constants of motion defined by the energies of each degree of freedom, it is well known that there is another integral of motion given by

$$\mathcal{C} = (\rho_1 P_2 - \rho_2 P_1)^2 + k_1 \frac{\rho_2^2}{\rho_1^2} + k_2 \frac{\rho_1^2}{\rho_2^2}. \tag{10}$$

Thus our Hamiltonian defines a superintegrable system, but we will not elaborate on this issue here. Just note that those systems, apart from being integrable in the Liouville-Arnold sense, they possess more constants of motion than degrees of freedom. Moreover, if the number  $N$  of independent constants takes the value  $N = 2n - 1$ , then the system is called maximally superintegrable. Apart from the three well known classic cases, more recently other less simple models such as the Calogero-Moser, the Smorodinsky-Winternitz and the hyperbolic Calogero-Sutherland-Moser have been identified as superintegrable  $n$ -dimensional systems ([López et al. 1999](#)).

### 2.2 The explicit solution

The system is made separable in two subsystems of one degree of freedom defined by the Hamiltonian functions

$$\mathcal{H}_a = \frac{1}{2} \left( P_1^2 + \frac{A_1^2 + a}{\rho_1^2} + \omega \rho_1^2 \right), \quad \mathcal{H}_b = \frac{1}{2} \left( P_2^2 + \frac{A_2^2 + b}{\rho_2^2} + \omega \rho_2^2 \right), \tag{11}$$

such that

$$\mathcal{H}_O = \mathcal{H}_a + \mathcal{H}_b. \tag{12}$$

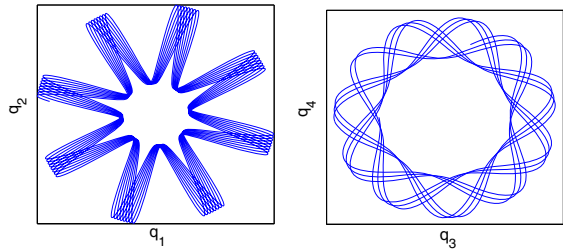
We integrate the differential system defined by (6) immediately, following closely the approach given by [Deprit \(1991\)](#). It is based on a slight variation of the Hamilton-Jacobi scheme that he calls *Delaunay transformation*. Introducing

$$\tilde{Q} = 2\mathcal{H}_a - \omega \rho_1^2 - \frac{A_1^2 + a}{\rho_1^2},$$

and the quantities  $a_1$  and  $b_1$

$$a_1 + b_1 = \sqrt{2 \left( \frac{\mathcal{H}_a}{\omega} + \sqrt{\frac{A_1^2 + a}{\omega}} \right)}, \quad a_1 - b_1 = \sqrt{2 \left( \frac{\mathcal{H}_a}{\omega} - \sqrt{\frac{A_1^2 + a}{\omega}} \right)}, \tag{13}$$

**Fig. 1** Generic solution of the 4-D Hamiltonian (1). *Left* projection  $(q_1, q_2)$ , *right*  $(q_3, q_4)$ .



we may write

$$\tilde{Q} = \frac{\omega}{\rho_1^2} (a_1^2 - \rho_1^2) (\rho_1^2 - b_1^2).$$

We see that the equation  $\tilde{Q} = 0$  has real roots when  $\mathcal{H}_a \geq \sqrt{\omega (A_1^2 + a)}$ . The system defined by the Hamiltonian  $\mathcal{H}_a$  is reduced to

$$\frac{d\rho_1}{d\tau} = P_1 = \sqrt{\tilde{Q}}, \quad \frac{d\alpha_1}{d\tau} = \frac{A_1}{\rho_1^2},$$

i.e., to two quadratures whose integrals are

$$\rho_1(\tau) = \sqrt{a_1^2 \sin^2 \sqrt{\omega} \tau + b_1^2 \cos^2 \sqrt{\omega} \tau}, \quad \alpha_1(\tau) = \frac{A_1}{a_1 b_1 \sqrt{\omega}} \tan^{-1} \left( \frac{a_1}{b_1} \tan \sqrt{\omega} \tau \right).$$

Note that in the literature this solution may be also obtained as a particular case of the so-called Pinney-Ermakov equation (see [Cariñena et al. 2007](#)).

Analogous computations for the Hamiltonian  $\mathcal{H}_b$  in Eq. 11, with quantities  $a_2$  and  $b_2$  given by

$$a_2 + b_2 = \sqrt{2 \left( \frac{\mathcal{H}_b}{\omega} + \sqrt{\frac{A_2^2 + b}{\omega}} \right)}, \quad a_2 - b_2 = \sqrt{2 \left( \frac{\mathcal{H}_b}{\omega} - \sqrt{\frac{A_2^2 + b}{\omega}} \right)}, \quad (14)$$

produce

$$\rho_2(\tau) = \sqrt{a_2^2 \sin^2 \sqrt{\omega} \tau + b_2^2 \cos^2 \sqrt{\omega} \tau}, \quad \alpha_2(\tau) = \frac{A_2}{a_2 b_2 \sqrt{\omega}} \tan^{-1} \left( \frac{a_2}{b_2} \tan \sqrt{\omega} \tau \right).$$

The equilibrium points of the reduced systems are the zeros of the potential functions, i.e.  $\rho_i^4 = k_i/\omega$ , ( $i = 1, 2$ ), which correspond to circular orbits with minimal energies  $\mathcal{H}_a = \sqrt{\omega k_1}$  and  $\mathcal{H}_b = \sqrt{\omega k_2}$ .

In each of the two separable subsystems the generic solution is a rosetta type-orbit, periodic or not, see Fig. 1. In particular, circular orbits appear as a limit case of the rosetta, but considering here as another orbit type due to their simplicity.

Therefore, it can be stated that the generic solution of the Hamiltonian system (1) is a pair of non-periodic rosetta orbits. Three different family classes are obtained: two rosetta orbits, one circular and one rosetta, or two circular orbits.

As our Hamiltonian system is separable in two subsystems in  $(\rho_1, \alpha_2)$  and  $(\rho_2, \alpha_2)$ , related by means of the energy of the complete system, we may take advantage of this property to obtain periodic orbits of the complete system, as we will see in the next section.

### 3 Analysis of periodic orbits

In this section we present an analysis of the orbits of the 4-D dynamical system, studying the resonances needed to obtain a periodic orbit. From the solutions given in Sect. 2.2 note that the radii  $\rho_1$  and  $\rho_2$  are periodic quantities with period  $T_\rho = \pi/\sqrt{\omega}$ . The angular quantities  $\alpha_1$  and  $\alpha_2$  are also periodic with periods  $T_{\alpha_1}$  and  $T_{\alpha_2}$ , respectively. Nevertheless, note that in general

$$\alpha_i(n T_\rho) = n \pi \frac{A_i}{a_i b_i \sqrt{\omega}} \neq 2m \pi, \quad (i = 1, 2),$$

where  $m, n$ , are integers.

Therefore, we will only find periodic solutions of the 4-D Hamiltonian (1) when the two resonance conditions

$$a_1 b_1 \sqrt{\omega}/A_1 \in \mathbb{Q}, \quad a_2 b_2 \sqrt{\omega}/A_2 \in \mathbb{Q}, \tag{15}$$

are satisfied.

In case of a pair of circular orbits only one commensurability relation is needed. By means of Eqs. 13 and 14 we can rewrite  $a_1 b_1 \sqrt{\omega}/A_1 = \left(\sqrt{A_1^2 + a/\sqrt{\omega}}\right) \sqrt{\omega}/A_1$ , and so the resonance condition for circular–circular orbits is

$$m A_2 \sqrt{A_1^2 + a} = n A_1 \sqrt{A_2^2 + b}, \quad m, n \in \mathbb{Z}, \tag{16}$$

while Eq. 15 must be satisfied for all other rosetta–circular or rosetta–rosetta pairs of orbits.

#### 3.1 Computing periodic solutions

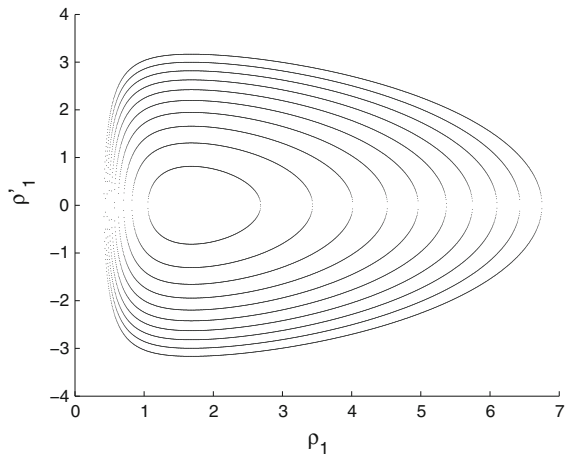
In order to search for periodic orbits, we take fixed values of the parameters  $a, b$  and total energy  $\mathcal{H}_0 (= \mathcal{H}_a + \mathcal{H}_b)$ , and compute closed trajectories for each subsystems  $\mathcal{H}_a, \mathcal{H}_b$  separately.

The simplest case is to consider circular orbits, whose radii are  $\rho_i = (k_i/\omega)^{1/4}$ . We illustrate them in Fig. 2 by plotting the phase portrait of any of these subsystems,  $\mathcal{H}_a$  or  $\mathcal{H}_b$ , for some given value of the parameters. The phase portrait represents multiple curves corresponding to different initial conditions in the same phase plane, and provides useful information about the system’s trajectories. Indeed Fig. 2 depicts a stable steady state in a phase space, where the fixed point corresponds to a circular orbit. The existence of this orbit in one of the separable subsystems (11) results in a 3-dimensional family of invariant tori, or a 2-tori out of a pair of circular orbits. Note that these tori may be seen as foliations of the Lagrangian tori associated to the system (1) when we take into account the existence of the reduced integral (10) of the reduced system (9) which has all orbits periodic. For more details see Balsas et al. (2009).

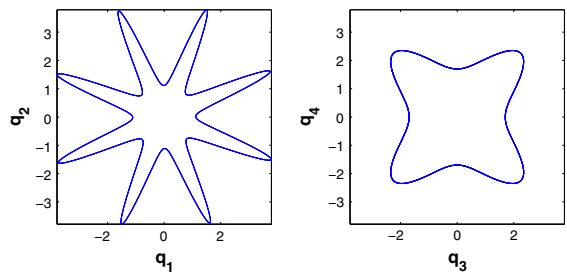
In order to compute the periodic orbit for given values of the parameters of the problem  $a$  and  $b$ , we consider a certain value of the energy of the complete system and we take some initial conditions  $(q_1, q_2)$  of a circular orbit for the system  $\mathcal{H}_a$ . The energy of this orbit imposes a value for  $\mathcal{H}_b$ , leading to obtain a second circular orbit  $(q_3, q_4)$  which is related to the first one by means of the energy.

The largest circular orbit  $(q_1, q_2)$  implies a greater value of the energy  $\mathcal{H}_a$ , leading to a smaller  $\mathcal{H}_b$  value for the circular orbit in the  $(q_3, q_4)$  plane. These orbits do not produce in general a 4-D periodic orbit. We must choose the appropriated initial conditions of both

**Fig. 2** Phase portrait of a typical subsystem with Hamiltonian  $\mathcal{H}_a$  or  $\mathcal{H}_b$ , where a center is depicted



**Fig. 3** Example of 4-D periodic orbit out of two rosetta 2-D periodic orbits. *Left*  $(q_1, q_2)$ , *right*  $(q_3, q_4)$



orbits so that the resonance condition (16) is satisfied, and so we can obtain a periodic orbit in the complete system with period the product of the period of the orbits  $(q_1, q_2)$  and  $(q_3, q_4)$ .

Remind that the commensurability conditions (15) apply to rosetta-type orbits, and therefore this same procedure applies for all orbits type. Figure 3 depicts another example of a 4-D periodic orbit whose projections  $(q_1, q_2)$  and  $(q_3, q_4)$  are a pair of rosetta orbits.

Hence, thanks to the separability property of the 4-D Hamiltonian system, we have a simple way of computing 4-D periodic orbits of the complete problem out of 2-D orbits of its subsystems satisfying the periodicity relations.

**4 Hartmann and isotropic potentials. Families of periodic orbits**

In this section are presented two families of three dimensional systems involving ring shaped potentials, the Hartmann and the Isotropic systems, which are reductions of the initial Hamiltonian (1). We also describe some particular periodic orbits of both potentials and the generation of initial conditions of 4-D orbits of (1) through the computation of periodic orbits of the 3-D reduced systems.

**4.1 The oscillator and the generalized Hartmann potentials**

We show first the relation of the Hamiltonian system (1) and the generalized Hartmann potentials  $V_1$  defined by Eq. 2. In order to do that we will use the transformation  $(r, \phi, \lambda, \psi) \rightarrow (q_1, q_2, q_3, q_4)$  given by



$$\begin{aligned}
 q_1 &= \sqrt{r} \sin \frac{\phi}{2} \cos \frac{\lambda - \psi}{2}, & q_3 &= \sqrt{r} \cos \frac{\phi}{2} \sin \frac{\lambda + \psi}{2}, \\
 q_2 &= \sqrt{r} \sin \frac{\phi}{2} \sin \frac{\lambda - \psi}{2}, & q_4 &= \sqrt{r} \cos \frac{\phi}{2} \cos \frac{\lambda + \psi}{2},
 \end{aligned}
 \tag{17}$$

with  $(r, \phi, \lambda, \psi) \in R^+ \times (0, \pi) \times [0, 2\pi] \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ , and whose Jacobian is  $-r \sin \phi/8$ . Later on we will need the inverse transformation written as

$$\begin{aligned}
 r &= q_1^2 + q_2^2 + q_3^2 + q_4^2, \\
 \sin \phi &= \frac{2\sqrt{(q_1^2 + q_2^2)(q_3^2 + q_4^2)}}{q_1^2 + q_2^2 + q_3^2 + q_4^2}, & \cos \phi &= \frac{q_3^2 + q_4^2 - q_1^2 - q_2^2}{q_1^2 + q_2^2 + q_3^2 + q_4^2}, \\
 \sin \lambda &= \frac{q_1 q_3 + q_2 q_4}{\sqrt{(q_1^2 + q_2^2)(q_3^2 + q_4^2)}}, & \cos \lambda &= \frac{q_1 q_4 - q_2 q_3}{\sqrt{(q_1^2 + q_2^2)(q_3^2 + q_4^2)}}, \\
 \sin \psi &= \frac{q_1 q_3 - q_2 q_4}{\sqrt{(q_1^2 + q_2^2)(q_3^2 + q_4^2)}}, & \cos \psi &= \frac{q_1 q_4 + q_2 q_3}{\sqrt{(q_1^2 + q_2^2)(q_3^2 + q_4^2)}}.
 \end{aligned}
 \tag{18}$$

These variables are well known in the literature. See the works of Kibler and Négadi, Ikeda and Miyachi (1971), and the classical physics book of Synge (1960). In Cornish (1984), we find a reference to the work of Barut et al. (1979), where the variables are introduced starting from the transformation  $(\zeta_A, \zeta_B) \rightarrow (x, y, z, \sigma)$

$$x + iy = 2\zeta_A \bar{\zeta}_B, \quad z = \zeta_A \bar{\zeta}_A - \zeta_B \bar{\zeta}_B, \quad \sigma = \arg \zeta_A \zeta_B,$$

where  $\zeta_A$  y  $\zeta_B$  are two complex variables. We find them also in Stiefel and Scheifele (1971), although no further use of them is done. As these variables are related to Euler angles of rotation, we propose to denote them as *Euler projective variables*.

The canonical extension associated to the transformation (17) is obtained as a Mathieu transformation, satisfying  $\sum Q_i dq_i = Rdr + \Phi d\phi + \Lambda d\lambda + \Psi d\psi$ . Therefore, the relations among the momenta are given by

$$\begin{aligned}
 R &= \frac{1}{2 \sum q_i^2} (q_1 Q_1 + q_2 Q_2 + q_3 Q_3 + q_4 Q_4), \\
 \Phi &= \frac{(q_1 Q_1 + q_2 Q_2)(q_3^2 + q_4^2) - (q_3 Q_3 + q_4 Q_4)(q_1^2 + q_2^2)}{2\sqrt{(q_1^2 + q_2^2)(q_3^2 + q_4^2)}}, \\
 \Lambda &= \frac{1}{2} (-q_2 Q_1 + q_1 Q_2 + q_4 Q_3 - q_3 Q_4), \\
 \Psi &= \frac{1}{2} (q_2 Q_1 - q_1 Q_2 + q_4 Q_3 - q_3 Q_4).
 \end{aligned}
 \tag{19}$$

The Hamiltonian (1) in the new variables may be written as

$$\mathcal{H} = 4r \left[ \frac{\omega}{8} + \frac{1}{2} \left( R^2 + \frac{\Phi^2}{r^2} + \frac{\Lambda^2}{r^2 \sin^2 \phi} \right) + \frac{\Psi^2 - 2 \Lambda \Psi \cos \phi}{2r^2 \sin^2 \phi} + \frac{c + d \cos \phi}{2r^2 \sin^2 \phi} \right], \tag{20}$$

where

$$c = \frac{a + b}{2}, \quad d = \frac{a - b}{2}.$$

Note that  $\lambda$  and  $\psi$  are cyclic variables, and so  $\Lambda$  and  $\Psi$  are first integrals. In other words the differential systems is described by

$$\frac{dr}{d\tau} = \frac{\partial \mathcal{H}}{\partial R}, \quad \frac{d\phi}{d\tau} = \frac{\partial \mathcal{H}}{\partial \Phi}, \quad \frac{dR}{d\tau} = -\frac{\partial \mathcal{H}}{\partial r}, \quad \frac{d\Phi}{d\tau} = -\frac{\partial \mathcal{H}}{\partial \phi},$$

and two quadratures  $\lambda = \int (\partial \mathcal{H} / \partial \Lambda) d\tau$  and  $\psi = \int (\partial \mathcal{H} / \partial \Psi) d\tau$ .

Using Poincaré notation and introducing a change of independent variable  $\tau \rightarrow s$  given by  $d\tau = 4r ds$ , the Hamiltonian  $\mathcal{K} = (\mathcal{H} - h) / (4r)$  takes the expression

$$\mathcal{K} = \frac{\omega}{8} + \frac{1}{2} \left( R^2 + \frac{\Phi^2}{r^2} + \frac{\Lambda^2}{r^2 \sin^2 \phi} \right) + \frac{\Psi^2 - 2\Lambda\Psi \cos \phi}{2r^2 \sin^2 \phi} + \frac{c + d \cos \phi}{2r^2 \sin^2 \phi} - \frac{h}{4r}, \tag{21}$$

where  $h$  is a fixed value of the Hamiltonian  $\mathcal{H}$  for chosen initial conditions, and the flow is defined now on the manifold  $\mathcal{K} = 0$ . We prefer to use a slightly different form, and we consider the Hamiltonian

$$\tilde{\mathcal{K}} = \frac{1}{2} \left( R^2 + \frac{\Phi^2}{r^2} + \frac{\Lambda^2}{r^2 \sin^2 \phi} \right) - \frac{h}{4r} + \frac{(\Psi^2 + c)/2}{r^2 \sin^2 \phi} + \frac{(d/2 - \Lambda\Psi) \cos \phi}{r^2 \sin^2 \phi} \tag{22}$$

in the manifold  $\tilde{\mathcal{K}} = -\frac{\omega}{8}$ .

Notice that this system has two cyclic variables  $\lambda$  and  $\psi$ . Therefore, denoting

$$\mathcal{K}^* = \frac{1}{2} \left( R^2 + \frac{\Phi^2}{r^2} + \frac{\Lambda^2}{r^2 \sin^2 \phi} \right) - \frac{h}{4r}$$

the differential system defined by (22) is given by

$$\begin{aligned} \frac{dr}{ds} &= \frac{\partial \tilde{\mathcal{K}}}{\partial R} = R, \\ \frac{d\phi}{ds} &= \frac{\partial \tilde{\mathcal{K}}}{\partial \Phi} = \frac{\Phi}{r^2}, \\ \frac{dR}{ds} &= -\frac{\partial \tilde{\mathcal{K}}}{\partial r} = -\frac{\partial \mathcal{K}^*}{\partial r} + \frac{\Psi^2 + c + (d - 2\Lambda\Psi) \cos \phi}{r^3 \sin^2 \phi}, \\ \frac{d\Phi}{ds} &= -\frac{\partial \tilde{\mathcal{K}}}{\partial \phi} = -\frac{\partial \mathcal{K}^*}{\partial \phi} - \frac{(\Psi^2 + c)/2}{r^2} \frac{\partial}{\partial \phi} \left( \frac{1}{\sin^2 \phi} \right) - \frac{(d/2 - \Lambda\Psi)}{r^2} \frac{\partial}{\partial \phi} \left( \frac{\cos \phi}{\sin^2 \phi} \right), \end{aligned} \tag{23}$$

and two quadratures

$$\lambda = \int \frac{\partial \tilde{\mathcal{K}}}{\partial \Lambda} ds = \int \frac{\Lambda - \Psi \cos \phi}{r^2 \sin^2 \phi} ds, \tag{24}$$

$$\psi = \int \frac{\partial \tilde{\mathcal{K}}}{\partial \Psi} ds = \int \frac{\Psi - \Lambda \cos \phi}{r^2 \sin^2 \phi} ds. \tag{25}$$

If we consider now the differential system defined by the Hamiltonian function with potential  $V_1$  in spherical variables  $(r, \phi, \lambda)$ ,

$$x_1 = r \sin \phi \cos \lambda, \quad x_2 = r \sin \phi \sin \lambda, \quad x_3 = r \cos \phi, \tag{26}$$

and their momenta  $(R, \Phi, \Lambda)$ , we will have

$$\mathcal{H}_1 = \mathcal{H}_0 + \frac{P + Q \cos \phi}{r^2 \sin^2 \phi} \quad \text{where } \mathcal{H}_0 = \frac{1}{2} \left( R^2 + \frac{\Phi^2}{r^2} + \frac{\Lambda^2}{r^2 \sin^2 \phi} \right) - \frac{\mu}{r},$$

and the differential system will be written as

$$\begin{aligned}
 \frac{dr}{ds} &= \frac{\partial \mathcal{H}_1}{\partial R} = R, \\
 \frac{d\phi}{ds} &= \frac{\partial \mathcal{H}_1}{\partial \Phi} = \frac{\Phi}{r^2}, \\
 \frac{dR}{ds} &= -\frac{\partial \mathcal{H}_1}{\partial r} = -\frac{\partial \mathcal{H}_0}{\partial r} + \frac{P + Q \cos \phi}{r^3 \sin^2 \phi}, \\
 \frac{d\Phi}{ds} &= -\frac{\partial \mathcal{H}_1}{\partial \phi} = -\frac{\partial \mathcal{H}_0}{\partial \phi} - \frac{P}{r^2} \frac{\partial}{\partial \phi} \left( \frac{1}{\sin^2 \phi} \right) - \frac{Q}{r^2} \frac{\partial}{\partial \phi} \left( \frac{\cos \phi}{\sin^2 \phi} \right),
 \end{aligned}
 \tag{27}$$

and the quadrature

$$\lambda = \int \frac{\partial \mathcal{H}_1}{\partial \Lambda} ds = \int \frac{\Lambda}{r^2 \sin^2 \phi} ds.
 \tag{28}$$

We check that these last equations coincide with Eqs. 23 and 24 when we restrict to the manifold  $\Psi = 0$ , taking  $h = 4\mu, c = 2P, d = 2Q$ , and we identify the variable  $s$  with the physical time  $t$ . Moreover, in this case the quadrature (25) takes the form

$$\psi = \int \frac{\partial \tilde{\mathcal{K}}}{\partial \Psi} ds = - \int \frac{\Lambda \cos \phi}{r^2 \sin^2 \phi} ds.
 \tag{29}$$

Thus, we have shown that the dynamics of the oscillator defined by the Hamiltonian function (1) includes the family of the generalized Hartmann potentials. Note that we may interchange the role played by  $\Lambda$  and  $\Psi$ . Further, when we restrict to the Hartmann case ( $Q = 0$ ), we may simplify the 4-D potential reducing it to only one parameter  $a = b$ .

### 4.2 Relation with generalized 3-D isotropic potentials

There is still another family of potentials related to our system (1), the isotropis potential  $V_2$ , defined by Eq. 3. Let us consider now the transformation

$$q_1 = r \cos \alpha \cos \beta, \quad q_2 = r \cos \alpha \sin \beta, \quad q_3 = r \sin \alpha \cos \gamma, \quad q_4 = r \sin \alpha \sin \gamma, \tag{30}$$

with Jacobian  $-r^3 \sin 2\alpha/2$  i.e.,  $\alpha \in (0, \pi/2) \cup (\pi/2, \pi)$ . This transformation was used by Kibler and Negali (1984a).

The associated canonical extension  $(q, Q) \rightarrow (r, \alpha, \beta, \gamma, R, A, B, C)$  reads

$$\begin{aligned}
 R &= \frac{1}{\sqrt{\sum q_i^2}} (q_1 Q_1 + q_2 Q_2 + q_3 Q_3 + q_4 Q_4), \\
 A &= \frac{(q_3 Q_3 + q_4 Q_4) (q_1^2 + q_2^2) - (q_1 Q_1 + q_2 Q_2) (q_3^2 + q_4^2)}{\sqrt{(q_1^2 + q_2^2) (q_3^2 + q_4^2)}}, \\
 B &= -q_2 Q_1 + q_1 Q_2, \\
 C &= -q_4 Q_3 + q_3 Q_4,
 \end{aligned}
 \tag{31}$$

while the inverse transformation, needed for the construction of the explicit transformation with the old variables, takes the form

$$\begin{aligned}
 r &= \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2}, \\
 \cos \alpha &= \sqrt{\frac{q_1^2 + q_2^2}{q_1^2 + q_2^2 + q_3^2 + q_4^2}}, & \sin \alpha &= \sqrt{\frac{q_3^2 + q_4^2}{q_1^2 + q_2^2 + q_3^2 + q_4^2}} \\
 \cos \beta &= \frac{q_1}{\sqrt{q_1^2 + q_2^2}}, & \sin \beta &= \frac{q_2}{\sqrt{q_1^2 + q_2^2}}, \\
 \cos \gamma &= \frac{q_3}{\sqrt{q_3^2 + q_4^2}}, & \sin \gamma &= \frac{q_4}{\sqrt{q_3^2 + q_4^2}}.
 \end{aligned}
 \tag{32}$$

The Hamiltonian (1) in these variables is given by

$$\mathcal{H} = \frac{1}{2} \left( R^2 + \frac{A^2}{r^2} + \frac{C^2}{r^2 \sin^2 \alpha} \right) + \frac{\omega}{2} r^2 + \frac{B^2 + a}{2r^2 \cos^2 \alpha} + \frac{b}{2r^2 \sin^2 \alpha}.
 \tag{33}$$

Note that  $\beta$  and  $\gamma$  are cyclic, thus  $B$  and  $C$  are first integrals. In other words the differential systems is described by

$$\frac{dr}{d\tau} = \frac{\partial \mathcal{H}}{\partial R}, \quad \frac{d\alpha}{d\tau} = \frac{\partial \mathcal{H}}{\partial A}, \quad \frac{dR}{d\tau} = -\frac{\partial \mathcal{H}}{\partial r}, \quad \frac{dA}{d\tau} = -\frac{\partial \mathcal{H}}{\partial \alpha}$$

and two quadratures

$$\beta = \int \frac{\partial \mathcal{H}}{\partial B} d\tau = \int \frac{B}{r^2 \cos^2 \alpha} d\tau, \quad \gamma = \int \frac{\partial \mathcal{H}}{\partial C} d\tau = \int \frac{C}{r^2 \sin^2 \alpha} d\tau.
 \tag{34}$$

If we restrict to the subsystem defined by  $(r, \alpha, \gamma, R, A, C)$ , and we consider the transformation defined by

$$x_1 = r \sin \alpha \cos \gamma, \quad x_2 = r \sin \alpha \sin \gamma, \quad x_3 = r \cos \alpha,
 \tag{35}$$

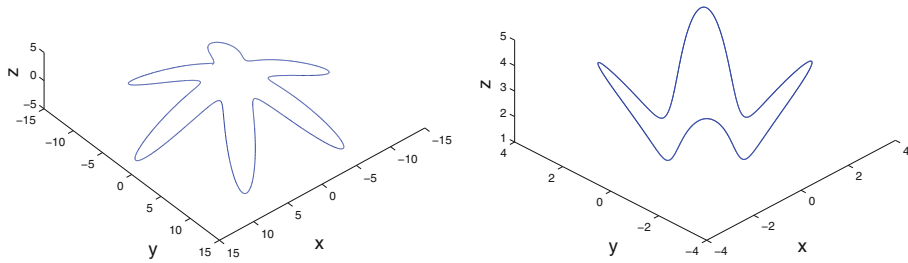
the defined system corresponds to the one given by the family of potentials  $V_2$ , choosing the constants as follows

$$\omega = \Omega^2, \quad a = P - B^2 \quad y \quad b = Q,$$

and identifying  $\tau$  with the physical time  $t$ .

Only when  $B = -C$  the dynamics of the 4-D oscillator belongs to the manifold  $q_2 Q_1 - q_1 Q_2 + q_4 Q_3 - q_3 Q_4 = 0$ . Notice that the 3-D isotropic oscillator is obtained either choosing  $B = a = b = 0$ , or if  $b = 0$  and  $a = -B^2$ . Kibler and Winternitz (1990) studied the case  $P = 0$ , and a similar analysis for the general case may be found in Kibler et al. (1996).

Finally, as it has been described along this section, differential systems with potentials  $V_1$  and  $V_2$  are related to our system (1). Therefore, once we get a 4-D periodic orbit of the system (1) we can apply the relations between the 4-D oscillator and the family of Hartmann and isotropic potentials, to obtain the corresponding 3-D periodic orbits. Figure 4 shows the orbits obtained for both potentials, when we apply the transformations to the initial condition of a particular 4-D periodic orbit. This 4-D orbit was depicted in Fig. 3.



**Fig. 4** Periodic orbits obtained through 4-D orbit shown in Fig. 3. *Left* Hartmann orbit in cylindric coordinates (26). *Right* Isotropic orbit in spherical coordinates (35).

### 4.3 Some particular periodic orbits of the Hartmann and isotropic potentials

Finally, we search for some particular periodic orbits of the Hartmann and Isotropic potentials, and their relation with the 4-D trajectories of the initial Hamiltonian (1).

In order to identify the special ring-shaped solutions, it is convenient to express the Hamiltonian defined by the potentials  $V_1$  and  $V_2$  in symplectic cylindric variables  $(\lambda, \rho, z, \Lambda, P, Z)$ ,

$$\mathcal{H}_1 = \frac{1}{2} \left( P^2 + Z^2 + \frac{\Lambda^2}{\rho^2} \right) - \frac{\mu}{\sqrt{\rho^2 + z^2}} + \frac{\tilde{P}}{\rho^2} + \frac{Qz}{\rho^2 \sqrt{\rho^2 + z^2}}, \tag{36}$$

and

$$\mathcal{H}_2 = \frac{1}{2} \left( P^2 + Z^2 + \frac{\Lambda^2}{\rho^2} \right) + \frac{\Omega^2}{2} (\rho^2 + z^2) + \frac{\tilde{P}}{2z^2} + \frac{Q}{2\rho^2}, \tag{37}$$

where we have denoted the parameter  $P \equiv \tilde{P}$  to avoid confusion with the cylindric momentum  $P$ . Note that both potentials are axially symmetric with the cyclic variable  $\lambda = \int (\Lambda/\rho^2) dt$ .

Then, we will focus on the characterization of three type of possible periodic solutions, namely the ones defined by  $\rho$  constant,  $z$  constant, or both.

*(i)  $(\rho, z)$  constant*

Related to  $V_2$  potentials, the family of orbits is composed of circles around the  $Oz$  axis and is given by

$$\dot{P} = \frac{\partial \mathcal{H}_2}{\partial \rho} = \frac{\Lambda^2}{\rho^3} - \Omega^2 \rho + \frac{Q}{\rho^3} = 0, \quad \dot{Z} = \frac{\partial \mathcal{H}_2}{\partial z} = -\Omega^2 z + \frac{\tilde{P}}{z^3} = 0. \tag{38}$$

The solution is  $(\rho_0, z_0) = \left( \sqrt[4]{(\Lambda^2 + Q)/\Omega^2}, \sqrt[4]{\tilde{P}/\Omega^2} \right)$ . Now  $\lambda = \int (\Lambda/\rho_0^2) dt$ , when  $\Lambda = 0$  it reduces to a circle of equilibria.

Looking for similar type of solutions in  $V_1$  potentials we find

$$\dot{P} = \frac{\partial \mathcal{H}_1}{\partial \rho} = \frac{\Lambda^2 + 2\tilde{P}}{\rho^3} - \frac{\mu \rho}{(\rho^2 + z^2)^{3/2}} + \frac{Qz(3\rho^2 + z^2)}{\rho^3(\rho^2 + z^2)^{3/2}} = 0, \tag{39}$$

$$\dot{Z} = \frac{\partial \mathcal{H}_1}{\partial z} = \frac{Q - \mu z}{(\rho^2 + z^2)^{3/2}} = 0. \tag{40}$$

Then, solutions  $(\rho_0, z_0)$  are  $z_0 = Q/\mu$  and  $\rho_0 = \sqrt{x}/\mu$ , where  $x$  is a positive root of the polynomial equation which may be readily obtained from Eq. 39.

(ii)  $\rho$  constant

A second special type of solutions are those satisfying that  $\rho$  is constant i.e. trajectories on a cylinder. Searching for them in the isotropic family  $V_2$  we get  $\rho_0 = \sqrt[4]{(\Lambda^2 + Q)}/\Omega^2$  and the differential equation

$$\ddot{z} = -\Omega^2 z + \frac{\tilde{P}}{z^3}. \tag{41}$$

By means of its Hamiltonian function Eq. 37, an alternative to Eq. 41 can be written as

$$\dot{z} = Z = \sqrt{D - \Omega^2 z^2 - \frac{\tilde{P}}{z^2}},$$

where  $D = 2(\mathcal{H}_2 - \Omega\sqrt{\Lambda^2 + Q})$ . Then, introducing the variable  $u = z^2$ , and differentiating we obtain

$$\int \frac{du}{\sqrt{-\Omega^2 u^2 + Du - \tilde{P}}} = 2(t - t_0),$$

i.e. we get a solution that oscillates on the cylinder in the circumferences  $\rho_0$ .

Likewise, trying to identify this type of periodic solutions in the Hartmann family, we find that the answer is negative.

(iii)  $z$  constant

A third special type of orbits are planar trajectories with  $z$  constant. Again, for the isotropic family they are easily identified in the plane  $z_0 = \sqrt[4]{\tilde{P}/\Omega^2}$  with  $\rho$  satisfying

$$\ddot{\rho} = -\Omega^2 \rho + \frac{\Lambda^2 + Q}{\rho^3}.$$

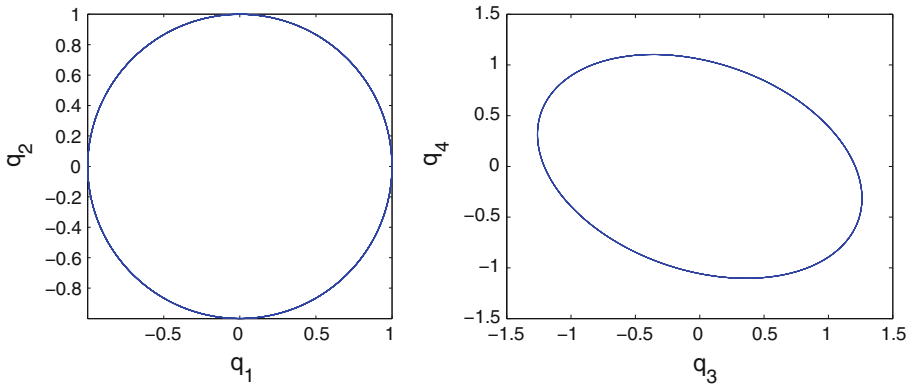
With respect to the possible solution for the Hartmann family, in the plane  $z_0 = Q/\mu$ , the differential equation to be studied is reduced to the analysis of the elliptic integral

$$- \int \frac{dv}{v^2 \sqrt{-4\mu Q^2 v^3 - (2\mathcal{H}_1 Q^2 + \mu^2(2\tilde{P} + \Lambda^2)) v^2 + 2\mu^3 v + 2\mu^2 \mathcal{H}_1}} = \frac{1}{\mu}(t - t_0), \tag{42}$$

where we have made use of the intermediary variable  $v$  related to  $\rho$  by  $v^2 = (\rho^2 + z_0^2)^{-1}$ .

4.4 From three to four dimensions

At last, note that any of these particular periodic orbits or a generic one can be used in order to generate initial conditions of a 4-D periodic orbit of the Hamiltonian system (1). Here is briefly described the procedure performed in the case of Isotropic potentials. Analogous treatment has to be followed for the Hartmann potential family.



**Fig. 5** Corresponding 4-D trajectory of  $V_2$  orbit with  $z$  constant

Once we have a set of initial conditions of a periodic orbit for the potential  $V_2$  with a given value of the parameters  $P$  and  $Q$ , the inverse transformation of the relation between the 4-D oscillator and the 3-D isotropic potential,

$$x = r \sin \alpha \cos \gamma, \quad y = r \sin \alpha \sin \gamma, \quad z = r \cos \alpha,$$

described in Sect. 4.2, allows us to obtain the spherical variables  $(r, \alpha, \gamma)$  and their derivatives. By means of the formulas detailed in that section, Eqs. 30 and 31, the coordinates and moments  $(q_3, q_4, Q_3, Q_4)$  are derived straightaway.

Our goal is to obtain a 4-D periodic orbit, thence it is mandatory to impose the commensurability conditions (15) for both separable systems  $(q_1, q_2, Q_1, Q_2)$  and  $(q_3, q_4, Q_3, Q_4)$ ,

$$\frac{\sqrt{A_1^2 + a}}{A_1} = \frac{m_1}{n_1}, \quad \frac{\sqrt{A_2^2 + b}}{A_2} = \frac{m_2}{n_2},$$

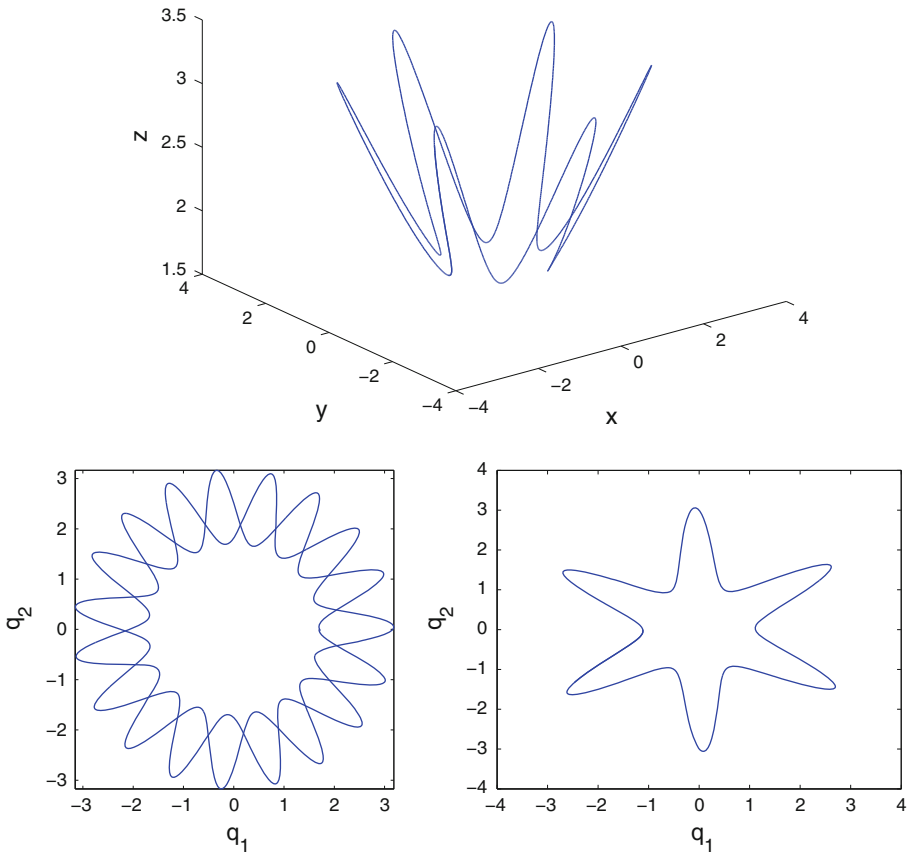
where we have been set the parameter of the initial Hamiltonian (1),  $\omega = 1$ , for the sake of simplicity. Note that each set of initial conditions in cartesian coordinates determines a value of  $A_1 = q_1 Q_2 - q_2 Q_1$  and  $A_2 = q_3 Q_4 - q_4 Q_3$ .

On the other hand, as we are jumping from 3-D to 4-D motion, the variables  $\beta, B$  in Eqs. 30 and 31 are undetermined, from which it follows that  $(q_1, q_2, Q_1, Q_2)$  joint with the parameter  $a$ , will be determined only taking into account the fulfillment of the commensurability relation. In this way we obtain a complete set of variables and parameters needed to integrate the 4-D Hamiltonian Eq. 1 providing a 4-D periodic orbit.

Next we present two figures depicting two examples related to the previous study. In Fig. 5 we represent the 4-D orbit obtained through one of the particular orbits of the  $V_2$  family, with  $z$  constant, above described. Figure 6 shows the 4-D orbit originated out of a generic periodic orbit in which none of  $(\rho, z)$  remain constant.

**Conclusion and future work**

We have established the relation of two families of ring-shaped type systems with a generalized integrable 4-D isotropic oscillator. This allows a unified treatment which is of interest both in quantum and classical studies. In particular conditions for periodic families have been



**Fig. 6**  $V_2$  generic periodic orbit (*top*) used to generate a 4-D orbit (*bottom*)

established. Topological aspects of this system, including its foliation by means of the first integrals will be presented in another paper (see Balsas et al.).

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