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► **To cite this version:**

Frédéric Klopp. Inverse tunneling estimates and applications to the study of spectral statistics of random operators on the real line. 2011. hal-00551931v1

**HAL Id: hal-00551931**

**<https://hal.science/hal-00551931v1>**

Preprint submitted on 5 Jan 2011 (v1), last revised 21 Jan 2012 (v4)

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# INVERSE TUNNELING ESTIMATES AND APPLICATIONS TO THE STUDY OF SPECTRAL STATISTICS OF RANDOM OPERATORS ON THE REAL LINE

FRÉDÉRIC KLOPP

ABSTRACT. We present a proof of Minami type estimates for one dimensional random Schrödinger operators valid at all energies in the localization regime provided a Wegner estimate is known to hold. The Minami type estimates are then applied to various models to obtain results on their spectral statistics.

The heuristics underlying our proof of Minami type estimates is that close by eigenvalues of a one-dimensional Schrödinger operator correspond either to eigenfunctions that live far away from each other in space or they come from some tunneling phenomena. In the second case, one can undo the tunneling and thus construct quasi-modes that live far away from each other in space.

RÉSUMÉ. Nous démontrons des inégalités de type Minami pour des opérateurs de Schrödinger aléatoires uni-dimensionnel dans toute la région localisée si une estimée de Wegner est connue. Ces estimées de type de Minami sont alors appliquées pour obtenir les statistiques spectrales pour divers modèles.

L'heuristique qui guide ce travail est que des valeurs propres proches pour un opérateur de Schrödinger sur un intervalle sont soit localisées loin l'une de l'autre soit sont la conséquence d'un phénomène d'"effet tunnel". Dans le second cas, on peut en "défaisant" cet effet tunnel construire des quasi-modes qui sont localisés loin l'un de l'autre.

## 0. INTRODUCTION

Consider the following two random operators on the real line

- the Anderson model

$$(0.1) \quad H_\omega^A = -\frac{d^2}{dx^2} + W(\cdot) + \sum_{n \in \mathbb{Z}} \omega_n V(\cdot - n)$$

where

- $W : \mathbb{R} \rightarrow \mathbb{R}$  is a bounded, continuous,  $\mathbb{Z}$ -periodic function;
- $V : \mathbb{R} \rightarrow \mathbb{R}$  is a bounded, continuous, compactly supported, non negative, not identically vanishing function;
- $(\omega_n)_{n \in \mathbb{Z}}$  are bounded i.i.d random variables, the common distribution of which admits a continuous density.

- the random displacement model

$$(0.2) \quad H_\omega^D = -\frac{d^2}{dx^2} + \sum_{n \in \mathbb{Z}} V(\cdot - n - \omega_n)$$

where

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The author is partially supported by the grant ANR-08-BLAN-0261-01.

- $V : \mathbb{R} \rightarrow \mathbb{R}$  is a bounded, odd function that has a fixed sign and is compactly supported in  $(-r_0, r_0)$  for some  $0 < r_0 < 1/2$ ;
- $(\omega_n)_{n \in \mathbb{Z}}$  are bounded i.i.d random variables, the common distribution of which admits a continuously differentiable density supported in  $[-r, r] \subset [-1/2 + r_0, 1/2 - r_0]$  and which support contains  $\{-r, r\}$ .

Let  $\bullet \in \{A, D\}$ . For  $L > 0$ , consider  $H_{\omega, L}^\bullet$  the operator  $H_\omega^\bullet$  restricted to the interval  $[-L, L]$  with Dirichlet boundary conditions. The spectrum of this operator is discrete and accumulates at  $+\infty$ ; we denote it by

$$E_1^\bullet(\omega, L) < E_2^\bullet(\omega, L) \leq \dots \leq E_n^\bullet(\omega, L) \leq \dots$$

It is well known (see e.g. [30]) that,  $\omega$  almost surely, the limit

$$N^\bullet(E) = \lim_{L \rightarrow +\infty} \frac{\#\{n; E_n^\bullet(\omega, L) \leq E\}}{2L}$$

exists and is independent of  $\omega$ .  $N^\bullet$  is the *integrated density of states* of the operator  $H_\omega^\bullet$ . This non decreasing function is the distribution function of a non negative measure, say,  $dN^\bullet$  supported on  $\Sigma^\bullet$ , the almost sure spectrum of  $H_\omega^\bullet$ .

Moreover, it is known (see e.g. [9, 24]) that there exists  $\tilde{E}^\bullet \in \mathbb{R}$  and  $\tilde{E}^\bullet > \inf \Sigma^\bullet$  such that  $N^\bullet$  is Lipschitz continuous on  $(-\infty, \tilde{E}^\bullet)$ . When  $\bullet = A$ , one can take  $\tilde{E}^\bullet = +\infty$ . For a fixed energy  $E_0$ , one defines the *locally unfolded levels* to be the points

$$\xi_n^\bullet(E_0, \omega, L) = 2L [N^\bullet(E_n^\bullet(\omega, L)) - N^\bullet(E_0)].$$

Out of these points form the point process

$$\Xi^\bullet(\xi; E_0, \omega, L) = \sum_{n \geq 1} \delta_{\xi_n^\bullet(E_0, \omega, L)}(\xi),$$

The local level statistics are described by

**Theorem 0.1.** *There exists an energy  $E^\bullet \leq \tilde{E}^\bullet$  such that  $(-\infty, E^\bullet) \cap \Sigma^\bullet \neq \emptyset$  and such that, if  $E_0 \in (-\infty, E^\bullet) \cap \Sigma^\bullet$  satisfies, for some  $\rho \geq 1$ , one has*

$$(0.3) \quad \forall a > b, \exists C > 0, \exists \varepsilon_0 > 0, \forall \varepsilon \in (0, \varepsilon_0), |N^\bullet(E_0 + a\varepsilon) - N^\bullet(E_0 + b\varepsilon)| \geq C\varepsilon^\rho$$

*then, when  $L \rightarrow +\infty$ , the point process  $\Xi(E_0, \omega, L)$  converges weakly to a Poisson process on  $\mathbb{R}$  with intensity the Lebesgue measure.*

To the best of our knowledge, Theorem 0.1 gives the first instance of a model that is not of alloy type for which local Poisson statistics have been proved.

As is to be expected from e.g. [29, 26, 15] and as we shall see in section 1, the local Poisson statistics property holds over the localized region of the spectrum i.e. the energy  $E^\bullet$  is the energy such that  $H_\omega^\bullet$  is localized in  $(-\infty, E^\bullet)$ . In particular, the conclusions of Theorem 0.1 also holds in any region of localization of  $H_\omega^\bullet$  where a Wegner type estimate is known to hold.

When  $\bullet = A$ , extending the analysis done in [11], we show that the localized region extends over the whole real axis, see section 5.2. Thus, we can take  $E^\bullet = +\infty$ . When  $\bullet = D$ , it is known that the localization region also extends over the whole real axis except for possibly a discrete set (see [3]); moreover, this analysis works under assumptions less restrictive than those made above. Unfortunately, this cannot be used directly as another crucial ingredient of our analysis is a Wegner type estimate that  $H_\omega^\bullet$  must satisfy in the localized region (see section 1). The existence of such a Wegner type estimate for  $H_\omega^D$  was proved near the bottom of the spectrum in [24].

Note that, to obtain the local Poisson statistics near an energy  $E_0$ , we do not require the density of states, i.e. the derivative of  $N^\bullet$ , not to vanish at  $E_0$ ; we only require that  $N^\bullet$  not be infinitely flat near  $E_0$  (i.e. in Theorem 0.1,  $\rho$  can be taken arbitrary positive). Such results were obtained for Anderson type models in [14] in all dimensions, at all energies in the discrete case and near the bottom of the spectrum in the continuous case.

Following the ideas of [15, 22], using the Minami type estimates that we present in section 1, we can obtain a host of other results on the asymptotics of the statistics of the eigenvalues of the random operator  $H_{\omega,L}^\bullet$  in the localized regime. We now give a few of those.

Fix  $\bullet \in \{A, D\}$ . For  $J = [a, b]$ , a compact interval s.t.  $N^\bullet(J) := N^\bullet(b) - N^\bullet(a) > 0$  and a fixed configuration  $\omega$ , consider the point process

$$\Xi_J^\bullet(\omega, t, \Lambda) = \sum_{E_n^\bullet(\omega, \Lambda) \in J} \delta_{N^\bullet(J) |\Lambda| [N_J^\bullet(E_n^\bullet(\omega, \Lambda)) - t]}$$

under the uniform distribution in  $[0, 1]$  in  $t$ ; here we have set

$$N_J^\bullet(\cdot) := \frac{N^\bullet(\cdot) - N^\bullet(a)}{N^\bullet(b) - N^\bullet(a)}.$$

This process was introduced in [27, 28]; we refer to these papers for more references, in particular, for references to the physics literature. The values  $(N^\bullet(E_n^\bullet(\omega, L)))_{n \geq 1}$  are called the *unfolded eigenvalues* of the operator  $H_{\omega,L}^\bullet$ .

Following [22], one proves

**Theorem 0.2.** *Fix  $J = [a, b] \subset (-\infty, E^\bullet) \cap \Sigma^\bullet$  a compact interval such that  $N^\bullet(b) - N^\bullet(a) = N^\bullet(J) > 0$ . Then,  $\omega$ -almost surely, the probability law of the point process  $\Xi_J^\bullet(\omega, \cdot, \Lambda)$  under the uniform distribution  $\mathbf{1}_{[0,1]}(t)dt$  converges to the law of the Poisson point process on the real line with intensity 1.*

As is shown in [28], Theorem 0.2 implies the convergence of the unfolded level spacings distributions for the levels in  $J$ . More precisely, define the  $n$ -th unfolded eigenvalue spacings

$$(0.4) \quad \delta N_n^\bullet(\omega, L) = 2L(N^\bullet(E_{n+1}(\omega, L)) - N^\bullet(E_n(\omega, L))) \geq 0.$$

Define the empirical distribution of these spacings to be the random numbers, for  $x \geq 0$

$$(0.5) \quad DLS^\bullet(x; J, \omega, L) = \frac{\#\{j; E_n^\bullet(\omega, L) \in J, \delta N_n^\bullet(\omega, L) \geq x\}}{N^\bullet(J, \omega, L)}$$

where  $N^\bullet(J, \omega, L) := \#\{E_n^\bullet(\omega, L) \in J\}$ .

**Theorem 0.3.** *Under the assumptions of Theorem 0.2,  $\omega$ -almost surely, as  $L \rightarrow +\infty$ ,  $DLS^\bullet(x; J, \omega, L)$  converges uniformly to the distribution  $x \mapsto e^{-x}$ .*

One can also obtain results for the eigenvalues themselves i.e. when they are not unfolded; we refer to [15, 22] for more details.

Finally we turn to results for level spacings that are local in energy (in the sense of Theorem 0.1). Fix  $E_0 \in (-\infty, E^\bullet) \cap \Sigma^\bullet$ . Pick  $I_L = [a_L, b_L]$ , a small interval. With the same notations as above (see (0.4)), define the empirical distribution of these spacings to be the random numbers, for  $x \geq 0$

$$(0.6) \quad DLS^\bullet(x; I_L, \omega, L) = \frac{\#\{j; E_j^\bullet(\omega, L) - E_0 \in I_L, \delta N_j^\bullet(\omega, L) \geq x\}}{N^\bullet(I_L, L, \omega)}.$$

We prove

**Theorem 0.4.** *Assume that  $E_0$  is such that, for some  $\rho \geq 1$ , there exists a neighborhood of  $E_0$ , say,  $U$  such that  $(x, y) \in U^2$ ,  $|N^\bullet(x) - N^\bullet(y)| \geq |x - y|^\rho$ .*

*Fix  $(I_L)_L$  a sequence of intervals such that  $\sup_{x \in I_L} |x| \xrightarrow{|L| \rightarrow +\infty} 0$ .*

*Assume that, for some  $\delta > 0$ , one has*

$$(0.7) \quad \begin{aligned} & \star L^{1-\delta} \cdot N^\bullet(E_0 + I_L) \xrightarrow{|L| \rightarrow +\infty} +\infty, \\ & \star \text{ if } \ell' = o(L) \text{ then } \frac{N^\bullet(E_0 + I_{L+\ell'})}{N^\bullet(E_0 + I_L)} \xrightarrow{|L| \rightarrow +\infty} 1. \end{aligned}$$

*Then, with probability 1, as  $|L| \rightarrow +\infty$ ,  $DLS(x; I_L, \omega, L)$  converges uniformly to the distribution  $x \mapsto e^{-x}$ , that is, with probability 1,*

$$(0.8) \quad \sup_{x \geq 0} |DLS(x; I_L, \omega, L) - e^{-x}| \xrightarrow{|L| \rightarrow +\infty} 0.$$

The condition (0.7) on the intervals  $(I_L)_L$  is to ensure that they contain sufficiently many eigenvalues for the empirical distribution to make sense and that this number does not vary too wildly when one slightly changes the size of  $I_L$ .

The main technical results of the present paper that we turn to below entail a number of other consequences about the spectral statistics in the localized region. We refer to [15, 22] for more such examples and more references.

## 1. THE SETTING AND THE RESULTS

Let us now turn to the main results of this paper. They concern random operators on the real line and consist in Minami type estimates valid for all energies in the localization region of general one dimensional random operator satisfying a Wegner estimate. As we shall see, we obtain two different estimates of the same type having each their own advantages and drawbacks.

Let us start with a description of our setting. From now on, on  $L^2(\mathbb{R})$ , we consider consider random Schrödinger operator of the form

$$(1.1) \quad H_\omega u = -\frac{d^2}{dx^2} u + q_\omega u$$

where  $q_\omega$  is an almost surely bounded  $\mathbb{Z}^d$ -ergodic random potential.

**Remark 1.1.** The boundedness assumption may be relaxed so as to allow local singularities and growth at infinity. We make it to keep our proofs as simple as possible.

It is well known (see e.g. [30]) that  $H_\omega$  then admits an integrated density of states, say,  $N$ , and, an almost sure spectrum, say,  $\Sigma$ . We now fix  $I$  an open interval in  $\Sigma$  and the subsequent assumptions and statements will be made on energies in  $I$ .

Let  $H_\omega(\Lambda)$  be the random Hamiltonian  $H_\omega$  restricted to the interval  $\Lambda := [0, L]$  with periodic boundary conditions.

We now state our main assumptions and comment on the validity of these assumptions for the models  $H_\omega^{A,D}$  defined respectively in (0.1) and (0.2).

Our first assumption will be a independence assumption for local Hamiltonians that are far away from each other, that is,

**(IAD):** There exists  $R_0 > 0$  such that for  $\text{dist}(\Lambda, \Lambda') > R_0$ , the random Hamiltonians  $H_\omega(\Lambda)$  and  $H_\omega(\Lambda')$  are independent.

**Remark 1.2.** This assumption may be relaxed to asking some control on the correlations between the random Hamiltonians restricted to different cubes. To keep the proofs as simple as possible, we assume (IAD).

Next, we assume that

**(W):** a Wegner type estimate holds i.e. there exists  $C > 0$ ,  $s \in (0, 1]$  and  $\rho \geq 1$  such that, for  $J \subset I$ , and  $\Lambda$ , an interval in  $\mathbb{R}$ , one has

$$(1.2) \quad \mathbb{E} [\text{tr}(\mathbf{1}_J(H_\omega(\Lambda)))] \leq C|J|^s|\Lambda|^\rho.$$

Here,  $|\cdot|$  denotes the length of the interval  $\cdot$ .

**Remark 1.3.** In many cases e.g. for the operators  $H_\omega^{A,D}$ , assumption (W) is known to hold for  $s = 1$  and  $\rho = 1$ . In the case of  $H_\omega^A$ , we can take  $I = \Sigma^A$  (see e.g. [9]). For Anderson type Hamiltonians with single site potentials that are not of fixed sign, Wegner estimates have been proved near the bottom of the spectrum and at spectral edges (see [21, 20]).

In the case of  $H_\omega^D$ , it holds near the bottom of  $\Sigma^D$  (see [24]).

The second assumption crucial to our study is the existence of a localization region to which  $I$  belongs i.e. we assume

**(Loc):** there exists  $\xi > 0$  such that

$$(1.3) \quad \sup_{\substack{L > 0 \\ \text{supp } f \subset I \\ |f| \leq 1}} \mathbb{E} \left( \sum_{n \in \mathbb{Z}} e^{\xi|n|} \|\mathbf{1}_{[-1/2, 1/2]} f(H_\omega(\Lambda_L)) \mathbf{1}_{[n-1/2, n+1/2]}\|_2 \right) < +\infty.$$

**Remark 1.4.** For the models  $H_\omega^{A,D}$ , the spectral theory has been studied under various assumptions on  $V$  and  $(\omega_\gamma)_\gamma$  (see e.g. [4, 10, 31, 17]). The existence of a region of localized states is well known and, in many cases, this region extends over the whole spectrum. In the case of  $H_\omega^A$ , in [11], this is proved under a more restrictive support condition on  $V$ , namely, that the support of  $V$  is contained in  $(-1/2, 1/2)$ ; that this condition can be removed is proved in section 5.2.

For Anderson type Hamiltonians with single site potentials that are not of fixed sign, localization has been proved in [21, 20] at the bottom of the spectrum

In the case of model  $H_\omega^D$ , localization has been proved at all energies except possibly at a discrete set (see [3]).

**1.1. An Minami type estimate in the localization region.** Our main technical results are the following Minami type estimates

**Theorem 1.1.** *Assume (W) and (Loc). Fix  $J$  compact in  $I$  the region of localization. Then, there exists  $\beta > 0$ , such that, for any  $q > 0$ , there exists  $L_q > 0$  s.t., for  $E \in J$ ,  $L \geq L_q$  and  $\varepsilon \in [L^{-q}, (\log L)^{-1-\beta}]$ , one has*

$$(1.4) \quad \sum_{k \geq 2} \mathbb{P} (\text{tr} [\mathbf{1}_{[E-\varepsilon, E+\varepsilon]}(H_\omega(\Lambda_L))] \geq k) \leq \beta (\varepsilon^s L)^2 (\log L)^\beta e^{\beta \varepsilon L / \log L} + L^{-q}.$$

The estimate (1.4) only becomes useful when  $\varepsilon^s L$  is small; as  $\rho \geq 1$ , this is also the case for the Wegner type estimate (W). Note that, as  $s \leq 1$ ,  $\varepsilon^s L$  is small only when  $\varepsilon L$  is small. Finally, note that, as  $\rho \geq 1$ , the factor  $(\varepsilon^s L)^2$  is better i.e. smaller than  $(\varepsilon^s L^\rho)^2$ , the square of the upper bound obtained by the Wegner type estimate (W). This improvement is a consequence of localization; this is analogous to the

improvement upon (W) given by Theorem 3.1.

The estimate (1.4) is stronger than the standard Minami estimate (see [26, 1, 16]) as the standard estimate only bounds the first term in the left hand side of (1.4). It is weaker than the Minami type estimate found in [8] which gives a bound on  $\sum_{k \geq 2} k \mathbb{P}(\text{tr}[\mathbf{1}_{[E-\varepsilon, E+\varepsilon]}(H_\omega(\Lambda_L))] \geq k)$ . The estimate (1.4) is nevertheless sufficient

to repeat the analysis done in [15, 22]. In particular, it is sufficient to obtain the description of the eigenvalues of  $H_\omega(\Lambda_L)$  in terms of the ‘‘approximated eigenvalues’’ i.e. the eigenvalues of  $H_\omega$  restricted to smaller cubes and to compute the law of those approximated eigenvalues (see [15, Lemma 2.1, Theorem 1.15 and 1.16], [22]).

One checks that Theorem 1.1 implies that, for any  $\rho \in (0, 1)$ , there exists  $\beta > 0$ , such that, for any  $q > 0$ , there exists  $L_q > 0$  s.t., for  $E \in J$ ,  $L \geq L_q$  and  $\varepsilon \in [L^{-q}, \beta^{-1}L^{-1/s}]$ , one has

$$(1.5) \quad \sum_{k \geq 2} \mathbb{P}(\text{tr}[\mathbf{1}_{[E-\varepsilon, E+\varepsilon]}(H_\omega(\Lambda_L))] \geq k) \leq \beta(\varepsilon^s L)^{1+\rho}.$$

Except for the condition on  $\varepsilon$ , this estimate is the right replacement of the assumption (M) made in [15, 22] if, in the Wegner type estimate (W), one has  $s = 1$ . One checks that the condition on  $\varepsilon$  is satisfied in the use made of this estimate in [15, 22].

**Remark 1.5.** The proof shows that one can take the exponent of the logarithmic error term in (1.4) to be  $4s + 2\rho$  where  $s$  and  $\rho$  are defined by (W).

As the density of states may vanish locally, it is useful to obtain an estimate of the left-hand side of (1.4) in terms of the weight the integrated density of states puts on a small interval around  $E$ . To obtain the results given in the introduction (in particular, to be able to allow  $\rho > 0$  to be arbitrary in (0.3) or in the assumption of Theorem 0.4), such an improvement is crucial. Ideally, one would like the estimate to involve  $N(E + \varepsilon) - N(E - \varepsilon)$  as is done in [14] for certain models.

We prove

**Theorem 1.2.** *Assume (IAD), (W) and (Loc). Fix  $J$  compact in  $I$  the region of localization. Then, there exists  $\beta > 0$  such that, for any  $q > 0$ , there exists  $L_q > 0$  s.t., for  $E \in J$ ,  $L \geq L_q$  and  $\varepsilon \in [L^{-q}, (\log L)^{-\beta}]$ , one has*

$$(1.6) \quad \sum_{k \geq 2} \mathbb{P}(\text{tr}[\mathbf{1}_{[E-\varepsilon, E+\varepsilon]}(H_\omega(\Lambda_L))] \geq k) \\ \leq L^{-q} + \beta(\log L)^\beta L^2 \varepsilon^s \left[ N(E + \varepsilon \cdot \log^\beta L) - N(E - \varepsilon \cdot \log^\beta L) \right] e^{\varepsilon^s L \log^\beta L}.$$

Let us compare (1.4) and (1.6). When  $\rho = 1$  in the Wegner type estimate (1.2),  $N(E)$  is Hölder continuous with exponent  $s$ . Thus, up to the error terms, (1.6) implies (1.4). When  $N$  is differentiable at  $E$  and the derivative of  $N$  does not vanish at  $E$ , (1.4) is essentially equivalent to (1.6). If the derivative of  $N$  vanishes at  $E$  then, (1.6) is stronger than (1.4).

One should also compare (1.6) with the improved Minami estimate obtained in [14]. The estimate (1.6) is worse as, in [14], the upper bound is obtained in terms of  $N(E + \varepsilon) - N(E - \varepsilon)$ . Nevertheless, if  $E$  satisfies the assumptions of Theorem 0.4, then estimate (1.6) is sufficient to repeat the analysis done in [14] and obtain the spectral statistics for (not too fast) vanishing density of states.

**Remark 1.6.** The proof shows that, in (1.6), one can take the exponent of the logarithmic error term to be

- $\rho + 2 + s$  in the volume factor where  $s$  and  $\rho$  are defined by (W),
- 4 in the term involving the integrated density of states  $N$ ,
- $2\rho$  in the term in the exponential.

To the best of our knowledge, up to the present work, the availability of decorrelation estimates of the type (1.4) or (1.6) relied on the fact that the single site potential was rank one ([26, 1, 16, 8]) or had the effective weight of a rank one potential as was shown in [7] in the Lifshitz tails regime. In the present paper, we exhibit a heuristic why such estimates should hold quite generally and use it to develop a different approach. This approach makes crucial use of localization to reduce the complexity of the problem i.e. to study the random Hamiltonian restricted to some much smaller cube. Such ideas were already used in [23] to study spectral correlations at distinct energies.

**1.2. Some consequences.** Before explaining the heuristics guiding the proofs of Theorems 1.1 and 1.2, let us very briefly describe some consequences for random operators. Essentially, all the conclusions described for the models  $H_\omega^\bullet$  ( $\bullet \in \{A, D\}$ ) in the introduction hold under the assumptions (IAD), (W) and (Loc). As said in the introduction, following [15, 22], more results on the spectral statistics can be obtained. As assumptions (IAD) and (Loc) have been proved for many models e.g. the Poisson model (see e.g. [3, 13, 12] or the Bernoulli Anderson model (see e.g. [2]), it remains to understand Wegner type estimates (W) or replacements of such estimates for those models.

**1.3. Inverse tunneling and the Minami type estimates.** We now turn to the heuristic we referred to earlier. The basic mechanism at work in our heuristics is what we call “inverse tunneling”. Let us explain this and therefore, first recall some facts on tunneling.

Fix  $\ell \in \mathbb{R}$  and  $q : [0, \ell] \rightarrow \mathbb{R}$  a real valued function bounded by  $Q > 0$ . On  $[0, \ell]$ , consider the Dirichlet eigenvalue problem defined by the differential expression  $-u'' + qu$  i.e. the eigenvalue problem

$$(1.7) \quad -\frac{d^2}{dx^2}u(x) + q(x)u(x) = Eu(x), \quad u(0) = u(\ell) = 0.$$

Tunneling estimates can be described as follows. Assume that the interval  $[0, \ell]$  can be split into two intervals, say,  $[0, \ell']$  and  $[\ell', \ell]$ , such that the Dirichlet eigenvalue problem for each of those intervals share a common eigenvalue and such that the associated eigenfunctions are “very” (typically exponentially) small near  $\ell'$  then the eigenvalue problem (1.7) has two eigenvalues that are “very” close together. The closeness of the eigenvalues and the smallness of the eigenfunctions are related; they are in general measured in terms of some parameter e.g. a coupling constant in front of the potential  $q$ , a semi-classical parameter in front of the kinetic energy  $-d^2/dx^2$  or the length of the interval  $\ell$  (see e.g. [25, 19, 5, 6]). The tunneling effect is well illustrated by the double well problem (see e.g. [18]).

In the present paper, we discuss a converse to the above description i.e. we assume that the eigenvalue problem (1.7) has two (or more) close together eigenvalues, say, 0 and  $E$  small associated respectively to  $u$  and  $v$ . Let  $r_u := \sqrt{|u|^2 + |u'|^2}$  and  $r_v := \sqrt{|v|^2 + |v'|^2}$ . Then, either of two things happen:



- (1) no tunneling occurs i.e.  $r_u \cdot r_v$  is small on the whole interval  $[0, \ell]$ . In this case, the eigenfunctions  $u$  and  $v$  live in separate space regions and, thus, don't see each other.
- (2) tunneling occurs i.e.  $r_u \cdot r_v$  becomes large in some region of space. In the connected components of such regions,  $u$  and  $v$  are roughly proportional. Thus, we show that it is possible to construct linear combinations of  $u$  and  $v$  that live in distinct space regions (i.e. undo the tunneling); these linear combinations are not true eigenfunctions anymore but they almost satisfy the eigenvalue equation as  $E$  is close to 0.

In both cases, we construct almost eigenfunctions that are living in distinct space regions (see section 2.2). Thus, we derive (1.4) using the Wegner type estimate (W) in each of these regions (see section 3). This yields Theorem 1.1. To obtain Theorem 1.2, one notices that, as at least one of the two regions is large, in this region, one can replace one of the Wegner type estimates by an improved estimate given by Theorem 3.1.

**1.4. Universal estimates.** For the sake of simplicity, assume  $q : [0, \ell] \rightarrow \mathbb{R}$  is bounded. On  $[0, \ell]$ , consider the operator  $Hu = -u'' + qu$  with self-adjoint Robin boundary conditions at 0 and  $\ell$  (i.e.  $u(0) \cos \alpha + u'(0) \sin \alpha = 0$ ). Then, one has

**Theorem 1.3.** *Fix  $J$  compact. There exists a constant  $C > 0$  (depending only on  $\|q\|$  and  $J$ ) such that, for  $\ell \geq 1$ , if  $\varepsilon \in (0, 1)$  is such that  $|\log \varepsilon| \geq C\ell$ , then, for any  $E \in J$ , the interval  $[E - \varepsilon, E + \varepsilon]$  contains at most a single eigenvalue of  $H$ .*

One can generalize this to

**Theorem 1.4.** *Fix  $\nu > 2$  and  $J$  compact. There exists  $\ell_0 > 1$  and  $C > 0$  (depending only on  $\|q\|_\infty$  and  $J$ ) such that, for  $\ell \geq \ell_0$ , if  $\varepsilon \in (0, \ell^{-\nu})$  then, for  $E \in J$ , the number of eigenvalues of  $H$  in the interval  $[E - \varepsilon, E + \varepsilon]$  is bounded by  $\max(1, C\ell/|\log \varepsilon|)$ .*

These a-priori bounds prove that there is some repulsion between the level for arbitrary Schrödinger operators in dimension one. For random systems in the localized phase, this repulsion takes place at a length scale of size  $e^{-C\ell}$ ; it is much smaller than the typical level spacings that is of size  $1/\ell$ .

## 2. INVERSE TUNNELING ESTIMATES

Fix  $\ell \in \mathbb{R}$  and  $q : [0, \ell] \rightarrow \mathbb{R}$  a real valued function bounded by  $Q > 0$ . On  $[0, \ell]$ , consider the Sturm-Liouville eigenvalue problem defined by

$$(2.1) \quad (Hu)(x) := -\frac{d^2}{dx^2}u(x) + q(x)u(x) = Eu(x), \quad u(0) = 0 = u(\ell).$$

**Remark 2.1.** Here, we use Dirichlet boundary conditions; the same analysis goes through with general Robin boundary conditions.

For  $u$ , a solution to (2.1), define the Prüfer variables (see e.g. [32]) by

$$r_u(x) \begin{pmatrix} \sin(\varphi_u(x)) \\ \cos(\varphi_u(x)) \end{pmatrix} := \begin{pmatrix} u(x) \\ u'(x) \end{pmatrix}, \quad r_u(x) > 0, \quad \varphi_u(x) \in \mathbb{R}$$

By the Cauchy-Lipschitz Theorem, if  $u$  does not vanish identically,  $r_u$  does not vanish.  $\varphi_u$  is chosen so as to be continuous. If  $u$  is a solution to (2.1), then we set

$\varphi_u(0) = 0$  and  $\varphi_u(\ell) = k\pi$  (for some  $k \in \mathbb{N}^*$ ). In the Prüfer variables, the eigenvalue equation in (2.1) becomes

$$(2.2) \quad \varphi'_u(x) = 1 - (1 + (q(x) - E)) \sin^2(\varphi_u(x))$$

$$(2.3) \quad \frac{r'_u(x)}{r_u(x)} = (1 + (q(x) - E)) \sin(\varphi_u(x)) \cos(\varphi_u(x)).$$

**2.1. General estimates for eigenfunctions associated to nearby eigenvalues.** Consider now  $u$  and  $v$  two normalized eigenfunctions of the Sturm-Liouville problem (2.1) associated to two consecutive eigenvalues, say, 0 and  $E$ . We assume  $0 < E \ll 1$ . Sturm's oscillation theorem then guarantees that  $\varphi_u(x) < \varphi_v(x)$  for  $x \in (0, \ell)$  and  $\varphi_v(\ell) = \varphi_u(\ell) + \pi$  (see e.g. [32]). Define

$$(2.4) \quad \delta\varphi(x) = \varphi_v(x) - \varphi_u(x).$$

Thus,  $\delta\varphi(0) = 0$ ,  $\delta\varphi(\ell) = \pi$  and  $\delta\varphi(x) \in (0, \pi)$  for  $x \in (0, \ell)$ .

The function  $\delta\varphi$  satisfies the following differential equation

$$(2.5) \quad \begin{aligned} (\delta\varphi)'(x) &= (1 + q(x))[\sin^2(\varphi_v(x)) - \sin^2(\varphi_u(x))] - E \sin^2(\varphi_v(x)) \\ &= (1 + q(x)) \sin(\delta\varphi(x)) \sin(2\varphi_v(x) - \delta\varphi(x)) - E \sin^2(\varphi_v(x)). \end{aligned}$$

On intervals where  $\sin(\delta\varphi(x))$  is small,  $r_u$  and  $r_v$  are essentially proportional to each other. We prove

**Lemma 2.1.** *Fix  $\varepsilon > 0$ . Assume that, for  $x \in [x_-, x_+]$ , one has  $\sin(\delta\varphi(x)) \leq \varepsilon$ . Then, there exists  $\lambda > 0$  such that, for  $x \in [x_-, x_+]$ , one has*

$$(2.6) \quad e^{-[(Q+1)\varepsilon+E]\ell} \leq \frac{r_v(x)}{r_u(x)} \frac{1}{\lambda} \leq e^{[(Q+1)\varepsilon+E]\ell}.$$

*Proof.* Comparing (2.3) for  $u$  and  $v$  yields

$$(2.7) \quad \left[ \log \left( \frac{r_v(x)}{r_u(x)} \right) \right]' = (1 + q(x)) \sin(\delta\varphi(x)) \cos(2\varphi_v(x) - \delta\varphi(x)) - E \sin(2\varphi_v(x))$$

As, for  $x \in [x_-, x_+]$  one has  $0 \leq \sin(\delta\varphi(x)) \leq \varepsilon$ , (2.7) yields, for  $x \in [x_-, x_+]$ ,

$$\left| \left[ \log \left( \frac{r_v(x)}{r_u(x)} \right) \right]' \right| \leq (1 + Q)\varepsilon + E.$$

Integrating this equation, for  $(x, y) \in [x_-, x_+]^2$ , one obtains

$$(2.8) \quad e^{-[(Q+1)\varepsilon+E](y-x)} \frac{r_v(y)}{r_u(y)} \leq \frac{r_v(x)}{r_u(x)} \leq e^{[(Q+1)\varepsilon+E](y-x)} \frac{r_v(y)}{r_u(y)}.$$

This in particular immediately yields (2.6) and completes the proof of Lemma 2.1.  $\square$

On intervals where  $\sin(\delta\varphi(x))$  is “large”, the Wronskian of  $u$  and  $v$  does not vary much on intervals over which  $\sin(\delta\varphi(x))$  is “large”. We prove

**Lemma 2.2.** *Fix  $\varepsilon > 0$  such that  $E < \varepsilon < 1$ . Assume that*

- for  $x \in [x_-, x_+]$ , one has  $\sin(\delta\varphi(x)) \geq \varepsilon$ ;
- $\sin(\delta\varphi(x_{\pm})) = \varepsilon$ .

Then, for  $x \in [x_-, x_+]$ , one has  $w(u, v)(x) > 0$  and

$$\max_{x \in [x_-, x_+]} \left[ 1 - \frac{w(u, v)(x)}{\max_{x \in [x_-, x_+]} w(u, v)(x)} \right] \leq \frac{E}{\varepsilon} (x_+ - x_-) \leq \frac{E\ell}{\varepsilon}.$$

*Proof.* Consider the Wronskian of  $u$  and  $v$ , say,  $w(u, v)(x) = u'(x)v(x) - v'(x)u(x)$ . As  $u$  and  $v$  are eigenfunctions for the same Sturm-Liouville problem for the eigenvalues 0 and  $E$ ,  $w(u, v)$  satisfies the equation  $[w(u, v)]' = Euv$ . Thus, for  $(x, y) \in [0, \ell]^2$ , one has

$$(2.9) \quad w(v, u)(x) = r_u(x)r_v(x) \sin(\delta\varphi(x)),$$

and

$$(2.10) \quad \begin{aligned} & r_u(x)r_v(x) \sin(\delta\varphi(x)) - r_u(y)r_v(y) \sin(\delta\varphi(y)) \\ &= E \int_y^x r_u(t)r_v(t) \sin(\varphi_u(t)) \sin(\varphi_v(t)) dt. \end{aligned}$$

The first statement is a direct consequence of (2.9) and the assumption on  $[x_-, x_+]$ . As for  $x \in [x_-, x_+]$ , one has  $\sin(\delta\varphi(x)) \geq \varepsilon$ , using (2.9), for  $(x, y) \in [x_-, x_+]^2$ , one can rewrite (2.10) as

$$w(u, v)(y) - w(u, v)(x) = \frac{E}{\varepsilon} \int_x^y w(u, v)(t)g(t)dt \quad \text{where} \quad \|g\| \leq 1.$$

Thus, for  $y$  such that  $w(u, v)(y) = \max_{x \in [x_-, x_+]} w(u, v)(x)$ , we obtain

$$0 \leq 1 - \frac{w(u, v)(x)}{\max_{x \in [x_-, x_+]} w(u, v)(x)} = \frac{E}{\varepsilon} \int_x^y \frac{w(u, v)(t)}{\max_{x \in [x_-, x_+]} w(u, v)(x)} g(t)dt \leq \frac{E}{\varepsilon} |y - x|$$

This completes the proof of Lemma 2.2.  $\square$

We prove

**Lemma 2.3.** Fix  $\varepsilon > 0$  such that  $E < \varepsilon < 1$ . Assume that

- for  $x \in [x_-, x_+]$ , one has  $\sin(\delta\varphi(x)) \geq \varepsilon$ ;
- $\sin(\delta\varphi(x_{\pm})) = \varepsilon$ .

Then, for any  $a > 1$ ,

- either  $r_u(x_-)r_v(x_+) + r_u(x_+)r_v(x_-) \leq 2a\ell(E/\varepsilon)^2$ ,
- or

$$1 - \frac{1}{1+a} \leq \frac{r_u(x_-)r_v(x_-)}{r_u(x_+)r_v(x_+)} \leq 1 + \frac{1}{a}.$$

*Proof.* Recall that the system  $(u, v)$  is orthonormal in  $L^2([0, \ell])$ ; thus, one has

$$(2.11) \quad \int_0^\ell r_u^2(x) \sin^2(\varphi_u(x)) dx = 1 = \int_0^\ell r_v^2(x) \sin^2(\varphi_v(x)) dx,$$

$$(2.12) \quad \int_0^\ell r_u(x)r_v(x) \sin(\varphi_u(x)) \sin(\varphi_v(x)) dx = 0.$$

As  $w(v, u)(0) = w(v, u)(\ell) = 0$ , this and (2.10) implies that

$$(2.13) \quad 0 < \max_{x \in [0, \ell]} r_u(x)r_v(x) \sin(\delta\varphi(x)) \leq E.$$

As for  $x \in [x_-, x_+]$ , one has  $\sin(\delta\varphi(x)) \geq \varepsilon/2$ , one obtains

$$(2.14) \quad 0 < \max_{x \in [x_-, x_+]} r_u(x)r_v(x) \leq 2E/\varepsilon.$$

Inserting this estimate into (2.10) for  $(x, y) = (x_-, x_+)$  and using the fact that  $\sin(\delta\varphi(x_-)) = \varepsilon = \sin(\delta\varphi(x_+))$ , one obtains

$$|r_u(x_-)r_v(x_-) - r_u(x_+)r_v(x_+)| \leq 2\ell(E/\varepsilon)^2.$$

This implies the alternative asserted by Lemma 2.3.  $\square$

**2.2. An inverse "splitting" result.** We prove

**Theorem 2.1.** *Fix  $S > 0$  arbitrary and  $J \subset \mathbb{R}$  a compact interval. There exists  $\varepsilon_0 > 0$  and  $\ell_0 > 0$  (depending only on  $\|q\|_\infty$ ,  $J$  and  $S$ ) such that, for  $\ell \geq \ell_0$  and  $0 < \varepsilon\ell^4 \leq \varepsilon_0$ , for  $E \in J$ , if the operator  $H$  defined in (2.1) has two eigenvalues in  $[E - \varepsilon, E + \varepsilon]$ , then there exists two points  $x_+$  and  $x_-$  in the lattice segment  $\varepsilon_0\mathbb{Z} \cap [0, \ell]$  satisfying  $S < x_+ - x_- < 2S$  such that, if  $H_-$ , resp.  $H_+$ , denotes the second order differential operator  $H$  defined by (1.7) and Dirichlet boundary conditions on  $[0, x_-]$ , resp. on  $[x_+, \ell]$ , then  $H_-$  and  $H_+$  each have an eigenvalue in the interval  $[E - \varepsilon\ell^4/\varepsilon_0, E + \varepsilon\ell^4/\varepsilon_0]$ .*

Theorem 2.1 is a consequence of Propositions 2.1 and 2.3 that are respectively proved in sections 2.2.1 and 2.2.2. Let us now explain the ideas guiding the proof of Theorem 2.1.

Up to a shift in energy and potential  $q$ , we may assume that  $E = 0$  and that, changing the notations, the eigenvalues considered in Theorem 2.1 are 0 and  $E > 0$ . All the estimates we will prove are uniform in  $\|q\|_\infty$  in this new setting, thus, uniform in  $\|q\|_\infty$  and  $J$  in the old setting. Note that, in the new notations we have  $E \leq \varepsilon$ . Let  $u$  and  $v$  be the eigenfunctions associated respectively to 0 and  $E$ . The goal is then to prove that we can find two independent linear combinations of  $u$  and  $v$  such that

- they vanish at two points, say,  $x_-$  and  $x_+$  satisfying the statement of Theorem 2.1,
- in each of these intervals  $[0, x_-]$  and  $[x_+, \ell]$ , the masses of the combinations are of size of order  $\ell^{-\alpha}$  (for some  $\alpha > 0$ ).

Therefore, we consider to cases:

- (1) if  $r_u \cdot r_v$  becomes "large" over  $[0, \ell]$  which we dub the "tunneling case".
- (2) if  $r_u \cdot r_v$  stays "small" over  $[0, \ell]$  which we dub the "non tunneling case".

In the first case,  $u$  and  $v$  put mass at the same locations in  $[0, \ell]$ . This is typically what happens in a tunneling situation (see e.g. [25, 19, 5, 6]). In this case, there is a strong "interaction" between  $u$  and  $v$  and the estimates obtained in section 2 enable us to show that  $u$  and  $v$  are quite similar up to a phase change. Although they are linearly independent (as they are eigenfunctions associated to distinct eigenvalues of the same self-adjoint operator), they are similar in the sense that  $r_u$  and  $r_v$  are similar (see Lemma 2.1). Their orthogonality comes mainly from the phase difference. We analyze this phase difference to prove that the claims of Theorem 2.1 hold in this case.

In the second case,  $u$  and  $v$  live "independent lives"; they are of course orthogonal but  $|u|$  and  $|v|$  (actually,  $r_u$  and  $r_v$  too) are also almost orthogonal. So,  $u$  and  $v$  roughly live on disjoint sets; this makes it quite simple to construct the functions

whose existence is claimed in Theorem 2.1: one just needs to restrict  $u$  and  $v$  to their “essential supports” to find the functions the existence of which is claimed in Theorem 2.1.

2.2.1. *When there is tunneling.* The case when there is tunneling can be described by the fact that the function  $u$  and  $v$  are “large” at the same location or equivalently by the fact that  $r_u \cdot r_v$  becomes “large” at some point of the interval  $[0, \ell]$ . Clearly, as  $u$  and  $v$  are normalized,  $r_u$  and  $r_v$  need each to be at most only of size  $1/\sqrt{\ell}$ . So one can say that  $r_u \cdot r_v$  becomes “large” if and only if  $r_u \cdot r_v \gtrsim \ell^{-1}$  in somewhere in  $[0, \ell]$ .

We prove

**Proposition 2.1.** *Fix  $S > 0$  arbitrary. There exists  $\eta_0 > 0$  (depending only on  $S$  and  $\|q\|_\infty$ ) such that, for  $\eta \in (0, \eta_0)$  and  $\ell$  sufficiently large (depending only on  $\eta$ ,  $S$  and  $\|q\|_\infty$ ), if  $u$  and  $v$  are as in section 2, that is, eigenfunctions of  $H$  associated respectively to the eigenvalues 0 and  $E$ , and, assume that  $E\ell^4 \leq \eta^4$  and that one has*

$$(2.15) \quad \exists x_0 \in [0, \ell], \quad r_u(x_0) \cdot r_v(x_0) \geq \frac{\eta}{\ell},$$

then, there exists two points  $x_+$  and  $x_-$  in the lattice segment  $\eta\mathbb{Z} \cap [0, \ell]$  satisfying

$$(2.16) \quad |\log(E\ell^2)|/C < x_- < x_+ < \ell - |\log(E\ell^2)|/C \quad \text{and} \quad S < x_+ - x_- < 2S$$

such that, if  $H_-$  (resp.  $H_+$ ) denotes the second order differential operator  $H$  defined by (2.1) and Dirichlet boundary conditions on  $[0, x_-]$  (resp. on  $[x_+, \ell]$ ), then  $H_-$  and  $H_+$  have an eigenvalue in the interval  $[-E\ell^4\eta^{-4}, E\ell^4\eta^{-4}]$ .

*Proof.* We keep the notations of section 2.1. By (5.2) for  $u$  and  $v$ , (2.15) implies that there exists  $C > 0$  (depending only on  $\|q\|_\infty$ ) such that

$$(2.17) \quad \forall x \in [x_0 - 1, x_0 + 1] \cap [0, \ell] \quad r_u(x) \cdot r_v(x) \geq \frac{\eta}{C\ell},$$

Note that, by (2.15) and (2.13), one has

$$(2.18) \quad \forall x \in [x_0 - 1, x_0 + 1] \cap [0, \ell], \quad \sin(\delta\varphi(x)) \leq E\ell/\eta$$

For the sake of definiteness, we assume that

$$(2.19) \quad \forall x \in [x_0 - 1, x_0 + 1] \cap [0, \ell], \quad 0 \leq \delta\varphi(x) \leq 2E\ell/\eta,$$

the case  $\pi - 2E\ell/\eta \leq \delta\varphi(x) \leq \pi$  being dealt with in the same way.

As  $u$  and  $v$  are normalized and orthogonal to each other, one proves

**Lemma 2.4.** *There exists  $C > 0$  (depending only on  $\|q\|_\infty$ ) and  $x_2 \in [0, \ell]$  such that, for  $x \in [x_2 - 1, x_2 + 1] \cap [0, \ell]$ , one has*

$$(2.20) \quad r_u(x) \cdot r_v(x) \geq \frac{\eta}{C\ell^2} \quad \text{and} \quad \pi - 2CE\ell^2/\eta \leq \delta\varphi(x) \leq \pi.$$

*Proof.* Indeed, by (2.17) and (2.19), one has, for some  $C > 0$ ,

$$(2.21) \quad \left| \int_{[x_0-1, x_0+1] \cap [0, \ell]} r_u(x)r_v(x) \sin^2(\varphi_u(x)) dx \right. \\ \left. + \int_{[0, \ell] \setminus [x_0-1, x_0+1]} r_u(x)r_v(x) \sin(\varphi_u(x)) \sin(\varphi_v(x)) dx \right| \leq C \frac{E\ell}{\eta}.$$

Hence, by (5.3) in Lemma 5.1 and (2.17), as  $E\ell^4 \leq \eta_0$ , we get that, for some  $C > 0$  (depending only on  $\|q\|_\infty$ ), one has

$$(2.22) \quad \int_{[0,\ell] \setminus [x_0-1, x_0+1]} r_u(x)r_v(x) \sin(\varphi_u(x)) \sin(\varphi_v(x)) dx \leq -\frac{\eta}{C\ell} \left(1 - \frac{C\eta^2}{\ell^2}\right) \leq -\frac{\eta}{2C\ell}$$

for  $\ell$  sufficiently large.

Write

$$(2.23) \quad \begin{aligned} & \int_{[0,\ell] \setminus [x_0-1, x_0+1]} r_u(x)r_v(x) \sin(\varphi_u(x)) \sin(\varphi_v(x)) dx \\ &= \int_{\substack{x \in [0,\ell] \setminus [x_0-1, x_0+1] \\ r_u(x)r_v(x) \leq \eta/\ell^2}} r_u(x)r_v(x) \sin(\varphi_u(x)) \sin(\varphi_v(x)) dx \\ & \quad + \int_{\substack{x \in [0,\ell] \setminus [x_0-1, x_0+1] \\ r_u(x)r_v(x) > \eta/\ell^2}} r_u(x)r_v(x) \sin(\varphi_u(x)) \sin(\varphi_v(x)) dx \end{aligned}$$

By (2.13), on the set  $\{x \in [0, \ell]; r_u(x)r_v(x) > \eta/\ell^2\}$ , one has  $\sin(\delta\varphi(x)) \leq E\ell^2/\eta$ . Thus, as  $E\ell^4 \leq \eta$ , (2.23) yields

$$(2.24) \quad \int_{\substack{x \in [0,\ell] \setminus [x_0-1, x_0+1] \\ r_u(x)r_v(x) > \eta/\ell^2 \\ \sin(\delta\varphi(x)) \leq E\ell^2/\eta}} r_u(x)r_v(x) \sin^2(\varphi_u(x)) \cos(\delta\varphi(x)) dx \leq -\frac{\eta}{2C\ell} \left(1 - \frac{C\eta^2}{\ell^2}\right) \leq -\frac{\eta}{3C\ell}.$$

This and (5.2) then proves Lemma 2.4.  $\square$

Clearly, by (2.19) and (2.20), one has  $[x_0 - 1, x_0 + 1] \cap [x_2 - 1, x_2 + 1] = \emptyset$ . For the sake of definiteness, assume that  $x_0 + 1 < x_2 - 1$ . By (2.19) and (2.22), as  $x \mapsto \delta\varphi(x)$  is continuous, there exists  $x_0 + 1 < x_1 < x_2 - 1$  such that  $\delta\varphi(x_1) = \pi/2$  that is  $\sin(\delta\varphi(x_1)) = 1$ .

Fix now  $\varepsilon = \eta/(C\ell^2)$ . By (2.18) and (2.20), there exists two intervals  $[x_0^-, x_0^+]$  and  $[x_2^-, x_2^+]$  such that,

- $[x_0 - 1, x_0 + 1] \cap [0, \ell] \subset [x_0^-, x_0^+] \subset [0, \ell]$ ;
- $[x_2 - 1, x_2 + 1] \cap [0, \ell] \subset [x_2^-, x_2^+] \subset [0, \ell]$ ;
- for  $x \in [x_0^-, x_0^+] \cup [x_2^-, x_2^+]$ , one has  $\sin(\delta\varphi(x)) \leq \varepsilon$ ;
- $\sin(\delta\varphi(x_0^\pm)) = \sin(\delta\varphi(x_2^\pm)) = \varepsilon$ .

As  $x_0 + 1 < x_2 - 1$ , one has  $0 < x_0^- < x_0^+ < x_2^- < x_2^+ < \ell$ . This also implies that  $[x_0, x_0 + 1] \subset [0, \ell]$  and  $[x_2 - 1, x_2] \subset [0, \ell]$ . Moreover, there exists a segment  $[x_1^-, x_1^+]$  such that

- $x_1 \in [x_1^-, x_1^+] \subset [x_0^+, x_2^-] \subset [0, \ell]$ ,
- for  $x \in [x_1^-, x_1^+]$ , one has  $\sin(\delta\varphi(x)) \geq \varepsilon$  and  $\sin(\delta\varphi(x_1^\pm)) = \varepsilon$ .

As  $\sin(\delta\varphi(x_1)) = 1$ , by Lemma 5.2, for some  $C > 0$  (depending only on  $S$  and  $\|q\|_\infty$ ), one has

$$(2.25) \quad \min_{x \in [x_1 - 2S, x_1 + 2S]} \sin(\delta\varphi(x)) \geq \frac{1}{C}.$$

Thus, for  $\ell$  sufficiently large, as  $\varepsilon < 1/C$ , one has  $[x_1 - 2S, x_1 + 2S] \in [x_1^-, x_1^+]$ .  
By Lemma 5.2, we know that

$$(2.26) \quad C^{-1}|\log(E\ell)| \leq x_0^+ - x_0^- \quad \text{and} \quad C^{-1}|\log(E\ell^2)| \leq x_2^+ - x_2^-$$

We apply Lemma 2.1 to  $[x_0^-, x_0^+]$  and  $[x_2^-, x_2^+]$ . Hence, for  $\ell$  sufficiently large, (2.6) implies that there exists  $\lambda_0 > 0$  and  $\lambda_2 > 0$  such that, for  $i \in \{0, 2\}$ , one has

$$(2.27) \quad \frac{\lambda_i}{1 + C\eta_0/\ell} \leq \min_{x \in [x_i^-, x_i^+]} \left( \frac{r_u(x)}{r_v(x)} \right) \leq \max_{x \in [x_i^-, x_i^+]} \left( \frac{r_u(x)}{r_v(x)} \right) \leq [1 + C\eta_0/\ell] \lambda_i.$$

Moreover, as  $r_u$  and  $r_v$  are bounded by a constant depending only on  $\|q\|_\infty$ , by (2.26),

$$(2.17) \quad \text{and} \quad (2.20), \quad \text{one has} \quad \frac{\eta}{C\ell^2} \leq \lambda_0, \lambda_2 \leq \frac{C\ell^2}{\eta}.$$

By Lemma 2.2, on  $[x_1^-, x_1^+]$ , one has

$$(2.28) \quad w(u, v)(x) = M \left( 1 + O\left(\frac{E\ell}{\varepsilon}\right) \right) \quad \text{where} \quad M := \max_{x \in [x_1^-, x_1^+]} w(u, v)(x).$$

We prove

**Lemma 2.5.** *There exists  $\eta_0 > 0$  (depending only on  $\|q\|_\infty$ ) such that, for  $\eta \in (0, \eta_0)$ , there exists  $(k_-, k_+) \in \mathbb{N}^2$  such that*

- (1)  $\frac{2S}{3} < x_1 - k_- \eta < \frac{3S}{4}$  and  $\frac{2S}{3} < k_+ \eta - x_1 < \frac{3S}{4}$ ;
- (2) *there exists  $\lambda_\pm \in \mathbb{R}$  s.t. for  $\bullet \in \{+, -\}$ , one has*
  - *either  $u(k_\bullet \eta) = \lambda_\bullet v(k_\bullet \eta)$  and*
    - $|\lambda_- - \lambda_0| \geq \eta_0 \eta$ ,
    - $|\lambda_+ - \lambda_2| \geq \eta_0 \eta$ .
  - *or  $v(k_\bullet \eta) = \lambda_\bullet u(k_\bullet \eta)$  and*
    - $|\lambda_- - 1/\lambda_0| \geq \eta_0 \eta$ ,
    - $|\lambda_+ - 1/\lambda_2| \geq \eta_0 \eta$ .

*Proof.* The proofs of the existence of  $k_-$  and  $k_+$  being the same up to obvious modifications, we only give the details for  $k_-$ .

By Lemma 5.3, there exists  $\eta_0 > 0$  (depending only on  $\|q\|_\infty$ ) such that, for  $\eta \in (0, \eta_0)$ ,  $|\sin(\varphi_u(x))|$  and  $|\sin(\varphi_v(x))|$  can stay smaller than  $\eta$  only on intervals of length less than  $\eta/\eta_0$ . Thus, there exists  $\eta_0 > 0$  such that, for  $\eta \in (0, \eta_0)$ , one can find an integer  $k$  such that

$$(2.29) \quad \frac{2S}{3} < x_1 - k\eta < \frac{3S}{4} \quad \text{and} \quad \begin{array}{l} |\sin(\varphi_u(x))| \geq \eta \\ |\sin(\varphi_v(x))| \geq \eta \end{array} \quad \text{for } x \in [(k-1)\eta, (k+1)\eta].$$

This, in particular, implies that  $u(x) \neq 0 \neq v(x)$  for  $x \in [(k-1)\eta, (k+1)\eta]$ . Note also that, by (2.26) and (2.29), for  $\ell$  large, one has  $k\eta \in [x_1 - 2S, x_1 + 2S] \subset [x_1^-, x_1^+]$ .

To fix ideas, assume moreover that  $r_u(k\eta) \geq r_v(k\eta)$ ; the reverse case is treated similarly interchanging  $u$  and  $v$ , and,  $\lambda_0$  and  $1/\lambda_0$ . Assume that the first point of (2) in Lemma 2.5 does not hold i.e. assume now that

$$(2.30) \quad \exists \lambda \in [\lambda_0 - \eta_0 \eta, \lambda_0 + \eta_0 \eta] \quad \text{such that} \quad u(k\eta) = \lambda v(k\eta).$$

Using (2.28), we compute

$$\begin{aligned} \left| \begin{array}{cc} u(k\eta + \eta) - \lambda v(k\eta + \eta) & 0 \\ v(k\eta + \eta) & v(k\eta) \end{array} \right| &= \left| \begin{array}{cc} u(k\eta + \eta) & u(k\eta) \\ v(k\eta + \eta) & v(k\eta) \end{array} \right| = \eta \int_0^1 w(u, v)(k\eta + \eta t) dt \\ &= M\eta(1 + O(E\ell/\varepsilon)). \end{aligned}$$

Thus, one obtains

$$u(k\eta + \eta) = \left( \lambda + \frac{M(1 + O(E\ell/\varepsilon))}{v(k\eta + \eta)v(k\eta)} \eta \right) v(k\eta + \eta),$$

that is,  $u(k\eta + \eta) = (\lambda + \delta\lambda)v(k\eta + \eta)$  where

$$\begin{aligned} |\delta\lambda| &= \left| \frac{M(1 + O(E\ell/\varepsilon))}{v(k\eta + \eta)v(k\eta)} \eta \right| \geq \frac{M\eta}{r_v(k\eta + \eta)r_v(k\eta)} \geq \frac{M\eta}{Cr_v^2(k\eta)} \\ &\geq \frac{M\eta}{Cr_v(k\eta)r_u(k\eta)} \geq \frac{\eta}{C} \frac{M}{w(u, v)(k\eta)} \geq 2\eta_0\eta. \end{aligned}$$

Here, we have successively used

- the uniform estimate on the growth rate of  $r_v$  given by equation (2.3),
- the assumption  $r_u(k\eta) \leq r_v(k\eta)$ ,
- the bound (2.25),
- we have presumably reduced  $\eta_0$ .

Thus, one has  $|\lambda + \delta\lambda - \lambda_0| \geq |\delta\lambda| - |\lambda - \lambda_0| \geq \eta\eta_0$  and we set  $k_- = k + 1$ .

If (2.30) does not hold, it suffices to set  $k_- = k$ .

This completes the proof of Lemma 2.5.  $\square$

To complete the proof of Proposition 2.3, we check the assertion about  $H_-$ ; the one about  $H_+$  is checked likewise except for the fact that  $\ell$  has to be replaced by  $\ell^2$ , compare (2.20) in Lemma 2.4 with (2.17).

The proof of Proposition 2.3 now depends on which of the alternatives of Lemma 2.5 is realized. First, assume that, in Lemma 2.5, it is the function  $u - \lambda_-v$  that vanishes at  $x_- = k_- \eta$ . So, the function  $u - \lambda_-v$  satisfies Dirichlet boundary conditions on the interval  $[0, x_-]$ . One computes

$$\|(H_- - E)(u - \lambda_-v)\|_{L^2([0, x_-])} = E\|u\|_{L^2([0, x_-])} \leq E.$$

Moreover, by the defining property of  $[x_0^-, x_0^+]$  and Lemma 2.1, as  $|\lambda_- - \lambda_0| \geq \eta_0\eta$ , using (2.27), for  $x \in [x_0^-, x_0^+]$ , one has

$$\begin{aligned} u(x) - \lambda_-v(x) &= r_u(x) \sin(\varphi_u(x)) - \lambda_-r_v(x) \sin(\varphi_v(x)) \\ &= [r_u(x) - \lambda_-r_v(x)] \sin(\varphi_v(x)) + O(E|u'(x)|) \\ &= [\lambda_0 - \lambda_-]v(x) + O(E|u'(x)|) + O(E^2|u(x)|) + O(\eta_0/\ell|v(x)|) \end{aligned}$$

Possibly reducing  $\eta_0$ , one then computes

$$\|u - \lambda_-v\|_{L^2([0, x_-])} \geq \eta_0(\eta - 1/\ell)\|v\|_{L^2([x_0-1, x_0+1])} - \sqrt{E}\ell \geq \eta_0\eta^3\ell^{-2} - E.$$

Thus, we know that  $H_-$  has an eigenvalue at distance at most  $E\ell^2/(2\eta_0\eta^3)$  from  $E$  if  $\eta_0\eta^3\ell^{-2} \geq 2E > 0$ .

When, in Lemma 2.5, it is the function  $v - \lambda_-u$  that vanishes at  $x_- = k_- \eta$ , one computes  $\|H_-(v - \lambda_-u)\|_{L^2([0, x_-])}$  and the remaining part of the proof is unchanged. This completes the proof of Proposition 2.1.  $\square$

Minor changes in the proof of Proposition 2.1 also yields the following result

**Proposition 2.2.** *Fix  $S > 0$  arbitrary. There exists  $\eta_0 > 0$  such that, for  $\eta \in (0, \eta_0)$  and  $\ell$  sufficiently large (depending only on  $\eta, S$  and  $\|q\|_\infty$ ), if  $u$  and  $v$  are as section 2 and such that (2.15) is satisfied, then, there exists a points  $\bar{x}$  in the lattice  $\eta\mathbb{Z}$  satisfying*

$$|\log \eta|/C < \bar{x} < \ell - |\log \eta|/C$$



such that, if  $H_-$  (resp.  $H_+$ ) denotes the second order differential operator  $H$  defined by (2.1) and Dirichlet boundary conditions on  $[0, \bar{x}]$  (resp. on  $[\bar{x}, \ell]$ ), then  $H_-$  and  $H_+$  have an eigenvalue in the interval  $[-E\ell^4\eta^{-4}, E\ell^4\eta^{-4}]$ .

For the application presented in section 0, it will be of importance to have two points  $x_-$  and  $x_+$  that are well separated from each other.

2.2.2. *When there is no tunneling.* The case when there is no tunneling can be described by the fact that both function  $u$  and  $v$  are “large” only at distinct location or equivalently by the fact that  $r_u \cdot r_v$  stays small all over the interval  $[0, \ell]$ . Clearly, as  $u$  and  $v$  are normalized,  $r_u$  and  $r_v$  need each to be at most only of size  $1/\sqrt{\ell}$ . So one can say that  $r_u \cdot r_v$  stays small if and only if  $r_u \cdot r_v \ll \ell^{-1}$  all over  $[0, \ell]$ . We prove

**Proposition 2.3.** *Fix  $S > 0$  arbitrary. There exists  $\eta_0 > 0$  (depending only on  $\|q\|_\infty$ ) such that, for  $\eta \in (0, \eta_0)$  and  $\ell$  sufficiently large (depending only on  $\eta$ ,  $S$  and  $\|q\|_\infty$ ), if  $u$  and  $v$  are as section 2 that is, eigenfunctions of  $H$  associated respectively to the eigenvalues 0 and  $E$ , and if  $E\ell \leq \eta^{1/4}$  and one has that*

$$(2.31) \quad \forall x \in [0, \ell], \quad r_u(x) \cdot r_v(x) \leq \frac{\eta}{\ell},$$

then, there exists two points  $x_+$  and  $x_-$  in the lattice  $\eta\mathbb{Z}$  satisfying

$$(2.32) \quad |\log \eta|/C < x_- < x_+ < \ell - |\log \eta|/C \quad \text{and} \quad S < x_+ - x_- < 2S$$

such that, if  $H_-$  (resp.  $H_+$ ) denotes the second order differential operator  $H$  defined by (2.1) and Dirichlet boundary conditions on  $[0, x_-]$  (resp. on  $[x_+, \ell]$ ), then  $H_-$  and  $H_+$  have an eigenvalue in the interval  $[-E\ell\eta^{-1/4}, E\ell\eta^{-1/4}]$ .

*Proof.* As  $u$  and  $v$  are normalized, one can pick  $x_u$  (resp.  $x_v$ ) s.t.  $r_u(x_u) \geq \ell^{-1/2}$  (resp.  $r_v(x_v) \geq \ell^{-1/2}$ ). Thus, by (2.31), one has  $r_u(x_v) \leq \eta\ell^{-1/2}$  and  $r_v(x_u) \leq \eta\ell^{-1/2}$ . To fix ideas, assume  $x_u < x_v$ . Note that, as  $r_u$  satisfies equation (2.3), one has  $|\log \eta|/C \leq x_v - x_u$  (for some  $C$  depending only on  $\|q\|_\infty$ ). Hence, as  $x \mapsto (r_u/r_v)(x)$  is continuous, there exists  $x_u < x_0 < x_v$  such that  $r_u(x_0) = r_v(x_0)$ . Define  $x_\pm$  to be respectively the points in the lattice  $\eta\mathbb{Z}$  closest to  $x_0 \pm S/2$ . Then, there exists  $C > 0$  (depending only on  $S$  and  $\|q\|_\infty$ ) such that

$$(2.33) \quad \frac{1}{C} \leq (r_u/r_v)(x_\pm) \leq C \quad \text{and} \quad |\log \eta|/C \leq \inf(x_v - x_+, x_- - x_u).$$

We will start with  $H_-$  on the interval  $[0, x_-]$ ; the case of  $H_+$  on the interval  $[x_+, \ell]$  is dealt with in the same way.

Assume that  $|\sin(\varphi_v(x_-))| \geq \sqrt[4]{\eta}$ . Then, we pick  $\lambda = \frac{\sin(\varphi_u(x_-))}{\sin(\varphi_v(x_-))}$ . We now set  $w_- = u - \lambda v$ . Thus,  $w$  vanishes at the points 0 and  $x_-$  and, one computes

$$\|H_- w_-\|_{L^2([0, x_-])} \leq E\lambda \leq \frac{E}{\sqrt[4]{\eta}}$$

and, using  $r_u(x_u) \geq \ell^{-1/2}$  and (5.1) in Lemma 5.1, for  $\eta$  sufficiently small

$$\begin{aligned} \|w\|_{L^2([0, x_-])}^2 &= \int_0^{x_-} (u(x) - \lambda v(x))^2 dx \\ &\geq \int_0^{x_-} u^2(x) dx - 2\lambda \int_0^{x_-} r_u(x)r_v(x) dx \geq \ell^{-1}/C - \eta^{3/4}\ell^{-1} \geq \frac{1}{2C\ell} \end{aligned}$$

for  $\eta$  sufficiently small. Hence, as  $H_-$  is self-adjoint, we have proved the statement of Proposition 2.3 if  $|\sin(\varphi_v(x_-))| \geq \sqrt[4]{\eta}$ .

Assume now that  $|\sin(\varphi_v(x_-))| \leq \sqrt[4]{\eta}$ . Then, for  $\eta$  sufficiently small, point (2) of Lemma 5.3 for  $\varphi_v$  guarantees that, for some  $x_0 \in \eta\mathbb{Z}$  such that  $x_- - 2\sqrt[8]{\eta} \leq x_0 \leq x_- - \sqrt[8]{\eta}$ , one has  $|\sin(\varphi_v(x_0))| \geq \sqrt[4]{\eta}$ . So, we can do the computations done above replacing  $x_-$  with  $x_0$ .

To obtain the counterpart of this analysis for  $H_+$  on  $[x_+, \ell]$ , we proceed as above except for the fact that we set  $w_+ = v - \lambda^{-1}u$  where  $\lambda$  is chosen as before and estimate  $\|(H_+ - E)w_+\|_{L^2([x_+, \ell])}$ .

This completes the proof of Proposition 2.3.  $\square$

Clearly, slight changes in the proof of Proposition 2.3 also yields the following result

**Proposition 2.4.** *There exists  $\eta_0 > 0$  (depending only on  $\|q\|_\infty$ ) such that, for  $\eta \in (0, \eta_0)$  and  $\ell$  sufficiently large, if  $u$  and  $v$  are as section 2 and if (2.31) is satisfied, then, there exists a points  $\bar{x}$  in the lattice  $\eta\mathbb{Z}$  satisfying*

$$|\log \eta|/C < \bar{x} < \ell - |\log \eta|/C$$

such that, if  $H_-$  (resp.  $H_+$ ) denotes the second order differential operator  $H$  defined by (2.1) and Dirichlet boundary conditions on  $[0, \bar{x}]$  (resp. on  $[\bar{x}, \ell]$ ), then  $H_-$  and  $H_+$  have an eigenvalue in the interval  $[-E\ell\eta^{-1/4}, E\ell\eta^{-1/4}]$ .

For the applications presented in section 0, it will be of importance to have two points  $x_-$  and  $x_+$  that are well separated from each other.

2.2.3. *Completing the proof of Theorem 2.1.* It suffices to pick  $\eta$  so small that both Propositions 2.1 and 2.3 hold. Note that there is a change of notations between Theorem 2.1 and Propositions 2.1 - 2.3. In Theorem 2.1,  $E - \varepsilon$  (resp.  $E + \varepsilon$ ) plays the role that 0 (resp.  $E$ ) plays in Propositions 2.1 and 2.3,  $2\varepsilon$  that of  $E$  and  $\varepsilon_0$  that of a power of  $\eta$  that is now fixed.  $\square$

### 3. THE PROOF OF THEOREMS 1.1 AND 1.2

The basic idea of the proof follows the basic idea of [23] i.e. use localization to reduce the complexity of the problem by reducing it to studying eigenvalues of  $H_\omega$  restricted to cubes of size roughly  $L$  or  $(\log L)^C$ .

**3.1. Reduction to localization cubes.** Pick  $J$  a compact interval where (Loc) is satisfied. Thus, we know

**Lemma 3.1** ([15]). *Under assumption (W) and (Loc), for any  $p > 0$ , there exists  $p' > 0$  such that, for  $L \geq 1$  large enough, with probability larger than  $1 - L^{-p}$ , if*

- (1)  $\varphi_{n,\omega}$  is a normalized eigenvector of  $H_\omega(\Lambda_L)$  associated to  $E_{n,\omega} \in J$ ,
- (2)  $x_n(\omega) \in \Lambda_L$  is a maximum of  $x \mapsto \|\varphi_{n,\omega}\|_x^2 = \int_{x-1}^{x+1} |\varphi_{n,\omega}(y)|^2 dy$  in  $\Lambda_L$ ,

then, for  $x \in \Lambda_L$ , one has

$$(3.1) \quad \|\varphi_{n,\omega}\|_x \leq L^{p'} e^{-\xi|x-x_n(\omega)|}$$

where  $\xi$  is given by assumption (Loc).

So, with good probability, all the eigenfunctions essentially live in cubes of size of order  $L$ . Thus, they only see the configuration  $\omega$  in such cubes. To fix ideas, we define the center of localization of an eigenfunction  $\varphi$  as the left most maximum of

$x \mapsto \|\varphi\|_x$ .

We prove

**Lemma 3.2.** *Assume (W) and (Loc). Fix  $J$  compact in  $\mathring{I}$ . Then, for any  $q > 0$ , there exists  $C > 0$  and  $L_q > 0$  s.t., for  $E \in J$ ,  $L \geq L_q$  and  $\varepsilon \in [L^{-q}, (C \log L)^{-2}]$ , one has*

$$(3.2) \quad \sum_{k \geq 2} \mathbb{P} \left( \text{tr} [\mathbf{1}_{[E-\varepsilon, E+\varepsilon]}(H_\omega(\Lambda_L))] \geq k \right) \\ \leq L^{-q} + \frac{L^2}{\ell} \mathbb{P}_{2,6\ell, \ell}(\varepsilon) + \left( \frac{L}{\ell} \right)^2 \left( \mathbb{P}_{1,3\ell/2, \ell}(\varepsilon) + L^{-q} \right)^2 e^{L \mathbb{P}_{1,3\ell/2, \ell}(\varepsilon)/\ell}$$

where either  $\ell = C \log L$  or  $L = (\log L)^C$  and, for  $j \geq 1$ , one has set

$$(3.3) \quad \mathbb{P}_{j, \ell, \ell'}(\varepsilon) := \sup_{\gamma \in \ell' \mathbb{Z} \cap [0, L]} \mathbb{P} \left( \text{tr} [\mathbf{1}_{[E-\varepsilon, E+\varepsilon]}(H_\omega(\Lambda_\ell(\gamma)))] \geq j \right).$$

*Proof of Lemma 3.2.* We prove Lemma 3.2 for the scales  $\ell = C \log L$ . In most of the proof, one easily checks that the argument works in the same way for scales  $\ell = (\log L)^C$  if  $C > 1$ ; we will indicate the changes needed to treat the scales  $\ell = (\log L)^C$  when necessary.

Pick  $E \in J$ . First, by standard bounds on the eigenvalue counting function of  $-\Delta$ , there exists  $C > 0$  depending only on  $J$  such that, for  $\varepsilon \in (0, 1)$ , one has

$$(3.4) \quad \text{tr} [\mathbf{1}_{[E-\varepsilon, E+\varepsilon]}(H_\omega(\Lambda_L))] \leq CL.$$

Let  $\mathcal{Z}_q$  be the set of configurations  $\omega$  defined by Lemma 3.1 for the exponent  $p = q+1$ . It has probability at least  $1 - L^{-q-1}$ . Thus, by (3.2), we estimate

$$(3.5) \quad \sum_{k \geq 2} \mathbb{P} \left( \{ \omega \notin \mathcal{Z}_q; \text{tr} [\mathbf{1}_{[E-\varepsilon, E+\varepsilon]}(H_\omega(\Lambda_L))] \geq k \} \right) \leq CL^{-q}$$

Let us now estimate  $\mathbb{P}(\{ \omega \in \mathcal{Z}_q; \text{tr} [\mathbf{1}_{[E-\varepsilon, E+\varepsilon]}(H_\omega(\Lambda_L))] \geq k \})$ . Let  $\ell = C \log L$  for some  $C > 0$  to be chosen later on.

For  $\omega \in \mathcal{Z}_q$ , by Lemma 3.1, for each  $\varphi$  eigenfunction of  $H_\omega(\Lambda_L)$  associated to an eigenvalue  $E \in J$ , we define the center of localization associated to  $\varphi$  as in the remarks following Lemma 3.1. We consider the events

$$\Omega_q^g = \left\{ \omega \in \mathcal{Z}_q; \begin{array}{l} \text{no two centers of localization of eigenfunctions} \\ \text{associated to eigenvalues in } [E - \varepsilon, E + \varepsilon] \\ \text{are at a distance less than } 4\ell \text{ from each other} \end{array} \right\} \text{ and } \Omega_q^b = \mathcal{Z}_q \setminus \Omega_q^g.$$

Note that, for  $\omega \in \Omega_q^g$ ,  $H_\omega(\Lambda_L)$  has at most  $[L/\ell]$  eigenvalues in  $[E - \varepsilon, E + \varepsilon]$ ; here,  $[\cdot]$  denotes the integer part of  $\cdot$ .

We prove

**Lemma 3.3.** *For  $q > 0$ , there exists  $C_q > 0$  and  $L_q > 0$  such that, if  $C > C_q$  such that if  $\ell = C \log L$  or  $\ell = (\log L)^C$ , for  $L \geq L_q$  and  $k \geq 2$ , one has*

$$(3.6) \quad \mathbb{P} \left( \left\{ \omega \in \Omega_q^b; \text{tr} [\mathbf{1}_{[E-\varepsilon, E+\varepsilon]}(H_\omega(\Lambda_L))] \geq k \right\} \right) \leq \frac{L}{\ell} \mathbb{P}_{2,6\ell, \ell}(\varepsilon) + L^{-q}$$

and, for  $k \leq [L/\ell]$ ,

$$(3.7) \quad \mathbb{P} \left( \left\{ \omega \in \Omega_q^g; \text{tr} [\mathbf{1}_{[E-\varepsilon, E+\varepsilon]}(H_\omega(\Lambda_L))] \geq k \right\} \right) \leq \binom{[L/\ell]}{k} \left( \mathbb{P}_{1,3\ell/2, \ell}(\varepsilon) + L^{-q} \right)^k$$

where  $\mathbb{P}_{j,\ell,\ell'}(\varepsilon)$  is defined in (3.3).

We postpone the proof of Lemma 3.3 to complete that of Lemma 3.2. We pick  $q \geq 1$  and sum (3.6) and (3.7) for  $k \geq 2$  to get, for some  $C > 0$

$$\begin{aligned} & \frac{1}{C} \sum_{k \geq 2} \mathbb{P} \left( \text{tr} [\mathbf{1}_{[E-\varepsilon, E+\varepsilon]}(H_\omega(\Lambda_L))] \geq k \right) \\ & \leq L^{-q} + \frac{L^2}{\ell} \mathbb{P}_{2,6\ell,\ell}(\varepsilon) + \left( \frac{L}{\ell} \right)^2 (\mathbb{P}_{1,3\ell/2,\ell}(\varepsilon) + L^{-q})^2 (1 + \mathbb{P}_{1,3\ell/2,\ell}(\varepsilon) + L^{-q})^{L/\ell} \\ & \leq C \left( L^{-q} + \frac{L^2}{\ell} \mathbb{P}_{2,6\ell,\ell}(\varepsilon) + \left( \frac{L}{\ell} \right)^2 (\mathbb{P}_{1,3\ell/2,\ell}(\varepsilon) + L^{-q})^2 e^{L \mathbb{P}_{1,3\ell/2,\ell}(\varepsilon)/\ell} \right). \end{aligned}$$

Here, we have used the bound, for  $(x, y) \in (\mathbb{R}^+)^2$  and  $m \leq n$  integers,

$$(3.8) \quad \sum_{k=m}^n \binom{n}{k} x^k y^{n-k} \leq \binom{n}{m} x^m (x+y)^{n-m}.$$

This completes the proof of Lemma 3.2.  $\square$

*Proof of Lemma 3.3.* We will use

**Lemma 3.4.** *For  $q > 0$ , there exists  $C_q > 0$  and  $L_q > 0$  such that, if  $C > C_q$  such that if  $\ell = C \log L$  or  $\ell = (\log L)^C$ , for  $L \geq L_q$  and  $\omega \in \mathcal{Z}_q$ , for any  $\gamma \in \Lambda_L$ , if  $H_\omega(\Lambda_L)$  has  $k$  eigenvalues in  $[E - \varepsilon, E + \varepsilon]$  with localization center in  $\Lambda_\ell(\gamma)$ , then  $H_\omega(\Lambda_{3\ell/2}(\gamma))$  has  $k$  eigenvalues in  $[E - \varepsilon - L^{-q}, E + \varepsilon + L^{-q}]$ .*

We postpone the proof of Lemma 3.4 to complete that of Lemma 3.3. Let us just say here that the proof shows that, when  $\ell = (\log L)^C$ ,  $C_q$  can be taken arbitrary in  $(1, +\infty)$ .

Let  $k \geq 2$ . We first estimate  $\mathbb{P} \left( \left\{ \omega \in \Omega_q^b; \text{tr} [\mathbf{1}_{[E-\varepsilon, E+\varepsilon]}(H_\omega(\Lambda_L))] \geq k \right\} \right)$ . Clearly, one has

$$\begin{aligned} & \mathbb{P} \left( \left\{ \omega \in \Omega_q^b; \text{tr} [\mathbf{1}_{[E-\varepsilon, E+\varepsilon]}(H_\omega(\Lambda_L))] \geq k \right\} \right) \\ & \leq \mathbb{P} \left( \left\{ \omega \in \Omega_q^b; \text{tr} [\mathbf{1}_{[E-\varepsilon, E+\varepsilon]}(H_\omega(\Lambda_L))] \geq 2 \right\} \right). \end{aligned}$$

Thus, we take  $k = 2$ .

Pick  $q' > (q+2)/s$  where  $s$  is given by (W). By the definition of  $\Omega_q^b$  and Lemma 3.4, one clearly has

$$\begin{aligned} & \mathbb{P} \left( \left\{ \omega \in \Omega_q^b; \text{tr} [\mathbf{1}_{[E-\varepsilon, E+\varepsilon]}(H_\omega(\Lambda_L))] \geq 2 \right\} \right) \\ & \leq \mathbb{P} \left( \left\{ \exists \gamma \in \ell\mathbb{Z} \cap [0, L]; \text{tr} [\mathbf{1}_{[E-\varepsilon-L^{-q'}, E+\varepsilon+L^{-q'}]}(H_\omega(\Lambda_{6\ell}(\gamma)))] \geq 2 \right\} \right) \\ & \leq \sum_{\gamma \in \ell\mathbb{Z} \cap [0, L]} \mathbb{P} \left( \left\{ \text{tr} [\mathbf{1}_{[E-\varepsilon-L^{-q'}, E+\varepsilon+L^{-q'}]}(H_\omega(\Lambda_{6\ell}(\gamma)))] \geq 2 \right\} \right) \\ & \leq \frac{L}{\ell} \mathbb{P}_{2,6\ell,\ell}(\varepsilon + L^{-q'}) \leq \frac{L}{\ell} \mathbb{P}_{2,6\ell,\ell}(\varepsilon) + L^{-q'} \end{aligned}$$

for  $L$  sufficiently large; in the last step, we have used the Wegner estimate (W),  $q' > (q+2)/s$  and  $\ell \leq (\log L)^C$ . This completes the proof of (3.6).

Let us now estimate  $\mathbb{P} \left( \left\{ \omega \in \Omega_q^g; \text{tr} [\mathbf{1}_{[E-\varepsilon, E+\varepsilon]}(H_\omega(\Lambda_L))] \geq k \right\} \right)$ . We partition the cube  $\Lambda_L$  into cubes  $(\Lambda_\ell(\gamma))_{\gamma \in \Gamma}$  so that  $\#\Gamma \asymp L/\ell$ .

Assume now that  $\omega \in \Omega_q^g$  is such that  $\text{tr}[\mathbf{1}_{[E-\varepsilon, E+\varepsilon]}(H_\omega(\Lambda_L))] \geq k$ . Thus, the localization centers for any two eigenfunctions being at least  $4\ell$  away from each other, by Lemma 3.4, we can find  $k$  points in  $\Gamma$ , say  $(\gamma_j)_{1 \leq j \leq k}$  such that

- for  $1 \leq j \leq k$ ,  $H_\omega(\Lambda_{3\ell/2}(\gamma_j))$  has exactly 1 eigenvalue in the interval  $\left[E - \varepsilon - L^{-q'}, E + \varepsilon + L^{-q'}\right]$ ;
- for  $1 \leq j < j' \leq k$ , one has  $\text{dist}(\Lambda_{3\ell/2}(\gamma_j), \Lambda_{3\ell/2}(\gamma_{j'})) > \ell/2$ .

Hence, by (IAD), for  $\ell$  sufficiently large, the operators  $(H_\omega(\Lambda_{(1+\eta)\ell}(\gamma_j)))_{1 \leq j \leq k}$  are stochastically independent. Hence, we have the bound

$$\mathbb{P}(\{\omega \in \Omega_q^g; \text{tr}[\mathbf{1}_{[E-\varepsilon, E+\varepsilon]}(H_\omega(\Lambda_L))] \geq k\}) \leq \binom{\#\Gamma}{k} (\mathbb{P}_{1,3\ell/2,\ell}(\varepsilon) + L^{-q})^k.$$

This completes the proof of (3.7) and, thus, of Lemma 3.3.  $\square$

*Proof of Lemma 3.4.* Analogous results can be found in [23, 15].

If  $\varphi$  is an eigenfunction of  $H_\omega(\Lambda_L)$  associated to  $e$  an eigenvalue in  $[E - \varepsilon, E + \varepsilon]$  that has localization center in  $\Lambda_\ell(\gamma)$ , then, by Lemma 3.1, we have that, for  $\chi$  a smooth cut-off that is 1 on  $\Lambda_{10\ell/9}(\gamma)$  and vanishing outside  $\Lambda_{3\ell/2}(\gamma)$ , one has

$$\|H_\omega(\Lambda_{3\ell/2}(\gamma)) - e)(\chi\varphi)\| \leq e^{-\xi\ell/8} \leq L^{-q}$$

if  $\ell = C \log L$  and  $C > 0$  is chosen sufficiently large. On the other hand, if one has  $k$  such eigenvalues, say,  $(\varphi_j)_{1 \leq j \leq k}$ , then  $k \leq CL$  and one computes the Gram matrix

$$(\langle \chi\varphi_j, \chi\varphi_{j'} \rangle)_{1 \leq j, j' \leq k} = (\langle \varphi_j, \varphi_{j'} \rangle)_{1 \leq j, j' \leq k} + O(k^2 L^{-q}) = \text{Id}_k + O(k^2 L^{-q}).$$

This completes the proof of Lemma 3.4.  $\square$

**3.2. The proof of Theorem 0.1.** We use Lemma 3.2. We pick  $\ell = C \log L$ . In (3.2), to estimate  $\mathbb{P}_{1,3\ell/2,\ell}(2\varepsilon)$ , we use the Wegner type estimate (W) and obtain

$$(3.9) \quad \mathbb{P}_{1,3\ell/2,\ell}(\varepsilon) \leq C\varepsilon^s \log^\rho L.$$

To estimate  $\mathbb{P}_{2,6\ell,\ell}(2\varepsilon)$ , we use Theorem 2.1 and the Wegner type estimate (W). The point  $(x_\pm)$  are not known but we know then belong to the lattice segment  $\varepsilon_0\mathbb{Z} \cap [0, \ell]$  (independent of the potential  $q_\omega$ ) so there are at most  $(\ell/\varepsilon_0)^2$  possible pairs of points. We choose the constant  $S > R_0$  defined by (IAD); hence, as the points  $x_+ - x_- \geq S$ , the operators  $H_- := H_\omega^D|_{[0, x_-]}$  and  $H_+ := H_\omega^D|_{[x_+, \ell]}$  are stochastically independent. Thus, applying the Wegner type estimate (W) for the operators  $H_\pm$  and summing over the pairs of points in  $\varepsilon_0\mathbb{Z} \cap [0, \ell]$  yields

$$(3.10) \quad \mathbb{P}_{2,6\ell,\ell}(2\varepsilon) \leq C(\log L)^{2\rho} (\varepsilon \log^4 L)^{2s}.$$

Plugging this and (3.9) into (3.2) yields Theorem 1.1.  $\square$

**3.3. The proof of Theorem 0.4.** In this case we want to improve our estimate by obtaining a bound in terms of the integrated density of states. We pick the scale  $\ell = (\log L)^C$  for some  $C > 1$  large so as to be able to apply Lemma 3.2. We are going to refine our estimate of  $\mathbb{P}_{1,3\ell/2,\ell}(2\varepsilon)$  and  $\mathbb{P}_{2,6\ell,\ell}(2\varepsilon)$  using [15, Theorem 1.1]. First notice that, as the points satisfy  $x_+ - x_- \leq 2S$ , one of the intervals  $I_- = [0, x_-]$  or  $I_+ = [x_+, \ell]$  is of length at least  $(\ell - S)/2$ . We recall

**Theorem 3.1** ([15]). *Assume (W) and (Loc) holds in  $I$ . Fix  $\nu \in (0, 1)$ . Then, for  $L > 1$  and  $J \subset I$  a compact interval such that  $N(J) \geq e^{-L^\nu}$ . Then,*

$$(3.11) \quad \mathbb{P}(\text{tr}(\mathbf{1}_J(H_\omega(\Lambda))) \geq 1) \leq \mathbb{E}[\text{tr}(\mathbf{1}_J(H_\omega(\Lambda)))] = N(J)|\Lambda| (1 + O(|\Lambda|^{-1})).$$

Strictly speaking Theorem 3.1 was proved under an assumption slightly stronger than (W), namely, (W) when  $s = 1$  and  $\rho = 1$ . One easily check in the proof that this is not necessary.

Hence, as  $C > 1$ , we pick  $\nu \in (0, 1)$  in Theorem 3.1 such that  $C\nu > 1$ . Thus, for  $L$  sufficiently large, one has  $e^{-\ell^\nu} \leq L^{-q}$  and, the arguments of section 3.2 yield that

$$(3.12) \quad \mathbb{P}_{1,3\ell/2,\ell}(2\varepsilon) \leq C [N(E - 2\varepsilon) - N(E - 2\varepsilon)] (\log L)^C$$

$$(3.13) \quad \mathbb{P}_{2,6\ell,\ell}(2\varepsilon) \leq C(\log L)^{\rho+C} |\varepsilon \log L|^s \times [N(E - 2\varepsilon \log^4 L) - N(E - 2\varepsilon \log^4 L)].$$

Notice that, when estimating  $\mathbb{P}_{2,6\ell,\ell}(2\varepsilon)$ , Theorem 2.1 only enables us to apply Theorem 3.1 to one of the intervals  $I_\pm$  as we only know that of them is of length at least  $\ell/2 = \log^C L/2$ .

Plugging (3.12) and (3.13) into (3.2) and using (W) for the term in the exponential in (3.2) yields Theorem 1.2.  $\square$

#### 4. PROOFS OF THE UNIVERSAL ESTIMATES

We now prove Theorems 1.3 and 1.4. By a shift in energy, it suffices to prove the results for  $E = 0$  and see that the constants only depend on  $\|q\|_\infty$ . From now on, we assume the energy interval under consideration is centered at  $E = 0$ .

*Proof of Theorem 1.3.* Assume  $H$  has at least two eigenvalues, say,  $E$  and  $\tilde{E}$  in  $[-\varepsilon, \varepsilon]$ . By shifting the potential by a constant less than 1, without loss of generality, we may assume that  $\tilde{E} = 0$  and  $E > 0$ . Let  $v$  and  $w$  be the fundamental solutions to the equation  $-u'' + qu = 0$  (i.e.  $v(0) = 1 = w'(0)$  and  $v'(0) = 0 = w(0)$ ) and let  $S_0(y, x)$  be the resolvent matrix associated to  $(v, w)$  i.e.

$$S_0(y, x) = \begin{pmatrix} v(y) & w(y) \\ v'(y) & w'(y) \end{pmatrix} \begin{pmatrix} w'(x) & -w(x) \\ -v'(x) & v(x) \end{pmatrix}.$$

Clearly  $S_0$  solves

$$\frac{d}{dy} S_0(y, x) = \begin{pmatrix} 0 & 1 \\ q(y) & 0 \end{pmatrix} S_0(y, x), \quad S_0(x, x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Obviously, as  $q$  is bounded, for some  $C$  depending only on  $\|q\|_\infty$ , one has

$$(4.1) \quad \|S_0(y, x)\| \leq e^{C|y-x|}.$$

Let  $u$  be a  $L^2([0, \ell])$ -normalized solution to  $Hu = Eu$ . Hence, we have

$$(4.2) \quad \begin{pmatrix} u(x) \\ u'(x) \end{pmatrix} = S_0(x, 0) \begin{pmatrix} u(0) \\ u'(0) \end{pmatrix} + \int_0^x S_0(y, 0) B(y) dy$$

where

$$B(y) = E \begin{pmatrix} 0 \\ u(x) \end{pmatrix}.$$

The eigenfunction, say,  $u_0$ , associated to  $H$  and 0 can be written as

$$\begin{pmatrix} u_0(x) \\ u_0'(x) \end{pmatrix} = S_0(x, 0) \begin{pmatrix} u_0(0) \\ u_0'(0) \end{pmatrix}.$$

As  $u$  and  $u_0$  satisfy the same boundary conditions, using (4.2), (4.1) and the normalization of  $u$ , we get that, for some  $\lambda > 0$ , one has

$$(4.3) \quad \left\| \begin{pmatrix} u \\ u' \end{pmatrix} - \lambda \begin{pmatrix} u_0 \\ u_0' \end{pmatrix} \right\|_{\infty} \leq C\varepsilon e^{C\ell}.$$

If  $\varepsilon \in (0, 1)$  such that  $|\log \varepsilon| \geq K\ell$  where  $K$  is taken such that, for  $\ell \geq 1$ , one has  $Ce^{(C-K)\ell} < 1$ . By (4.3), as, on  $[0, \ell]$ ,  $u$  and  $u_0$  are normalized and orthogonal to each other, we get  $\lambda^2 + 1 < 1$  which is absurd. This completes the proof of Lemma 1.3.  $\square$

*Proof of Theorem 1.4.* Assume  $H$  has  $N + 1$  eigenvalues in  $[-\varepsilon, \varepsilon]$ . As  $q$  is bounded, standard comparison with the Laplacian  $H_0 = -d^2/dx^2$  implies that  $N \leq C\ell$  for some  $C > 0$  depending only  $\|q\|_{\infty}$ .

As in the proof of Lemma 1.3, we may assume that the smallest one of them be 0, thus, that the other be positive. Let  $(u_j)_{0 \leq j \leq N}$  be the associated normalized eigenfunctions,  $u_0$  being the one associated to the eigenvalue 0.

Fix  $1 \leq \tilde{\ell} < \ell$  to be chosen later. Partition the interval  $[0, \ell]$  into  $A$  intervals of length approximately  $\tilde{\ell}$  i.e.  $[0, \ell] = \cup_{1 \leq \alpha \leq A} I_{\alpha}$  where  $I_{\alpha} = [x_{\alpha}, x_{\alpha+1}]$  and  $x_{\alpha+1} - x_{\alpha} \asymp \tilde{\ell}$ ; hence,  $A \asymp \ell/\tilde{\ell}$ .

As in Lemma 1.3, let  $(v, w)$  be the fundamental solutions to  $-u'' + qu = 0$ . Formula (4.2) and (4.1) show that there exists constants  $((\lambda_j^{\alpha}))_{\substack{1 \leq j \leq N \\ 1 \leq \alpha \leq A}}$  and  $((\beta_j^{\alpha}))_{\substack{1 \leq j \leq N \\ 1 \leq \alpha \leq A}}$  such that, for  $0 \leq j \leq N$  and  $1 \leq \alpha \leq A$ , we have

$$(4.4) \quad \sup_{x \in I_{\alpha}} \left| \begin{pmatrix} u_j(x) \\ u_j'(x) \end{pmatrix} - \lambda_j^{\alpha} \begin{pmatrix} v(x) \\ v'(x) \end{pmatrix} + \beta_j^{\alpha} \begin{pmatrix} w(x) \\ w'(x) \end{pmatrix} \right| \leq C\varepsilon e^{C\tilde{\ell}}.$$

Let  $\langle \cdot, \cdot \rangle$  denote the standard scalar product on  $L^2([0, \ell])$  and  $\langle \cdot, \cdot \rangle_{\alpha}$  that on  $L^2(I_{\alpha})$ . One has

$$(4.5) \quad \text{Id}_{N+1} = ((\langle u_i, u_j \rangle))_{\substack{0 \leq i \leq N \\ 0 \leq j \leq N}} = \sum_{\alpha=1}^A ((\langle u_i, u_j \rangle_{\alpha}))_{\substack{0 \leq i \leq N \\ 0 \leq j \leq N}}.$$

Using (4.4), we compute

$$(4.6) \quad M_{\alpha} := ((\langle u_i, u_j \rangle_{\alpha}))_{\substack{0 \leq i \leq N \\ 0 \leq j \leq N}} = \sum_{n=1}^4 M_{\alpha, n} + S_{\alpha}$$

where

$$(4.7) \quad M_{\alpha, 1} = \langle v, v \rangle_{\alpha} ((\lambda_i^{\alpha} \lambda_j^{\alpha}))_{\substack{0 \leq i \leq N \\ 0 \leq j \leq N}}, \quad M_{\alpha, 2} = \langle v, w \rangle_{\alpha} ((\lambda_i^{\alpha} \beta_j^{\alpha}))_{\substack{0 \leq i \leq N \\ 0 \leq j \leq N}},$$

$$(4.8) \quad M_{\alpha, 3} = \langle w, v \rangle_{\alpha} ((\beta_i^{\alpha} \lambda_j^{\alpha}))_{\substack{0 \leq i \leq N \\ 0 \leq j \leq N}}, \quad M_{\alpha, 4} = \langle w, w \rangle_{\alpha} ((\beta_i^{\alpha} \beta_j^{\alpha}))_{\substack{0 \leq i \leq N \\ 0 \leq j \leq N}},$$

$$(4.9) \quad \|S_{\alpha}\| \leq C\varepsilon N e^{C\tilde{\ell}} \tilde{\ell} \leq C\varepsilon \ell \tilde{\ell} e^{C\tilde{\ell}}.$$

Pick  $\tilde{\ell} = |\log \varepsilon|/K$  for some  $K$  sufficiently large; as  $0 < \varepsilon \leq \ell^{-\nu}$  with  $\nu > 2$ , for  $\ell$  sufficiently large, by (4.9), one has

$$\sum_{\alpha=1}^A \|S_{\alpha}\| \leq C\varepsilon \ell^2 e^{C\tilde{\ell}} \leq C\ell^{2-\nu(1-C/K)} \leq 1/2.$$

By (4.7) and (4.8), the matrices  $(M_{\alpha,n})_{\alpha,n}$  are all of rank at most 1. Hence, equation (4.5) implies that  $4A \geq N + 1$  which yields  $N + 1 \leq C\ell/|\log \varepsilon|$  for some  $C > 0$ . This completes the proof of Theorem 1.4.  $\square$

One can wonder whether the bounds given in Theorems 1.3 and 1.4 are optimal. Examples build using semi-classical ideas show that the orders of magnitudes are. The precise values of the constants depend on the details of the potential  $q$ .

## 5. APPENDIX

In this appendix, we collect various technical results useful for our study.

**5.1. Some results on differential equations.** We recall some standard estimates on ordinary differential equations that are immediate consequences of equations (2.2) and (2.3), and, presumably well known (see e.g. [33, 11]). We use the notation of section 2.

**Lemma 5.1.** *There exists a constant  $C > 0$  (depending only on  $\|q\|_\infty$ ) such that, for  $u$  a solution to  $Hu = 0$  (see (2.1)), if  $I(x) := [x - 1/2, x + 1/2] \cap [0, \ell]$ , one has*

$$(5.1) \quad \forall x \in [0, \ell], \quad \frac{1}{C} \int_{I(x)} u^2(y) dy \leq r_u^2(x) \leq C \int_{I(x)} u^2(y) dy,$$

$$(5.2) \quad \forall x \in [0, \ell], \quad \min_{I(x)} r_u \leq \max_{I(x)} r_u \leq C \min_{I(x)} r_u,$$

$$(5.3) \quad \forall x \in [0, \ell], \quad \|\sin(\varphi_u(\cdot))\|_{L^2(I(x))} \geq \frac{1}{C}.$$

**Lemma 5.2.** *Let  $\delta\varphi$  be a solution to the equation (2.5). There exists  $C > 0$  (depending only on  $\|q\|_\infty$ ) such that, for  $x_0 \in [0, \ell]$ , one has*

$$\forall x \in [0, \ell], \quad |\sin(\delta\varphi(x))| \leq [|\sin(\delta\varphi(x_0))| + E]e^{C|x-x_0|}.$$

*Proof.* Write  $s(x) = |\sin(\delta\varphi(x))|$  and note that, integrating equation (2.5) implies that

$$s(x) \leq s(x_0) + E + C \int_{x_0}^x s(t) dt.$$

The statement of Lemma 5.2 then follows from Gronwall's Lemma (see e.g. [32]).  $\square$

**Lemma 5.3.** *There exists  $\eta_0 > 0$  depending only on  $\|q\|_\infty$  such that, for  $\eta \in (0, \eta_0)$  and  $\varphi_u$ , a solution to equation (2.2), one has*

(1) *if  $y < y'$  are such that  $\max_{x \in [y, y']} |\sin \varphi_u(x)| \leq \eta$ , then  $|y - y'| \leq \eta/\eta_0$ ;*

(2) *if  $|\sin(\varphi_u(y))| \leq \eta$  then, for  $4\eta \leq |x - y| \leq \sqrt{\eta}$ , one has*

$$|\sin(\varphi_u(y))| \geq |x - y|/2.$$

(3) *if  $y < y'$  are such that*

$$|\sin \varphi_u(y)| = |\sin \varphi_u(y')| = \eta \text{ and } \min_{x \in [y, y']} |\sin \varphi_u(x)| \geq \eta$$

$$\text{then } |y - y'| \geq (\eta_0 - \eta)\eta_0.$$

*Proof.* First, by equation (2.2), for some  $C > 0$  depending only on  $\|q\|_\infty$ , one has  $|\varphi'_u(x)| \leq C$  and, if  $|\sin(\varphi_u(x))| \leq \eta$  then  $1 - C\eta^2 \leq |\cos \varphi_u(x)|\varphi'_u(x)$ . Pick  $\eta_0 \in (0, 1)$  such that  $1 - C\eta_0^2 \geq 1/2$ .



To prove point (1), consider  $y < y'$  such that  $\max_{x \in [y, y']} |\sin \varphi_u(x)| \leq \eta$ . As  $\eta < \eta_0 < 1$ ,  $\cos \varphi_u(x)$  does not change sign on  $[y, y']$ . Thus, one computes

$$2\eta \geq |\sin \varphi_u(y') - \sin \varphi_u(y)| = \int_y^{y'} |\cos \varphi_u(x)| \varphi'_u(x) dx \geq |y - y'|/2.$$

This proves (1) possibly diminishing the value of  $\eta_0$ .

To prove point (2), as by equation (2.2), for some  $C > 0$  depending only on  $\|q\|_\infty$ , one has  $|\varphi'_u(x)| \leq C$ , there exists  $\eta_0 > 0$  such that, for  $\eta \in (0, \eta_0]$ , if  $|\sin(\varphi_u(y))| \leq \eta$ , one has  $|\sin(\varphi_u(x))| \leq \eta_0$  for  $|x - y| \leq \eta_0$ . Thus, at the cost of reducing possibly  $\eta_0$ ,  $x \mapsto |\cos(\varphi_u(x))|$  stays larger than  $9/10$  on  $[y - \eta_0, y + \eta_0]$  and by equation (2.2), one has  $d/dx[\sin(\varphi_u(x))] \geq 3/4$  on  $[y - \eta_0, y + \eta_0]$ . This, the assumption  $|\sin(\varphi_u(y))| \leq \eta$  and the Taylor formula immediately entail point (2).

To prove point (3), note that, as  $|\varphi'_u(x)| \leq C$ , for  $y < z < y + (\eta_0 - \eta)/C$ , one has  $|\sin(\varphi_u(z))| \leq \eta_0$ . Thus,  $x \mapsto \cos \varphi_u(x)$  keeps a constant sign on the interval  $[y, \min(y', y + (\eta_0 - \eta)/C)]$ . Moreover, as  $\min_{x \in [y, y']} |\sin \varphi_u(x)| \geq \eta$ , so does  $x \mapsto \sin \varphi_u(x)$  and both signs are the same. Thus, for  $y < z < y + (\eta_0 - \eta)/C$ , we know that

$$\begin{aligned} |\sin \varphi_u(z)| &= |\sin \varphi_u(y)| + \int_y^z |\cos \varphi_u(x)| \varphi'_u(x) dx \\ &\geq \eta + \sqrt{1 - (\eta')^2}(z - y)/2 > \eta. \end{aligned}$$

Hence, one has  $y' > y + (\eta_0 - \eta)/C$ . This proves (2) at the expense of possibly changing  $\eta_0$  again. This completes the proof of Lemma 5.3.  $\square$

**5.2. Localization for the model  $H_\omega^A$ .** In the present section, we show how to extend the results of [11] to our assumptions.

Let

$$\tilde{H}_\omega = -\frac{d^2}{dx^2} + \tilde{W}(\cdot) + \sum_{n \in \mathbb{Z}} \tilde{\omega}_n \tilde{V}(\cdot - n) + W(x)$$

where

- $(\tilde{\omega}_n)_{n \in \mathbb{Z}}$  and  $\tilde{V}$  satisfy the assumptions that  $(\omega_n)_{n \in \mathbb{Z}}$  and  $V$  satisfy for  $H_\omega^A$  in section 0,
- $\tilde{V}$  has its support in  $(-1/2, 1/2)$ ,
- $\tilde{W}$  is uniformly continuous on  $\mathbb{R}$ .

Then, the main result of [11] can be rephrased in the following way:  $H_\omega$  satisfies (Loc) (see (1.3)) for any compact interval  $I$  (see Lemma 2.1 and Proposition 2.2 in [11]).

Consider now  $H_\omega^A$  as defined in section 0. Let  $n_0 \in \mathbb{N}$  be such that  $\text{supp} V \subset (-n_0/2, n_0/2)$ . Doing the change of variable  $x = n_0 y$ , we can rewrite

$$(5.4) \quad H_\omega^A = n_0^{-2} \left( -\frac{d^2}{dy^2} + \tilde{W}(\cdot) + \sum_{n \in \mathbb{Z}} \tilde{\omega}_n \tilde{V}(\cdot - n) \right)$$

where

- $\tilde{V}(\cdot) = n_0^2 V(n_0 \cdot)$ , thus,  $\tilde{V}$  has its support in  $(-1/2, 1/2)$ ,
- $\tilde{\omega}_n = \omega_{n \cdot n_0}$  for  $n \in \mathbb{Z}$ ,

- $\tilde{W}(\cdot) = n_0^2 \sum_{n \in \mathbb{Z} \setminus n_0 \mathbb{Z}} \omega_n V(n_0 \cdot -n)$ , thus,  $\tilde{W}$  is uniformly continuous on  $\mathbb{R}$  for any choice of  $(\omega_n)_{n \in \mathbb{Z} \setminus n_0 \mathbb{Z}}$  (as the random variables are bounded).

So, for any choice of  $(\omega_n)_{n \in \mathbb{Z} \setminus n_0 \mathbb{Z}}$ , we know that  $H_\omega^A$  satisfies (Loc) on any compact interval  $I$  when the expectation is taken with respect to the random variables  $(\omega_n)_{n \in n_0 \mathbb{Z}}$ . A priori, the constant in the right hand side of (1.3) may depend on the choice of  $(\omega_n)_{n \in \mathbb{Z} \setminus n_0 \mathbb{Z}}$ . The proof of Theorem 1 in [11] shows that this is not the case. More precisely, as  $\tilde{W}$  stays uniformly bounded independently of the choice of  $(\omega_n)_{n \in \mathbb{Z} \setminus n_0 \mathbb{Z}}$ , the estimates of the operator  $T_1$  and its continuity with respect to the potential  $\tilde{W}$  ( $W_0$  in [11]) yield that the right hand side of (1.3) is bounded uniformly in  $(\omega_n)_{n \in \mathbb{Z} \setminus n_0 \mathbb{Z}}$ . Thus,  $H_\omega^A$  satisfies (Loc) on any compact interval  $I$ .

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