Uniting two Control Lyapunov Functions for affine systems

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Abstract

The problem of piecing together two Control Lyapunov Functions (CLFs) is addressed. The first CLF characterizes a local asymptotic controllability property toward the origin, whereas the second CLF is related to a global asymptotic controllability property with respect to a compact set. A sufficient condition is expressed to obtain an explicit solution. This sufficient condition is shown to be always satisfied for a linear second order controllable system. In a second part, it is shown how this uniting CLF problem can be used to solve the problem of piecing together two stabilizing control laws. Finally, this framework is applied on a numerical example to improve local performance of a globally stabilizing state feedback.

I. Introduction

Smooth Control Lyapunov Functions (CLFs) are instrumental in many feedback control designs and can be traced back to Artstein who introduced this Lyapunov characterization of asymptotic controllability in [4]. For instance, one of the useful characteristic of smooth CLFs is the existence of *universal formulas* for stabilization of nonlinear affine (in the control) systems (see [5], [7]). Numerous tools for the design of global CLF are now available (for instance by backstepping [6], or by forwarding [9], [14]). On another hand, via linearization (or other local approaches), one may design local CLF yielding locally stabilizing controllers. This leads to the idea of uniting a local CLF with a global CLF. In Section II a sufficient condition to piece together a pair of CLFs is given.

This issue is closely related to the ability to piece together a local controller and a global one. This problem of unification of control laws was introduced in [16]. It has been subsequently

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developed in [11] where this problem has been solved by considering controllers with continuous and discrete dynamics (namely hybrid controller). As shown in Section III below, solving the uniting CLF problem provides a simple solution to the uniting control problem without employing discrete dynamics. Some related results concerning the unification of different controllers can be found in [13], [17] where hybrid controllers are used, or in [1] where the patchy feedbacks design has been studied.

A numerical example is given in Section IV showing how this framework can be used to modify the local behavior of the trajectories of a nonlinear system in order to minimize a cost function. In contrast to the solution by means of hybrid controllers (see e.g. [12]), this approach allows the design of a continuous global control and locally optimal.

II. Problem statement and main result

A. Problem formulation

The nonlinear systems under consideration in this paper have the following form:

$$\dot{x} \;=\; f(x) + g(x)\,u \;, \tag{1}$$

where x in $\mathbb{R}^n$ is the state, u in $\mathbb{R}^p$ is the control input, and $f:\mathbb{R}^n \to \mathbb{R}^n$ and $g\;:\mathbb{R}^n \to \mathbb{R}^p$ are locally Lipschitz functions such that $f(0) = 0$.

For system (1), two CLFs V_0 and V_∞ satisfying the Artstein condition (see [4]) on specific sets are given. More precisely, the following assumption holds.

Assumption 1: *There exist a positive definite and continuously differentiable function $V_0 : \mathbb{R}^n \to \mathbb{R}_+$, a positive semi-definite, proper and continuously differentiable function $V_\infty\, :\mathbb{R}^n \to \mathbb{R}_+$, and positive values R_0 and r_∞ such that :*

- Local CLF: $\{x : 0 < V_0(x) \leq R_0, L_gV_0(x) = 0\} \;\subseteq\; \{x : L_fV_0(x) \;<\; 0\}$; (2)

- Non-local CLF: $\{x : V_\infty(x) \geq r_\infty,\; L_gV_\infty(x) = 0\} \;\subseteq\; \{x : L_fV_\infty(x) \;<\; 0\}$; (3)

- Covering assumption: $\{x : V_\infty(x) > r_\infty\} \cup \{x : V_0(x) < R_0\} \;=\; \mathbb{R}^n$.

The function V_∞ characterizes the global asymptotic controllability toward the set $\{x : V_\infty \leq R_0\}$ for system (1). Hence, this function is proper but not necessarily positive definite.

Roughly speaking the Covering assumption means that the two sets, in which the asymptotic controllability property holds (the two sets in which each CLF satisfies Artstein condition), overlap and cover the entire domain.

The problem addressed in this paper can be formalized as follows:

Uniting CLF problem: The uniting CLF problem is to find a proper, positive definite and continuously differentiable function $V:\mathbb{R}^n \to \mathbb{R}_+$ such that:

- *Global CLF:* $\{x : x \neq 0,\, L_gV(x) = 0\} \subseteq \{x : L_fV(x) < 0\}$; (4)

- *Local property:* $\{x : V_\infty(x) \leq r_\infty\} \subseteq \{x : V(x) = r_\infty V_0(x)\}$; (5)

- *Non-local property:* $\{x : V_0(x) \geq R_0\} \subseteq \{x : V(x) = R_0 V_\infty(x)\}$. (6)

As shown in Section III, one of the main interest of solving the uniting CLF problem is that it provides a way to piece together (continuously) some specific stabilizing controllers.

B. A sufficient condition and a constructive theorem

The first result establishes that, with the following additional assumption, the existence of a solution to the uniting CLF problem is obtained.

Assumption 2: *Given two positive values r_∞ and R_0 and two functions $V_0:\mathbb{R}^n \to \mathbb{R}_+$ and $V_\infty:\mathbb{R}^n \to \mathbb{R}_+$, for all x in $\{x : V_\infty(x) > r_\infty,\, V_0(x) < R_0\}$, the following implication holds:*

$$\exists\, \lambda_x > 0 : L_gV_0(x) = -\lambda_x L_gV_\infty(x) \;\Rightarrow\; L_fV_0(x)|L_gV_\infty(x)| + L_fV_\infty(x)|L_gV_0(x)| < 0\ . \tag{7}$$

The first result can now be stated.

Theorem 2.1: *Under Assumptions 1 and 2, there exists a solution to the uniting CLF problem. More precisely, the function $V:\mathbb{R}^n \to \mathbb{R}_+$ defined, for all x in $\mathbb{R}^n$, by*

$$V(x) \;=\; R_0\Big[\varphi_0(V_0(x)) + \varphi_\infty(V_\infty(x))\Big]V_\infty(x) \;+\; r_\infty\Big[1 - \varphi_0(V_0(x)) - \varphi_\infty(V_\infty(x))\Big]V_0(x)\ , \tag{8}$$

where $\varphi_0:\mathbb{R}_+ \to [0,1]$ and $\varphi_\infty:\mathbb{R}_+ \to [0,1]$ are two continuously differentiable non-decreasing functions satisfying[1]*:*

$$\varphi_0(s)\begin{cases} =\ 0 & \forall s \leq r_0 \\ >\ 0 & \forall r_0 < s < R_0 \\ =\ \frac{1}{2} & \forall s \geq R_0 \end{cases}\ ,\ \varphi_\infty(s)\begin{cases} =\ 0 & \forall s \leq r_\infty \\ >\ 0 & \forall r_\infty < s < R_\infty \\ =\ \frac{1}{2} & \forall s \geq R_\infty \end{cases}\ , \tag{11}$$

[1]For instance, φ_0 and φ_∞ can be defined as:

$$\varphi_0(s) \;=\; \frac{3}{2}\left(\frac{s-r_0}{R_0-r_0}\right)^2 - \left(\frac{s-r_0}{R_0-r_0}\right)^3 \quad , \qquad s \in [r_0, R_0]\ , \tag{9}$$

$$\varphi_\infty(s) \;=\; \frac{3}{2}\left(\frac{s-r_\infty}{R_\infty-r_\infty}\right)^2 - \left(\frac{s-r_\infty}{R_\infty-r_\infty}\right)^3 \quad , \qquad s \in [r_\infty, R_\infty]\ . \tag{10}$$

and where $r_0 = \max_{\{x:V_\infty(x)\leq r_\infty\}} V_0(x)$ *and*[2] $R_\infty = \min_{\{x:V_0(x)\geq R_0\}} V_\infty(x)$, *is a proper, positive definite continuously differentiable function satisfying (4), (5), and (6).*

The structure of the function V is inspired by the construction given in [2].

Proof: The first part of the proof is devoted to show that the positive real numbers r_0 and R_∞ are properly defined. Indeed, the function V_∞ being positive semi-definite and proper, the set $\{x : V_\infty(x) \leq r_\infty\}$ is a non empty compact subset and r_0 can be properly defined. For R_∞, two cases need to be considered:

- If $\{x : V_0(x) \geq R_0\} \neq \emptyset$, pick any element x^* in $\{x : V_0(x) \geq R_0\}$. Since the function V_∞ is proper, it yields that $\{x : V_\infty(x) \leq V(x^*)\}$ is a compact set and $\min_{\{x:V_0(x)\geq R_0\}} V_\infty(x) = \min_{\{x:V_0(x)\geq R_0, V_\infty(x)\leq V(x^*)\}} V_\infty(x)$. Therefore in this case, R_∞ can be defined.
- In the case where $\{x : V_0(x) \geq R_0\} = \emptyset$, let R_∞ be any positive real number such that $R_\infty > r_\infty$.

Note that with the Covering assumption, it yields that:

$$r_0 < R_0 \quad , \qquad r_\infty < R_\infty \ . \tag{12}$$

Indeed if one of the two inequalities in (12) is not satisfied then this implies the existence of x^* in $\mathbb{R}^n$ such that $V_\infty(x^*) \leq r_\infty$ and $V_0(x^*) \geq R_0$ and consequently x^* is not in the set $\{x : V_\infty(x) > r_\infty\} \cup \{x : V_0(x) < R_0\}$ which contradicts the Covering assumption.

The function V_0 being positive definite and the function V_∞ being proper, it can be checked that V is positive definite and proper. Moreover it satisfies the local and asymptotic properties given in Equations (5) and (6).

It remains to show that V satisfies the Artstein condition for all x in $\mathbb{R}^n \setminus \{0\}$. Note that the functions V_0 and V_∞ satisfying the implications (2) and (3), it yields that the function V satisfies the Artstein condition on the set $\{x : V_0(x) \geq R_0\} \cup \{x \neq 0 : V_\infty(x) \leq r_\infty\}$.

Note that in the set $\{x : V_0(x) < R_0 \, , \, V_\infty(x) > r_\infty\}$, the following inequality holds:

$$R_0 V_\infty(x) - r_\infty V_0(x) \ > \ 0 \ . \tag{13}$$

Furthermore,

$$\begin{aligned} L_f V(x) &= A(x) L_f V_0(x) + B(x) L_f V_\infty(x) \ , \\ L_g V(x) &= A(x) L_g V_0(x) + B(x) L_g V_\infty(x) \ , \end{aligned}$$

[2]In the case where $\{x : V_0(x) \geq R_0\} = \emptyset$ let R_∞ be such that $R_\infty > r_\infty$.

where the continuous functions $A : \mathbb{R}^n \to \mathbb{R}_+$ and $B : \mathbb{R}^n \to \mathbb{R}_+$ are defined as, for all x in $\mathbb{R}^n$,

$$A(x) = [R_0 V_\infty(x) - r_\infty V_0(x)]\, \varphi_0'(V_0(x)) + r_\infty\, [1 - \varphi_0(V_0(x)) - \varphi_\infty(V_\infty(x))] \ ,$$

$$B(x) = [R_0 V_\infty(x) - r_\infty V_0(x)]\, \varphi_\infty'(V_\infty(x)) + R_0\, [\varphi_0(V_0(x)) + \varphi_\infty(V_\infty(x))] \ .$$

In the set $\{x : V_0(x) < R_0\, ,\, V_\infty(x) > r_\infty\}$ it holds that $A(x) > 0$ and $B(x) > 0$. Suppose there exists x^* in this set such that $L_g V(x^*) = 0$. Two cases have to be considered:

- If $L_g V_0(x^*) = 0$, then $L_g V_\infty(x^*) = 0$, and since V_0 and V_∞ satisfy the Artstein condition, this implies that $L_f V(x^*) < 0$;
- If $L_g V_0(x^*) \neq 0$, this implies:

$$L_g V_0(x^*) = -\frac{B(x^*)}{A(x^*)} L_g V_\infty(x^*) \ , \tag{14}$$

  and

$$A(x^*) = \frac{B(x^*)\, |L_g V_\infty(x^*)|}{|L_g V_0(x^*)|} \ .$$

  Consequently,

$$L_f V(x^*) = \frac{B(x^*)}{|L_g V_0(x^*)|} \Big[L_f V_0(x^*)\, |L_g V_\infty(x^*)| + L_f V_\infty(x)\, |L_g V_0(x^*)| \Big] \ ,$$

  and with (14) and Assumption 2, it yields $L_f V(x^*) < 0$.

Hence, the function V satisfies the Artstein condition for all x in $\mathbb{R}^n \setminus \{0\}$. This concludes the proof of Theorem 2.1. ■

C. About Assumption 2

Another formulation of Assumption 2 can be given as stated in the following proposition the proof of which can be found in [3].

Proposition 2.2: *Given two continuously differentiable functions $V_0 : \mathbb{R}^n \to \mathbb{R}_+$ and $V_\infty : \mathbb{R}^n \to \mathbb{R}_+$, and a state x in $\mathbb{R}^n \setminus \{0\}$ such that Artstein condition is satisfied for both functions, the implication (7) is equivalent to the existence of a control u_x in $\mathbb{R}^p$ such that:*

$$L_f V_0(x) + L_g V_0(x)\, u_x < 0 \ , \ L_f V_\infty(x) + L_g V_\infty(x)\, u_x < 0 \ . \tag{15}$$

III. Application to the design of a uniting controller

Theorem 2.1 can be used to design stabilizing controllers with a prescribed behavior around the equilibrium, and another behavior for large values of the state. In other words Theorem 2.1 gives a solution to the uniting control problem. This problem has been introduced in [16] and further developed in [11]. In the present context, the following theorem is obtained.

Theorem 3.1: *Consider two functions $V_0 : \mathbb{R}^n \to \mathbb{R}_+$ and $V_\infty : \mathbb{R}^n \to \mathbb{R}_+$ and two positive real numbers R_0 and r_∞ satisfying Assumptions 1 and 2. Assume that V_0 is proper. For any continuous function $\phi_0 : \mathbb{R}^n \to \mathbb{R}^p$ satisfying, for all x in $\{x : 0 < V_0(x) \leq R_0\}$,*

$$L_f V_0(x) + L_g V_0(x) \phi_0(x) < 0 \,, \tag{16}$$

and any continuous function $\phi_\infty : \mathbb{R}^n \to \mathbb{R}^p$ satisfying for all x in $\{x : V_\infty(x) \geq r_\infty\}$

$$L_f V_\infty(x) + L_g V_\infty(x) \phi_\infty(x) < 0 \,, \tag{17}$$

there exists a continuous function $\phi : \mathbb{R}^n \to \mathbb{R}^p$ which solves the uniting controller problem, i.e. such that

1) *$\phi(x) = \phi_0(x)$ for all x such that $V_\infty(x) \leq r_\infty$;*
2) *$\phi(x) = \phi_\infty(x)$ for all x such that $V_0(x) \geq R_0$;*
3) *the origin of the system $\dot{x} = f(x) + g(x)\phi(x)$ is a globally asymptotically stable equilibrium.*

The idea of the proof is to design a controller which is a continuous path going from $\phi_0(x)$ for x small to $\phi_\infty(x)$ for larger values of the state. The good behavior of the trajectories in between is ensured by adding a sufficiently large term which depends on the uniting control Lyapunov function. More precisely, the function $\phi : \mathbb{R}^n \to \mathbb{R}^m$ obtained from Theorem 3.1 and which is a solution to the uniting controller problem is defined as

$$\phi(x) = H(x) - k\, c(x) L_g V(x)^T \ , \forall x \in \mathbb{R}^n \,, \tag{18}$$

where $V : \mathbb{R}^n \to \mathbb{R}_+$ is the Control Lyapunov Function obtained from Theorem 2.1, and with $H(x) = \gamma(x)\phi_0(x) + [1 - \gamma(x)]\phi_\infty(x)$ where γ is any continuous function[3] such that

$$\gamma(x) = \begin{cases} 1 & \text{if } V_\infty(x) \leq r_\infty \,, \\ 0 & \text{if } V_0(x) \geq R_0 \,, \end{cases}$$

[3]For instance, giving φ_0 and φ_∞ defined in (11), a possible choice is: $\gamma(x) = 1 - \varphi_0(V_0(x)) - \varphi_\infty(V_\infty(x))$. (19)

and the function c is any continuous function such that[4]

$$c(x)\begin{cases} = 0 & \text{if } V_0(x) \geq R_0 \text{ or } V_\infty(x) \leq r_\infty \ , \\ > 0 & \text{if } V_0(x) < R_0 \text{ and } V_\infty(x) > r_\infty \ , \end{cases} \tag{21}$$

and k is a positive real number sufficiently large to ensure that V is a Lyapunov function of the closed-loop system. The existence of k is obtained employing compactness arguments (see analogous arguments in [2, Lemma 2.13]).

Proof: Note that the function ϕ satisfies item 1) and 2) of Theorem 3.1. It remains to show item 3). Taking the function V as a candidate Lyapunov function obtained in (8), the continuous function $\mathring{V} : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}$ can be introduced as, for all (x,k) in $\mathbb{R}^n \times \mathbb{N}$,

$$\mathring{V}(x,k) \ = \ \frac{\partial V}{\partial x}(x)f(x) \ + \ \frac{\partial V}{\partial x}(x)g(x)H(x) \ - \ k\,c(x)\left|\frac{\partial V}{\partial x}(x)g(x)\right|^2 \ . \tag{22}$$

With the local and non-local properties of the function V (as stated in (5) and (6) respectively), for all x in $\{x \neq 0 : V_\infty(x) \ \leq \ r_\infty \text{ or } V_0(x) \ \geq \ R_0\}$ and all k in $\mathbb{N}$,

$$\mathring{V}(x,k) < 0 \ . \tag{23}$$

It is now shown that if k is selected sufficiently large then we have the negativeness of $\mathring{V}(x,k)$ for all $x \neq 0$. To prove that, suppose the assertion is wrong and suppose for each k in $\mathbb{N}$, there exists x_k in $\mathbb{R}^n \setminus \{0\}$ such that

$$\mathring{V}(x_k,k) \geq 0 \quad , \qquad \forall\, k \ \in \ \mathbb{N} \ . \tag{24}$$

First note that with (23), for all k, x_k is in the set $\{x : V_\infty(x) \geq r_\infty\} \cap \{x : V_0(x) \leq R_0\}$ which is compact since V_0 is proper and V_∞ is continuous. With (22) and (24), it yields that,

$$0 \leq c(x_k)\left|L_gV(x_k)\right|^2 \leq \frac{M}{k} \ , \tag{25}$$

with $M = \max_{\{x : V_\infty(x) \geq r_\infty\} \cap \{x : V_0(x) \leq R_0\}} \left\{L_fV(x) + L_gV(x)H(x)\right\}$. Moreover, $(x_k)_{k\in\mathbb{N}}$ is a sequence living in a compact set, thus there exists a subsequence $(x_{k_\ell})_{\ell\in\mathbb{N}}$ which converges to a point denoted $x^* \neq 0$. With (25), it implies that $c(x^*)\left|L_gV(x^*)\right|^2 = 0$ and consequently,

$$\mathring{V}(x^*,k) = w(x^*)$$

where $w(x^*) = \frac{\partial V}{\partial x}(x^*)f(x^*) \ + \ \frac{\partial V}{\partial x}(x^*)g(x^*)H(x^*)$. From the fact that $c(x^*)\left|L_gV(x^*)\right|^2 = 0$, two cases may be distinguished,

[4]For instance, a possible choice is $c(x) = \max\{0, (R_0 - V_0(x))(V_\infty(x) - r_\infty)\}$ (20)

- if $L_gV(x^*) = 0$, by the Artstein condition, then $L_fV(x^*) < 0$ and thus $w(x^*) < 0$;
- if $c(x^*) = 0$, then by (21), x^* is in the set $\{x \neq 0 : V_\infty(x) \leq r_\infty \text{ or } V_0(x) \geq R_0\}$. With (23), this implies $w(x^*) < 0$.

Since the function w is continuous at x^*, $w(x^*) < 0$, and the sequence $(x_{k_\ell})_{\ell \in \mathbb{N}}$ converges to x^*, there exists ℓ_∞ such that, for all $\ell > \ell_\infty$, $\mathring{V}(x_{k_\ell}, k_\ell) \leq w(x_{k_\ell}) < 0$. This contradicts (24). Therefore there exists $k > 0$ such that (23) is satisfied for all $x \neq 0$. Hence, item 3) is also satisfied. ■

This theorem shows that as soon as the uniting CLF problem is solved, a continuous solution to the uniting controller problem is obtained. Note also, that if discontinuous controllers with discrete dynamics (not only continuous static controllers) are allowed, the existence of a hybrid controller solving the problem is obtained under Assumption 1 only (see [11], [13]).

IV. Illustration on an example

To illustrate the interest of the uniting controller solution developed in this paper, a numerical example is provided in this section. Consider the nonlinear system (1) when $n = 3$, $p = 1$, and the vector fields f and g defined by, for all $x = (x_1, x_2, x_3)$ in $\mathbb{R}^3$,

$$f(x) = \begin{bmatrix} -x_1 + x_3 \\ x_1^2 - x_2 - 2x_1x_3 + x_3 \\ -x_2 \end{bmatrix} \quad , \qquad g(x) = Gx \quad , \qquad G = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} . \tag{26}$$

Let V_∞ be the continuously differentiable positive definite and proper function defined by

$$V_\infty(x_1, x_2, x_3) = x_1^2 + \left(x_1^2 + x_2\right)^2 + x_3^2 \ , \ \forall x \in \mathbb{R}^3 \ .$$

Along the vector fields f and g defined in (26), the Lie derivatives of the function V_∞ is

$$L_fV_\infty(x) = -2x_1^2 - 2\left(x_1^2 + x_2\right)^2 + 2x_3\left(x_1^2 + x_1\right) \ , \ L_gV_\infty(x) = 2x_3 \ , \ \forall x \in \mathbb{R}^3 \ .$$

Note that, for all x in $\mathbb{R}^3 \setminus \{0\}$, the Artstein condition is satisfied (i.e. $L_gV_\infty(x) = 0 \Rightarrow L_fV_\infty < 0$). Consequently, V_∞ is a global CLF and the control law $u = \phi_\infty(x)$ with

$$\phi_\infty(x) = -x_1^2 - x_1 - x_3 \quad , \qquad \forall x \in \mathbb{R}^3 \ , \tag{27}$$

is such that, along the trajectories of the system (1) in closed loop with ϕ_∞, for all x in $\mathbb{R}^3$, $\dot{\overbrace{V_\infty(x)}} \leq -2x_1^2 - 2\left(x_1^2 + x_2\right)^2 - 2x_3^2$. Hence the function ϕ_∞ defined in (27) ensures global asymptotic stability of the origin of the system defined in (1) and (26).

Note however that despite the global asymptotic stability of the origin is obtained with this control law, there is no guarantee that the performance obtained is satisfactory. For instance, it may be interesting that the controller locally minimizes a criterium defined as the limit, when $t \to \infty$, of the operator $J : L^2\left(\mathbb{R}_+;\mathbb{R}^3\right) \times L^2\left(\mathbb{R}_+;\mathbb{R}\right) \times \mathbb{R}_+ \to \mathbb{R}_+$ defined by, for all (x,u,t) in $L^2\left(\mathbb{R}_+;\mathbb{R}^3\right) \times L^2\left(\mathbb{R}_+;\mathbb{R}\right) \times \mathbb{R}_+$,

$$J(x,u,t) \;=\; \int_0^t x(s)^T Q x(s) + R u(s)^2 ds \;, \tag{28}$$

where Q is a symmetric positive definite matrix in $\mathbb{R}^3$ and R is a positive real number.

The techniques developed in this paper may be instrumental to modify the stabilizing controller $u = \phi_\infty$ such that the criterium J is minimized around the origin. A similar problem has been addressed in [10] where a general cost function depending on exogenous disturbances is considered. In [10], using a backstepping approach for upper triangular systems, a controller, which matches the optimal control law up to a desired order, is extended to a global stabilizer. In the uniting CLF approach, the global controller is computed independently from the optimal problem and an upper triangular structure is not required. However an assumption (namely Assumption 2) is needed. Using the first order approximation, this assumption can be rewritten in terms of an LMI (see Proposition 4.1 below).

The first order approximation around the origin of system (1) with f and g defined in (26) is

$$\dot{x} \;=\; F\,x \;+\; G\,u \quad , \qquad F \;=\; \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix} . \tag{29}$$

The system (29) being linear, an LQ controller minimizing the criterium defined in (28), is given by $\phi_0(x) = -R^{-1} G^T P_0 x$, where P_0 is the symmetric positive definite solution of the Riccati equation:

$$P_0 F \,+\, F^T P_0 - P_0 G R^{-1} G^T P_0 + Q \;=\; 0 \;. \tag{30}$$

The tools developed in this paper provides a sufficient condition guaranteeing the existence of a continuous state feedback $u = \phi(x)$ which unites the optimal local controller ϕ_0 and the global one ϕ_∞ while ensuring global asymptotic stability of the origin. Indeed, this proposition can be obtained (its proof is given in [3]).

Proposition 4.1: *Assume there exists a matrix K in $\mathbb{R}^{1\times 3}$ satisfying the following LMI:*

$$\left\{\begin{array}{lcl}(F+GK)^T P_0 + P_0(F+GK) & < & 0\ ,\\ (F+GK)^T P_\infty + P_\infty(F+GK) & < & 0\ ,\end{array}\right. \tag{31}$$

where $P_\infty = diag\,(1,1,1)$. Then there exists a continuous function $\phi : \mathbb{R}^3 \to \mathbb{R}$ such that the control law $u = \phi(x)$ makes the origin of the system (1) a globally asymptotically stable equilibrium and such that $\phi(x) = \phi_0(x)$ in a neighborhood of the origin.

For the numerical illustration, the matrix Q is randomly selected as :

$$Q = \begin{bmatrix} 0.8 & 0.6 & 0.3 \\ \star & 0.6 & 0.5 \\ \star & \star & 1 \end{bmatrix}\ , \tag{32}$$

and $R = 1$. The matrix P_0 and the optimal local controller $\phi_0 = K_0 x$ obtained solving the associated Riccati equation can be computed employing the *care* routine in *Matlab* :

$$P_0 = \begin{bmatrix} 0.3389 & 0.1412 & 0.3496 \\ \star & 0.3870 & -0.0912 \\ \star & \star & 1.2316 \end{bmatrix}\ , \qquad K_0 = \begin{bmatrix} 0.3496 & -0.0912 & 1.2316 \end{bmatrix}\ . \tag{33}$$

Employing the *Matlab* package *Yalmip* ([8]) in combination with the solver *Sedumi* ([15]), it can be checked[5] that the LMI condition (31) is satisfied for a particular K in $\mathbb{R}^{1\times 3}$. Consequently, Proposition 4.1 applies and a controller which unites the optimal local one ϕ_0 and the global one ϕ_∞ can be constructed.

The uniting controller is given in (18) where the uniting CLF V is obtained from Theorem 2.1, and the functions φ_0, φ_∞, γ, and c are respectively defined by (9), (10), (19) and (20), with the following tuning parameters $R_0 = 0.88$, $r_\infty = 0.35$, $r_0 = 0.4739$, $R_\infty = 0.65$ and $k = 1$.

Figure 1 compares the time-evolution of the cost J defined in (28) when considering the nominal control law $u = \phi_\infty$ and the uniting one $u = \phi$, with the initial condition $x(0) = [1\ 1\ 1]^T$. Figure 2 shows the time-evolution of the control values $u = \phi$. With this approach, there is no guarantee that, for all initial conditions, the cost obtained employing the uniting controller will be lower than the one obtained using the global one. More precisely, there exist initial conditions for which the use of the interpolation between both controllers affects too strongly the cost.

[5]The Matlab files can be downloaded from `http://homepages.laas.fr/~vandrieu/Publication.htm`

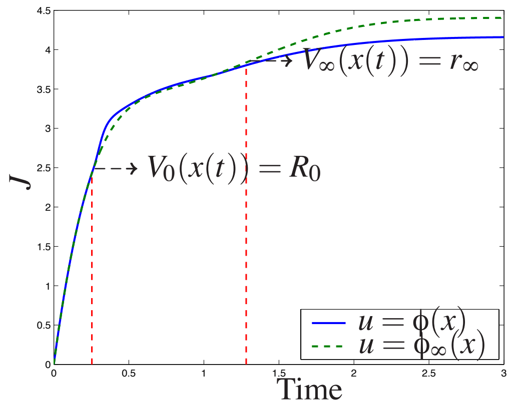

Fig. 1. Time-evolution of the cost function J with the controls ϕ (in plain line) and ϕ_∞ (in dashed line).

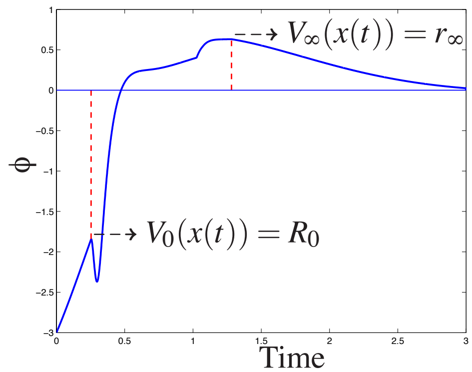

Fig. 2. Time-evolution of the uniting controller ϕ.

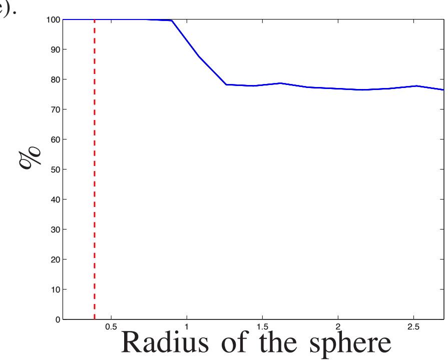

Fig. 3. Percentage of initial conditions for which the cost with the uniting controller ϕ is better than the global controller ϕ_∞. The left part of the dashed line is included in the set $\{x, V_\infty(x) \leq r_\infty\}$.

To check if the uniting controller is statistically better than the global one, a set of initial conditions is considered. This set is uniformly distributed on spheres with different radius. Figure 3 plots the percentage of initial conditions for which the cost has been improved when using the uniting controller. For more than 75% of initial conditions the cost is lower with the uniting controller than with the global controller. Note that for small radius, the corresponding initial conditions are inside the set $\{x, V_\infty(x) \leq r_\infty\}$ and consequently the uniting controller is exactly the optimal one. Hence, it is not surprising that the percentage of improvement is 100%.

V. Conclusion

In this paper, the problem of piecing together two Control Lyapunov Functions is considered. Solving this one provides a simple solution to the uniting controllers problem. Two characterizations of a sufficient condition guaranteeing the solvability of the united CLF problem are given. As shown on a numerical illustration, it allows to exhibit a sufficient condition to improve the qualitative behavior of the trajectories of nonlinear systems around the equilibrium.

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