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Basic Structure of Inconsistent Mathematics

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Dedicated to Marie-Louise Nykamp

Abstract

Contrary to the general awareness, we have been heavily involved in, and also dependent upon, inconsistent mathematics ever since the introduction of electronic digital computers in the 1940s. Here, a basic general structure is presented for various inconsistent mathematical theories. This structure, which is a significant extension of the one in [3], offers one of the effective ways for a safe use of inconsistent mathematical theories.

1. Amusing Example of Basic Failure of General Human Awareness

It may not be so easy to find a more blatant example of rather blindly rushing into a major new venture without a proper theoretical understanding of some of its critically important features, than it has been the case with our more and more massive use of electronic digital computers ever since the 1940s. Indeed, even when considered to operate only on the natural integers, our electronic digital computers function

without any exception whatsoever - and at that, inevitably so - upon the following *trivially inconsistent* system of axioms which we denote by (PA-MI) :

- (PA) : the usual Peano Axioms for \mathbb{N} ,

plus the *ad-hock axiom*, according to which :

- (MI) : there exists $M \in \mathbb{N}$, $M \gg 1$, such that $M + 1 = M$

Such an M , called “machine infinity”, is usually larger than 10^{100} , however, it is inevitably inherent in every electronic digital computer, due to obvious unavoidable physical limitations. And clearly, the above mix of (PA) and (MI) axioms, a mix which we denoted by (PA-MI), is *inconsistent*. Yet we do not mind, among others, flying on planes designed and built with the use of such electronic digital computers.

Needless to say, there has from the beginning been a keen awareness of the undesirable consequences of the inevitable presence of such a “machine infinity” when operating an electronic digital computer. And the consequent care has always been taken to avoid computations which may reach “overflow” or “underflow”.

On the other hand, ever since the 1940s, it has for some decades not been realized in the least the *major novelty* on theoretical level which got inaugurated in mathematics at large by the more and more massive use of the inconsistent (PA-MI) system of axioms. Indeed, the general human awareness simply failed to note that the ancient taboo of “No Contradictions” was being more and more massively set aside with the ever increasing role electronic digital computers were playing in everyday life.

Amusingly in this regard, when as late as in 1995, the first more important work, [2], on inconsistent mathematics was published, it missed mentioning the above by then long term, massive and ever increasing presence in everyday life of the inconsistent axiomatic system (PA-MI). What makes that omission even more amusing is that the major motivation for the introduction of studies in inconsistent mathematics

has been claimed to be nothing else but theoretical computer science ...

As for the setting aside of the ancient taboo of “No Contradictions”, it is worth mentioning that the similarly ancient taboo of “No Self-Reference” was also set aside in mathematics, this time in the 1980s, [1]. And in case anyone may find such a venture exotic, it is again motivated by important considerations in theoretical computer science ...

Further related details and references can be found in [3,4].

2. A Basic Structure of Inconsistent Mathematics

Let F be any nonvoid set which we consider to represent the well formed formulas in a given formal language. Let $\mathcal{P}_f(F)$ be set of all finite subsets of F . Let $\delta : \mathcal{P}_f(F) \longrightarrow \mathcal{P}(F)$ be a mapping which we consider to represent the logical deductions possible in a formal theory built upon the formal language considered. More precisely, given $S \in \mathcal{P}_f(F)$, then any $f \in \delta(S)$ is supposed to be a logical deduction obtained from S , and in fact, $\delta(S)$ is supposed to be the set of all such possible logical deductions. Obviously, we can assume that

$$(2.1) \quad S \subseteq \delta(S), \quad S \in \mathcal{P}_f(F)$$

an assumption which simplifies a lot in the subsequent notations.

An essential feature of well formed formulas in formal languages is the presence of the *negation* operator $\nu : F \longrightarrow F$ which has the property that

$$(2.2) \quad \nu(\nu(f)) = f, \quad \nu(f) \neq f, \quad f \in F$$

Let now $A \subset F$ be a set of well formed formulas which play the role of *axioms* of a formal theory in the given formal language.

We associate with A the set

$$(2.3) \quad T_A = T_0 \cup T_1 \cup T_2 \cup \dots$$

where

$$(2.4) \quad \begin{aligned} T_0 &= A \\ T_1 &= \bigcup_{S \in \mathcal{P}_f(T_0)} \delta(S) \\ T_2 &= \bigcup_{S \in \mathcal{P}_f(T_1)} \delta(S) \\ &\vdots \end{aligned}$$

We call T_A the *theory*, or more appropriately, the *theorems* resulting from the axioms A .

Definition 2.1.

The axioms A , and thus the theory T_A , are called *consistent*, if and only if

$$(2.5) \quad \forall f \in T_A : \nu(f) \notin T_A$$

Definition 2.2.

Given an inconsistent system A of axioms, the theorem $f \in T_A$ is called *safe*, if and only if

$$(2.6) \quad \nu(f) \notin T_A$$

□

An important role, as seen in the sequel, in being safely able to use inconsistent theories by identifying some of their safe theorems is played by the concept of *length of proof* of a theorem, [3], given in :

Definition 2.3.

For a system A of axioms we define the respective *A-length of proof* of theorems in T_A , namely

$$(2.7) \quad l_A : T_A \longrightarrow \mathbb{N}$$

as follows. Given $f \in T_A$, then $l_A(f)$ is the smallest $n \in \mathbb{N}$, such that $f \in T_n$ in (2.3), (2.4).

3. Equivalent Axioms

As we shall see, a useful feature in safely dealing with inconsistent mathematical theories is the fact that one and the same theory can in general be given by different systems of axioms. Within the above general setup, this is formulated as follows :

Definition 3.1.

Two sets of axioms A and B are called *equivalent*, if and only if

$$(3.1) \quad T_A = T_B$$

in which case we denote $A \approx B$.

□

Based on the above, we can define the *minimum length of proof* of theorems according to :

Definition 3.2.

Given a system A of axioms, then the respective *minimum length of proof* of theorems in T_A is the mapping

$$(3.2) \quad ml_A : T_A \longrightarrow \mathbb{N}$$

where, for $f \in T_A$, we have

$$(3.3) \quad ml_A(f) = \min l_B(f)$$

with the minimum taken over all axiomatic systems B , for which $B \approx A$.

Remark 3.1.

As recalled in [3], a feature of special interest here of usual formal theories is that the difference between ml_A and l_A can be arbitrary large. In fact, one typically has the following stronger property. Given $f \in T_A$, one can find an equivalent axiomatic system B , such that $l_B(f)$ can be arbitrary larger than $l_A(f)$, and thus also than $ml_A(f)$. This fact is at the basis of one possible way for safely dealing with inconsistent theories, as seen in the next section.

4. One Way for Safely Dealing with Inconsistent Theories

Definition 4.1.

Given a system of axioms A , we denote by

$$(4.1) \quad T_A^c$$

the set of all $f \in T_A$ for which we also have $\nu(f) \in T_A$.

□

Clearly, a system A of axioms is inconsistent, if and only if

$$(4.2) \quad T_A^c \neq \phi$$

Definition 4.2.

Given an inconsistent system of axioms A , we define

$$(4.3) \quad L(A) = \min l_A(f)$$

where $f \in T_A^c$.

□

It follows that we have :

Lemma 4.1.

If A is an inconsistent system of axioms and $f \in T_A$, then

$$(4.4) \quad l_A(f) < L(A) \implies \nu(f) \notin T_A$$

thus the inequality

$$(4.4^*) \quad l_A(f) < L(A)$$

is sufficient for f to be safe.

Definition 4.3.

Given an inconsistent system A of axioms, we denote by

$$(4.5) \quad ST_A$$

the set of all $f \in T_A$ which satisfy (4.4*).

Further, we denote by

$$(4.6) \quad ST(A) = \bigcup ST_B$$

where the union is taken over all axiomatic systems B , with $B \approx A$. □

One obtains now easily :

Theorem 4.1.

If A is an inconsistent axiomatic system and $f \in T_A$, then the following are equivalent :

$$(4.7) \quad f \in ST(A)$$

$$(4.8) \quad \exists B \approx A : f \in ST_B$$

$$(4.9) \quad \exists B \approx A : l_B(f) < L(B)$$

Each of the above conditions implies that

$$(4.10) \quad \nu(f) \notin T_A$$

which means that f is safe.

Remark 4.1.

1) In view of the above, clearly, the theorems $f \in ST(A)$ are safe within the inconsistent theory T_A which corresponds to the inconsistent system A of axioms.

2) In general, we have the *strict* inclusion

$$(4.11) \quad ST(A) \subsetneq \{f \in T_A \mid \nu(f) \notin T_A\}$$

thus the relation $f \in ST(A)$ is in general only *sufficient* in order for f to be safe.

3) Certainly, safe theorems $f \in T_A$ may exist in general, for which one has $l_A(f) > L(A)$.

4) The above way, through membership in $ST(A)$, to obtain safe theorems in an inconsistent theory is obviously but only one of the many possible ways, and as such, it may appear to be the simplest one.

5) In specific cases, due to (4.8) or (4.9), it may be highly inefficient to use $ST(A)$ in order to identify certain safe theorems $f \in T_A$ in an inconsistent theory T_A .

Thus the above method in finding safe theorems in an inconsistent mathematical theory is of interest rather in principle.

References

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