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► **To cite this version:**

| El Hadj Aly Dia. Simulation of Lévy processes and option pricing. 2010. hal-00551972v1

**HAL Id: hal-00551972**

**<https://hal.science/hal-00551972v1>**

Preprint submitted on 5 Jan 2011 (v1), last revised 28 Nov 2012 (v4)

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# SIMULATION OF LÉVY PROCESSES AND OPTION PRICING

EL HADJ ALY DIA\*

**Abstract.** We use Monte-Carlo methods to value exotic options in the exponential Lévy model when the Lévy process has an infinite activity. We first approximate the Lévy process by either truncating its small jumps or replacing them by a Brownian motion with the same variance. Then we use an original method to simulate the big jumps to avoid rejection sampling. The errors resulting from the small jump approximations are also derived for some exotic options (barrier, lookback, American and Asian). Further we propose a simple method to value Asian options.

**Key words.** Exponential Lévy model, Exotic options, Monte-Carlo method

**AMS subject classifications.** 60G51, 65C05, 65N15, 60J75, 91G20

**JEL classification.** C02, C63, G13

**1. Introduction.** General exponential Lévy models (see [3, 5, 10]) are widely used, nowadays, in option valuation. Many numerical methods based on Fourier analysis have been subsequently developed to value exotic options (see [4, 11, 12, 14]). However, in many situations, Monte-Carlo methods have to be used, especially when the underlying Lévy process has an infinite Lévy measure. Because the simulation of such a Lévy process is not straightforward, except in some special cases (for example Gamma or Inverse Gaussian processes), in practise, the small jumps of the Lévy process are either truncated or replaced by a Brownian motion with the same variance. The latter approach was introduced by Asmussen and Rosinsky [2], who showed that, under suitable conditions, the normalized cumulated small jumps asymptotically behave like Brownian motion.

The purpose of this article is to derive bounds for the errors generated by these two methods of approximation in the valuation of exotic options in exponential Lévy models, and to propose new Monte-Carlo methods to value these options.

The paper is organized as follows. In the next section, we recall some basic facts about real Lévy processes and give some useful notations. In section 3 we will study the errors resulting from the small jump approximations for barrier, lookback, American and Asian options. The results of this section are the applications of [9]. In Section 4 we will propose a new method to simulate Lévy processes with infinite Lévy measure. This method will be used to value exotic options. The main idea is to approximate the inverse of the cumulative distribution function of the big jumps to avoid acceptance-rejection method which is time-consuming. In Section 5, we will show how to value Asian options when there is no Brownian component. The last section is devoted to the numerical study of the optimality of the bounds derived in Section 3.

**2. Preliminaries.** Consider a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ . The distribution of a Lévy process  $X$  is characterized by its generating triplet  $(\gamma, \sigma^2, \nu)$  where  $(\gamma, \sigma) \in \mathbb{R} \times \mathbb{R}^+$ , and  $\nu$  is a Radon measure on  $\mathbb{R} \setminus \{0\}$  satisfying

$$\int_{\mathbb{R}} (1 \wedge x^2) \nu(dx) < \infty.$$

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By the Lévy-Itô decomposition,  $X$  can be written in the form

$$X_t = \gamma t + \sigma B_t + \int_{|x|>1, s \in [0, t]} x J_X(dx \times ds) + \lim_{\delta \downarrow 0} \int_{\delta \leq |x| \leq 1, s \in [0, t]} x \tilde{J}_X(dx \times ds) \quad (2.1)$$

Here  $J_X$  is a Poisson measure on  $\mathbb{R} \times [0, \infty)$  with intensity  $\nu(dx)dt$ ,  $\tilde{J}_X(dx \times ds) = J_X(dx \times ds) - \nu(dx)ds$  and  $B$  is a standard Brownian motion. The process  $X$  has finite activity if  $\nu(\mathbb{R}) < \infty$ . If  $\nu(\mathbb{R}) = +\infty$ , the process  $X$  is called an infinite activity Lévy process. Given  $\epsilon > 0$ , we define the process  $R^\epsilon$  by

$$R_t^\epsilon = \int_{0 \leq |x| \leq \epsilon, s \in [0, t]} x \tilde{J}_X(dx \times ds), \quad t \geq 0. \quad (2.2)$$

Note that we have

$$\begin{aligned} \mathbb{E}R_t^\epsilon &= 0 \\ \text{Var}(R_t^\epsilon) &= (\sigma^X(\epsilon))^2 t, \end{aligned}$$

where

$$\sigma^X(\epsilon) = \sqrt{\int_{|x| \leq \epsilon} x^2 \nu(dx)}.$$

The process  $X^\epsilon$  is then defined by

$$X_t^\epsilon = X_t - R_t^\epsilon, \quad t \geq 0. \quad (2.3)$$

We also define the processes  $\hat{X}^\epsilon$  by

$$\hat{X}_t^\epsilon = X_t^\epsilon + \sigma(\epsilon)\hat{W}_t, \quad t \geq 0, \quad (2.4)$$

where  $\hat{W}$  is a standard Brownian motion independent of  $X$ . Set, for any  $t \geq 0$ ,

$$\begin{aligned} M_t^X &= \sup_{0 \leq s \leq t} X_s, \quad M_t^{\epsilon, X} = \sup_{0 \leq s \leq t} X_s^\epsilon, \quad \hat{M}_t^{\epsilon, X} = \sup_{0 \leq s \leq T} \hat{X}_s^\epsilon, \\ m_t^X &= \inf_{0 \leq s \leq t} X_s, \quad m_t^{\epsilon, X} = \inf_{0 \leq s \leq t} X_s^\epsilon, \quad \hat{m}_t^{\epsilon, X} = \inf_{0 \leq s \leq T} \hat{X}_s^\epsilon. \end{aligned}$$

The following notations will be used for the results in the next section. We define

$$\begin{aligned} \sigma_0^X(\epsilon) &= \max(\sigma^X(\epsilon), \epsilon). \\ \beta^X(\epsilon) &= \frac{\int_{|x| < \epsilon} x^4 \nu(dx)}{(\sigma_0(\epsilon))^4} \\ \beta_1^X(\epsilon) &= \beta(\epsilon)^{\frac{1}{6}} \left( \sqrt{\log \left( \frac{t}{\beta(\epsilon)^{\frac{1}{3}}} + 3 \right)} + 1 \right) \\ \beta_{p, \theta}^X(\epsilon) &= (\beta^X(\epsilon))^{\frac{p\theta}{p+4\theta}} \left( \log \left( \frac{1}{(\beta^X(\epsilon))^{\frac{p\theta}{p+4\theta}}} \right) \right)^p \\ \tilde{\beta}_{p, \theta}^X(\epsilon) &= \beta_{\frac{1}{p}-1, \theta}^X(\epsilon). \end{aligned}$$

When there is no ambiguity we can remove the super index  $X$ . Recall that Asmussen and Rosinski proved ([2], Theorem 2.1) that, if  $X$  is a Lévy process, then the process

$\sigma(\epsilon)^{-1}R^\epsilon$  converges in distribution to a standard Brownian motion, when  $\epsilon \rightarrow 0$ , if and only if for any  $k > 0$

$$\lim_{\epsilon \rightarrow 0} \frac{\sigma(k\sigma(\epsilon) \wedge \epsilon)}{\sigma(\epsilon)} = 1. \quad (2.5)$$

This result is implied by the condition

$$\lim_{\epsilon \rightarrow 0} \frac{\sigma(\epsilon)}{\epsilon} = +\infty. \quad (2.6)$$

The conditions (2.5) and (2.6) are equivalent, if  $\nu$  does not have atoms in some neighborhood of zero ([2], Proposition 2.1). On the other hand we approximate  $X$  by  $\tilde{X}$  when the condition (2.6) is satisfied. In this case the functions  $\beta$ ,  $\beta_1$ ,  $\beta_{p,\theta}$  and  $\tilde{\beta}_{p,\theta}$  converge to 0 when  $\epsilon$  goes to 0.

**3. Error bounds for exotic options.** Let  $(S_t)_{t \in [0, T]}$  be the price of a security. The  $\sigma$ -algebra  $\mathcal{F}_t$  will represent here the historical information on the price until time  $t$ . Under the exponential Lévy model, the process  $S$  behaves like an exponential of a Lévy process

$$S_t = S_0 e^{X_t}, \quad \forall t \geq 0.$$

The considered probability will be a risk-neutral probability, under which the process  $(e^{-(r-\delta)t} S_t)_{t \in [0, T]}$  is a martingale. The parameter  $r$  is the risk-free interest rate, and  $\delta$  is the dividend rate. The options we will consider in the sequel will have as underlying the asset with price  $S$ . The option price will be denoted by  $V$ . Set  $V^\epsilon$  (resp.  $\hat{V}^\epsilon$ ) as the price of the option obtained by replacing  $X$  by  $X^\epsilon$  (resp.  $\hat{X}^\epsilon$ ). We will call  $V^\epsilon$  (resp.  $\hat{V}^\epsilon$ ) the approximated price by truncation (resp. the approximated price by Brownian approximation).

**3.1. Barrier options.** The approximation error is bigger in the barrier case compared to the other types of option we will study. This is due to the fact that the payoff is less smooth.

Barrier	call	put
<i>Up Out</i>	$(S_T - K)^+ \mathbf{1}_{\{S_0 e^{M_T} < H\}}$	$(K - S_T)^+ \mathbf{1}_{\{S_0 e^{M_T} < H\}}$
<i>Up In</i>	$(S_T - K)^+ \mathbf{1}_{\{S_0 e^{M_T} \geq H\}}$	$(K - S_T)^+ \mathbf{1}_{\{S_0 e^{M_T} \geq H\}}$
<i>Down Out</i>	$(S_T - K)^+ \mathbf{1}_{\{S_0 e^{m_T} > H\}}$	$(K - S_T)^+ \mathbf{1}_{\{S_0 e^{m_T} > H\}}$
<i>Down In</i>	$(S_T - K)^+ \mathbf{1}_{\{S_0 e^{m_T} \leq H\}}$	$(K - S_T)^+ \mathbf{1}_{\{S_0 e^{m_T} \leq H\}}$

FIG. 3.1. *Payoffs of barrier options*

**PROPOSITION 3.1.** *Let  $X$  be an integrable Lévy process with generating triplet  $(\gamma, \sigma^2, \nu)$ . We assume that  $M_T$  has a locally bounded probability density function. Then the price of a continuous barrier option and its approximated value by truncation satisfy*

1. *For the put, for any  $q \in (0, 1)$*

$$V = V^\epsilon + O(\sigma_0(\epsilon)^{1-q}).$$

2. *For the call, if  $\mathbb{E}e^{X_T} < \infty$ , then for any  $q \in (0, 1)$*

$$V = V^\epsilon + O((\sigma_0(\epsilon))^{1-q}).$$

PROPOSITION 3.2. *Let  $X$  be an integrable Lévy process with generating triplet  $(\gamma, \sigma^2, \nu)$ . We assume that  $M_T$  has a locally bounded probability density function, then for any  $\rho, \theta \in (0, 1)$ , the price of a continuous barrier option and its approximated value by Brownian approximation satisfy*

1. *For the put*

$$V = \hat{V}^\epsilon + O\left(\sigma_0(\epsilon)^{1-\rho} (\tilde{\beta}_{\rho, \theta}(\epsilon))^\rho\right).$$

2. *For the call, if  $\mathbb{E}e^{X_T} < \infty$ , then*

$$V = \hat{V}^\epsilon + O\left((\sigma_0(\epsilon))^{1-\rho} (\tilde{\beta}_{\rho, \theta}(\epsilon))^\rho\right).$$

The existence of a probability density function for  $M_T$  and its regularity are studied in [6, 8].

REMARK 3.3. *If  $\mathbb{E}e^{X_t} < \infty$ , then*

$$\mathbb{E}\left|e^{X_t} - e^{X_t^\epsilon}\right| \leq C\sigma_0(\epsilon),$$

where  $C$  is a constant independent of  $\epsilon$ . Indeed, we have

$$\begin{aligned} \mathbb{E}\left|e^{X_t} - e^{X_t^\epsilon}\right| &= \mathbb{E}\left|e^{X_t^\epsilon + R_t^\epsilon} - e^{X_t^\epsilon}\right| \\ &= \mathbb{E}e^{X_t^\epsilon} \mathbb{E}\left|e^{R_t^\epsilon} - 1\right| \\ &\leq \mathbb{E}e^{X_t^\epsilon} \mathbb{E}|R_t^\epsilon| e^{(R_t^\epsilon)^+} \\ &\leq \mathbb{E}e^{X_t^\epsilon} \mathbb{E}|R_t^\epsilon| \left(e^{R_t^\epsilon} + 1\right). \end{aligned}$$

And we can prove that for any  $\beta \in \mathbb{R}$ ,  $\left(e^{\beta R_t^\delta}\right)_{0 \leq \delta \leq 1}$  is uniformly integrable. And if  $\int_{x>1} e^x \nu(dx) < \infty$  ( $\iff \mathbb{E}e^{X_s} < \infty$  for some  $s > 0$ ) then  $\left(e^{X_t^\delta}\right)_{0 \leq \delta \leq 1}$  is also uniformly integrable. The proof of these results are left to the reader. Then, we use Hölder inequality and Proposition 3.1 of [9].

*Proof of Proposition 3.1.* We denote by  $C$  any constant independent of  $\epsilon$ . In the *Up and Out* put case, we have

$$\begin{aligned} |V - V^\epsilon| &\leq \left| \mathbb{E}e^{-rT} (K - S_0 e^{X_T})^+ \mathbf{1}_{S_0 e^{M_T} < H} - \mathbb{E}e^{-rT} (K - S_0 e^{X_T^\epsilon})^+ \mathbf{1}_{S_0 e^{M_T^\epsilon} < H} \right| \\ &\leq \left| \mathbb{E}e^{-rT} (K - S_0 e^{X_T})^+ \left( \mathbf{1}_{S_0 e^{M_T} < H} - \mathbf{1}_{S_0 e^{M_T^\epsilon} < H} \right) \right| \\ &\quad + \left| \mathbb{E}e^{-rT} \left( (K - S_0 e^{X_T})^+ - (K - S_0 e^{X_T^\epsilon})^+ \right) \mathbf{1}_{S_0 e^{M_T^\epsilon} < H} \right|. \end{aligned}$$

We define by  $I_1$  (resp.  $I_2$ ) the first (resp. the second) term in the right hand side of the last inequality. Set  $h = \log\left(\frac{H}{S_0}\right)$ , we have

$$\begin{aligned} I_1 &= \left| \mathbb{E}e^{-rT} (K - S_0 e^{X_T})^+ (\mathbf{1}_{M_T < h} - \mathbf{1}_{M_T^\epsilon < h}) \right| \\ &= \left| \mathbb{E}e^{-rT} (K - S_0 e^{X_T})^+ (\mathbf{1}_{M_T < h, M_T^\epsilon \geq h} - \mathbf{1}_{M_T \geq h, M_T^\epsilon < h}) \right| \\ &\leq K\mathbb{P}[M_T < h, M_T^\epsilon \geq h] + K\mathbb{P}[M_T \geq h, M_T^\epsilon < h] \\ &= O(\sigma_0(\epsilon)^{1-q}), \text{ see the proof of Proposition 3.12 of [9].} \end{aligned}$$

On the other hand

$$\begin{aligned} I_2 &\leq S_0 \mathbb{E} \left| e^{X_T} - e^{X_T^\epsilon} \right| \\ &= O(\sigma_0(\epsilon)), \text{ by Remark 3.3.} \end{aligned}$$

Hence

$$|V - V^\epsilon| = O(\sigma_0(\epsilon)^{1-q}).$$

For the *Down and Out* put, we have

$$\begin{aligned} |V - V^\epsilon| &\leq \left| \mathbb{E} e^{-rT} (K - S_0 e^{X_T})^+ \mathbf{1}_{S_0 e^{m_T} > H} - \mathbb{E} e^{-rT} (K - S_0 e^{X_T^\epsilon})^+ \mathbf{1}_{S_0 e^{m_T^\epsilon} > H} \right| \\ &\leq \left| \mathbb{E} e^{-rT} (K - S_0 e^{X_T})^+ (\mathbf{1}_{S_0 e^{m_T} < H} - \mathbf{1}_{S_0 e^{m_T^\epsilon} < H}) \right| \\ &\quad + \left| \mathbb{E} e^{-rT} \left( (K - S_0 e^{X_T})^+ - (K - S_0 e^{X_T^\epsilon})^+ \right) \mathbf{1}_{S_0 e^{m_T^\epsilon} < H} \right|. \end{aligned}$$

Set again  $I_1$  (resp.  $I_2$ ) the first (resp. the second) term in the right hand side of the last inequality, but this time  $h = \log(\frac{S_0}{H})$ , then we have

$$\begin{aligned} I_2 &\leq S_0 \mathbb{E} \left| e^{X_T} - e^{X_T^\epsilon} \right| \\ &= O(\sigma_0(\epsilon)), \text{ by Remark 3.3.} \end{aligned}$$

We set

$$\tilde{X} = -X, \quad \tilde{M}_T = \sup_{0 \leq s \leq T} (-X_s), \quad \tilde{M}_T^\epsilon = \sup_{0 \leq s \leq T} (-X_s^\epsilon).$$

Thus

$$\begin{aligned} I_1 &= \left| \mathbb{E} e^{-rT} (K - S_0 e^{X_T})^+ (\mathbf{1}_{\tilde{M}_T < h} - \mathbf{1}_{\tilde{M}_T^\epsilon < h}) \right| \\ &= \left| \mathbb{E} e^{-rT} (K - S_0 e^{X_T})^+ (\mathbf{1}_{\tilde{M}_T < h, \tilde{M}_T^\epsilon \geq h} - \mathbf{1}_{\tilde{M}_T \geq h, \tilde{M}_T^\epsilon < h}) \right| \\ &\leq K \mathbb{P} [\tilde{M}_T < h, \tilde{M}_T^\epsilon \geq h] + K \mathbb{P} [\tilde{M}_T \geq h, \tilde{M}_T^\epsilon < h] \\ &= O(\sigma_0(\epsilon)^{1-q}), \text{ see the proof of Proposition 3.12 of [9].} \end{aligned}$$

So, we have

$$|V - V^\epsilon| = O(\sigma_0(\epsilon)^{1-q}).$$

For the *Down and Out* call, we have

$$\begin{aligned} |V - V^\epsilon| &\leq \left| \mathbb{E} e^{-rT} (S_0 e^{X_T} - K)^+ \mathbf{1}_{S_0 e^{m_T} > H} - \mathbb{E} e^{-rT} (S_0 e^{X_T^\epsilon} - K)^+ \mathbf{1}_{S_0 e^{m_T^\epsilon} > H} \right| \\ &\leq \left| \mathbb{E} e^{-rT} (S_0 e^{X_T} - K)^+ (\mathbf{1}_{S_0 e^{m_T} > H} - \mathbf{1}_{S_0 e^{m_T^\epsilon} > H}) \right| \\ &\quad + \left| \mathbb{E} e^{-rT} \left( (S_0 e^{X_T} - K)^+ - (S_0 e^{X_T^\epsilon} - K)^+ \right) \mathbf{1}_{S_0 e^{m_T^\epsilon} > H} \right|. \end{aligned}$$

Note by  $I_1$  (resp.  $I_2$ ) the first (resp. the second) term in the right hand side of the last inequality, we keep  $h = \log\left(\frac{S_0}{H}\right)$ . So, we have

$$\begin{aligned} I_1 &= \left| \mathbb{E} e^{-rT} (S_0 e^{X_T} - K)^+ (\mathbf{1}_{-m_T < h} - \mathbf{1}_{-m_T^\epsilon < h}) \right| \\ &= e^{-rT} \left| \mathbb{E} e^{X_T} (S_0 - K e^{-X_T})^+ (\mathbf{1}_{-m_T < h, -m_T^\epsilon \geq h} - \mathbf{1}_{-m_T \geq h, -m_T^\epsilon < h}) \right| \\ &= e^{-rT} \mathbb{E} e^{X_T} \left| \bar{\mathbb{E}} \left( S_0 - K e^{\tilde{X}_T} \right)^+ \left( \mathbf{1}_{\tilde{M}_T < h, \tilde{M}_T^\epsilon \geq h} - \mathbf{1}_{\tilde{M}_T \geq h, \tilde{M}_T^\epsilon < h} \right) \right|, \end{aligned}$$

where  $\bar{\mathbb{E}}$  is the expectation under  $\bar{\mathbb{P}}$ , an equivalent measure of  $\mathbb{P}$  which comes from the Esscher transform with Radon-Nikodym derivative

$$\frac{d\bar{\mathbb{P}}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \frac{e^{X_t}}{\mathbb{E} e^{X_t}}.$$

We will denote by  $(\bar{\gamma}, \bar{\sigma}^2, \bar{\nu})$  the generating triplet of  $\tilde{X}$  under  $\bar{\mathbb{P}}$ . In particular we have

$$\bar{\nu}(dx) = e^{-x} \nu(-dx). \quad (3.1)$$

See Section 9.4 and Section 9.5 of [7] for more details. So

$$\begin{aligned} I_1 &\leq e^{-rT} \mathbb{E} e^{X_T} S_0 \bar{\mathbb{P}} [\tilde{M}_T < h, \tilde{M}_T^\epsilon \geq h] + e^{-rT} \mathbb{E} e^{X_T} S_0 \bar{\mathbb{P}} [\tilde{M}_T \geq h, \tilde{M}_T^\epsilon < h] \\ &= O\left(\left(\sigma_0^{\tilde{X}}(\epsilon)\right)^{1-q}\right), \text{ see the proof of Proposition 3.12 of [9]} \\ &= O\left(\left(\sigma_0(\epsilon)\right)^{1-q}\right). \end{aligned}$$

The  $\sigma_0(\epsilon)$  obtained by the proof of Proposition 3.12 of [9], should be that obtained by using  $\bar{\nu}$ . But due to (3.1), the result remains true when we use  $\nu$ . We also have

$$\begin{aligned} I_2 &\leq S_0 \mathbb{E} \left| e^{X_T} - e^{X_T^\epsilon} \right| \\ &= O(\sigma_0(\epsilon)), \text{ by Remark 3.3.} \end{aligned}$$

So, we have

$$|V - V^\epsilon| = O(\sigma_0(\epsilon)^{1-q}).$$

For the *Up and Out* call, we have

$$\begin{aligned} |V - V^\epsilon| &\leq \left| \mathbb{E} e^{-rT} (S_0 e^{X_T} - K)^+ \mathbf{1}_{S_0 e^{M_T} < H} - \mathbb{E} e^{-rT} (S_0 e^{X_T^\epsilon} - K)^+ \mathbf{1}_{S_0 e^{M_T^\epsilon} < H} \right| \\ &\leq \left| \mathbb{E} e^{-rT} (S_0 e^{X_T} - K)^+ \left( \mathbf{1}_{S_0 e^{M_T} < H} - \mathbf{1}_{S_0 e^{M_T^\epsilon} < H} \right) \right| \\ &\quad + \left| \mathbb{E} e^{-rT} \left( (S_0 e^{X_T} - K)^+ - (S_0 e^{X_T^\epsilon} - K)^+ \right) \mathbf{1}_{S_0 e^{M_T^\epsilon} < H} \right|. \end{aligned}$$

Note by  $I_1$  (resp.  $I_2$ ) the first (resp. the second) term in the right hand side of the

last inequality. Here,  $h = \log\left(\frac{H}{S_0}\right)$ .

$$\begin{aligned}
I_1 &= \left| \mathbb{E} e^{-rT} (S_0 e^{X_T} - K)^+ (\mathbf{1}_{M_T < h} - \mathbf{1}_{M_T^\epsilon < h}) \right| \\
&= e^{-rT} \left| \mathbb{E} e^{X_T} (S_0 - K e^{-X_T})^+ (\mathbf{1}_{M_T < h, M_T^\epsilon \geq h} - \mathbf{1}_{M_T \geq h, M_T^\epsilon < h}) \right| \\
&= e^{-rT} \mathbb{E} e^{X_T} \left| \bar{\mathbb{E}} \left( S_0 - K e^{\bar{X}_T} \right)^+ \left( \mathbf{1}_{M_T^{-\bar{X}} < h, M_T^{-\bar{X}, \epsilon} \geq h} - \mathbf{1}_{M_T^{-\bar{X}} \geq h, M_T^{-\bar{X}, \epsilon} < h} \right) \right| \\
&\leq e^{-rT} \mathbb{E} e^{X_T} S_0 \left( \bar{\mathbb{P}} \left[ M_T^{-\bar{X}} < h, M_T^{-\bar{X}, \epsilon} \geq h \right] + \bar{\mathbb{P}} \left[ M_T^{-\bar{X}} \geq h, M_T^{-\bar{X}, \epsilon} < h \right] \right) \\
&= O \left( \left( \sigma_0^{-\bar{X}}(\epsilon) \right)^{1-q} \right), \text{ see the proof of Proposition 3.12 in [9]} \\
&= O \left( (\sigma_0(\epsilon))^{1-q} \right).
\end{aligned}$$

We also have

$$\begin{aligned}
I_2 &\leq S_0 \mathbb{E} \left| e^{X_T} - e^{X_T^\epsilon} \right| \\
&= O(\sigma_0(\epsilon)), \text{ by Remark 3.3.}
\end{aligned}$$

So, we have

$$|V - V^\epsilon| = O(\sigma_0(\epsilon)^{1-q}).$$

The *In* case can be deduced from the relation between *In* and *Out* options.  $\diamond$

The proof of Proposition 3.2 is similar to the above case. We will consider the discrete prices (i.e. we use the discrete supremum). And we will build on the proof of the third result of Proposition 4.12 of [9].

**3.2. Lookback and hindsight options.** We will focus only on the lookback case. For the hindsight options we will use the relations between lookback and hindsight options.

Option	call	put
Lookback	$S_T - S_0 e^{mT}$	$S_0 e^{M_T} - S_T$
Hindsight	$(S_0 e^{M_T} - K)^+$	$(K - S_0 e^{mT})^+$

FIG. 3.2. *The payoffs of lookback and hindsight options.*

**PROPOSITION 3.4.** *Let  $X$  be an infinite activity Lévy process with generating triplet  $(\gamma, \sigma^2, \nu)$ . We assume that there exists  $p > 1$  such that  $\int_{|x|>1} e^{px} \nu(dx) < \infty$ . Then the price of a continuous lookback call option and its approximation satisfy*

$$V(S_-) = V^\epsilon(S_-) + O(\sigma_0(\epsilon))$$

$$V(S_-) = \hat{V}^\epsilon(S_-) + O(\sigma_0(\epsilon)\beta_1(\epsilon)).$$

**PROPOSITION 3.5.** *Let  $X$  be an infinite activity Lévy process with generating triplet  $(\gamma, \sigma^2, \nu)$ . We assume that there exists  $p > 1$  such that  $\int_{|x|>1} e^{p|x|} \nu(dx) < \infty$ .*



Then the price of a continuous lookback put option and its approximation satisfy

$$V(S_+) = V^\epsilon(S_+) + O(\sigma_0(\epsilon))$$

$$V(S_+) = \hat{V}^\epsilon(S_+) + O\left(\sigma_0(\epsilon) \left(\beta_{\frac{p}{p-1}, \theta}(\epsilon)\right)^{1-\frac{1}{p}}\right) \quad \forall \theta \in (0, 1).$$

The above results show that replacing small jumps by a Brownian motion leads to a better approximation, if the condition (2.5) is satisfied.

*Proof of Proposition 3.4.* To simplify the proof we assume that  $r = 0$  and  $S_0 = 1$ . Recall that  $m_T = -\tilde{M}_T$ . We have

$$\begin{aligned} |V(S_-) - V^\epsilon(S_-)| &\leq \mathbb{E} \left| e^{-\tilde{M}_T} - e^{-\tilde{M}_T^\epsilon} \right| + \mathbb{E} \left| e^{X_T} - e^{X_T^\epsilon} \right| \\ &= O(\sigma_0(\epsilon)), \text{ by Proposition 3.4 of [9] and Remark 3.3.} \end{aligned}$$

On the other hand, we have

$$\begin{aligned} |V(S_-) - \hat{V}^\epsilon(S_-)| &\leq \left| \mathbb{E} \left( e^{-\tilde{M}_T} - S_- \right)^+ - \mathbb{E} \left( e^{-\tilde{M}_T^{\hat{X}, \epsilon}} - S_- \right)^+ \right| + \left| \mathbb{E} \left( e^{X_T} - e^{\hat{X}_T^\epsilon} \right) \right| \\ &= O(\sigma_0(\epsilon)\beta_1(\epsilon)), \text{ by Theorem 4.5 and Proposition 4.1 of [9].} \end{aligned}$$

◇

*Proof of Proposition 3.5.* To simplify the proof we assume that  $r = 0$  and  $S_0 = 1$ . We have

$$\begin{aligned} |V(S_+) - V^\epsilon(S_+)| &\leq \mathbb{E} \left| e^{M_T} - e^{M_T^\epsilon} \right| + \mathbb{E} \left| e^{X_T} - e^{X_T^\epsilon} \right| \\ &= O(\sigma_0(\epsilon)), \text{ by Proposition 3.8 of [9] and Remark 3.3.} \end{aligned}$$

On the other hand, we have

$$\begin{aligned} |V(S_+) - \hat{V}^\epsilon(S_+)| &\leq \left| \mathbb{E} \left( e^{M_T} - S_+ \right)^+ - \mathbb{E} \left( e^{\hat{M}_T^\epsilon} - S_+ \right)^+ \right| + \left| \mathbb{E} \left( e^{X_T} - e^{\hat{X}_T^\epsilon} \right) \right| \\ &= O\left(\sigma_0(\epsilon) \left(\beta_{\frac{p}{p-1}, \theta}(\epsilon)\right)^{1-\frac{1}{p}}\right) \quad \forall \theta \in (0, 1), \end{aligned}$$

thanks to Propositions 4.11 and Proposition 4.1 of [9].

◇

**COROLLARY 3.6.** *Propositions 3.4 and 3.5 are true for hindsight option provided that we replace, call by put, and put by call.*

**3.3. American options.** Recall that the price of an American option with strike  $K$  and maturity  $T$  is given by

$$\begin{aligned} &\sup_{\tau \in \mathcal{T}_{[0, T]}} \mathbb{E} e^{-r\tau} (S_0 e^{X_\tau} - K)^+, \text{ for the call} \\ &\sup_{\tau \in \mathcal{T}_{[0, T]}} \mathbb{E} e^{-r\tau} (K - S_0 e^{X_\tau})^+, \text{ for the put,} \end{aligned}$$

where  $\mathcal{T}_{[0, T]}$  is the of stopping times with values in  $[0, T]$ .

**PROPOSITION 3.7.** *Let  $X$  be an infinite activity Lévy process with generating triplet  $(\gamma, \sigma^2, \nu)$ . Then the price of a continuous American option and its approximation by truncation satisfy*

1. For the put, we have

$$V = V^\epsilon + O(\sigma(\epsilon)).$$

2. For the call, if  $\mathbb{E}e^{X_T} < \infty$ , then

$$V = V^\epsilon + O(\sigma(\epsilon)).$$

**PROPOSITION 3.8.** *Let  $X$  an infinite activity Lévy process with generating triplet  $(\gamma, \sigma^2, \nu)$ . Then of a continuous American option and its Brownian approximation satisfy*

1. For the put, we have

$$V = \hat{V}^\epsilon + O(\sigma_0(\epsilon)\beta_1(\epsilon)).$$

2. For the call, if  $\mathbb{E}e^{X_T} < \infty$ , then

$$V = \hat{V}^\epsilon + O(\sigma_0(\epsilon)\beta_1(\epsilon)).$$

*Proof of Proposition 3.7 and Proposition 3.8.* For the put, we use Remark 3.5 and Proposition 4.10 of [9]. Furthermore, using the proof of Proposition 3.1, the call price is given by

$$\sup_{\tau \in \mathcal{T}_{[0, T]}} e^{-r\tau} \mathbb{E}e^{X_\tau} \bar{\mathbb{E}} \left( S_0 - Ke^{\tilde{X}_\tau} \right)^+,$$

and the Lévy measure of  $\tilde{X}$  under the new probability is given by (3.1). Hence

$$\begin{aligned} |V - \hat{V}^\epsilon| &= \left| \sup_{\tau \in \mathcal{T}_{[0, T]}} e^{-r\tau} \mathbb{E}e^{X_\tau} \bar{\mathbb{E}} \left( S_0 - Ke^{\tilde{X}_\tau} \right)^+ - \sup_{\tau \in \mathcal{T}_{[0, T]}} e^{-r\tau} \mathbb{E}e^{X_\tau^\epsilon} \bar{\mathbb{E}} \left( S_0 - Ke^{\tilde{X}_\tau^\epsilon} \right)^+ \right| \\ &= \left| \sup_{\tau \in \mathcal{T}_{[0, T]}} e^{-r\tau} \mathbb{E}e^{X_\tau} \bar{\mathbb{E}} \left( S_0 - Ke^{\tilde{X}_\tau^\epsilon} \right)^+ - \sup_{\tau \in \mathcal{T}_{[0, T]}} e^{-r\tau} \mathbb{E}e^{X_\tau^\epsilon} \bar{\mathbb{E}} \left( S_0 - Ke^{\tilde{X}_\tau^\epsilon} \right)^+ \right| \\ &\quad + \left| \sup_{\tau \in \mathcal{T}_{[0, T]}} e^{-r\tau} \mathbb{E}e^{X_\tau} \bar{\mathbb{E}} \left( S_0 - Ke^{\tilde{X}_\tau} \right)^+ - \sup_{\tau \in \mathcal{T}_{[0, T]}} e^{-r\tau} \mathbb{E}e^{X_\tau} \bar{\mathbb{E}} \left( S_0 - Ke^{\tilde{X}_\tau^\epsilon} \right)^+ \right|. \end{aligned}$$

Note by  $I_1$  (resp.  $I_2$ ) the first (resp. the second) term in the right hand side of the last equality. Set

$$\begin{aligned} I_1^1 &= \sup_{\tau \in \mathcal{T}_{[0, T]}} e^{-r\tau} \left( \mathbb{E}e^{X_\tau} - \mathbb{E}e^{X_\tau^\epsilon} \right) \bar{\mathbb{E}} \left( S_0 - Ke^{\tilde{X}_\tau^\epsilon} \right)^+ \\ I_1^2 &= \sup_{\tau \in \mathcal{T}_{[0, T]}} e^{-r\tau} \left( \mathbb{E}e^{X_\tau^\epsilon} - \mathbb{E}e^{X_\tau} \right) \bar{\mathbb{E}} \left( S_0 - Ke^{\tilde{X}_\tau^\epsilon} \right)^+. \end{aligned}$$

We can show that

$$|I_1| \leq |\max(I_1^1, I_1^2)|.$$

Using the proof of Proposition 3.2 of [8], we get

$$I_1 = O(\sigma(\epsilon)^2).$$

Furthermore by Remark 3.5 of [9], we have

$$I_1 = O(\sigma(\epsilon)).$$

This proves the second result of Proposition 3.7. For the second result of Proposition 3.8, we use the same arguments, but we use this time the proof of Proposition 3.2 of [8] by replacing  $X^\epsilon$  by  $\hat{X}^\epsilon$  and Proposition 4.10 of [9].  $\diamond$

**3.4. Asian options.** The approximation errors, in this case, are similar to the lookback case. The payoffs of arithmetic Asian options with fixed strike  $K$  are given by

$$\begin{aligned} & \left( \frac{1}{T} \int_0^T S_0 e^{X_s} ds - K \right)^+, \text{ for the call} \\ & \left( K - \frac{1}{T} \int_0^T S_0 e^{X_s} ds \right)^+, \text{ for the put.} \end{aligned}$$

Call	fixed strike	floating strike
Arithmetic mean	$\left( \frac{1}{T} \int_0^T S_0 e^{X_s} ds - K \right)^+$	$\left( S_T - \frac{1}{T} \int_0^T S_0 e^{X_s} ds \right)^+$
Geometric mean	$\left( S_0 \exp\left(\frac{1}{T} \int_0^T X_s ds\right) - K \right)^+$	$\left( S_T - S_0 \exp\left(\frac{1}{T} \int_0^T X_s ds\right) \right)^+$

FIG. 3.3. Payoffs of Asian call options

Put	fixed strike	floating strike
Arithmetic mean	$\left( K - \frac{1}{T} \int_0^T S_0 e^{X_s} ds \right)^+$	$\left( \frac{1}{T} \int_0^T S_0 e^{X_s} ds - S_T \right)^+$
Geometric mean	$\left( K - S_0 \exp\left(\frac{1}{T} \int_0^T X_s ds\right) \right)^+$	$\left( S_0 \exp\left(\frac{1}{T} \int_0^T X_s ds\right) - S_T \right)^+$

FIG. 3.4. Payoffs of Asian put options

**PROPOSITION 3.9.** *Let  $X$  be an infinite activity Lévy process with generating triplet  $(\gamma, \sigma^2, \nu)$ . We assume there exists  $p > 1$  such that  $\mathbb{E}e^{pM_T} < \infty$ . Then, the price of a continuous arithmetic Asian option with fixed strike its approximation by truncation satisfy*

$$V = V^\epsilon + O(\sigma_0(\epsilon)).$$

*Proof.* To simplify the proof we assume that  $r = 0$  and  $S_0 = 1$ . We have

$$\begin{aligned} |V - V^\epsilon| & \leq \mathbb{E} \frac{1}{T} \int_0^T \left| e^{X_s} - e^{X_s^\epsilon} \right| ds \\ & = \mathbb{E} \frac{1}{T} \int_0^T |X_s - X_s^\epsilon| e^{\tilde{X}_s^\epsilon} ds, \end{aligned}$$

where  $\bar{X}_s^\epsilon$  is between  $X_s$  and  $X_s^\epsilon$ . Define  $p^*$  such that  $\frac{1}{p} + \frac{1}{p^*} = 1$ . We have

$$\begin{aligned} |V - V^\epsilon| &\leq \mathbb{E} \sup_{0 \leq s \leq T} |R_s^\epsilon| \max(e^{M_T}, e^{M_T^\epsilon}) \\ &\leq \mathbb{E} \left( \sup_{0 \leq s \leq T} |R_s^\epsilon|^{p^*} \right)^{\frac{1}{p^*}} \left( \mathbb{E} \max(e^{pM_T}, e^{pM_T^\epsilon}) \right)^{\frac{1}{p}} \\ &\leq \frac{p^*}{p^* - 1} \mathbb{E} \left( |R_T^\epsilon|^{p^*} \right)^{\frac{1}{p^*}} \left( \mathbb{E} e^{pM_T} + \mathbb{E} e^{pM_T^\epsilon} \right)^{\frac{1}{p}} \\ &= O(\sigma_0(\epsilon)), \text{ by Proposition 3.1 and Lemma 3.9 of [8].} \end{aligned}$$

◇

**PROPOSITION 3.10.** *Let  $X$  be an infinite activity Lévy process with generating triplet  $(\gamma, \sigma^2, \nu)$ . We assume that there exists  $p > 1$  such that  $\mathbb{E} e^{pM_T} < \infty$ . Then the price of a continuous arithmetic Asian option with fixed strike and its Brownian approximation satisfy, for any  $\theta \in (0, 1)$*

$$V = \hat{V}^\epsilon + O\left(\sigma_0(\epsilon) \left(\beta_{\frac{p}{p-1}, \theta}(\epsilon)\right)^{1-\frac{1}{p}}\right).$$

*Proof.* To simplify the proof we assume that  $r = 0$  and  $S_0 = 1$ . Define by  $f$  the payoff function of the Asian option, and set

$$\begin{aligned} V_n &= \mathbb{E} f \left( \frac{1}{n} \sum_{k=1}^n e^{X_{\frac{kT}{n}}} \right) \\ V_n^\epsilon &= \mathbb{E} f \left( \frac{1}{n} \sum_{k=1}^n e^{X_{\frac{kT}{n}}^\epsilon + \sigma(\epsilon) \hat{W}_{\frac{kT}{n}}} \right). \end{aligned}$$

The sequence  $V_n$  (resp.  $V_n^\epsilon$ ) converges to  $V$  (resp.  $V^\epsilon$ ). On the other hand, for  $k \in \{1, \dots, n\}$ , we have

$$R_{\frac{kt}{n}}^\epsilon = \frac{1}{\sqrt{n}} \sum_{j=1}^k V_j^n,$$

where

$$V_j^n = \sqrt{n} \left( R_{\frac{jt}{n}}^\epsilon - R_{(j-1)\frac{t}{n}}^\epsilon \right).$$

The r.v.  $(V_j^n)_{j \in \{1, \dots, n\}}$  are i.i.d. and have the same distribution as  $\sqrt{n} R_{\frac{t}{n}}^\epsilon$ . But  $\mathbb{E} V_1 = 0$ , and  $\text{var}(V_1) = \sigma(\epsilon)^2 t$ , by Theorem 1 of [16] (see pp. 163) there exists positive i.i.d. r.v.,  $(\tau_j)_{j \in \{1, \dots, n\}}$ , and a standard Brownian motion,  $\hat{B}$ , such that  $(\sum_{j=1}^k V_j^n, k \in \{1, \dots, n\})$  and  $(\hat{B}_{\tau_1 + \dots + \tau_k}, k \in \{1, \dots, n\})$  have the same joint distribution. Thus  $(R_{\frac{kt}{n}}^\epsilon, k \in \{1, \dots, n\})$  and  $(\hat{B}_{\frac{\tau_1 + \dots + \tau_k}{n}}, k \in \{1, \dots, n\})$  have the same joint distribution. We also have

$$\left( \sigma(\epsilon) \hat{W}_{\frac{kt}{n}}, k \in \{1, \dots, n\} \right) =^d \left( \hat{B}_{\frac{\sigma(\epsilon)^2 kt}{n}}, k \in \{1, \dots, n\} \right)$$

Set

$$T_k = \frac{\tau_1 + \cdots + \tau_k}{n}$$

$$T_k^\epsilon = \frac{\sigma(\epsilon)^2 kt}{n}.$$

Thus

$$\sum_{k=1}^n e^{X_{\frac{kT}{n}}} = d \sum_{k=1}^n e^{X_{\frac{kT}{n}} + \hat{B}_{T_k}}$$

$$\sum_{k=1}^n e^{X_{\frac{kT}{n}} + \sigma(\epsilon) \hat{W}_{\frac{kT}{n}}} = d \sum_{k=1}^n e^{X_{\frac{kT}{n}} + \hat{B}_{T_k}^\epsilon}.$$

So

$$|V_n - V_n^\epsilon| \leq \mathbb{E} \frac{1}{n} \sum_{k=1}^n \left| e^{X_{\frac{kT}{n}} + \hat{B}_{T_k}} - e^{X_{\frac{kT}{n}} + \hat{B}_{T_k}^\epsilon} \right|$$

$$= \mathbb{E} \frac{1}{n} \sum_{k=1}^n \left| \hat{B}_{T_k} - \hat{B}_{T_k}^\epsilon \right| e^{\bar{X}_k^\epsilon},$$

where  $\bar{X}_k^\epsilon$  is between  $X_{\frac{kT}{n}} + \hat{B}_{T_k}$  and  $X_{\frac{kT}{n}} + \hat{B}_{T_k}^\epsilon$ . Define  $p^*$  such that  $\frac{1}{p} + \frac{1}{p^*} = 1$ .

We have

$$|V_n - V_n^\epsilon| \leq \left( \mathbb{E} \frac{1}{n} \sum_{k=1}^n \left| \hat{B}_{T_k} - \hat{B}_{T_k}^\epsilon \right|^{p^*} \right)^{\frac{1}{p^*}} \left( \mathbb{E} \frac{1}{n} \sum_{k=1}^n e^{p \bar{X}_k^\epsilon} \right)^{\frac{1}{p}}$$

$$\leq \left( \mathbb{E} \sup_{1 \leq k \leq n} \left| \hat{B}_{T_k} - \hat{B}_{T_k}^\epsilon \right|^{p^*} \right)^{\frac{1}{p^*}} \left( \mathbb{E} \frac{1}{n} \sum_{k=1}^n e^{p \bar{X}_k^\epsilon} \right)^{\frac{1}{p}}.$$

But

$$\mathbb{E} \frac{1}{n} \sum_{k=1}^n e^{p \bar{X}_k^\epsilon} \leq \mathbb{E} \frac{1}{n} \sum_{k=1}^n e^{p \left( X_{\frac{kT}{n}} + \hat{B}_{T_k} \right)} + \mathbb{E} \frac{1}{n} \sum_{k=1}^n e^{p \left( X_{\frac{kT}{n}} + \hat{B}_{T_k}^\epsilon \right)}$$

$$= \mathbb{E} \frac{1}{n} \sum_{k=1}^n e^{p \left( X_{\frac{kT}{n}} + R_{\frac{kT}{n}}^\epsilon \right)} + \mathbb{E} \frac{1}{n} \sum_{k=1}^n e^{p \left( X_{\frac{kT}{n}} + \sigma(\epsilon) \hat{W}_{\frac{kT}{n}} \right)}$$

$$\leq \mathbb{E} e^{pM_T} + \mathbb{E} e^{pM_T^\epsilon}$$

$$\leq \mathbb{E} \left( e^{pM_t} + e^{p\sigma(\epsilon) \sup_{0 \leq s \leq T} W_s} e^{pM_T^\epsilon} \right)$$

$$\leq \mathbb{E} e^{pM_T} + 2e^{\frac{p^2}{2} \sigma(\epsilon)^2 T} \mathbb{E} e^{pM_T^\epsilon}$$

$$\leq 2e^{\frac{p^2}{2} \sigma(\epsilon)^2 T} \mathbb{E} \left( e^{pM_T} + e^{pM_T^\epsilon} \right).$$

The term in the right hand side of the last inequality is bounded by a constant, say  $C$ , thanks to Lemma 3.9 of [9]. So

$$|V_n - V_n^\epsilon| \leq C \left( \mathbb{E} \sup_{1 \leq k \leq n} \left| \hat{B}_{T_k} - \hat{B}_{T_k}^\epsilon \right|^{p^*} \right)^{\frac{1}{p^*}}.$$

Hence by Theorem 4.6 of [9].

$$|V - V^\epsilon| = O\left(\sigma_0(\epsilon) \left(\beta_{\frac{p}{p-1}, \theta}(\epsilon)\right)^{1-\frac{1}{p}}\right) \quad \forall \theta \in (0, 1).$$

◇

The above results on fixed strike Asian options are, obviously applicable for floating strike Asian options. If the mean is geometric, we will also get the same results.

**COROLLARY 3.11.** *The results of Proposition 3.9 and Proposition 3.10 are true for either arithmetic or geometric Asian options whether the strike is fixed or floating.*

The proof is similar to the proofs of the above propositions.

**4. Simulation of infinite activity Lévy processes.** In this section we will show a new method to simulate an infinite activity Lévy process. The first step is to approximate the Lévy process by a finite activity Lévy process as seen above. Then we will simulate the big jumps.

**4.1. Simulation method.** We have (see Section 2)

$$\begin{aligned} X_t^\epsilon &= X_t - R_t^\epsilon \\ &= \gamma t + \int_{|x|>1, s \in [0, t]} x J_X(dx \times ds) + \int_{\epsilon \leq |x| \leq 1, s \in [0, t]} x J_X(dx \times ds) \\ &= \left( \gamma - \int_{\epsilon \leq |x| \leq 1} x \nu(dx) \right) t + \int_{|x|>\epsilon, s \in [0, t]} x J_X(dx \times ds) \\ &= \left( \gamma - \int_{\epsilon \leq |x| \leq 1} x \nu(dx) \right) t + \int_{x>\epsilon, s \in [0, t]} x J_X(dx \times ds) \\ &\quad + \int_{x<-\epsilon, s \in [0, t]} x J_X(dx \times ds) \\ &= \gamma_0^\epsilon t + \sum_{i=1}^{N_t^+} Y_i^+ - \sum_{i=1}^{N_t^-} Y_i^-, \end{aligned}$$

where

$$\gamma_0^\epsilon = \gamma - \int_{\epsilon \leq |x| \leq 1} x \nu(dx). \quad (4.1)$$

The r.v.  $(Y_i^+)_{i \geq 1}$  are i.i.d. with common law  $\frac{\nu_\epsilon^+(dx)}{\nu(\epsilon, +\infty)}$ , the r.v.  $(Y_i^-)_{i \geq 1}$  are i.i.d. with common law  $\frac{\nu_\epsilon^-(-dx)}{\nu(-\infty, \epsilon)}$ . The measures  $\nu_\epsilon^+$  and  $\nu_\epsilon^-$  correspond respectively to  $\nu$  restricted on  $(0, +\infty)$  and on  $(-\infty, 0)$ . The process  $X^\epsilon$  is a compound Poisson process. The simulation of the jump times is relatively simple. The Problem is, because the numbers of jumps on  $[0, T]$  can be large, how to quickly simulate the size of the jumps. We will focus on the simulation of the positive jumps  $(Y_i^+)_{i \geq 1}$ . The simulation of  $(Y_i^-)_{i \geq 1}$  will be similar. Let  $\lambda_+^\epsilon = \nu(\epsilon, \infty)$ . The cumulative distribution function (cdf) of  $Y_1^+$  cannot be determined explicitly, its inverse either. So one way to simulate  $Y_1^+$  is to use rejection sampling. This is time consuming, especially since it will make on average  $\lambda_+^\epsilon T$  simulations. The alternative is to make a *discrete inversion* of the cdf,  $F_+$ , of  $Y_1^+$ . We have, for all  $x > \epsilon$

$$F_+(x) = \frac{1}{\lambda_+^\epsilon} \int_\epsilon^x \nu(dx).$$

We will define a positive real  $A$  in order to have  $\nu(A, +\infty)$  very small, in order of  $10^{-16}$  for example (that is what we choose in our simulations). We suppose then that the r.v.  $Y_1^+$  is in  $[\epsilon, A]$ . Set for any  $k \in \{0, \dots, n\}$

$$x_k = k \frac{A - \epsilon}{n} + \epsilon$$

$$y_k = \frac{F_+(x_k)}{F_+(A)}.$$

Where  $n$  is the number of the discretization points on  $[\epsilon, A]$ . Note that  $y_0 = 0$ . How do we compute  $(F_+(x_k))_{1 \leq k \leq n}$ ? Notice that for any  $k \in \{1, \dots, n\}$ , we have

$$F_+(x_k) = \sum_{j=1}^k (F_+(x_j) - F_+(x_{j-1})),$$

with

$$(F_+(x_j) - F_+(x_{j-1})) = \int_{x_{j-1}}^{x_j} \nu(dx).$$

Depending on the Lévy measure, we will define an approximation method for the integrale  $\int_{x_{j-1}}^{x_j} \nu(dx)$ . We define the function  $G_+$  by, for any  $y \in [0, 1]$

$$G_+(y) = x,$$

where  $x$  is the unique real satisfying

$$\frac{F_+(x)}{F_+(A)} = y.$$

Let  $y \in [0, 1]$ , to compute  $G_+(y)$ , we use the following method. We have to find first the integer  $k > 1$  satisfying  $y_{k-1} \leq y < y_k$ . Then we have

$$yF_+(A) = y_{k-1} + \int_{x_{k-1}}^{G_+(y)} \nu(dy).$$

We must approximate the above integrale depending on  $G_+(y)$ , and express the latter as a function of  $y$ . We will call  $G_+$ , the *discrete inverse function* of  $F_+$ . When  $n$  and  $A$  go to the infinity, we will get the inverse function of  $F_+$ . So we state that to simulate  $Y_1^+$ , we just need to simulate

$$G_+(U), \text{ where } U \text{ is a uniform r.v. on } [0, 1].$$

**4.2. Estimates of the inverse cdf of the jumps.** We will, for some popular models, estimate the function  $G_+$ . The models that we consider in this section are VG, CGMY and NIG. Nonetheless, using the same methodology we can estimate the function  $G_+$  for any other model.

**4.2.1. The Variance-Gamma case.** Let  $G$  be a gamma process with de parameters  $(\mu, \kappa) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$  (see [13]), satisfying  $G_0 = 0$  and for any  $t \geq 0$  and  $h > 0$ ,  $G_{t+h} - G_t$  have a gamma distribution with parameters  $\left(h \frac{\mu^2}{\kappa}, \frac{\kappa}{\mu}\right)$ . In fact in financial

applications  $\mu = 1$ , and the process  $(W_{G_t})_{t \geq 0}$  is a VG process VG with parameter  $(\theta, \sigma, \kappa)$ . Its characteristic exponent is given by

$$\varphi(u) = \log \left( \left( 1 - i\theta\kappa u + \frac{\sigma^2}{2}\kappa u^2 \right)^{-\frac{1}{\kappa}} \right).$$

The process  $(W_{G_t})_{t \geq 0}$ , can be defined by its Lévy measure  $\nu$ . Indeed

$$\nu(dx) = C \frac{e^{-Mx}}{x} \mathbb{1}_{x>0} dx + C \frac{e^{-G|x|}}{|x|} \mathbb{1}_{x<0} dx,$$

where

$$\begin{aligned} C &= \frac{1}{\kappa} \\ M &= \frac{1}{\sigma} \sqrt{\frac{2}{\kappa} + \frac{\theta^2}{\sigma^2}} - \frac{\theta}{\sigma^2} \\ G &= \frac{1}{\sigma} \sqrt{\frac{2}{\kappa} + \frac{\theta^2}{\sigma^2}} + \frac{\theta}{\sigma^2}. \end{aligned}$$

This is a particular case of the CGMY process (by taking  $Y = 0$ , see [5]). The probability density function of  $Y_1^+$  is then

$$f_+(x) = \frac{C}{\lambda_+^\epsilon} \frac{e^{-Mx}}{x}, \quad x > \epsilon.$$

Then for any  $x > \epsilon$

$$F_+(x) = \frac{C}{\lambda_+^\epsilon} \int_\epsilon^x \frac{e^{-My}}{y} dy.$$

Hence

$$F_+(x_k) - F_+(x_{k-1}) = \frac{C}{\lambda_+^\epsilon} \int_{x_{k-1}}^{x_k} \frac{e^{-My}}{y} dy.$$

We approximate this integral by

$$\frac{C}{\lambda_+^\epsilon} e^{-Mx_{k-1}} \int_{x_{k-1}}^{x_k} \frac{dy}{y} = \frac{C}{\lambda_+^\epsilon} e^{-Mx_{k-1}} \log \left( \frac{x_k}{x_{k-1}} \right).$$

The function  $G_+$  satisfies

$$yF_+(A) = y_{k-1} + \frac{C}{\lambda_+^\epsilon} \int_{x_{k-1}}^{G_+(y)} \frac{e^{-My}}{y} dy.$$

As previously the above integral is approximated by

$$\frac{C}{\lambda_+^\epsilon} e^{-Mx_{k-1}} \log \left( \frac{G_+(y)}{x_{k-1}} \right).$$

Hence  $G_+(y)$  can be approximated by

$$x_{k-1} \exp \left[ \frac{\lambda_+^\epsilon}{C} (yF_+(A) - y_{k-1}) e^{-Mx_{k-1}} \right]. \quad (4.2)$$



**4.2.2. The CGMY case.** It is a pure jump Lévy process (see [5]), with Lévy measure

$$\nu(dx) = C \frac{e^{-Mx}}{x^{1+Y}} \mathbf{1}_{x>0} dx + C \frac{e^{-G|x|}}{|x|^{1+Y}} \mathbf{1}_{x<0} dx$$

Where  $C$ ,  $G$  and  $M$  are positive, and  $Y \in (0, 2)$ . When  $Y = 0$ , we get the Variance-Gamma model. Its characteristic exponent is given by

$$\varphi(u) = C \left( (M - iu) \log \left( 1 - \frac{iu}{M} \right) + (G + iu) \log \left( 1 + \frac{iu}{G} \right) \right),$$

if  $Y = 1$ , or

$$\varphi(u) = C \Gamma(-Y) \left[ M^Y \left( \left( 1 - \frac{iu}{M} \right)^Y - 1 + \frac{iuY}{M} \right) + G^Y \left( \left( 1 + \frac{iu}{G} \right)^Y - 1 - \frac{iuY}{G} \right) \right],$$

if  $Y \neq 1$ . In the CGMY model, the probability density function of  $Y_1^+$  is

$$f_+(x) = \frac{C}{\lambda_+^\epsilon} \frac{e^{-Mx}}{x^{1+x}}, \quad x > \epsilon.$$

Then its cdf is

$$F_+(x) = \frac{C}{\lambda_+^\epsilon} \int_\epsilon^x \frac{e^{-My}}{y^{1+Y}} dy.$$

Hence

$$F_+(x_k) - F_+(x_{k-1}) = \frac{C}{\lambda_+^\epsilon} \int_{x_{k-1}}^{x_k} \frac{e^{-My}}{y^{1+Y}} dy.$$

Then we approximate  $F_+(x_k) - F_+(x_{k-1})$  by

$$\frac{C}{\lambda_+^\epsilon} e^{-Mx_{k-1}} \int_{x_{k-1}}^{x_k} \frac{dy}{y^{1+Y}} = \frac{C}{\lambda_+^\epsilon Y} e^{-Mx_{k-1}} \left( \frac{1}{x_{k-1}^Y} - \frac{1}{x_k^Y} \right).$$

So  $G_+$  is solution of the equation

$$yF_+(A) = y_{k-1} + \frac{C}{\lambda_+^\epsilon} \int_{x_{k-1}}^{G_+(y)} \frac{e^{-My}}{y^{1+Y}} dy.$$

We approximate the above integrals by

$$\frac{C}{\lambda_+^\epsilon Y} e^{-Mx_{k-1}} \left( \frac{1}{x_{k-1}^Y} - \frac{1}{(G_+(y))^Y} \right).$$

Hence  $G_+(y)$  can be approximated by

$$\left[ \frac{1}{x_{k-1}^Y} - \frac{\lambda_+^\epsilon Y}{C} e^{Mx_{k-1}} (yF_+(A) - y_{k-1}) \right]^{-\frac{1}{Y}}. \quad (4.3)$$

**4.2.3. The NIG case.** Like the VG model, the NIG (Normal Inverse Gaussian) model (see [3]) is a particular case of the hyperbolic models. It is chartered by four parameters :  $\alpha$ ,  $\beta$ ,  $\hat{\delta}$  and  $\mu$ . Where  $0 \leq |\beta| \leq \alpha$ ,  $\hat{\delta} > 0$  and  $\mu \in \mathbb{R}$ . Its generating triplet are  $(\gamma, 0, \nu)$ , where

$$\gamma = \mu + 2 \frac{\alpha \hat{\delta}}{\pi} \int_0^1 \sinh(\beta x) K_1(\alpha x) dx$$

$$\nu(dx) = \frac{\alpha \hat{\delta}}{\pi |x|} K_1(\alpha |x|) e^{\beta x} dx,$$

with

$$K_\lambda(z) = \frac{1}{2} \int_{\mathbb{R}^+} y^{\lambda-1} \exp\left(-\frac{1}{2}z\left(y + \frac{1}{y}\right)\right) dy.$$

In financial applications we set  $\mu = 0$ . Then the NIG is represented by three parameters :  $(\alpha, \beta, \hat{\delta})$ . The cdf of  $Y_1^+$  is

$$f_+(x) = \frac{\alpha \hat{\delta}}{\pi x} K_1(\alpha x) e^{\beta x}, \quad x > \epsilon.$$

And then its cdf is given by

$$F_+(x) = \frac{\alpha \hat{\delta}}{\pi} \int_\epsilon^x \frac{K_1(\alpha y)}{y} e^{\beta y} dy.$$

Therefore

$$F_+(x_k) - F_+(x_{k-1}) = \frac{\alpha \hat{\delta}}{\pi} \int_{x_{k-1}}^{x_k} \frac{K_1(\alpha y)}{y} e^{\beta y} dy.$$

To approximate the above integrale, we need to study the asymptotic behaviour of  $K_1$ . We have (see [1], Formula 9.7.2 and Formula 9.8.7)

$$K_1(x) \underset{x \downarrow 0}{\sim} \frac{C}{x}, \quad \text{for a given } C > 0$$

$$K_1(x) \underset{x \rightarrow +\infty}{\sim} \sqrt{\frac{\pi}{2x}} e^{-x}.$$

Hence the following approximation

$$\frac{\alpha \hat{\delta}}{\pi} x_{k-1} K_1(\alpha x_{k-1}) e^{\beta x_{k-1}} \int_{x_{k-1}}^{x_k} \frac{dy}{y^2} = \frac{\alpha \hat{\delta}}{\pi} x_{k-1} K_1(\alpha x_{k-1}) e^{\beta x_{k-1}} \left( \frac{1}{x_{k-1}} - \frac{1}{x_k} \right).$$

In NIG case  $G_+$  satisfy

$$yF_+(A) = y_{k-1} + \frac{\alpha \hat{\delta}}{\pi} \int_{x_{k-1}}^{G_+(y)} \frac{K_1(\alpha y)}{y} e^{\beta y} dy.$$

So we approximate  $G_+(y)$  by

$$\left( \frac{1}{x_{k-1}} - \frac{\pi}{\alpha \hat{\delta}} \frac{yF_+(A) - y_{k-1}}{x_{k-1} K_1(\alpha x_{k-1})} e^{-\beta x_{k-1}} \right)^{-1}. \quad (4.4)$$

The  $Y_1^-$  case is treated in the same way, we only need to substitute  $\beta$  by  $-\beta$ .

**5. Valuation of Asian options.** We will focus on the fixed strike Asian put option. The call case can be easily deduced. Floating Asian options, can be valued using fixed strike options and symmetry. Consider the following payoffs

$$\left( K - \frac{1}{T} \int_0^T S_0 e^{X_s} ds \right)^+, \text{ arithmetic Asian put}$$

$$\left( K - S_0 e^{\frac{1}{T} \int_0^T X_s ds} \right)^+, \text{ geometric Asian put.}$$

We set

$$V_a = e^{-rT} \mathbb{E} \left( K - \frac{1}{T} \int_0^T S_0 e^{X_s} ds \right)^+$$

$$V_g = e^{-rT} \mathbb{E} \left( K - S_0 e^{\frac{1}{T} \int_0^T X_s ds} \right)^+.$$

The Lévy process  $X$  considered has the generating triplet  $(\gamma, 0, \nu)$ . In fact we will estimate the quantities  $V_a^\epsilon$ ,  $\hat{V}_a^\epsilon$ ,  $V_g^\epsilon$  and  $\hat{V}_g^\epsilon$  obtained by replacing  $X$  by  $X^\epsilon$  or  $\hat{X}^\epsilon$ . Let  $(T_j^\epsilon)_{j \geq 1}$  be jump times of  $X^\epsilon$ . Note

$$\hat{T}_0^\epsilon = 0$$

$$\hat{T}_j^\epsilon = T_j^\epsilon \wedge T.$$

We have

$$V_a^\epsilon = e^{-rT} \mathbb{E} \left( K - \frac{1}{T} \int_0^T S_0 e^{X_s^\epsilon} ds \right)^+$$

$$= e^{-rT} \mathbb{E} \left( K - \frac{S_0}{T} \sum_{j=1}^{N_T^\epsilon+1} \int_{\hat{T}_{j-1}^\epsilon}^{\hat{T}_j^\epsilon} e^{X_s^\epsilon} ds \right)^+$$

$$= e^{-rT} \mathbb{E} \left( K - \frac{S_0}{T} \sum_{j=1}^{N_T^\epsilon+1} \int_{\hat{T}_{j-1}^\epsilon}^{\hat{T}_j^\epsilon} e^{\gamma_0^\epsilon s + \sum_{i=1}^{j-1} Y_i^\epsilon} ds \right)^+, \text{ see (4.1).}$$

So

$$V_a^\epsilon = e^{-rT} \mathbb{E} \left( K - \frac{S_0}{T} \sum_{j=1}^{N_T^\epsilon+1} e^{\sum_{i=1}^{j-1} Y_i^\epsilon} \int_{\hat{T}_{j-1}^\epsilon}^{\hat{T}_j^\epsilon} e^{\gamma_0^\epsilon s} ds \right)^+$$

$$= e^{-rT} \mathbb{E} \left( K - \frac{S_0}{T} \sum_{j=1}^{N_T^\epsilon+1} e^{\sum_{i=1}^{j-1} Y_i^\epsilon} \frac{e^{\gamma_0^\epsilon \hat{T}_j^\epsilon} - e^{\gamma_0^\epsilon \hat{T}_{j-1}^\epsilon}}{\gamma_0^\epsilon} \right)^+$$

$$= e^{-rT} \mathbb{E} \left( K - \frac{S_0}{T} \sum_{j=1}^{N_T^\epsilon+1} \frac{e^{\gamma_0^\epsilon \hat{T}_j^\epsilon + \sum_{i=1}^{j-1} Y_i^\epsilon} - e^{\gamma_0^\epsilon \hat{T}_{j-1}^\epsilon + \sum_{i=1}^{j-1} Y_i^\epsilon}}{\gamma_0^\epsilon} \right)^+.$$

Hence

$$V_a^\epsilon = e^{-rT} \mathbb{E} \left( K - \frac{S_0}{T} \sum_{j=1}^{N_T^\epsilon+1} \frac{e^{X_{\hat{T}_j^\epsilon}^\epsilon} - e^{X_{\hat{T}_{j-1}^\epsilon}^\epsilon}}{\gamma_0^\epsilon} \right)^+. \quad (5.1)$$

By the same way, we get

$$V_g^\epsilon = e^{-rT} \mathbb{E} \left( K - S_0 \exp \left( \frac{1}{T} \sum_{j=1}^{N_T^\epsilon+1} \sum_{i=1}^{j-1} Y_i^\epsilon + \frac{1}{2} \left( (\hat{T}_j^\epsilon)^2 - (\hat{T}_{j-1}^\epsilon)^2 \right) \right) \right)^+. \quad (5.2)$$

In geometric case when we replace  $X^\epsilon$  by  $\hat{X}^\epsilon$ , we get

$$\begin{aligned} \hat{V}_g^\epsilon = e^{-rT} \mathbb{E} \left( K - S_0 \exp \left( \frac{1}{T} \sum_{j=1}^{N_T^\epsilon+1} \sum_{i=1}^{j-1} Y_i^\epsilon + \frac{1}{2} \left( (\hat{T}_j^\epsilon)^2 - (\hat{T}_{j-1}^\epsilon)^2 \right) \right. \right. \\ \left. \left. + \sigma(\epsilon) \int_{\hat{T}_{j-1}^\epsilon}^{\hat{T}_j^\epsilon} \hat{W}_s ds \right) \right)^+. \end{aligned} \quad (5.3)$$

And given  $N_T^\epsilon$  and  $(\hat{T}_j^\epsilon)_{1 \leq j \leq N_T^\epsilon}$ , the r.v.  $\left( \int_{\hat{T}_{j-1}^\epsilon}^{\hat{T}_j^\epsilon} \hat{W}_s ds \right)_{1 \leq j \leq N_T^\epsilon+1}$  are independent with gaussian distribution with mean 0 and variance  $\frac{1}{3} (\hat{T}_j^\epsilon - \hat{T}_{j-1}^\epsilon)^3$ . In the arithmetic case, we have the following result.

**PROPOSITION 5.1.** *Let  $X$  be an infinite activity Lévy process with generating triplet  $(\gamma, 0, \nu)$  and  $f$  be a Lipschitz function. We assume that  $\mathbb{E}e^{M_T} < \infty$ . Then*

$$\begin{aligned} \mathbb{E}f \left( \frac{1}{T} \int_0^T S_0 e^{\hat{X}_s^\epsilon} ds \right) = \mathbb{E}f \left( \frac{S_0}{T} \sum_{j=1}^{N_T^\epsilon+1} e^{\hat{X}_{\hat{T}_{j-1}^\epsilon}^\epsilon} \left( \frac{e^{\gamma_0^\epsilon (\hat{T}_j^\epsilon - \hat{T}_{j-1}^\epsilon)} - 1}{\gamma_0^\epsilon} + \sigma(\epsilon) g_j^\epsilon \right) \right) \\ + O(\sigma(\epsilon)^2), \end{aligned}$$

with

$$g_j^\epsilon = \int_{\hat{T}_{j-1}^\epsilon}^{\hat{T}_j^\epsilon} e^{\gamma_0^\epsilon (s - \hat{T}_{j-1}^\epsilon)} (\hat{W}_s - \hat{W}_{\hat{T}_{j-1}^\epsilon}) ds.$$

Given  $N_T^\epsilon$  and  $(\hat{T}_j^\epsilon)_{1 \leq j \leq N_T^\epsilon}$ , the r.v.  $(g_j^\epsilon)_{1 \leq j \leq N_T^\epsilon+1}$  are independent and gaussian, and

$$\text{var}(g_j^\epsilon) = \frac{1}{2(\gamma_0^\epsilon)^3} \left( (2\gamma_0^\epsilon (\hat{T}_j^\epsilon - \hat{T}_{j-1}^\epsilon) - 3) e^{2\gamma_0^\epsilon (\hat{T}_j^\epsilon - \hat{T}_{j-1}^\epsilon)} + 4e^{\gamma_0^\epsilon (\hat{T}_j^\epsilon - \hat{T}_{j-1}^\epsilon)} - 1 \right). \quad (5.4)$$

However  $g_j^\epsilon$  is correlated with  $\hat{W}_{\hat{T}_j^\epsilon} - \hat{W}_{\hat{T}_{j-1}^\epsilon}$ . Indeed, we have

$$\text{cov} \left( g_j^\epsilon, \hat{W}_{\hat{T}_j^\epsilon} - \hat{W}_{\hat{T}_{j-1}^\epsilon} \right) = \frac{\hat{T}_j^\epsilon - \hat{T}_{j-1}^\epsilon}{\gamma_0^\epsilon} e^{\gamma_0^\epsilon (\hat{T}_j^\epsilon - \hat{T}_{j-1}^\epsilon)} - \frac{e^{\gamma_0^\epsilon (\hat{T}_j^\epsilon - \hat{T}_{j-1}^\epsilon)} - 1}{(\gamma_0^\epsilon)^2}. \quad (5.5)$$

*Proof of Proposition 5.1.* Set

$$Z^\epsilon = \frac{1}{T} \int_0^T S_0 e^{\hat{X}_s^\epsilon} ds.$$

We have

$$\begin{aligned}
Z^\epsilon &= \frac{S_0}{T} \sum_{j=1}^{N_T^\epsilon+1} e^{\sum_{i=1}^{j-1} Y_i^\epsilon} \int_{\hat{T}_{j-1}^\epsilon}^{\hat{T}_j^\epsilon} e^{\gamma_0^\epsilon s + \sigma(\epsilon) \hat{W}_s} ds \\
&= \frac{S_0}{T} \sum_{j=1}^{N_T^\epsilon+1} e^{\sum_{i=1}^{j-1} Y_i^\epsilon + \gamma_0^\epsilon \hat{T}_{j-1}^\epsilon + \sigma(\epsilon) \hat{W}_{\hat{T}_{j-1}^\epsilon}} \int_{\hat{T}_{j-1}^\epsilon}^{\hat{T}_j^\epsilon} e^{\gamma_0^\epsilon (s - \hat{T}_{j-1}^\epsilon) + \sigma(\epsilon) (\hat{W}_s - \hat{W}_{\hat{T}_{j-1}^\epsilon})} ds \\
&= \frac{S_0}{T} \sum_{j=1}^{N_T^\epsilon+1} e^{\hat{X}_{\hat{T}_{j-1}^\epsilon}^\epsilon} \int_{\hat{T}_{j-1}^\epsilon}^{\hat{T}_j^\epsilon} e^{\gamma_0^\epsilon (s - \hat{T}_{j-1}^\epsilon) + \sigma(\epsilon) (\hat{W}_s - \hat{W}_{\hat{T}_{j-1}^\epsilon})} ds.
\end{aligned}$$

Note that using the integral form of Taylor Theorem, we have

$$\begin{aligned}
e^{\sigma(\epsilon) (\hat{W}_s - \hat{W}_{\hat{T}_{j-1}^\epsilon})} &= 1 + \sigma(\epsilon) (\hat{W}_s - \hat{W}_{\hat{T}_{j-1}^\epsilon}) \\
&\quad + \int_0^{\sigma(\epsilon) (\hat{W}_s - \hat{W}_{\hat{T}_{j-1}^\epsilon})} e^y (\sigma(\epsilon) (\hat{W}_s - \hat{W}_{\hat{T}_{j-1}^\epsilon}) - y) dy.
\end{aligned}$$

So, noting  $\alpha$  the Lipschitz constant of  $f$ , we have

$$\begin{aligned}
\delta_\epsilon &:= \left| \mathbb{E} f \left( \frac{1}{T} \int_0^T S_0 e^{\hat{X}_s^\epsilon} ds \right) - \mathbb{E} f \left( \frac{S_0}{T} \sum_{j=1}^{N_T^\epsilon+1} e^{\hat{X}_{\hat{T}_{j-1}^\epsilon}^\epsilon} \left( \frac{e^{\gamma_0^\epsilon (\hat{T}_j^\epsilon - \hat{T}_{j-1}^\epsilon)} - 1}{\gamma_0^\epsilon} + \sigma(\epsilon) g_j^\epsilon \right) \right) \right| \\
&\leq \alpha \mathbb{E} \frac{S_0}{T} \sum_{j=1}^{N_T^\epsilon+1} e^{\hat{X}_{\hat{T}_{j-1}^\epsilon}^\epsilon} \int_{\hat{T}_{j-1}^\epsilon}^{\hat{T}_j^\epsilon} ds e^{\gamma_0^\epsilon (s - \hat{T}_{j-1}^\epsilon)} \\
&\quad \times \left| \int_0^{\sigma(\epsilon) (\hat{W}_s - \hat{W}_{\hat{T}_{j-1}^\epsilon})} e^y (\sigma(\epsilon) (\hat{W}_s - \hat{W}_{\hat{T}_{j-1}^\epsilon}) - y) dy \right| \\
&\leq \alpha \frac{\sigma(\epsilon)^2}{2} \frac{S_0}{T} \mathbb{E} \sum_{j=1}^{N_T^\epsilon+1} e^{\hat{X}_{\hat{T}_{j-1}^\epsilon}^\epsilon} \int_{\hat{T}_{j-1}^\epsilon}^{\hat{T}_j^\epsilon} (\hat{W}_s - \hat{W}_{\hat{T}_{j-1}^\epsilon})^2 e^{\gamma_0^\epsilon (s - \hat{T}_{j-1}^\epsilon) + \sigma(\epsilon) (\hat{W}_s - \hat{W}_{\hat{T}_{j-1}^\epsilon})} ds.
\end{aligned}$$

Notice that for  $s$  in  $[\hat{T}_{j-1}^\epsilon, \hat{T}_j^\epsilon]$ , we have

$$\begin{aligned}
\hat{X}_{\hat{T}_{j-1}^\epsilon}^\epsilon + \gamma_0^\epsilon (s - \hat{T}_{j-1}^\epsilon) + \sigma(\epsilon) (\hat{W}_s - \hat{W}_{\hat{T}_{j-1}^\epsilon})^+ &= \sum_{i=1}^{j-1} Y_i^\epsilon + \gamma_0^\epsilon s + \sigma(\epsilon) \hat{W}_{\hat{T}_{j-1}^\epsilon} \\
&\quad + \sigma(\epsilon) (\hat{W}_s - \hat{W}_{\hat{T}_{j-1}^\epsilon})^+ \\
&\leq \sum_{i=1}^{j-1} Y_i^\epsilon + \gamma_0^\epsilon s + \sigma(\epsilon) \sup_{\hat{T}_{j-1}^\epsilon \leq \tau \leq \hat{T}_j^\epsilon} \hat{W}_\tau \\
&\leq \sum_{i=1}^{j-1} Y_i^\epsilon + \gamma_0^\epsilon s + \sigma(\epsilon) \sup_{0 \leq \tau \leq T} \hat{W}_\tau.
\end{aligned}$$

Thus

$$\begin{aligned}
\delta_\epsilon &\leq \alpha \frac{\sigma(\epsilon)^2}{2} \frac{S_0}{T} \mathbb{E} \sum_{j=1}^{N_T^\epsilon+1} e^{\sum_{i=1}^{j-1} Y_i^\epsilon} \int_{\hat{T}_{j-1}^\epsilon}^{\hat{T}_j^\epsilon} ds e^{\gamma_0^\epsilon s} \left( \sup_{0 \leq \tau \leq T} \hat{W}_\tau - \inf_{0 \leq \tau \leq T} \hat{W}_\tau \right)^2 \\
&\quad \times e^{\sigma(\epsilon) \sup_{0 \leq \tau \leq T} \hat{W}_\tau} \\
&\leq \alpha \frac{\sigma(\epsilon)^2}{2} \frac{S_0}{T} \mathbb{E} \sum_{j=1}^{N_T^\epsilon+1} e^{\sum_{i=1}^{j-1} Y_i^\epsilon} \left( e^{\gamma_0^\epsilon \hat{T}_j^\epsilon} + e^{\gamma_0^\epsilon \hat{T}_{j-1}^\epsilon} \right) \\
&\quad \times \int_{\hat{T}_{j-1}^\epsilon}^{\hat{T}_j^\epsilon} ds \mathbb{E} \left( \sup_{0 \leq \tau \leq T} \hat{W}_\tau - \inf_{0 \leq \tau \leq T} \hat{W}_\tau \right)^2 e^{\sigma(\epsilon) \sup_{0 \leq \tau \leq T} \hat{W}_\tau} \\
&\leq \alpha \frac{\sigma(\epsilon)^2}{2} \frac{S_0}{T} \mathbb{E} \sum_{j=1}^{N_T^\epsilon+1} \left( e^{X_{\hat{T}_j^\epsilon}^\epsilon} + e^{X_{\hat{T}_{j-1}^\epsilon}^\epsilon} \right) \left( \hat{T}_j^\epsilon - \hat{T}_{j-1}^\epsilon \right) \mathbb{E} \left( \sup_{0 \leq \tau \leq T} \hat{W}_\tau - \inf_{0 \leq \tau \leq T} \hat{W}_\tau \right)^2 \\
&\quad \times e^{\sigma(\epsilon) \sup_{0 \leq \tau \leq T} \hat{W}_\tau}.
\end{aligned}$$

So

$$\begin{aligned}
\delta_\epsilon &\leq \alpha \sigma(\epsilon)^2 \frac{S_0}{T} \mathbb{E} \sum_{j=1}^{N_T^\epsilon+1} e^{M_T^\epsilon} \left( \hat{T}_j^\epsilon - \hat{T}_{j-1}^\epsilon \right) \mathbb{E} \left( \sup_{0 \leq \tau \leq T} \hat{W}_\tau - \inf_{0 \leq \tau \leq T} \hat{W}_\tau \right)^2 e^{\sigma(\epsilon) \sup_{0 \leq \tau \leq T} \hat{W}_\tau} \\
&\leq \alpha \sigma(\epsilon)^2 S_0 \mathbb{E} e^{M_T^\epsilon} \mathbb{E} \left( \sup_{0 \leq \tau \leq T} \hat{W}_\tau - \inf_{0 \leq \tau \leq T} \hat{W}_\tau \right)^2 e^{\sigma(\epsilon) \sup_{0 \leq \tau \leq T} \hat{W}_\tau}.
\end{aligned}$$

Using Lemma 3.9 of [8], we conclude that

$$\delta_\epsilon = O(\sigma(\epsilon)^2).$$

◇

A natural control variate for  $\hat{V}_a^\epsilon$  (resp.  $V_a^\epsilon$ ) is  $\frac{1}{T} \int_0^T S_0 e^{\hat{X}_s^\epsilon} ds$  (resp.  $\frac{1}{T} \int_0^T S_0 e^{X_s^\epsilon} ds$ ). We must approximate their values.

**PROPOSITION 5.2.** *Let  $X$  be an infinite activity Lévy process with generating triplet  $(\gamma, \sigma^2, \nu)$  and  $f$  a Lipschitz function. We assume that  $\int_{|x|>1} e^x \nu(dx) < \infty$ . Then we have*

$$\begin{aligned}
\mathbb{E} \int_0^T e^{X_s} ds - \mathbb{E} \int_0^T e^{X_s^\epsilon} ds &= \frac{\sigma(\epsilon)^2}{2} \int_0^T s \mathbb{E} e^{X_s^\epsilon} ds + O(\sigma_0(\epsilon)^3) \\
\mathbb{E} \int_0^T e^{X_s} ds - \mathbb{E} \int_0^T e^{\hat{X}_s^\epsilon} ds &= O(\sigma_0(\epsilon)^3).
\end{aligned}$$

Recall that in the case where we replace  $X_s$  by  $X_s^\epsilon$ , the approximation error for Asian option is  $O(\sigma_0(\epsilon))$ . So we do not need to know the term  $\int_0^T s \mathbb{E} e^{X_s^\epsilon} ds$ . Furthermore under the condition  $(e^{(r-\delta)s+X_s})_{s \geq 0}$  is a martingale, we have

$$\mathbb{E} \int_0^T e^{X_s} ds = \begin{cases} \frac{e^{(r-\delta)T} - 1}{r - \delta} & \text{if } r - \delta \neq 0 \\ T & \text{otherwise} \end{cases}$$

*Proof of Proposition 5.2.* We have

$$\begin{aligned}\delta_\epsilon &:= \mathbb{E} \int_0^T e^{X_s} ds - \mathbb{E} \int_0^T e^{X_s^\epsilon} ds \\ &= \int_0^T \mathbb{E} \left( e^{X_s} - e^{X_s^\epsilon} \right) ds.\end{aligned}$$

Using the integral form of the Taylor Theorem, we get

$$\begin{aligned}\mathbb{E} \left( e^{X_s} - e^{X_s^\epsilon} \right) &= \frac{\sigma(\epsilon)^2 s}{2} \mathbb{E} e^{X_s^\epsilon} + \int_0^1 \mathbb{E} \left( e^{X_s^\epsilon + \theta R_s^\epsilon} - e^{X_s^\epsilon} \right) (1 - \theta) (R_s^\epsilon)^2 d\theta \\ &= \frac{\sigma(\epsilon)^2 s}{2} \mathbb{E} e^{X_s^\epsilon} + \mathbb{E} e^{X_s^\epsilon} \left( \mathbb{E} e^{R_s^\epsilon} - 1 - R_s^\epsilon - \frac{(R_s^\epsilon)^2}{2} \right).\end{aligned}$$

The last equality is obtained by a simple integration. On the other hand

$$\begin{aligned}\mathbb{E} e^{X_s^\epsilon} \left| \mathbb{E} e^{R_s^\epsilon} - 1 - R_s^\epsilon - \frac{(R_s^\epsilon)^2}{2} \right| &\leq \mathbb{E} e^{X_s^\epsilon} \mathbb{E} |R_s^\epsilon|^3 e^{\sup_{0 \leq u \leq s} R_u^\epsilon} \\ &\leq \mathbb{E} e^{X_s^\epsilon} \left( \mathbb{E} |R_s^\epsilon|^4 \right)^{\frac{3}{4}} \left( \mathbb{E} e^{4 \sup_{0 \leq u \leq s} R_u^\epsilon} \right)^{\frac{1}{4}} \\ &\leq \mathbb{E} e^{X_s^\epsilon} \left( \mathbb{E} |R_s^\epsilon|^4 \right)^{\frac{3}{4}} \left( \mathbb{E} e^{4 \sup_{0 \leq u \leq T} R_u^\epsilon} \right)^{\frac{1}{4}}.\end{aligned}$$

Using Proposition 4.1 of [9], we conclude that

$$\mathbb{E} e^{X_s^\epsilon} \left| \mathbb{E} e^{R_s^\epsilon} - 1 - R_s^\epsilon - \frac{(R_s^\epsilon)^2}{2} \right| = O(\sigma_0(\epsilon)^3).$$

uniformly with  $s \in [0, T]$ . Hence

$$\mathbb{E} \int_0^T e^{X_s} ds - \mathbb{E} \int_0^T e^{X_s^\epsilon} ds = \frac{\sigma(\epsilon)^2}{2} \int_0^T s \mathbb{E} e^{X_s^\epsilon} ds + O(\sigma_0(\epsilon)^3).$$

Replacing in the previous expressions  $X_s$  by  $\hat{X}_s^\epsilon$ , we will obtain  $\sigma(\epsilon)\hat{W}_s$  instead of  $R_s^\epsilon$ . Thus

$$\mathbb{E} \int_0^T e^{\hat{X}_s^\epsilon} ds - \mathbb{E} \int_0^T e^{X_s^\epsilon} ds = \frac{\sigma(\epsilon)^2}{2} \int_0^T s \mathbb{E} e^{X_s^\epsilon} ds + O(\sigma_0(\epsilon)^3).$$

Therefore

$$\mathbb{E} \int_0^T e^{X_s} ds - \mathbb{E} \int_0^T e^{\hat{X}_s^\epsilon} ds = O(\sigma_0(\epsilon)^3).$$

◇

**6. Numerical examples.** In this section we will study numerically the optimality of the bounds derived in Section 3. For example in Proposition 3.4, we have shown that (for the lookback call) we have

$$|V(S_-) - V^\epsilon(S_-)| \leq C\sigma_0(\epsilon).$$

So

$$\log(|V(S_-) - V^\epsilon(S_-)|) \leq \log(C) + \log(\sigma_0(\epsilon)).$$

Hence when  $\epsilon$  goes to 0, we will get

$$\frac{\log(|V(S_-) - V^\epsilon(S_-)|)}{\log(\sigma_0(\epsilon))} \geq \frac{\log(C)}{\log(\sigma_0(\epsilon))} + 1.$$

So we will represent in Figure 6.1 the function  $f$  defined by

$$f(\epsilon) = \frac{\log(|V(S_-) - V^{1-\epsilon}(S_-)|)}{\log(\sigma_0(1-\epsilon))}, \quad \forall \epsilon \in (0, 1).$$

The function  $f$  as expected converges to 1 when  $\epsilon$  goes to 1. The bound is, in the

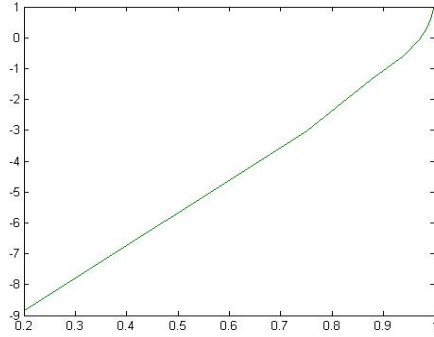


FIG. 6.1. *Optimality of the error bound for a lookback call in VG model. The parameters are  $S_0 = 100$ ,  $r = 0.0548$ ,  $\delta = 0$ ,  $T = 0.40504$ ,  $S_+ = 100$ ,  $\theta = -0.2859$ ,  $\kappa = 0.2505$ ,  $\sigma = 0.1927$ .*

case of truncation, optimal for lookback options. For Brownian approximation this not seems to be the case.

In Figure 6.2, we consider a lookback put. The function  $f$  is this time defined by

$$f(\epsilon) = \frac{\log(|V(S_+) - \hat{V}^{1-\epsilon}(S_+)|)}{\log(1-\epsilon)}, \quad \forall \epsilon \in (0, 1).$$

where  $V(S_+)$  is the price of the lookback put and  $\hat{V}^{1-\epsilon}(S_+)$  its Brownian approximation. The choose of the denominator in the definition of  $f$  is due the fact that we are looking for the best power of  $\epsilon$  (actually  $1 - \epsilon$  in this case) that we can have. We will compare this power with that obtained using the bounds in the second result of Proposition 3.5 (and in Proposition 3.10). In fact we will use the expression  $\sigma_0(1-\epsilon)\beta(1-\epsilon)^{\frac{1}{5}} \log\left(\frac{1}{\beta(1-\epsilon)^{\frac{1}{5}}}\right)$  (which is the best bound when  $p$  goes to  $+\infty$  in



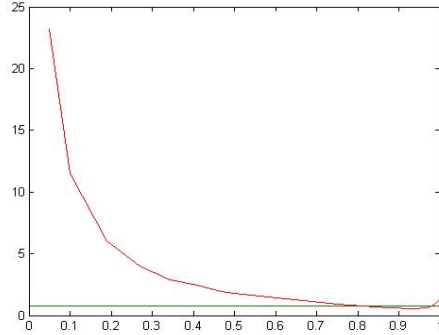


FIG. 6.2. *Optimality of the error bound for a lookback put in CGMY model. The parameters are  $S_0 = 100$ ,  $r = 0.05$ ,  $\delta = 0.02$ ,  $T = 1$ ,  $S_+ = 100$ ,  $C = 4$ ,  $G = 50$ ,  $M = 60$ ,  $Y = 0.7$ .*

Proposition 3.5). The power of  $1 - \epsilon$  in CGMY model is  $1 - 0.3Y$ . The function  $f$  converges to a constant bigger than  $1 - 0.3Y$  (which corresponds to the horizontal line) when  $\epsilon$  goes to 1. It seems that the function converges to  $1 + 0.5Y$  which corresponds to the bound  $\sigma_0(1 - \epsilon)\beta(1 - \epsilon)$ .

we observe the same phenomenon with the Asian options. In Figure 6.3, we consider an arithmetic Asian call with fixed strike. The function  $f$  is this time defined by

$$f(\epsilon) = \frac{\log\left(\left|V - \hat{V}^{1-\epsilon}\right|\right)}{\log(1 - \epsilon)}, \quad \forall \epsilon \in (0, 1).$$

where  $V$  is the price of the arithmetic Asian call and  $\hat{V}^{1-\epsilon}$  its Brownian approximation. We will also compare the limit of  $f$  with the power of  $1 - \epsilon$  in the expression  $\sigma_0(1 - \epsilon)\beta(1 - \epsilon)^{\frac{1}{5}} \log\left(\frac{1}{\beta(1 - \epsilon)^{\frac{1}{5}}}\right)$ . The power of  $1 - \epsilon$  in NIG model is 0.7. The function  $f$

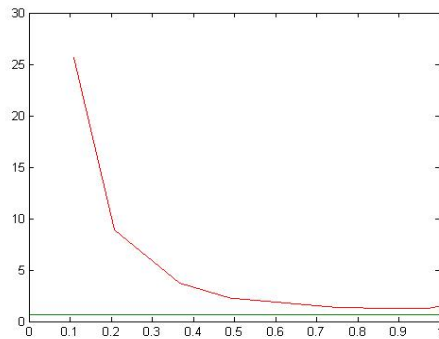


FIG. 6.3. *Optimality of the error bound for an arithmetic Asian call in NIG model. The parameters are  $S_0 = 100$ ,  $K = 90$ ,  $r = 0.0387$ ,  $\delta = 0$ ,  $T = 1$ ,  $\alpha = 6.1882$ ,  $\beta = -3.8941$ ,  $\hat{\delta} = 0.1622$ .*

converges again to a constant bigger than 0.7 (which corresponds to the horizontal line) when  $\epsilon$  goes to 1. The function seems to converge to 1.5 which corresponds to the bound  $\sigma_0(1 - \epsilon)\beta(1 - \epsilon)$ .

For barrier options, we have the same remarks as for Asian and lookback option. The bound for barrier options seems to be optimal in jump truncation case. In the Brownian approximation, the bound should be tighter, but is better than in case of jump truncation. In Figure 6.4 we consider the case of a *down and out* call in VG model. So the function is defined by

$$f(\epsilon) = \frac{\log(|V - V^{1-\epsilon}|)}{\log(1-\epsilon)}, \quad \forall \epsilon \in (0, 1).$$

It converges to 1 when  $\epsilon$  goes to 1, as expected. And in Figure 6.5 we consider the

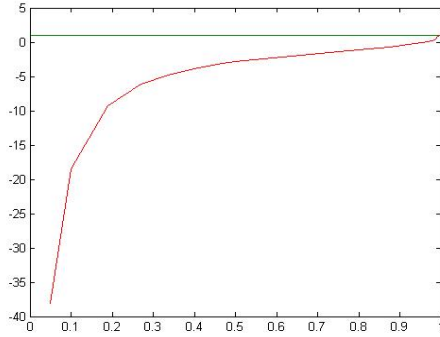


FIG. 6.4. *Optimality of the error bound for a down and out call option in VG model. The parameters are  $S_0 = 100$ ,  $K = 100$ ,  $H = 80$ ,  $r = 0.05$ ,  $\delta = 0.02$ ,  $T = 1$ ,  $\theta = -0.2$ ,  $\kappa = 0.1$ ,  $\sigma = 0.16$ .*

case of a *down and out* call in NIG model. The function  $f$  is thus defined by

$$f(\epsilon) = \frac{\log(|V - \hat{V}^{1-\epsilon}|)}{\log(1-\epsilon)}, \quad \forall \epsilon \in (0, 1).$$

The power of  $1 - \epsilon$  is bigger than that of the bound in Proposition 3.2, but it is better

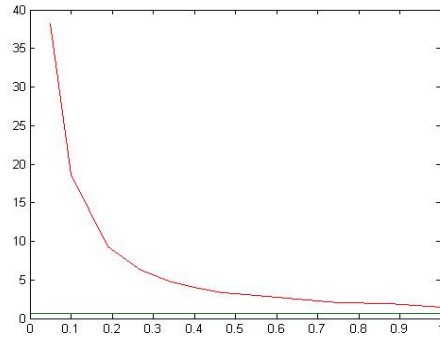


FIG. 6.5. *Optimality of the error bound for a down and out call option in NIG model. The parameters are  $S_0 = 100$ ,  $K = 100$ ,  $H = 80$ ,  $r = 0.05$ ,  $\delta = 0.02$ ,  $T = 1$ ,  $\alpha = 15$ ,  $\beta = -5$ ,  $\hat{\delta} = 0.5$ .*

than the case of jump truncation.

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