



On the instability of eigenvalues

Sylvain Golenia

► To cite this version:

Sylvain Golenia. On the instability of eigenvalues. Geometry–Exploratory Workshop on Differential Geometry and its Applications, 2009, Romania. 71-75, Cluj Univ. Press, Cluj-Napoca. hal-00552181

HAL Id: hal-00552181

<https://hal.science/hal-00552181>

Submitted on 5 Jan 2011

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ON THE INSTABILITY OF EIGENVALUES

SYLVAIN GOLÉLIA

ABSTRACT. This is the proceeding of a talk given in Workshop on Differential Geometry and its applications at Alexandru Ioan Cuza University Iași, Romania, September 2–4, 2009. I explain how positive commutator estimates help in the analysis of embedded eigenvalues in a geometrical setting. Then, I will discuss the disappearance of eigenvalues in the perturbation theory and its relation with the Fermi golden rule.

1. INTRODUCTION

Let $\mathbb{H} := \{(x, y) \in \mathbb{R}^2, y > 0\}$ be the Poincaré half-plane and we endow it with the metric $g := y^{-2}(dx^2 + dy^2)$. Consider the group $\Gamma := PSL_2(\mathbb{Z})$. It acts faithfully on \mathbb{H} by homographies, from the left. The interior of a fundamental domain of the quotient \mathbb{H}/Γ is given by $X := \{(x, y) \in \mathbb{H}, |x| < 1, x^2 + y^2 > 1\}$. Let $\mathcal{H} := L^2(X, g)$ be the set of L^2 integrable function acting on X , with respect to the volume element $dx dy/y^2$. Let $\mathcal{C}_b^\infty(X)$ be the restriction to X of the smooth bounded functions acting on \mathbb{H} which are \mathbb{C} -valued and invariant under Γ . The (non-negative) Laplace operator is defined as the closure of

$$\Delta := -y^2(\partial_x^2 + \partial_y^2), \text{ on } \mathcal{C}_b^\infty(X).$$

It is a (unbounded) self-adjoint operator on $L^2(X)$. Using Eisenstein series, for instance, one sees that its essential spectrum is given by $[1/4, \infty)$ and that it has no singularly continuous spectrum, with respect to the Lebesgue measure. It is well-known that Δ has infinitely many eigenvalues accumulating at $+\infty$ and that every eigenspace is of finite dimension. We refer to [5] for an introduction to the subject.

We consider the Schrödinger operator $H_\lambda := \Delta + \lambda V$, where V is the multiplication by a bounded, real-valued function and $k \in \mathbb{R}$. We focus on an eigenvalue $k > 1/4$ of Δ and assume that the following hypothesis of *Fermi golden rule* holds true. Namely, there is $c_0 > 0$ so that:

$$(1.1) \quad \lim_{\varepsilon \rightarrow 0^+} PV\overline{P} \operatorname{Im}(H_0 - k + i\varepsilon)^{-1} \overline{P}VP \geq c_0 P,$$

in the form sense and where $P := P_k$, the projection on the eigenspace of k , and $\overline{P} := 1 - P$. As P is of finite dimension, the limit can be taken in the weak or in the strong sense. At least formally, $\overline{P} \operatorname{Im}(H_0 - k + i\varepsilon)^{-1} \overline{P}$ tends to the Dirac mass $\pi \delta_k(\overline{P}H_0)$. Therefore, the potential V couples the eigenspace of k and $\overline{P}H_0$ over k in a non-trivial way. This is a key assumption in the second-order perturbation theory of embedded eigenvalues, e.g., [13], and all the art is to prove that it implies there is $\lambda_0 > 0$ that H_λ has no eigenvalue in a neighborhood of k for $\lambda \in (0, |\lambda_0|)$.

In [4], one shows that generically the eigenvalues disappear under the perturbation of a potential (or of the metric) on a compact set. In this note, we are interested about the optimal decay at infinity of the perturbation given by a potential. Using the general result obtained in [3] and under a hypothesis of Fermi golden rule, one is only able to cover the assumption $VL^3 = o(1)$, as $y \rightarrow +\infty$, where L denotes the operator of multiplication by $L := (x, y) \mapsto 1 + \ln(y)$. We give the main result:

Theorem 1.1. *Let $k > 1/4$ be an L^2 -eigenvalue of Δ . Suppose that $VL = o(1)$, as $y \rightarrow +\infty$ and that the Fermi golden rule (1.1) holds true, then there is $\lambda_0 > 0$, so that H_λ has no eigenvalue in a neighborhood of k , for all $\lambda \in (0, |\lambda_0|)$. Moreover, if $VL^{1+\varepsilon} = o(1)$, as $y \rightarrow +\infty$ for some $\varepsilon > 0$, then H_λ has no singularly continuous spectrum.*

We believe that the hypothesis $VL = o(1)$ is optimal in the scale of L . In our approach, we use the Mourre theory, see [1, 12] and establish a positive commutator estimate.

2. IDEA OF THE PROOF

Standardly, for y large enough and up to some isometry \mathcal{U} , see for instance [6, 9, 10] the Laplace operator can be written as

$$(2.2) \quad \tilde{\Delta} = (-\partial_r^2 + 1/4) \otimes P_0 + \tilde{\Delta}(1 \otimes P_0^\perp)$$

on $\mathcal{C}_c^\infty((c, \infty), dr) \otimes \mathcal{C}^\infty(S^1)$, for some $c > 0$ and where P_0 is the projection on constant functions and $P_0^\perp := 1 - P_0$. The Friedrichs extension of the operator $\tilde{\Delta}(1 \otimes P_0^\perp)$ has compact resolvent.

Then, as in [9, 10], we construct a conjugate operator. One chooses $\Phi \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\Phi(x) = x$ on $[-1, 1]$, and sets $\Phi_\Upsilon(x) := \Upsilon \Phi(x/\Upsilon)$, for $\Upsilon \geq 1$. Let $\tilde{\chi}$ be a smooth cut-off function being 1 for r big enough and 0 for r being close to c . We define on $\mathcal{C}_c^\infty((c, \infty) \times S^1)$ a micro-localized version of the generator of dilations:

$$(2.3) \quad S_{\Upsilon,0} := \tilde{\chi} \left((\Phi_\Upsilon(-i\partial_r)r + r\Phi_\Upsilon(-i\partial_r)) \otimes P_0 \right) \tilde{\chi}.$$

The operator $\Phi_\Upsilon(-i\partial_r)$ is defined on the real line by $\mathcal{F}^{-1}\Phi_\Upsilon(\cdot)\mathcal{F}$, where \mathcal{F} is the unitary Fourier transform. We also denote its closure by $S_{\Upsilon,0}$ and it is self-adjoint. In [6] for instance, one does not use a micro-localization and one is not able to deal with really singular perturbation of the metric as in [9, 10].

Now, one obtains

$$[\partial_r^2, \tilde{\chi}(\Phi_\Upsilon r + r\Phi_\Upsilon)\tilde{\chi}] = 4\tilde{\chi}\partial_r\Phi_\Upsilon\tilde{\chi} + \text{remainder}.$$

Using a cut-off function $\tilde{\mu}$ being 1 on the cusp and 0 for $y \leq 2$, we set

$$(2.4) \quad S_\Upsilon := \mathcal{U}^{-1}S_{\Upsilon,0}\mathcal{U}\tilde{\mu}$$

This is self-adjoint in $L^2(X)$. Now by taking Υ big enough, one can show, as in [9, 10] that given an interval \mathcal{J} around k , there exist $\varepsilon_\Upsilon > 0$ and a compact operator K_Υ such that the inequality

$$(2.5) \quad E_{\mathcal{J}}(\Delta)[\Delta, iS_\Upsilon]E_{\mathcal{J}}(\Delta) \geq (4\inf(\mathcal{J}) - \varepsilon_\Upsilon)E_{\mathcal{J}}(\Delta) + E_{\mathcal{J}}(\Delta)K_\Upsilon E_{\mathcal{J}}(\Delta)$$

holds in the sense of forms, and such that ε_Υ tends to 0 as Υ goes to infinity. Here, $E_{\mathcal{J}}(\cdot)$ denotes the spectral measure above the interval \mathcal{J} .

Now, we apply \overline{P} to the left and right of (2.5). Easily one has $\overline{P}E_{\mathcal{J}}(\Delta) = \overline{P}E_{\mathcal{J}}(\Delta\overline{P})$. We get:

$$\begin{aligned} \overline{P}E_{\mathcal{J}}(\overline{P}\Delta)[\overline{P}\Delta, i\overline{P}S_{\Upsilon}\overline{P}]E_{\mathcal{J}}(\overline{P}\Delta)\overline{P} &\geq (4\inf(\mathcal{J}) - \varepsilon_{\Upsilon})\overline{P}E_{\mathcal{J}}(\Delta\overline{P})\overline{P} \\ &\quad + \overline{P}E_{\mathcal{J}}(\overline{P}\Delta)K_{\Upsilon}E_{\mathcal{J}}(\overline{P}\Delta)\overline{P} \end{aligned}$$

One can show that $\overline{P}S_{\Upsilon}\overline{P}$ is self-adjoint in $\overline{P}L^2(X)$ and that $[P\overline{\Delta}, \overline{P}S_{\Upsilon}\overline{P}]$ extends to a bounded operator.

We now shrink the size of the interval \mathcal{J} . As $\overline{P}\Delta$ has no eigenvalue in \mathcal{J} , then the operator $\overline{P}E_{\mathcal{J}}(\overline{P}\Delta)K_{\Upsilon}E_{\mathcal{J}}(\overline{P}\Delta)\overline{P}$ tends to 0 in norm. Therefore, by shrinking enough, one obtains a smaller interval \mathcal{J} containing k and a constant $c > 0$ so that

$$(2.6) \quad \overline{P}E_{\mathcal{J}}(\overline{P}\Delta)[\overline{P}\Delta, i\overline{P}S_{\Upsilon}\overline{P}]E_{\mathcal{J}}(\overline{P}\Delta)\overline{P} \geq c\overline{P}E_{\mathcal{J}}(\Delta\overline{P})\overline{P}$$

holds true in the form sense on $\overline{P}L^2(X)$. At least formally, the positivity on $\overline{P}L^2(X)$ of the commutator $[H_{\lambda}, i\overline{P}S_{\Upsilon}\overline{P}]$, up to some spectral measure and to some small λ , should be a general fact and should not rely on the Fermi golden rule hypothesis.

We now try to extract some positivity on $PL^2(X)$. First, we set

$$(2.7) \quad R_{\varepsilon} := ((H_0 - k)^2 + \varepsilon^2)^{-1/2}, \quad \overline{R}_{\varepsilon} := \overline{P}R_{\varepsilon} \text{ and } F_{\varepsilon} := \overline{R}_{\varepsilon}^2.$$

Note that $\varepsilon R_{\varepsilon}^2 = \text{Im}(H_0 - k + i\varepsilon)^{-1}$ and that R_{ε} commutes with P . Using (1.1), we get:

$$(2.8) \quad (c_1/\varepsilon)P \geq PV\overline{P}F_{\varepsilon}\overline{P}VP \geq (c_2/\varepsilon)P,$$

for $\varepsilon_0 > \varepsilon > 0$.

We follow an idea of [2], which was successfully used in [8, 11] and set

$$B_{\varepsilon} := \text{Im}(\overline{R}_{\varepsilon}^2 VP).$$

It is a finite rank operator. Observe now that we gain some positivity as soon as $\lambda \neq 0$:

$$(2.9) \quad P[H_{\lambda}, i\lambda B_{\varepsilon}]P = \lambda^2 PVF_{\varepsilon}VP \geq (c_2\lambda^2/\varepsilon)P.$$

It is therefore natural to modify the conjugate operator S_{Υ} to obtain some positivity on $PL^2(X)$. We set

$$(2.10) \quad \hat{S}_{\Upsilon} := \overline{P}S_{\Upsilon}\overline{P} + \lambda\theta B_{\varepsilon}.$$

It is self-adjoint on $\mathcal{D}(S_{\Upsilon})$ and is diagonal with respect to the decomposition $\overline{P}L^2(X) \oplus PL^2(X)$.

Here $\theta > 0$ is a technical parameter. We choose ε and θ , depending on λ , so that $\lambda = o(\varepsilon)$, $\varepsilon = o(\theta)$ and $\theta = o(1)$ as λ tends to 0. We summarize this into:

$$(2.11) \quad |\lambda| \ll \varepsilon \ll \theta \ll 1, \text{ as } \lambda \text{ tends to } 0.$$

With respect to the decomposition $\overline{P}E_{\mathcal{J}}(\Delta) \oplus PE_{\mathcal{J}}(\Delta)$, as λ goes to 0, we have

$$\begin{aligned} E_{\mathcal{J}}(\Delta) [\lambda V, i\overline{P}S_{\mathcal{I}}\overline{P}] E_{\mathcal{J}}(\Delta) &= \begin{pmatrix} O(\lambda) & O(\lambda) \\ O(\lambda) & 0 \end{pmatrix}, \\ E_{\mathcal{J}}(\Delta) [\Delta, i\lambda\theta B_{\varepsilon}] E_{\mathcal{J}}(\Delta) &= \begin{pmatrix} 0 & O(\lambda\theta\varepsilon^{-1/2}) \\ O(\lambda\theta\varepsilon^{-1/2}) & 0 \end{pmatrix}, \\ \text{and } E_{\mathcal{J}}(\Delta) [\lambda V, i\lambda\theta B_{\varepsilon}] E_{\mathcal{J}}(\Delta) &= \begin{pmatrix} O(\lambda^2\theta\varepsilon^{-3/2}) & O(\lambda^2\theta\varepsilon^{-3/2}) \\ O(\lambda^2\theta\varepsilon^{-3/2}) & \lambda^2\theta F_{\varepsilon} \end{pmatrix}. \end{aligned}$$

Now comes the delicate point. Under the condition (2.11) and by choosing \mathcal{I} , slightly smaller than \mathcal{J} , we use the previous estimates and a Schur Lemma to deduce:

$$(2.12) \quad E_{\mathcal{I}}(H_{\lambda})[H_{\lambda}, i\hat{S}_{\mathcal{I}}]E_{\mathcal{I}}(H_{\lambda}) \geq \frac{c\lambda^2\theta}{\varepsilon}E_{\mathcal{I}}(H_{\lambda}),$$

for some positive c and as λ tends to 0.

We mention that only the decay of VL is used to establish the last estimate. In fact, one uses that $[V, i\hat{S}_{\mathcal{I}}](\Delta + 1)^{-1}$ is a compact operator.

Now it is a standard use of the Mourre theory to deduce Theorem 1.1 and refer to [1], see [9, 10] for some similar application of the theory. For the absence of eigenvalue, one relies on the fact that given an eigenfunction f of H_{λ} w.r.t. an eigenvalue $\kappa \in \mathcal{I}$, one has:

$$(2.13) \quad \langle f, [H_{\lambda}, i\hat{S}_{\mathcal{I}}]f \rangle = \langle f, [H_{\lambda} - \kappa, i\hat{S}_{\mathcal{I}}]f \rangle = 0.$$

Then, one applies f on the right and on the left of (2.12) and infers that $f = 0$ thanks to the fact that the constant $c\lambda^2\theta$ is non-zero.

In [9, 10], we prove that the C_0 -group $(e^{iS_{\mathcal{I}}t})_{t \in \mathbb{R}}$ stabilizes the domain $\mathcal{D}(H_{\lambda}) = \mathcal{D}(\Delta)$. By perturbation, we prove that this is also the case for $(e^{i\hat{S}_{\mathcal{I}}t})_{t \in \mathbb{R}}$. Thanks to this property, we can expand the commutator of (2.13) in a legal way. This is known as the Virial theorem in the Mourre Theory, see [1, 12].

REFERENCES

- [1] W. Amrein, A. Boutet de Monvel and V. Georgescu, *C₀-Groups, commutator methods and spectral theory of N-body Hamiltonians*, Birkhäuser, Basel-Boston-Berlin, 1996.
- [2] V. Bach, J. Fröhlich, I.M. Sigal and A. Soffer: *Positive commutators and Spectrum of Pauli-Fierz Hamiltonian of Atoms and Molecules*, Commun. Math. Phys. **207** (3), 1999, 557–587.
- [3] L. Cattaneo, G.M. Graf, and W. Hunziker *A general resonance theory based on Mourre's inequality* Ann. Henri Poincaré 7, No. 3, 583–601 (2006).
- [4] Y. Colin de Verdière, *Pseudo-laplaciens II*, Ann. Inst. Fourier 33, No.2, 87–113 (1983).
- [5] J. Fischer, *An approach to the Selberg trace formula via the Selberg zeta-function*, Lecture Notes in Mathematics, **1253**. Springer-Verlag, Berlin, 1987.
- [6] R. Froese and P. D. Hislop, *Spectral analysis of second-order elliptic operators on noncompact manifolds*, Duke Math. J. **58** (1989), No. 1, 103–129.
- [7] V. Georgescu and S. Golénia, *Decay Preserving Operators and stability of the essential spectrum*, J. Oper. Theory. **59** (2008), no. 1, 115–155.
- [8] S. Golénia: *Positive commutators, Fermi golden rule and the spectrum of the zero temperature Pauli-Fierz hamiltonians*. J. of Funct. Anal., Vol. 256, Issue 8, April 2009, 2587–2620.

- [9] S. Golénia and S. Moroianu, *Spectral analysis of magnetic Laplacians on conformally cusp manifolds*, Ann. H. Poincaré **9** (2008), 131–179.
- [10] S. Golénia and S. Moroianu, *The spectrum of Schrödinger operators and Hodge Laplacians on conformally cusp manifolds*. To appear in Trans. A.M.S.
- [11] M. Merlki, *Positive Commutators in Non-Equilibrium Statistical Mechanics*, Comm. Math. Phys. **223**, 327–362 (2001).
- [12] E. Mourre, *Absence of singular continuous spectrum for certain self-adjoint operators*, Comm. Math. Phys. **91** (1981) 391–408.
- [13] M. Reed and B. Simon, *Methods of modern mathematical physics I-IV: Fourier analysis, self-adjointness*, Academic Press, New York – San Francisco – London, 1975.

INSTITUT DE MATHÉMATIQUES DE BORDEAUX, UNIVERSITÉ BORDEAUX 1, 351, COURS DE LA LIBRA-
TION, 33405 TALENCE CEDEX, FRANCE

E-mail address: sylvain.golenia@u-bordeaux1.fr