Multi-operator Scaling Random Fields

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Abstract. In this paper, we define and study a new class of random fields called harmonizable multi-operator scaling stable random fields. These fields satisfy a local asymptotic operator scaling property which generalizes both the local asymptotic self-similarity property and the operator scaling property. Actually, they locally look like operator scaling random fields whose order is allowed to vary along the sample paths. We also give an upper bound of their modulus of continuity. Their pointwise Hölder exponents may also vary with the position x and their anisotropic behavior is driven by a matrix which may also depend on x.

1. Introduction

Self-similar random processes and fields are required to model numerous natural phenomena, e.g. in internet traffic, hydrology, geophysics or financial markets, see for instance [27, 18, 1]. A very important class of such fields is given by fractional stable random fields (see [25]). In particular, the well-known fractional Brownian field B_H is a Gaussian H-self-similar random field with stationary increments. It is an isotropic generalization of the famous fractional Brownian motion ([19, 12]). Self-similar isotropic α-stable fields have been extensively used to propose an alternative to Gaussian modeling (see [21, 27] for instance) to mimic heavy-tailed persistent phenomena.

However, isotropy property is a serious drawback for many applications in medicin [8], in geophysics [22, 9] and in hydrology [5], just to mention a few. Recently, an important class of anisotropic random fields has been studied in [7]. These fields are anisotropic generalizations of self-similar stable random fields. They satisfy an operator scaling property which generalizes the classical self-similarity property. More precisely, for E a real $d \times d$ matrix with positive real parts of the eigenvalues, a scalar valued random field $(X(x))_{x\in\mathbb{R}^d}$ is called *operator scaling* of order E and $H > 0$ if for every $c > 0$

$$\{X(c^E x); x \in \mathbb{R}^d\} \stackrel{(fdd)}{=} \{c^H X(x); x \in \mathbb{R}^d\}, \tag{1}$$

where $\stackrel{(fdd)}{=}$ means equality of finite dimensional distributions and as usual $c^E = \exp(E \log c)$. Let us recall that the self-similarity property corresponds to the case where E is the identity matrix. Let us also remark that up to consider the matrix E/H, we may and will assume, without loss of generality, that $H = 1$. The anisotropic behavior of operator scaling random fields with

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stationary increments is then driven by a matrix. In particular, when θ_j is an eigenvector of E associated with the eigenvalue λ_j, any operator scaling random field for E is $1/\lambda_j$-self-similar in direction θ_j. Furthermore, the critical global and directional Hölder exponents of harmonizable operator scaling stable random fields are given by the eigenvalues of E (see [6]). Let us emphasize that these exponents and the directions of self-similarity do not vary according to the position.

Moreover, the self-similarity is a global property which can be too restrictive for applications. Actually, numerous phenomena exhibit scale invariance that may vary according to the scale or to the position and are usually called multifractal (see [10, 24, 22] for examples). To allow more flexibility, [4] has introduced the local asymptotic self-similarity property. This property characterizes random fields that locally seem self-similar but whose local regularity properties evolve. Since then, many examples of locally asymptotically self-similar random fields have been introduced and studied, e.g. in [4, 23, 3, 2, 15, 26].

In this paper, we introduce the local asymptotic operator scaling property which generalizes both the local asymptotic self-similarity property and the operator scaling property. A scalar valued random field X is *locally asymptotically operator scaling at point x of order $A(x)$* if

$$\lim_{\varepsilon\to 0^+}\left(\frac{X(x+\varepsilon^{A(x)}u)-X(x)}{\varepsilon}\right)_{u\in\mathbb{R}^d}\overset{(fdd)}{=}(Z_x(u))_{u\in\mathbb{R}^d}\,, \tag{2}$$

with Z_x a non degenerate random field. Let us first remark that the local asymptotic self-similarity property of exponent $h(x)$ corresponds to the local asymptotic operator self-similarity of order $A(x) = I_d/h(x)$ with I_d the identity matrix of order d. Moreover, operator scaling random fields of order E are locally asymptotically operator scaling at point 0 of order E. Of course, if they have also stationary increments, they are locally asymptotically operator scaling at any point x. In addition, if (2) is fulfilled, the random field Z_x is operator scaling of order $A(x)$. In other words, a local asymptotic multi-operator random field locally looks like an operator scaling random field whose order is allowed to vary along the sample paths.

Then, we focus on harmonizable multi-operator scaling stable random fields, which generalize harmonizable operator scaling stable random fields. A harmonizable multi-operator scaling stable random field X satisfies the local asymptotic self-similarity property (2) with Z_x a harmonizable operator scaling stable random field of order $A(x)$. Moreover, its local sample path properties at point x are the same as those of Z_x which have been established in [7, 6]. Hence, its local regularity varies with the position x and its anisotropic behavior is driven by a matrix which depends on x. In particular, for any eigenvector $\theta_j(x)$ of $A(x)$ associated with the real eigenvalue $\lambda_j(x)$, the random field X admits $H_j(x) = 1/\lambda_j(x)$ as pointwise Hölder exponent in direction $\theta_j(x)$ at point x. Let us point out that we establish an accurrate upper bound for the modulus of continuity. Such upper bound has already been given for real harmonizable fractional stable

motions in [14], for some Gaussian random processes in [13] and for harmonizable operator scaling stable random fields in [6]. Then, in this paper, we generalize these results to harmonizable multi-operator scaling stable random fields. To study the sample paths in the case of α-stable random fields with $\alpha \in (0,2)$, we use a LePage series representation (see [17, 16] for details on such series) which is chosen to be conditionnally Gaussian as in [14, 6].

Harmonizable multi-operator scaling stable random fields are defined in Section 2. In this section, we also state all the assumptions we will need and present many examples that fulfill them. Section 3 is devoted to the properties of the polar coordinates: these coordinates are one of the main tools we use to study the sample paths as in [7, 6]. In Section 4, we state the sample path properties of the class of random fields under study (modulus of continuity and pointwise directional Hölder exponents). Section 5 is devoted to the local asymptotic operator self-similar property. Some technical proofs are postponed to the Appendix.

Throughout this paper, $B(x,\gamma)$ denotes the closed Euclidean ball of center x and radius γ.

2. Harmonizable representation

Harmonizable stable random fields are defined as stochastic integrals of deterministic kernels with respect to a stable random measure. In this paper we will always assume that the following assumption holds:

Assumption 1. *Let $\alpha \in (0,2]$ and W_α be a complex isotropic α-stable random measure with Lebesgue control measure (see* [25] *p.281 for details on such measures). Note that W_2 is an isotropic complex Gaussian random measure.*

Let us recall (see [25]) that the stochastic integral

$$W_\alpha(f) := \int_{\mathbb{R}^d} f(\xi) W_\alpha(d\xi)$$

is well-defined if and only if $f \in L^\alpha\big(\mathbb{R}^d\big)$. Furthermore, for $f \in L^\alpha\big(\mathbb{R}^d\big)$, $W_\alpha(f)$ is a stable complexed-valued random variable whose characteristic function is given by

$$\forall z \in \mathbb{C},\ \mathbb{E}(\exp\left(i\,\mathrm{Re}\left(\overline{z}W_\alpha(f)\right)\right)) = \exp\left(-s_\alpha \|W_\alpha(f)\|_\alpha^\alpha\, |z|^\alpha\right)$$

where

$$\|W_\alpha(f)\|_\alpha = \left(\int_{\mathbb{R}^d} |f(\xi)|^\alpha d\xi\right)^{1/\alpha} \text{ and } s_\alpha = \frac{1}{2\pi}\int_0^\pi |\cos(\xi)|^\alpha d\xi.$$

Note that if $\alpha = 2$, for each square integrable function f, the stochastic integral $W_2(f)$ is a centered Gaussian random variable.

According to [7], a harmonizable operator scaling stable random field $X = (X(x))_{x\in\mathbb{R}^d}$ is defined by

$$X(x) = \mathrm{Re}\int_{\mathbb{R}^d} \Big(\mathrm{e}^{i\langle x,\xi\rangle} - 1\Big)\psi(\xi)^{-1-\mathrm{trace}(E_0)/\alpha}\, W_\alpha(d\xi), \tag{3}$$

with E_0 a $d \times d$ real matrix whose eigenvalues have real parts greater than 1 and $\psi : \mathbb{R}^d \to [0, \infty)$ is a continuous E_0-homogeneous function, that is $\psi(c^{E_0}\xi) = c\psi(\xi)$ for all $c > 0$ and $x \in \mathbb{R}^d$, such that

$$\forall \xi \neq 0,\ \psi(\xi) \neq 0.$$

In order to obtain a field whose local behavior is given by a harmonizable operator scaling stable random field, we replace in (3) the matrix E_0 (respectively the function ψ) by a matrix $E(x)$ (respectively a function ψ_x) which depends on the position x. In this approach, the function ψ_x is $E(x)$-homogeneous. This leads us to consider

$$X_{\alpha,\psi}(x) = \mathrm{Re} \int_{\mathbb{R}^d} \Big(\mathrm{e}^{i\langle x,\xi\rangle} - 1 \Big) \psi_x(\xi)^{-1-\mathrm{trace}(E(x))/\alpha}\, W_\alpha(d\xi).$$

This approach has already been used to define the multifractional Brownian field in [4, 23]. To ensure that the field $X_{\alpha,\psi}$ is well-defined, we only have to assume that $(E(x), \psi_x)$ satisfies the assumptions of [7] for any x. Before we state these assumptions, let us introduce several notations we will use throughout the paper.

Notations. We denote $\mathcal{M}^{>0}\big(\mathbb{R}^d\big)$ the space of all $d \times d$ real matrix whose eigenvalues have positive real parts. In the following, for any $x \in \mathbb{R}^d$, $E(x) \in \mathcal{M}^{>0}\big(\mathbb{R}^d\big)$. The eigenvalues of $E(x)$ are denoted by $\lambda_1(x), \ldots, \lambda_d(x)$. For each $j = 1, \ldots, d$ and each $x \in \mathbb{R}^d$, we set

$$a_j(x) = \mathrm{Re}\,(\lambda_j(x)),\ H_j(x) = \frac{1}{a_j(x)},\ \overline{H}(x) = \max_{1\leq i\leq d} H_i(x) \ \text{ and } \ \underline{H}(x) = \min_{1\leq i\leq d} H_i(x). \tag{4}$$

The multi-operator scaling random field $X_{\alpha,\psi}$ is well-defined as soon as the two following assumptions are fulfilled. These assumptions come from [7] when E and ψ do not vary with the position x.

Assumption 2. *Assume that*

$$\forall x \in \mathbb{R}^d,\ \min_{1\leq j\leq d} a_j(x) > 1$$

with a_j defined by (4).

Assumption 3. *For every $x \in \mathbb{R}^d$, let $\psi_x : \mathbb{R}^d \to [0, +\infty)$ be a continuous function, $E(x)$-homogeneous which means according to Definition 2.6 of* [7] *that*

$$\psi_x(c^{E(x)}\xi) = c\psi_x(\xi) \text{ for all } c > 0 \text{ and } \xi \in \mathbb{R}^d.$$

Let us also assume that $\psi_x(\xi) \neq 0$ for $\xi \neq 0$.

Following ideas of [2], let us now define generalized multi-operator scaling stable random fields. These fields will be useful in the study the sample paths of harmonizable multi-operator scaling stable random fields.

Theorem 2.1. *Assume that Assumptions 1, 2 and 3 are fulfilled. Then, the random field*

$$Y_{\alpha,\psi}(x,y,z) = \mathrm{Re}\int_{\mathbb{R}^d}\Big(\mathrm{e}^{i\langle x,\xi\rangle}-1\Big)\psi_y(\xi)^{-\beta_\alpha(z)}\,W_\alpha(d\xi),\ x,y,z\in\mathbb{R}^d, \tag{5}$$

where

$$\beta_\alpha(z) = 1+\frac{q(z)}{\alpha}\quad \textit{with}\quad q(z)=\mathrm{trace}(E(z)) \tag{6}$$

is well-defined on the non empty set

$$U=\left\{(x,y,z)\in\mathbb{R}^{3d}\,:\,0<1+(q(z)-q(y))/\alpha<\min_{1\le j\le d}\mathrm{Re}\,(\lambda_j(y))=\frac{1}{\overline{H}(y)}\right\}.$$

The random field $Y_{\alpha,\psi}$ is called generalized multi-operator scaling stable random field.

Proof. Let $x,y,z\in\mathbb{R}^d$ and $H=1+(q(z)-q(y))/\alpha$. Since $\beta_\alpha(z)=H+q(y)/\alpha$, according to Theorem 4.1 of [7] (applied with $\psi=\psi_y$), the random variable $Y_{\alpha,\psi}(x,y,z)$ is well-defined as soon as

$$0<H<\min_{1\le j\le d}\mathrm{Re}\,(\lambda_j(y))=\frac{1}{\overline{H}(y)},$$

which holds for any $(x,y,z)\in U$. □

We now introduce the class of harmonizable multi-operator scaling random fields which will study in this paper.

Definition 2.1. *Assume that Assumptions 1, 2 and 3 are fulfilled. Then, the random field*

$$X_{\alpha,\psi}(x)=Y_{\alpha,\psi}(x,x,x)=\mathrm{Re}\int_{\mathbb{R}^d}\Big(\mathrm{e}^{i\langle x,\xi\rangle}-1\Big)\psi_x(\xi)^{-\beta_\alpha(x)}\,W_\alpha(d\xi)\ ,x\in\mathbb{R}^d, \tag{7}$$

with β_α defined by (6), is well-defined and is called harmonizable multi-operator scaling stable random field.

Remark 2.1. *If $\alpha=2$, $X_{\alpha,\psi}$ is a real-valued centered Gaussian random field.*

Remark 2.2. *Assume that Assumptions 1, 2 and 3 are fulfilled. Let*

$$\forall x\in\mathbb{R}^d,\ \forall\xi\in\mathbb{R}^d,\ \widetilde{\psi}_x(\xi)=\frac{1}{2}(\psi_x(\xi)+\psi_x(-\xi)).$$

Then, for each $x\in\mathbb{R}^d$, $\widetilde{\psi}_x$ is an even function. Since Assumption 3 is also fulfilled by $(x,\xi)\mapsto\widetilde{\psi}_x$, the harmonizable multi-operator scaling stable random fields $X_{\alpha,\psi}$ and $X_{\alpha,\widetilde{\psi}}$ are well-defined by (7). Moreover, one can easily see that

$$\Big\{X_{\alpha,\psi}(x);\,x\in\mathbb{R}^d\Big\}\overset{(fdd)}{=}\Big\{X_{\alpha,\widetilde{\psi}}(x);\,x\in\mathbb{R}^d\Big\}.$$

Therefore, the function ψ_x will be chosen even for any $x\in\mathbb{R}^d$ in all the examples we will give.

Let us emphasize that to study the sample paths of $X_{\alpha,\psi}$, we need the functions ψ and E to be sufficiently regular. We introduce now all the assumptions we will use in sequel.

Assumption 4. *Let $T = \prod_{i=1}^{d} [b_i, d_i]$ with $b_i < d_i$ for $1 \leq i \leq d$. Let us assume that the function $(x, \xi) \mapsto \psi_x(\xi)$ is locally Lipschitz on $T \times \mathbb{R}^d \backslash \{0\}$, that is for every compact set $\mathcal{K} \subset T \times \mathbb{R}^d \backslash \{0\}$, there exists a finite positive constant $c_{2,1} = c_{2,1}(\mathcal{K})$ such that*

$$|\psi_{x_1}(\xi_1) - \psi_{x_2}(\xi_2)| \leq c_{2,1} \left(\|x_1 - x_2\| + \|\xi_1 - \xi_2\| \right)$$

for every $(x_1, \xi_1), (x_2, \xi_2) \in \mathcal{K}$.

Assumption 5. *Let $T = \prod_{i=1}^{d} [b_i, d_i]$ with $b_i < d_i$ for $1 \leq i \leq d$. Let us assume that the map $E : x \longmapsto E(x)$ is a Lipschitz function on T: there exists a finite positive constant $c_{2,2} = c_{2,2}(T)$ such that, for $x_1, x_2 \in T$*

$$\|E(x_1) - E(x_2)\| \leq c_{2,2} \|x_1 - x_2\|.$$

Assumption 6. *Let $T = \prod_{i=1}^{d} [b_i, d_i]$ with $b_i < d_i$ for $1 \leq i \leq d$. Let us assume that for any $x, y \in T$, $E(x)$ and $E(y)$ are commuting matrices:*

$$E(x)E(y) = E(y)E(x).$$

We now conclude this section by several examples. We first give two straightforward classes of examples. The first one is given by harmonizable operator scaling stable random fields. The second one includes the classical multifractional Brownian field as defined in [4].

Example 2.1 (**Operator scaling random fields**). *Let E_0 be a $d \times d$ real matrix whose eigenvalues have real parts greater than 1. Let us consider a function $\psi : \mathbb{R}^d \to [0, \infty)$ E_0-homogeneous, locally Lipschitz on $\mathbb{R}^d \backslash \{0\}$ and such that*

$$\forall \xi \neq 0,\ \psi(\xi) \neq 0.$$

For all $x, \xi \in \mathbb{R}^d$, let

$$E(x) = E_0 \quad \text{and} \quad \psi_x = \psi.$$

Then, Assumptions 2-6 are fulfilled and under Assumption 1, $X_{\alpha,\psi}$ is a harmonizable operator scaling stable random field for E_0^t with stationary increments, see [7]*. In particular, $X_{\alpha,\psi}$ satisfies the operator-scaling property* (1) *for E_0^t (and $H = 1$).*

Example 2.2 (**Multifractional operator scaling random fields**). *Let E_0 and ψ be as in Example 2.1 and let $h : \mathbb{R}^d \longrightarrow (0, 1)$ be a locally Lipschitz function. For all $x \in \mathbb{R}^d$, let us define*

$$E(x) = \frac{1}{h(x)} E_0 \quad \text{and } \psi_x = \psi^{h(x)}.$$

Then, Assumptions 2-6 are fulfilled and under Assumption 1, the random field $X_{\alpha,\psi}$ given by (7) *is well-defined. In particular, if $E_0 = I_d$ is the identity matrix and if $\psi = \|\cdot\|_2$ is the Euclidean norm on $\mathbb{R}^d$, then $X_{\alpha,\psi}$ is a multifractional harmonizable stable random field, called multifractional Brownian field if $\alpha = 2$ (see* [4]*).*

Remark 2.3. *Let us focus on the special case $d = 1$. If we assume that ψ_x is an even function for any $x \in \mathbb{R}^d$, Assumption 3 implies that there exists a positive function c such that*

$$\psi_x(\xi) = c(x)|\xi|^{h(x)}, \text{ for any } x, \xi \in \mathbb{R}^d,$$

where $h = 1/E$. Hence, if $d = 1$, under Assumptions 1-3, the random process $X_{\alpha,\psi}$ is a multi-fractional harmonizable stable random motion, up to a deterministic multiplicative function.

Example 2.3. *For every $1 \leq j \leq d$, assume that H_j is a locally Lipschitz function on $\mathbb{R}^d$ with values in $(0, 1)$. Assume also that*

$$\inf_{x\in\mathbb{R}^d} \min_{1\leq j\leq d} H_j(x) > 0.$$

Consider the map

$$E = diag\,(1/H_1, \ldots, 1/H_d)$$

defined on $\mathbb{R}^d$ with values in the space of diagonal matrices. Let $\rho \in (0, \inf_{x\in\mathbb{R}^d} \min_{1\leq j\leq d} H_j(x)]$ and

$$\psi_x(\xi) = \left(|\xi_1|^{\frac{2H_1(x)}{\rho}} + \cdots + |\xi_d|^{\frac{2H_d(x)}{\rho}}\right)^{\rho/2},$$

for every $x, \xi \in \mathbb{R}^d$. Then, Assumptions 2-6 are fulfilled such that, under Assumption 1, the random field $X_{\alpha,\psi}$ given by (7) is well-defined.

Example 2.4. *Let E and ψ_x as in Example 2.3. Let $P \in GL_d(\mathbb{R})$ be an invertible matrix. Then the map*

$$x \mapsto P^{-1}E(x)P$$

satisfies Assumptions 2, 5 and 6. Moreover, the function

$$\varphi : (x, \xi) \mapsto \psi_x(P\xi)$$

satisfies Assumptions 3 and 4. Then, the harmonizable multi-operator scaling stable random field $X_{\alpha,\varphi}$ is well defined by (7).

Example 2.5. *Let $d = 2$. Let us consider the map*

$$x \mapsto E(x) = a(x) \begin{pmatrix} \cos(\theta(x)) & \sin(\theta(x)) \\ -\sin(\theta(x)) & \cos(\theta(x)) \end{pmatrix}$$

where a and θ are locally Lipschitz functions on $\mathbb{R}^d$. Assume that

$$\forall x \in \mathbb{R}^d, \ a(x)\cos(\theta(x)) > 1.$$

For every $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^d$, let

$$\psi_x(\xi) = \|\xi\|_2^{1/(a(x)\cos(\theta(x)))}.$$

Then Assumptions 2-6 are fulfilled such that, under Assumption 1, the random field $X_{\alpha,\psi}$ given by (7) is well-defined.

Example 2.6. *Let $E_i : \mathbb{R}^d \to \mathcal{M}^{>0}(\mathbb{R}^d)$ satisfy Assumption 2 for $i \in \{1,2\}$ and let $\psi^{(i)}$ satisfying Assumption 3 with respect to E_i for $i \in \{1,2\}$. Consider the map*

$$E = E_1 \mathbf{1}_{[0,1]^d} + E_2 \mathbf{1}_{\mathbb{R}^d \setminus [0,1]^d}.$$

and for any $x \in \mathbb{R}^d$, the function

$$\psi_x(\xi) = \psi_x^{(1)}(\xi)\mathbf{1}_{[0,1]^d}(x) + \psi_x^{(2)}(\xi)\mathbf{1}_{\mathbb{R}^d \setminus [0,1]^d}(x).$$

Then ψ satisfies Assumption 3 with respect to E. The random fields $X_{\alpha,\psi^{(1)}}$, $X_{\alpha,\psi^{(2)}}$ and $X_{\alpha,\psi}$ are well defined by (7) and

$$X_{\alpha,\psi} = X_{\alpha,\psi^{(1)}} \mathbf{1}_{[0,1]^d} + X_{\alpha,\psi^{(2)}} \mathbf{1}_{\mathbb{R}^d \setminus [0,1]^d}.$$

The approach proposed in this example allows to define harmonizable stable random fields which are piecewise operator scaling.

In the next section we recall one of the main tools needed to study operator scaling random fields, in particular a change of variables formula with respect to adapted polar coordinates.

3. Polar coordinates

Let us recall the main properties of polar coordinates adapted to a single matrix as introduced in [20]. Let $M \in \mathcal{M}^{>0}\big(\mathbb{R}^d\big)$. As in Chapter 6 of [20], let us consider the norm $\|\cdot\|_M$ defined by

$$\|x\|_M = \int_0^1 \big\|t^M x\big\| \frac{dt}{t}, \quad \forall x \in \mathbb{R}^d \tag{8}$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^d$. Then, according to Chapter 6 of [20], $\|\cdot\|_M$ is a norm on $\mathbb{R}^d$ such that the map

$$\begin{array}{rccc} \Psi_M : & (0,+\infty) \times S_M & \longrightarrow & \mathbb{R}^d \setminus \{0\} \\ & (r,\theta) & \longmapsto & r^M \theta \end{array}$$

is a homeomorphism, where

$$S_M = \{\xi \in \mathbb{R}^d : \|\xi\|_M = 1\} \tag{9}$$

is the unit sphere for $\|\cdot\|_M$. Hence we can write any $\xi \in \mathbb{R}^d \setminus \{0\}$ uniquely as

$$\xi = \tau_M(\xi)^M \ell_M(\xi) \tag{10}$$

with $\tau_M(\xi) > 0$ and $\ell_M(\xi) \in S_M$. Here, for any $\xi \in \mathbb{R}^d \setminus \{0\}$, $\tau_M(\xi)$ should be interpreted as the *radial part* of ξ with respect to M and $\ell_M(\xi) \in S_M$ as its *directional part* with respect to M.

Let us now recall the formula of integration in *polar coordinates* established in [7].

Proposition 3.1. *There exists a unique finite Radon measure σ_M on the unit sphere S_M defined by (9) such that for all $f \in L^1(\mathbb{R}^d, d\xi)$,*

$$\int_{\mathbb{R}^d} f(\xi)\, d\xi = \int_0^{+\infty} \int_{S_M} f(r^M \theta)\, \sigma_M(d\theta)\, r^{\mathrm{trace}(M)-1}\, dr.$$

The main difficulty in our setting is that we do not consider a single matrix but a family $(E(x))_{x\in\mathbb{R}^d}$ of matrices. Hence we need uniform controls on the polar coordinates. These will follow from the next Lemmas.

Lemma 3.2. *Let $T=\prod_{i=1}^{d}[b_i,d_i]$ with $b_i<d_i$ for $1\leq i\leq d$. Assume that $E:T\to\mathcal{M}^{>0}(\mathbb{R}^d)$ is continuous on T and satisfies Assumption 6 on T. Then the map*

$$\begin{array}{rccc}\mathcal{P}\ :\ & [0,+\infty)\times T & \longrightarrow & \mathcal{M}\big(\mathbb{R}^d\big)\\ & (t,x) & \longmapsto & t^{E(x)}\end{array}$$

is continuous on $[0,+\infty)\times T$ (with convention $0^{E(x)}=0$).

Proof. According to Proposition 2.2.11 of [20], since $E:T\to\mathcal{M}^{>0}\big(\mathbb{R}^d\big)$ is continuous on T, the map $\mathcal{P}$ is continuous on $(0,+\infty)\times T$. Therefore, the main problem is to prove that $\mathcal{P}$ is continuous at $(0,x)$ for any $x\in T$.
Let us fix $x\in T$. Then, let $\delta>0$ such that the real parts of all the eigenvalues of $E(x)$ are greater than 2δ, that is such that

$$\underline{H}(x)=\min_{1\leq j\leq d}H_j(x)>2\delta.$$

It follows from Theorem 2.2.4 of [20] that

$$\sup_{\|\theta\|=1}t^{2\delta}\|t^{-E(x)}\theta\|\xrightarrow[t\to+\infty]{}0.$$

Then, by continuity of $t\mapsto t^{-E(x)}$ on $[1,+\infty[$, one can find a finite positive constant c such that

$$\forall t\in[1,+\infty),\ \|t^{-E(x)}\|:=\sup_{\|\theta\|=1}\|t^{-E(x)}\theta\|\leq ct^{-2\delta}.$$

Now, since E is continuous on T, there exists $r_\delta\in(0,+\infty)$ such that

$$\forall y\in B(x,r_\delta)\cap T,\ \|E(x)-E(y)\|\leq\delta,$$

where $B(x,r_\delta)$ is the Euclidean closed ball centered at point x with radius r_δ.
Therefore for any $s\in(0,1]$ and any $y\in B(x,r_\delta)\cap T$, according to Assumption 6

$$\|s^{E(y)}\|=\|s^{E(y)-E(x)}s^{E(x)}\|\leq\|s^{E(y)-E(x)}\|\|s^{E(x)}\|\leq cs^{-\|E(y)-E(x)\|}s^{2\delta}.$$

Hence, for any $s\in(0,1]$ and any $y\in B(x,r_\delta)\cap T$,

$$\|s^{E(y)}\|\leq cs^{\delta},$$

which also holds for $s=0$ by convention and concludes the proof. □

Let us remark that one can establish the continuity of $\mathcal{P}$ on $[0,+\infty)\times T$ without Assumption 6. However, without Assumption 6, the proof is very long and this assumption will be needed in sequel.

Lemma 3.2 leads to an uniform control of $\left\|t^{E(x)}\right\|$ with respect to the eigenvalues of $E(x)$, which proof is postponed to the Appendix.

Lemma 3.3. *Let $T = \prod_{i=1}^{d}[b_i, d_i]$ with $b_i < d_i$ for $1 \leq i \leq d$. Let $\overline{H}$ and $\underline{H}$ be defined by (4). Assume that $E : T \to \mathcal{M}^{>0}(\mathbb{R}^d)$ is continuous on T and satisfies Assumption 6. Then, for any $\delta > 0$ and $r_0 > 0$, there exist some finite constants $c_{3,1} = c_{3,1}(T, \delta, r_0) > 0$ and $c_{3,2} = c_{3,2}(T, \delta, r_0)$ such that for any $x \in T$,*

(i) for all $t \in [0, r_0]$,

$$t^{1/\underline{H}(x)} \leq \|t^{E(x)}\| \leq c_{3,1}\, t^{1/\overline{H}(x)-\delta};$$

(ii) for all $t \in [r_0, +\infty)$,

$$t^{1/\overline{H}(x)} \leq \|t^{E(x)}\| \leq c_{3,2}\, t^{1/\underline{H}(x)+\delta}.$$

Moreover, Lemma 3.2 leads also to an uniform control of $\|\cdot\|_{E(x)}$ with respect to the Euclidean norm, stated in the next lemma, which proof is again postponed to the Appendix.

Lemma 3.4. *Let $T = \prod_{i=1}^{d}[b_i, d_i]$ with $b_i < d_i$ for $1 \leq i \leq d$. Assume that $E : T \to \mathcal{M}^{>0}(\mathbb{R}^d)$ is continuous on T and satisfies Assumption 6. Then there exist two finite positive constants $c_{3,3} = c_{3,3}(T)$ and $c_{3,4} = c_{3,4}(T)$ such that*

$$\forall x \in T,\, \forall \xi \in \mathbb{R}^d,\, c_{3,3}\|\xi\|_{E(x)} \leq \|\xi\| \leq c_{3,4}\|\xi\|_{E(x)}$$

and such that

$$\forall x \in T,\, c_{3,3} \leq \sigma_{E(x)}\left(S_{E(x)}\right) \leq c_{3,4}$$

with $\sigma_{E(x)}$ the measure introduced in Proposition 3.1.

Using Lemmas 3.3 and 3.4 we can compare uniformly the radial parts with the Euclidean norm. The following proposition, which proof is postponed to the Appendix, is one of the main tools to obtain Hölder regularity of multi-operator scaling stable random fields.

Proposition 3.5. *Let $T = \prod_{i=1}^{d}[b_i, d_i]$ with $b_i < d_i$ for $1 \leq i \leq d$. Let $\overline{H}$ and $\underline{H}$ be defined by (4). Assume that $E : T \to \mathcal{M}^{>0}(\mathbb{R}^d)$ is continuous on T and satisfies Assumption 6. Then, for any $\delta \in (0, \min_{x\in T}\underline{H}(x))$, there exist two finite positive constants $c_{3,5} = c_{3,5}(T, \delta)$ and $c_{3,6} = c_{3,6}(T, \delta)$ such that for all $x \in T$ and $\|\xi\| \leq 1$,*

$$c_{3,5}\|\xi\|^{\overline{H}(x)+\delta} \leq \tau_{E(x)}(\xi) \leq c_{3,6}\|\xi\|^{\underline{H}(x)-\delta}, \tag{11}$$

and, for all $\|\xi\| \geq 1$,

$$c_{3,5}\|\xi\|^{\underline{H}(x)-\delta} \leq \tau_{E(x)}(\xi) \leq c_{3,6}\|\xi\|^{\overline{H}(x)+\delta}. \tag{12}$$

Let us mention that for any fixed $x \in \mathbb{R}^d$, the inequality (11), respectively (12), holds true with $|\log(\|\xi\|)|^d$ instead of $\|\xi\|^{-\delta}$, respectively instead of $\|\xi\|^{\delta}$, with constants $c_{3,5}, c_{3,6}$ depending

on x (see [6] for a proof).

We end this section by comparing the radial parts $\tau_{E(x)}(\xi)$ and $\tau_{E(y)}(\xi)$, uniformly in ξ, if x and y are closed enough. This result will be useful to obtain an upper bound for the modulus of continuity of multi-operator scaling stable random fields. Its proof is postponed to the Appendix.

Proposition 3.6. *Let $T = \prod_{i=1}^{d}[b_i, d_i]$ with $b_i < d_i$ for $1 \leq i \leq d$. Assume that $E : T \to \mathcal{M}^{>0}(\mathbb{R}^d)$ is continuous on T and satisfies Assumption 6. Then, for any $\varepsilon \in (0,1)$, there exists $\gamma > 0$ such that for all $x, y \in T$ with $\|x - y\| \leq \gamma$,*

$$c_{3,7}\tau_{E(y)}(\xi)^{1+\varepsilon} \leq \tau_{E(x)}(\xi) \leq c_{3,8}\tau_{E(y)}(\xi)^{1-\varepsilon},\ \forall \|\xi\| \leq 1 \tag{13}$$

and,

$$c_{3,7}\tau_{E(y)}(\xi)^{1-\varepsilon} \leq \tau_{E(x)}(\xi) \leq c_{3,8}\tau_{E(y)}(\xi)^{1+\varepsilon},\ \forall \|\xi\| \geq 1 \tag{14}$$

where $c_{3,7} = c_{3,7}(T)$ and $c_{3,8} = c_{3,8}(T)$ are two finite positive constants that only depend on T.

Let us emphasize that all these results depend only on the eigenvalues of the matrices. Therefore they also hold when the map E is replaced by $E^t : x \mapsto E(x)^t$, where $E(x)^t$ is the transpose matrix of $E(x)$.

4. Sample paths Regularity of multi-operator scaling stable random fields

4.1. **Preliminary result on the scale parameter.** In order to study the regularity of the sample paths of $X_{\alpha,\psi}$ defined by (7), we consider the increments

$$X_{\alpha,\psi}(u) - X_{\alpha,\psi}(v) = Y_{\alpha,\psi}(u,u,u) - Y_{\alpha,\psi}(v,v,v), \forall u, v \in \mathbb{R}^d$$

with $Y_{\alpha,\psi}$ defined by (5). Observe that

$$X_{\alpha,\psi}(u) - X_{\alpha,\psi}(v) = Y_{1,\alpha,u}(u,v) + Y_{2,\alpha,u}(u,v) + Y_{3,\alpha,v}(u,v),$$

with

$$\left\{\begin{array}{rcl} Y_{1,\alpha,x}(u,v) & = & Y_{\alpha,\psi}(x,u,u) - Y_{\alpha,\psi}(x,u,v), \\ Y_{2,\alpha,x}(u,v) & = & Y_{\alpha,\psi}(x,u,v) - Y_{\alpha,\psi}(x,v,v), \\ Y_{3,\alpha,x}(u,v) & = & Y_{\alpha,\psi}(u,x,x) - Y_{\alpha,\psi}(v,x,x). \end{array}\right.$$

By Theorem 2.1, the random variables $Y_{1,\alpha,x}(u,v)$ and $Y_{2,\alpha,x}(u,v)$ are well-defined as soon as $x \in \mathbb{R}^d$ and

$$|q(v) - q(u)| < \alpha \min\big(1/\overline{H}(u) - 1, 1/\overline{H}(v) - 1, 1\big). \tag{15}$$

Note that for every $x, u, v \in \mathbb{R}^d$, $Y_{3,\alpha,x}(u,v)$ is also well-defined and is an increment of a harmonizable operator scaling stable random field with exponent $E = E(v)^t$ and kernel function $\psi(\xi) = \psi_v(\xi)$ (see [7]).

In this section, we compare the scale parameter

$$\|X_{\alpha,\psi}(u) - X_{\alpha,\psi}(v)\|_\alpha,$$

with $\tau_{E(v)^t}(u-v)$ uniformly in u, v. In order to obtain our estimates, we study the scale parameters of $Y_{1,\alpha,x}(u,v)$, $Y_{2,\alpha,x}(u,v)$ and $Y_{3,\alpha,x}(u,v)$. The controls of these parameters are stated in the three following lemmas, whose proofs are postponed to the Appendix.

Lemma 4.1. *Assume that Assumptions 1-6 are fulfilled and let $K \subset \mathbb{R}^d$ be a compact set of $\mathbb{R}^d$. Then, for $\gamma > 0$ small enough, there exists $c_{4,1} = c_{4,1}(K,T,\gamma)$ a finite positive constant such that, for every $x \in K$, $u, v \in T$ with $\|u-v\| \le \gamma$, $Y_{1,\alpha,x}(u,v)$ is well-defined and*

$$\|Y_{1,\alpha,x}(u,v)\|_\alpha^\alpha \le c_{4,1}\|u-v\|^\alpha.$$

Lemma 4.2. *Assume that Assumptions 1-6 are fulfilled and let $K \subset \mathbb{R}^d$ be a compact set of $\mathbb{R}^d$. Then, for $\gamma > 0$ small enough, there exists $c_{4,2} = c_{4,2}(K,T,\gamma)$ a finite positive constant such that, for every $x \in K$, $u, v \in T$ with $\|u-v\| \le \gamma$, $Y_{2,\alpha,x}(u,v)$ is well-defined and*

$$\|Y_{2,\alpha,x}(u,v)\|_\alpha^\alpha \le c_{4,2}\|u-v\|^\alpha.$$

Lemma 4.3. *Assume that Assumptions 1-6 are fulfilled and let $K \subset \mathbb{R}^d$ be a compact set of $\mathbb{R}^d$. Then, there exist two finite positive constants $c_{4,3} = c_{4,3}(K)$ and $c_{4,4} = c_{4,4}(K)$ such that for every $x \in K$ and every $u, v \in \mathbb{R}^d$,*

$$c_{4,3}\tau_{E(x)^t}(u-v)^\alpha \le \|Y_{3,\alpha,x}(u,v)\|_\alpha^\alpha \le c_{4,4}\tau_{E(x)^t}(u-v)^\alpha.$$

From the three previous lemmas, an uniform control of the scale parameter of

$$X_{\alpha,\psi}(u) - X_{\alpha,\psi}(v)$$

can be stated. The local behavior of this scale parameter is closely linked to the Hölder regularity of the sample paths of the multi-operator stable random field $X_{\alpha,\psi}$. Actually, in the Gaussian case the Hölder regularity is characterized by the local behavior of this scale parameter; in the α-stable case ($\alpha < 2$), the next theorem leads to an upper bound for the Hölder regularity.

Theorem 4.4. *Assume that Assumptions 1-6 are fulfilled. Then, for $\gamma > 0$ small enough there exist two finite positive constants $c_{4,5} = c_{4,5}(T,\gamma)$ and $c_{4,6} = c_{4,6}(T,\gamma)$ such that*

$$\|X_{\alpha,\psi}(u) - X_{\alpha,\psi}(v)\|_\alpha^\alpha \begin{cases} \ge & c_{4,5}\max\left(\tau_{E(v)^t}(u-v), \tau_{E(u)^t}(u-v)\right)^\alpha \\ \le & c_{4,6}\min\left(\tau_{E(v)^t}(u-v), \tau_{E(u)^t}(u-v)\right)^\alpha, \end{cases}$$

for every $u, v \in T$ such that $\|u-v\| \le \gamma$.

Proof of Theorem 4.4. Let $u, v \in T$ such that $\|u-v\| \le \gamma$ with $\gamma \in (0,1)$. Then, for γ small enough, by Lemmas 4.1 and 4.2, $Y_{1,\alpha,u}(u,v)$ and $Y_{2,\alpha,u}(u,v)$ are well-defined. Note that $Y_{3,\alpha,v}(u,v)$ is also well-defined. Then, we can write

$$X_{\alpha,\psi}(u) - X_{\alpha,\psi}(v) = Y_{1,\alpha,u}(u,v) + Y_{2,\alpha,u}(u,v) + Y_{3,\alpha,v}(u,v).$$

Hence, for γ small enough,

$$\|X_{\alpha,\psi}(u) - X_{\alpha,\psi}(v)\|_\alpha^\alpha \begin{cases} \ge & 2^{-2\alpha}\|Y_{3,\alpha,v}(u,v)\|_\alpha^\alpha - \|Y_{1,\alpha,u}(u,v)\|_\alpha^\alpha - \|Y_{2,\alpha,u}(u,v)\|_\alpha^\alpha \\ \le & 2^{2\alpha}\big(\|Y_{1,\alpha,u}(u,v)\|_\alpha^\alpha + \|Y_{2,\alpha,u}(u,v)\|_\alpha^\alpha + \|Y_{3,\alpha,v}(u,v)\|_\alpha^\alpha\big) \end{cases}$$

By applying Lemmas 4.1, 4.2 and 4.3, for γ small enough,

$$\|X_{\alpha,\psi}(u) - X_{\alpha,\psi}(v)\|_\alpha^\alpha \begin{cases} \geq & 2^{-2\alpha} c_{4,3} \tau_{E(v)^t}(u-v)^\alpha - (c_{4,1}+c_{4,2})\|u-v\|^\alpha \\ \leq & 2^{2\alpha}\left(c_{4,4}\tau_{E(v)^t}(u-v)^\alpha + (c_{4,1}+c_{4,2})\|u-v\|^\alpha\right). \end{cases}$$

Since $\overline{H} := \max_{z\in T} \overline{H}(z) < 1$, we can choose $\delta \in (0, \min_{z\in T} \underline{H}(z))$ such that

$$\forall z \in T,\ \overline{H}(z) + \delta \leq \overline{H} + \delta < 1.$$

By Proposition 3.5, there exists a finite constant $c_{3,5} = c_{3,5}(T,\delta)$, such that

$$\|u-v\|^\alpha \leq c_{3,5}^{-\alpha} \|u-v\|^{\alpha(1-(\overline{H}+\delta))} \tau_{E(v)^t}(u-v)^\alpha.$$

Then, one can choose γ small enough such that

$$(c_{4,1}+c_{4,2})\|u-v\|^\alpha \leq 2^{-2\alpha-1} c_{4,3} \tau_{E(v)^t}(u-v)^\alpha$$

for every $u, v \in T$ such that $\|u-v\| \leq \gamma$. Therefore we can choose $c_{4,5} = 2^{-2\alpha-1} c_{4,3}$ and $c_{4,6} = 2^{2\alpha} c_{4,4} + 2^{-2\alpha-1} c_{4,3}$. □

From the previous theorem, we easily deduce the stochastic continuity of a harmonizable multi-operator scaling stable random field.

Corollary 4.5. *Assume that Assumptions 1-6 are fulfilled. Then the harmonizable multi-operator scaling stable random field $X_{\alpha,\psi}$ defined by (7) is stochastically continuous on T.*

Proof. By Theorem 4.4, there exists $\gamma \in (0,1)$ and a finite positive constant $c_{4,6}$ such that

$$\|X_{\alpha,\psi}(u) - X_{\alpha,\psi}(v)\|_\alpha^\alpha \leq c_{4,6} \tau_{E(v)^t}(u-v)^\alpha.$$

for any $u, v \in T$ satisfying $\|u-v\| \leq \gamma$.
Let $\delta \in \min_{x\in T} \underline{H}(x)$. By Proposition 3.5, there exists a finite positive constant $c_{3,6} = c_{3,6}(T,\delta)$ such that

$$\|X_{\alpha,\psi}(u) - X_{\alpha,\psi}(v)\|_\alpha^\alpha \leq c_{4,6}\, c_{3,6}^\alpha \|u-v\|^{\alpha(\underline{H}(v)-\delta)}$$

for any $u, v \in T$ satisfying $\|u-v\| \leq \gamma$. In particular, since $\alpha(\underline{H}(v)-\delta) > 0$,

$$\forall v \in T,\ \lim_{u\to v} \|X_{\alpha,\psi}(u) - X_{\alpha,\psi}(v)\|_\alpha^\alpha = 0,$$

which implies the stochastic continuity of $X_{\alpha,\psi}$ on T (see Proposition 3.5.1 of [25]). □

Let us also mention that in a special case, when the field $X_{\alpha,\psi}$ has stationary increments, Yimin Xiao proves in Theorem 3.6 of [29] a strong local non-determinism property that enables him to study their local times.

4.2. **Modulus of continuity.** In this section, we give an upper bound on the modulus of continuity of a harmonizable multi-operator scaling stable random field $X_{\alpha,\psi}$ around the position x. Let us emphasize that we control the behavior of an increment

$$X_{\alpha,\psi}(x+u) - X_{\alpha,\psi}(x+v)$$

using the polar coordinate $\tau_{E(x)^t}$ with respect to the matrix $E(x)^t$, which takes into account the anisotropic behavior of $X_{\alpha,\psi}$ around x. As in [14, 6], one of the main tools we use is a LePage series representation (see [17, 16] for details on such series) which is a conditionally Gaussian series. Since E may vary with the position x, the main difference to [6] is that we need some uniform controls of the polar coordinates and an uniform comparison of the radial parts with respect to $E(x)^t$ and $E(y)^t$ (see Section 3). This leads to an upper bound less accurate than the upper bound given in [6] in the case of operator scaling harmonizable stable random fields. The difference is a log term but our upper bound is sufficient to obtain the pointwise Hölder exponents. Let us also point out that our modulus of continuity is local and not uniform in contrast to [28].

Theorem 4.6. *Assume that Assumptions 1-6 are fulfilled on T. There exists a modification $X^*_{\alpha,\psi}$ of $X_{\alpha,\psi}$ on T such that almost surely, for all $x \in T$, for all $\varepsilon > 0$,*

$$\lim_{\gamma \downarrow 0} \sup_{\substack{\|u\|\leq\gamma, \|v\|\leq\gamma \\ x+u, x+v \in T}} \frac{\left|X^*_{\alpha,\psi}(x+u) - X^*_{\alpha,\psi}(x+v)\right|}{\tau_{E(x)^t}(u-v)^{1-\varepsilon}} = 0.$$

Proof. For every $k \in \mathbb{N}\backslash\{0\}$ and $j = (j_1, \ldots, j_d) \in \mathbb{Z}^d$ we set

$$x_{k,j} = \frac{j}{2^k} \text{ and } \mathcal{D}_k = \left\{x_{k,j} : j \in \mathbb{Z}^d \cap 2^k T\right\}.$$

Let us remark that the sequence $(\mathcal{D}_k)_k$ is increasing and set $\mathcal{D} = \bigcup_{k=1}^{\infty} \mathcal{D}_k$, which is dense in T.

First Step: In this step, we assume that $\alpha \in (0, 2)$.

Let us fix $x_0 \in T \cap \mathcal{D}$. Following [14, 6], we consider a Lepage series representation of $X_{\alpha,\psi}$ which is a conditionally Gaussian series. This series depends on the position x_0 we have fixed.

Let $(T_n)_{n\geq 1}$, $(g_n)_{n\geq 1}$ and $(\xi_n)_{n\geq 1}$ be independent sequences of random variables.

- T_n is the nth arrival time of a Poisson process with intensity 1.
- $(g_n)_{n\geq 1}$ is a sequence of i.i.d. Gaussian complex isotropic random variables so that $g_n \overset{(d)}{=} \mathrm{e}^{i\theta} g_n$ for any $\theta \in \mathbb{R}$.
- $(\xi_n)_{n\geq 1}$ is a sequence of i.i.d. random variables with common law $\mu_{x_0}(d\xi) = m_{x_0}(\xi)d\xi$. with

$$m_{x_0}(\xi) = \frac{c_{a,x_0}}{\tau_{E(x_0)}(\xi)^{q(x_0)} \left|\log \tau_{E(x_0)}(\xi)\right|^{1+a}},$$

where $a > 0$, q is defined by (6) and c_{a,x_0} is chosen such that $\displaystyle\int_{\mathbb{R}^d} m_{x_0}(\xi)d\xi = 1$.

Let

$$d_\alpha = \mathbb{E}(|\mathrm{Re}\,(g_1)|^\alpha)^{-1/\alpha}\left(\frac{1}{2\pi}\int_0^\pi |\cos(x)|^\alpha dx\right)^{1/\alpha}\left(\int_0^{+\infty}\frac{\sin(x)}{x^\alpha}dx\right)^{-1/\alpha}$$

and

$$f_\alpha(x,\xi) = \Big(\mathrm{e}^{i\langle x,\xi\rangle} - 1\Big)\psi_x(\xi)^{-1-q(x)/\alpha},\ \forall x,\xi\in\mathbb{R}^d. \tag{16}$$

According to Proposition 4.1 of [6], for every $x\in\mathbb{R}^d$

$$Z_\alpha(x) = d_\alpha\,\mathrm{Re}\left(\sum_{n=1}^{+\infty} T_n^{-1/\alpha} m_{x_0}(\xi_n)^{-1/\alpha} f_\alpha(x,\xi_n) g_n\right),$$

converges almost surely and $Z_\alpha \overset{(fdd)}{=} X_{\alpha,\psi}$. Then, conditionally to $(T_n,\xi_n)_n$, $Z_\alpha(u) - Z_\alpha(v)$ is a real centered Gaussian random variable with variance

$$v_\alpha^2((u,v)\,|\,(T_n,\xi_n)_n) = \frac{d_\alpha^2}{2}\,\mathbb{E}\big(|g_1|^2\big)\sum_{n=1}^{+\infty} T_n^{-2/\alpha} m_{x_0}(\xi_n)^{-2/\alpha}|f_\alpha(u,\xi_n) - f_\alpha(v,\xi_n)|^2. \tag{17}$$

Second Step: Let us now assume that $\alpha\in(0,2]$ and set $Z_2 = X_{2,\psi}$. For $k\in\mathbb{N}\backslash\{0,1\}$ and $(i,j)\in\mathbb{Z}^d$, we consider the set

$$E_{i,j}^k = \begin{cases} \{\omega : |Z_\alpha(x_{k,i}) - Z_\alpha(x_{k,j})| > v_\alpha((x_{k,i},x_{k,j})\,|\,(T_n,\xi_n)_n)\,\varphi(\|x_{k,i}-x_{k,j}\|)\} & \text{if } \alpha\in(0,2) \\ \big\{\omega : |Z_2(x_{k,i}) - Z_2(x_{k,j})| > \|Z_2(x_{k,i}) - Z_2(x_{k,j})\|_2\,\varphi(\|x_{k,i}-x_{k,j}\|)\big\} & \text{if } \alpha = 2 \end{cases}$$

where, as in [13],

$$\varphi(t) = \sqrt{2d\log\frac{1}{t}},\ t>0.$$

Then, for every (k,i,j),

$$\mathbb{P}\big(E_{i,j}^k\big) = \mathbb{P}(|N| > \varphi(\|x_{k,i}-x_{k,j}\|)),$$

where N is a real centered Gaussian random variable with variance 1. Let us choose $\delta\in(0,1/3)$ and set for $k\in\mathbb{N}\backslash\{0,1\}$, $\delta_k = 2^{-(1-\delta)k}$ and

$$I_k = \big\{i\in\mathbb{Z}^d\cap 2^kT : \|x_{k,i}-x_0\|\le\delta_k\big\}.$$

For every $(i,j)\in I_k^2$, since φ is a decreasing function

$$\mathbb{P}\big(E_{i,j}^k\big) \le \mathbb{P}(|N|>\varphi(2\delta_k)) \le \sqrt{\frac{2}{\pi}}\frac{\mathrm{e}^{-\varphi^2(2\delta_k)/2}}{\varphi(2\delta_k)} = \frac{2^d 2^{-(1-\delta)kd}}{\sqrt{2d((1-\delta)k-1)\log 2}}.$$

Note that $\mathrm{card}\,I_k \le c_T 2^{\delta kd}$ with c_T a finite positive constant which only depends on T. Hence, since $\delta<1/3$

$$\sum_{k=2}^{+\infty}\sum_{(i,j)\in I_k}\mathbb{P}\left(E_{i,j}^k\right) \le \frac{2^d}{\sqrt{2d(1-2\delta)\log 2}}\sum_{k=2}^{+\infty} 2^{-(1-\delta)kd}\,\mathrm{card}\,I_k^2 < +\infty.$$

Therefore, by the Borel-Cantelli Lemma, almost surely there exists an integer $k^*(\omega)$ such that for every $k\ge k^*(\omega)$,

$$|Z_\alpha(u) - Z_\alpha(v)| \le v_\alpha((u,v)\,|\,(T_n,\xi_n)_n)\,\varphi(\|u-v\|) \tag{18}$$

as soon as $u,v\in\mathcal{D}_k$ with $\|u-x_0\|\le\delta_k$ and $\|v-x_0\|\le\delta_k$.

Third Step: We now give an upper bound of the conditional variance v_α when $\alpha \in (0,2)$ and of the variogram of Z_2. For the sake of clearness, the proof of the following lemma is postponed to the Appendix.

Lemma 4.7. *Let $\epsilon \in (0,1)$.*

(1) If $\alpha \in (0,2)$, almost surely there exists $\gamma > 0$, such that for all $u, v \in B(x_0, \gamma)$,

$$v_\alpha((u,v) \mid (T_n, \xi_n)_n) \leq \tau_{E(x_0)^t}(u-v)^{1-\epsilon}.$$

(2) If $\alpha = 2$, for all $u, v \in B(x_0, \gamma)$,

$$\|Z_2(u) - Z_2(v)\|_2 \leq \tau_{E(x_0)^t}(u-v)^{1-\epsilon}.$$

Let $\varepsilon \in (0,1)$. Let us apply the previous lemma with $\epsilon = \varepsilon/2$ and choose $\gamma = \gamma(\epsilon)$ small enough such that

$$\tau_{E(x_0)^t}(u-v)^{1-\varepsilon/2}\varphi(\|u-v\|) \leq \tau_{E(x_0)^t}(u-v)^{1-\varepsilon}$$

for all $u, v \in B(x_0, \gamma)$. Moreover, let us choose $k^*(\omega) = k^*(\omega, \varepsilon)$ such that $\delta_{k^*(\omega)} \leq \gamma$. Hence, for all $\alpha \in (0,2]$, by equation (18), almost surely, for every $k \geq k^*(\omega)$,

$$|Z_\alpha(u) - Z_\alpha(v)| \leq \tau_{E(x_0)^t}(u-v)^{1-\varepsilon}$$

as soon as $u, v \in \mathcal{D}_k$ with $\|u - x_0\| \leq \delta_k$ and $\|v - x_0\| \leq \delta_k$. Then, let

$$\Omega^*_{x_0} = \bigcap_{\varepsilon \in \mathbb{Q} \cap (0,1)} \bigcup_{n=1}^{+\infty} \bigcap_{k \geq n} \bigcap_{\substack{u,v \in \mathcal{D}_k \\ u,v \in B(x_0, \delta_k)}} \Big\{ |X_{\alpha,\psi}(u) - X_{\alpha,\psi}(v)| \leq \tau_{E(x_0)^t}(u-v)^{1-\varepsilon} \Big\}.$$

Since $X_{\alpha,\psi}$ and Z_α have the same finite dimensional margins, we have proved that $\mathbb{P}\Big(\Omega^*_{x_0}\Big) = 1$. Therefore, $\mathbb{P}(\Omega^*) = 1$ with $\Omega^* = \bigcap_{x_0 \in \mathcal{D}} \Omega^*_{x_0}$.

Similar arguments as in Step 4 of [6] lead to the existence of a finite positive constant $C = C(T) > 0$ such that almost surely, for all $x_0 \in \mathcal{D}$, for all $\varepsilon \in \mathbb{Q} \cap (0,1)$, there exists $k^* = k^*(\omega, \varepsilon, x_0) > 0$ such that for all $u, v \in \mathcal{D}$ with $u, v \in B(x_0, \delta_{k^*})$,

$$|X_{\alpha,\psi}(u) - X_{\alpha,\psi}(v)| \leq C\tau_{E(x_0)^t}(u-v)^{1-\varepsilon}. \tag{19}$$

Fourth Step: We now define a modification of $X_{\alpha,\psi}$. Let $\omega \in \Omega^*$. First, we set

$$X^*_{\alpha,\psi}(y)(\omega) = X_{\alpha,\psi}(y)(\omega), \forall y \in \mathcal{D} \cap T.$$

Let $\varepsilon \in \mathbb{Q} \cap (0,1)$ and $x \in T$. Then, one can find $x_0 \in \mathcal{D}$ and $\gamma = \gamma(x, \varepsilon, \omega) \in (0,1)$ such that

$$B(x, \gamma) \subset B(x_0, \delta_{k^*(\omega, \varepsilon, x_0)}).$$

Let us now consider $y \in B(x, \gamma)$ and $y^{(n)} \in \mathcal{D}$ such that $\lim_{n \to +\infty} y^{(n)} = y$. In view of (19), for all n, m such that $y^{(n)}, y^{(m)} \in B(x, \gamma)$,

$$\big|X^*_{\alpha,\psi}\big(y^{(n)}\big)(\omega) - X^*_{\alpha,\psi}\big(y^{(m)}\big)(\omega)\big| \leq C\tau_{E(x_0)^t}\big(y^{(n)} - y^{(m)}\big)^{1-\varepsilon},$$

such that $\left(X^*_{\alpha,\psi}\left(y^{(n)}\right)(\omega)\right)_n$ is a Cauchy sequence and hence converges. We set

$$X^*_{\alpha,\psi}(y)(\omega) = \lim_{n\to+\infty} X^*_{\alpha,\psi}\left(y^{(n)}\right)(\omega).$$

Remark that this limit does not depend on the choice of $\left(y^{(n)}\right)$, nor of the choice of $\varepsilon \in \mathbb{Q}\cap(0,1)$. Observe that $X^*_{\alpha,\psi}$ is then well defined on $B(x,\gamma)$. Then, by (19) and continuity of $\tau_{E(x_0)^t}$,

$$\forall u,v \in B(x,\gamma),\ \left|X^*_{\alpha,\psi}(u)(\omega) - X^*_{\alpha,\psi}(v)(\omega)\right| \le C\tau_{E(x_0)^t}(u-v)^{1-\varepsilon}. \tag{20}$$

By Proposition 3.6, up to change γ,

$$\forall u,v \in B(x,\gamma),\ \left|X^*_{\alpha,\psi}(u)(\omega) - X^*_{\alpha,\psi}(v)(\omega)\right| \le Cc_{3,8}\tau_{E(x)^t}(u-v)^{1-2\varepsilon}.$$

where $c_{3,8}$ does not depend on (u,v).
To conclude the proof, let us emphasize that $X^*_{\alpha,\psi}$ is a modification of $X_{\alpha,\psi}$ since $X_{\alpha,\psi}$ is stochastically continuous (by Corollary 4.5). □

4.3. **Hölder exponents.** In this section, we are interested in the global and directional Hölder regularity of the sample paths of a harmonizable multi-operator scaling stable random field $X_{\alpha,\psi}$. We first prove that $X_{\alpha,\psi}$ admits a modification whose sample paths are "globally" Hölder on T. This is a consequence of Theorem 4.6 and of the comparison of the radial part $\tau_{E^t(x)}$ with the Euclidean norm.

Corollary 4.8. *Assume that Assumptions 1-6 are fulfilled and let $X_{\alpha,\psi}$ be the harmonizable multi-operator scaling stable random field defined by* (7)*. Then, there exists a modification of $X_{\alpha,\psi}$ which has H-Hölder sample paths on the compact set T for any $H \in (0, \min_{y\in T}\underline{H}(y))$.*

Proof. It is sufficient to prove that for every $x \in T$ and every $H \in (0, \min_{y\in T}\underline{H}(y))$, there exists a modification of $X_{\alpha,\psi}$ which has H-Hölder sample paths on $B(x,\gamma_x)\cap T$ for a well chosen radius γ_x. Then, let us fix $x \in T$ and $\varepsilon \in (0,1)$. By Theorem 4.6, there exists a modification $X^*_{\alpha,\psi}$ of $X_{\alpha,\psi}$ on T, $\gamma = \gamma(x,\varepsilon) \in (0,1/2)$ and a finite positive constant $c = c(x,T,\gamma)$ such that

$$\left|X^*_{\alpha,\psi}(u) - X^*_{\alpha,\psi}(v)\right| \le c\tau_{E(x)^t}(u-v)^{1-\varepsilon}$$

for any $u,v \in T$ such that $\|x-u\| \le \gamma$ and $\|x-v\| \le \gamma$.
By Proposition 3.5, for any $\delta \in (0, \min_{v\in T}\underline{H}(v))$, there exists a finite positive constant $c_{3,6} = c_{3,6}(T,\delta)$ such that

$$\left|X^*_{\alpha,\psi}(u) - X^*_{\alpha,\psi}(v)\right| \le c\,c_{3,6}\|u-v\|^{(\underline{H}(x)-\delta)(1-\varepsilon)}$$

for any $u,v \in T$ such that $\|x-u\| \le \gamma$ and $\|x-v\| \le \gamma$.
Therefore, for any $u,v \in T$ such that $\|x-u\| \le \gamma$ and $\|x-v\| \le \gamma$,

$$\left|X^*_{\alpha,\psi}(u) - X^*_{\alpha,\psi}(v)\right| \le c\,c_{3,6}\|u-v\|^{(\min_{y\in T}\underline{H}(y)-\delta)(1-\varepsilon)}$$

since $\|u-v\| < 1$. Hence, $X^*_{\alpha,\psi}$ has H-Hölder sample paths on $B(x,\gamma(x,\varepsilon))\cap T$ for any $H < \min_{y\in T}\underline{H}(y)(1-\varepsilon)$. This leads to the conclusion. □

As already mentioned, the Hölder sample paths regularity of a continuous modification of $X_{\alpha,\psi}$ may vary both with the position and with the direction. At position x, the dependence on the directions is characterized by the Jordan decomposition of $E(x)$.

Notation Let $x \in \mathbb{R}^d$. Let us consider the Jordan decomposition of $E(x)$ as in [6]. Hence,

$$E(x) = P(x)^{-1} \begin{pmatrix} J_1(x) & 0 & \dots & 0 \\ 0 & J_2(x) & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & J_{p_x}(x) \end{pmatrix} P(x). \tag{21}$$

We can assume that each $J_j(x)$ is associated with $a_j(x) = 1/H_j(x)$, the real part of the eigenvalue $\lambda_j(x)$. Observe that

$$\underline{H}(x) = \min_{1\le j\le p_x} H_j(x) \quad \text{and} \quad \overline{H}(x) = \max_{1\le j\le p_x} H_j(x).$$

We denote by $(e_1, \dots, e_d)$ the canonical basis of $\mathbb{R}^d$ and set $f_j(x) = P(x)^{-1}e_j$ for every $j = 1, \dots, d$. Hence, $(f_1(x), \dots, f_d(x))$ is a basis of $\mathbb{R}^d$. For all $j = 1, \dots, p_x$, let

$$W_j(x) = \text{span}\left(f_k(x) \ ; \ \sum_{i=1}^{j-1} d_i + 1 \le k \le \sum_{i=1}^{j} d_i \right) \tag{22}$$

where d_i is the size of $J_i(x)$. Then, $\mathbb{R}^d = \bigoplus_{j=1}^{p_x} W_j(x)$. Moreover each $W_j(x)$ is an $E(y)$-invariant set when $y \in \mathbb{R}^d$ is such that $E(x)E(y) = E(y)E(x)$.

When v varies in $W_j(x)$, [6] proved that the behavior of the radial part $\tau_{E(x)^t}(v)$ around $v = 0$ is characterized by $H_j(x)$. Then, if we only consider $X^*_{\alpha,\psi}$ on a straight line driven by $u \in W_j(x)$, Corollary 4.8 can be strenghtened.

Corollary 4.9. *Let $x \in \mathbb{R}^d$. Assume that Assumptions 1-6 are fulfilled with $T = [x - \eta, x + \eta] = \prod_{j=1}^d [x_j - \eta, x_j + \eta]$ for $\eta > 0$. Let $u \in W_j(x)\backslash\{0\}$ where $W_j(x)$ is defined by (22) and $1 \le j \le p_x$. Then, there exists a modification $X^*_{\alpha,\psi}$ of $X_{\alpha,\psi}$ on T such that the random process $\big(X^*_{\alpha,\psi}(x + tu)\big)_{t\in\mathbb{R}}$ has H-Hölder sample paths in a neighborhood of $t = 0$ for any $H \in (0, H_j(x))$.*

Proof. Let $\varepsilon \in (0, 1)$. By Theorem 4.6, we can consider a modification $X^*_{\alpha,\psi}$ of $X_{\alpha,\psi}$ on T which has continuous sample paths on T. Moreover, by Theorem 4.6, there exists $\gamma = \gamma(x, \varepsilon) \in (0, 1/2)$ and a finite positive constant $c_1 = c_1(x, T, \gamma)$ such that

$$\left|X^*_{\alpha,\psi}(x + tu) - X^*_{\alpha,\psi}(x + su)\right| \le c_1 \tau_{E(x)^t}((t - s)u)^{1-\varepsilon}$$

for any $s, t \in \mathbb{R}$ such that $|t| \le \gamma/\|u\|$ and $|s| \le \gamma/\|u\|$.

Then by Corollary 3.4 of [6], applied to $E(x)^t$ and $r = 2\gamma$, there exists a finite positive constant $c_2 = c_2(x, \gamma)$ and $l_j = l_j(x) \in \mathbb{N}^*$ such that

$$\left|X^*_{\alpha,\psi}(x + tu) - X^*_{\alpha,\psi}(x + su)\right| \le c_1 c_2 \|u\|^{(1-\varepsilon)H_j(x)} |t - s|^{(1-\varepsilon)H_j(x)} \left|\log\left(|t - s|\|u\|\right)\right|^{(l_j-1)H_j(x)}$$

for any $s,t \in \mathbb{R}$ such that $|t| \leq \gamma/\|u\|$ and $|s| \leq \gamma/\|u\|$. Therefore, for $\delta > 0$ small enough, $\left(X^*_{\alpha,\psi}(x+tu)\right)_{t\in\mathbb{R}}$ has $(H_j(x) - \varepsilon H_j(x) - \delta)$-Hölder sample paths in a neighborhood of $t = 0$, which concludes the proof. □

We now focus on Hölder directional and global pointwise exponents. Let us first define these exponents.

Definition 4.1. *Let $x \in \mathbb{R}^d$, $(X(y))_{y\in\mathbb{R}^d}$ be a real-valued random field and S^{d-1} be the Euclidean unit sphere of $\mathbb{R}^d$. Assume that X^* is a modification of X which has continuous sample paths in a neighborhood of x.*

(1) The Hölder pointwise exponent of X at point x is

$$H_X(x) = \sup\left\{H > 0,\ \lim_{y\to 0} \frac{X^*(x+y) - X^*(x)}{\|y\|^H} = 0\right\}.$$

(2) Moreover, the directional Hölder pointwise exponent $H_X(x,u)$ of the random field X at point x in direction $u \in S^{d-1}$ is the Hölder pointwise exponent at point x of the process $(X(x+tu))_{t\in\mathbb{R}}$, that is

$$H_X(x,u) = \sup\left\{H > 0,\ \lim_{t\to 0} \frac{X^*(x+tu) - X^*(x)}{|t|^H} = 0\right\}.$$

Note that Corollaries 4.8 and 4.9 give lower bounds of these exponents. Moreover, since the harmonizable random field $X_{\alpha,\psi}$ is a stable random field, an upper bound can be deduced from the behavior of the scale parameter

$$\|X_{\alpha,\psi}(u) - X_{\alpha,\psi}(v)\|_\alpha$$

when u and v are close to x. More precisely, we use the lower bound stated in Theorem 4.4 and the comparison of the radial part $\tau_{E(x)^t}$ with the Euclidean norm.

Corollary 4.10. *Let $x \in \mathbb{R}^d$. Assume that Assumptions 1-6 are fulfilled with $T = [x-\eta, x+\eta] = \prod_{j=1}^d [x_j - \eta, x_j + \eta]$ for $\eta > 0$. Let us consider $X^*_{\alpha,\psi}$ a continuous modification of $X_{\alpha,\psi}$ on T.*

(1) Let $u \in W_j(x) \cap S^{d-1}$ where $W_j(x)$ is defined by (22), $1 \leq j \leq p_x$. Then the directional pointwise Hölder exponent of the random field $X^*_{\alpha,\psi}$ at point x in direction u is almost surely $H_j(x)$, that is

$$H_{X^*_{\alpha,\psi}}(x,u) = \sup\left\{H > 0,\ \lim_{\lambda\to 0} \frac{X^*_{\alpha,\psi}(x+\lambda u) - X^*_{\alpha,\psi}(x)}{\|\lambda\|^H} = 0\right\} = H_j(x) \quad \textit{almost surely.}$$

(2) Moreover, the pointwise Hölder exponent of the random field $X^*_{\alpha,\psi}$ at point x is almost surely $\underline{H}(x)$, that is

$$H_{X^*_{\alpha,\psi}}(x) = \sup\left\{H > 0,\ \lim_{y\to 0} \frac{X^*_{\alpha,\psi}(x+y) - X^*_{\alpha,\psi}(x)}{\|y\|^H} = 0\right\} = \underline{H}(x) = \min_{1\leq j\leq d} H_j(x) \quad \textit{almost surely.}$$

Proof. Let $X^*_{\alpha,\psi}$ be a modification of $X_{\alpha,\psi}$ which has continuous sample paths on T (see Theorem 4.6).

(1) Let $x \in T$ and $u \in W_j(x) \cap S^{d-1}$. By Corollary 4.9, it is clear that almost surely,

$$H_{X_{\alpha,\psi}}(x,u) \geq H_j(x).$$

By Theorem 4.4, there exists $\gamma \in (0,1)$ and $c_{4,1} = c_{4,1}(T,\gamma)$ a finite positive constant such that

$$c_{4,1}\tau_{E(x)^t}(tu)^\alpha \leq \|X_{\alpha,\psi}(x+tu) - X_{\alpha,\psi}(x)\|_\alpha^\alpha = \left\|X^*_{\alpha,\psi}(x+tu) - X^*_{\alpha,\psi}(x)\right\|_\alpha^\alpha$$

for any $t \in \mathbb{R}$ such that $|t| \leq \gamma$. Hence, by Corollary 3.4 of [6], for any $H > H_j(x)$, there exists a finite positive constant c such that

$$c|t|^{\alpha H} \leq \left\|X^*_{\alpha,\psi}(x+tu) - X^*_{\alpha,\psi}(x)\right\|_\alpha^\alpha$$

for any $t \in \mathbb{R}$ such that $|t| \leq \gamma$. Therefore, for any $H > H_j(x)$,

$$\lim_{t\to 0} \left\| \frac{X^*_{\alpha,\psi}(x+tu) - X^*_{\alpha,\psi}(x)}{|t|^H} \right\|_\alpha^\alpha = +\infty,$$

which implies that $\frac{X^*_{\alpha,\psi}(x+tu) - X^*_{\alpha,\psi}(x)}{|t|^H}$ is almost surely unbounded as $t \to 0$ since $X^*_{\alpha,\psi}$ is an α-stable random field. This leads to

$$H_{X_{\alpha,\psi}}(x,u) = H_j(x) \text{ almost surely.}$$

(2) Let $x \in T$. By Corollary 4.8 and continuity of $\underline{H}$, it is clear that almost surely,

$$H_{X_{\alpha,\psi}}(x) \geq \underline{H}(x) = \min_{1\leq j\leq d} H_j(x).$$

Moreover, by definition of the directional exponents $H_{X_{\alpha,\psi}}(x,u)$, $u \in S^{d-1}$,

$$H_{X_{\alpha,\psi}}(x) \leq \min_{u\in S^{d-1}} H_{X_{\alpha,\psi}}(x,u).$$

Then, since for any $1 \leq j \leq p_x$, $W_j \cap S^{d-1} \neq \emptyset$, assertion (i) implies

$$H_{X_{\alpha,\psi}}(x) \leq \min_{1\leq j\leq p_x} H_j(x) = \underline{H}(x)$$

almost surely, which concludes the proof.

□

Let us now illustrate the previous results.

Example 4.1. *Let E_0 be a matrix of size $d \times d$ whose eigenvalues have real parts greater than 1. We denote by $H_1^{(0)}, \ldots, H_d^{(0)}$ the inverse of the real parts of the eigenvalues of E_0. Let $W_1, \ldots, W_p$ the subspaces associated to the Jordan's decomposition of E_0 as (22) and let h, E and ψ be as in Example 2.2. Observe that for any $x \in \mathbb{R}^d$,*

$$W_j(x) = W_j$$

since $E(x) = E_0/h(x)$. Then, according to Corollary 4.10, if $u \in W_j \cap S^{d-1}$, for any $x \in \mathbb{R}^d$,

$$H_{X^*_{\alpha,\psi}}(x,u) = h(x)H_j^{(0)} \text{ almost surely.}$$

Note that the map

$$s \in \mathbb{R} \mapsto H_{X^*_{\alpha,\psi}}(x+su,u) = h(x+su)H_j^{(0)}$$

is positive and continuous on $\mathbb{R}$. *Therefore, according to Corollary 3.15 of* [11], *for the Gaussian process* $\left(X^*_{2,\psi}(x+su)\right)_{s\in\mathbb{R}}$,

$$\text{almost surely}, \forall s \in \mathbb{R},\ H_{X^*_{2,\psi}}(x+su,u) = h(x+su)H_j^{(0)}.$$

Similarly, for all $x \in \mathbb{R}^d$,

$$H_{X^*_{\alpha,\psi}}(x) = h(x)\min_{1\le j\le d} H_j^{(0)} \text{ almost surely}$$

such that, still according to Corollary 3.15 of [11],

$$\text{almost surely}, \forall x \in \mathbb{R}^d,\ H_{X^*_{2,\psi}}(x) = h(x)\min_{1\le j\le d} H_j^{(0)}.$$

Example 4.2. *Assume that for every* $1 \le j \le d$, H_j *is a locally Lipschitz function on* $\mathbb{R}^d$ *with values in* $(0,1)$. *We assume moreover that* $\inf_{x\in\mathbb{R}^d} \underline{H}(x) > 1$. *Let us consider*

$$E = P^{-1}\,diag\,(1/H_1,\dots,1/H_d)\,P,$$

with $P \in GL_d(\mathbb{R})$ *an invertible matrix, and* $X_{\alpha,\varphi}$ *as in Example 2.4. Let* $(e_j)_{1\le j\le d}$ *be the canonical basis of* $\mathbb{R}^d$ *and* $f_j = P^{-1}e_j$ *for all* $1 \le j \le d$. *Then, according to Corollary 4.10, for all* $1 \le j \le d$, *for any* $x \in \mathbb{R}^d$,

$$H_{X^*_{\alpha,\varphi}}(x, f_j/\|f_j\|) = H_j(x) \quad \text{almost surely.}$$

Once again we get in the Gaussian case that almost surely,

$$\forall s \in \mathbb{R},\ H_{X^*_{2,\varphi}}(x+sf_j, f_j/\|f_j\|) = H_j(x+sf_j).$$

Similarly, for all $x \in \mathbb{R}^d$, *almost surely* $H_{X^*_{\alpha,\varphi}}(x) = \underline{H}(x)$ *and almost surely*

$$\forall x \in \mathbb{R}^d,\ H_{X^*_{2,\varphi}}(x) = \underline{H}(x).$$

Example 4.3. *Let* $d = 2$. *Let us consider as in Example 2.5 the map*

$$x \mapsto E(x) = a(x)\begin{pmatrix} \cos(\theta(x)) & \sin(\theta(x)) \\ -\sin(\theta(x)) & \cos(\theta(x)) \end{pmatrix}$$

where a *and* θ *are locally Lipschitz functions on* $\mathbb{R}^d$ *such that* $\forall x \in \mathbb{R}^d,\ a(x)\cos(\theta(x)) > 1$, *and* $X_{\alpha,\psi}$ *the associated random field. Then, for every* $x \in \mathbb{R}^d$ *and* $u \in S^{d-1}$, *almost surely*

$$H_{X^*_{\alpha,\psi}}(x,u) = H_{X^*_{\alpha,\psi}}(x) = \frac{1}{a(x)\cos(\theta(x))}$$

with almost surely $\forall x \in \mathbb{R}^d,\ H_{X^*_{2,\psi}}(x) = 1/(a(x)\cos(\theta(x)))$.

5. Local Operator scaling property

In general, harmonizable multi-operator scaling random fields are not operator scaling: they do not satisfy the global property (1) for any fix matrix E. However, they satisfy a weak property we call *local asymptotic operator scaling property*, which we introduce in the next definition.

Definition 5.1. *Let $x \in \mathbb{R}^d$ and $A(x)$ be a $d \times d$ real matrix. A random field $(X(y))_{y\in\mathbb{R}^d}$ is locally asympotically operator scaling of order $A(x)$ at point x if*

$$\lim_{\varepsilon\to 0^+} \left(\frac{X(x+\varepsilon^{A(x)}u) - X(x)}{\varepsilon} \right)_{u\in\mathbb{R}^d} \overset{(fdd)}{=} (Z_x(u))_{u\in\mathbb{R}^d}\,, \tag{23}$$

with Z_x a non degenerate random field. Moreover a random field X which satisfies (23) *is called multi-operator random field of order A.*

As mentioned in the introduction, the local asymptotic operator scaling property generalizes both the operator scaling property and the local asymptotic self-similarity property. On the one hand, an operator scaling random field X with stationary increments is locally asymptotically operator scaling at any point x. On the other hand, a locally asymptotically self-similar random field at point x with order $h(x)$ is locally asymptotically operator scaling at point x of order $I_d/h(x)$.
Note also that the local asymptotic self-similarity property can not capture the operator scaling property since it only reveals local self-similarity which is not sufficient to characterize the anisotropy. Actually, let X be an operator scaling random field of order E_0. Assume that the Jordan's decomposition of E_0 is given by (21) with $\underline{H} = \min_{1\le j\le p} H_j = H_1$ such that $J_1 = \frac{1}{H_1} I_{d_1}$ and let W_1 be the corresponding eigenvector space (see (22)). Then, writing for $u \in \mathbb{R}^d = \bigoplus_{j=1}^{p} W_j$, $u = u_1 + v$ with $u_1 \in W_1$, it is clear that, for any $\varepsilon > 0$, $\varepsilon^{1/H_1}u = \varepsilon^{E_0}\left(u_1 + \varepsilon^{1/H_1 - E_0}v\right)$ with

$$\varepsilon^{1/H_1-E_0}v \underset{\varepsilon\to 0_+}{\longrightarrow} 0$$

since $v \in \bigoplus_{j=2}^{p} W_j$, with $\min_{2\le j\le p} H_j > H_1$. Then, by operator scaling property, if X is stochastically continuous,

$$\lim_{\varepsilon\to 0^+} \left(\frac{X(\varepsilon u)}{\varepsilon^{H_1}} \right)_{u\in\mathbb{R}^d} \overset{(fdd)}{=} (X(\pi_{W_1}u)))_{u\in\mathbb{R}^d}$$

with π_{W_1} the projection on W_1. In other words, if X is non degenerated on W_1, X is locally asymptotically self-similar of order H_1 at point 0 with tangent field $(X(\pi_{W_1}u))_{u\in\mathbb{R}^d}$.

The following remark gives some properties of the random field Z_x, which are immediate consequences of (23).

Remark 5.1. *Assume that* (23) *is fulfilled. Then Z_x is operator scaling of order $A(x)$, that is*

$$\forall c > 0,\ \left(Z_x\left(c^{A(x)}u\right)\right)_{u\in\mathbb{R}^d} \overset{(fdd)}{=} c\,(Z_x(u))_{u\in\mathbb{R}^d}\,.$$

Moreover, if $\theta \in \mathbb{R}^d$ is an eigenvector of $A(x)$ associated with a real eigenvalue λ, then

$$\lim_{\varepsilon\to 0^+} \left(\frac{X(x+\varepsilon t\theta) - X(x)}{\varepsilon^{1/\lambda}} \right)_{t\in\mathbb{R}} \overset{(fdd)}{=} (Z_x(t\theta))_{t\in\mathbb{R}}\,.$$

The main result of this section is stated in the next theorem. As expected, a harmonizable multi-operator scaling stable random field $X_{\alpha,\psi}$ locally looks like a harmonizable operator scaling stable random field.

Theorem 5.1. *Let $x \in \mathbb{R}^d$. Assume that Assumptions 1-6 are fulfilled on $T = [x-\eta, x+\eta] = \prod_{j=1}^d [x_j - \eta, x_j + \eta]$. Then, the random field $X_{\alpha,\psi}$ is locally asymptotically operator scaling at point x of order $E(x)^t$ in the sense that*

$$\lim_{\varepsilon \to 0^+} \left(\frac{X_{\alpha,\psi}(x + \varepsilon^{E(x)^t} u) - X_{\alpha,\psi}(x)}{\varepsilon} \right)_{u \in \mathbb{R}^d} \stackrel{(fdd)}{=} \left(X_{\psi_x}(u) \right)_{u \in \mathbb{R}^d}, \tag{24}$$

where X_{ψ_x} is a harmonizable α-stable operator scaling field with respect to $E(x)^t$ and ψ_x in the sense of Theorem 4.1 in [7].

Remark 5.2. *In the case where $\alpha = 2$, one can prove that* (24) *holds in distribution on the space of continuous functions endowed with the topology of the uniform convergence on compact sets. Actually, in this case, one can applied the classical criterion of tightness based on second moments of increments. However, if $\alpha \in (0,2)$, proving tightness is much harder and an open problem.*

Proof. Let $x \in \mathbb{R}^d$ and $u \in \mathbb{R}^d$. Then, for $\varepsilon > 0$ small enough, the random variables $Y_{1,\alpha,x+\varepsilon^{E(x)^t}u}\big(x + \varepsilon^{E(x)^t} u, x\big)$ and $Y_{2,\alpha,x+\varepsilon^{E(x)^t}u}\big(x + \varepsilon^{E(x)^t} u, x\big)$ are well-defined. Then, for $\varepsilon > 0$ small enough, using the notation of section 4.1, we get

$$\begin{aligned} X_{\alpha,\psi}(x + \varepsilon^{E(x)^t} u) - X_{\alpha,\psi}(x) &= Y_{1,\alpha,x+\varepsilon^{E(x)^t}u}\Big(x + \varepsilon^{E(x)^t} u, x\Big) + Y_{2,\alpha,x+\varepsilon^{E(x)^t}u}\Big(x + \varepsilon^{E(x)^t} u, x\Big) \\ &\quad + Y_{3,\alpha,x}\Big(x + \varepsilon^{E(x)^t} u, x\Big). \end{aligned} \tag{25}$$

By Lemma 4.1, for $\varepsilon > 0$ small enough,

$$\left\| Y_{1,\alpha,x+\varepsilon^{E(x)^t}u}\Big(x + \varepsilon^{E(x)^t} u, x\Big) \right\|_\alpha^\alpha \leq c_{4,1} \left\| \varepsilon^{E(x)^t} u \right\|^\alpha$$

where the finite positive constant $c_{4,1}$ does not depend on ε. Therefore, by Lemma 3.3, for $\varepsilon > 0$ small enough and $\delta > 0$ small enough

$$\left\| Y_{1,\alpha,x+\varepsilon^{E(x)^t}u}\Big(x + \varepsilon^{E(x)^t} u, x\Big) \right\|_\alpha^\alpha \leq c_{4,1} c_{3,1}^\alpha \|u\|^\alpha \varepsilon^{\alpha/\overline{H}(x) - \alpha\delta}$$

where the finite positive constant $c_{3,1}$ does not depend on ε. Since $Y_{1,\alpha,x+\varepsilon^{E(x)^t}u}\big(x + \varepsilon^{E(x)^t} u, x\big)$ is an stable random variable and since $\overline{H}(x) < 1$, the previous inequality leads to

$$\lim_{\varepsilon \to 0^+} \frac{Y_{1,\alpha,x+\varepsilon^{E(x)^t}u}\big(x + \varepsilon^{E(x)^t} u, x\big)}{\varepsilon} = 0 \quad \text{in probability.} \tag{26}$$

Using Lemma 4.2 and Proposition 3.3, the same arguments yield that

$$\lim_{\varepsilon \to 0^+} \frac{Y_{2,\alpha,x+\varepsilon^{E(x)^t}u}\big(x + \varepsilon^{E(x)^t} u, x\big)}{\varepsilon} = 0 \quad \text{in probability.} \tag{27}$$

Observe that the random field

$$\big(X_{\psi_x}(v)\big)_{v \in \mathbb{R}^d} = \big(Y_{\alpha,\psi}(v, x, x)\big)_{v \in \mathbb{R}^d}$$

is well-defined and is is a harmonizable α-stable operator scaling field with respect to $E(x)^t$ and ψ_x in the sense of Theorem 4.1 in [7]. Moreover,

$$\forall v \in \mathbb{R}^d,\ Y_{3,\alpha,x}(v,x) = X_{\psi_x}(v) - X_{\psi_x}(x).$$

Then, by stationarity of the increments of X_{ψ_x} and the operator scaling property (see Corollary 4.2 of [7]),

$$\Big(Y_{3,\alpha,x}\Big(x+\varepsilon^{E(x)^t}v,x\Big)\Big)_{v\in\mathbb{R}^d} \overset{(fdd)}{=} \varepsilon\big(X_{\psi_x}(v)\big)_{v\in\mathbb{R}^d}. \tag{28}$$

From Equations (25), (26), (27) and (28), one easily deduces that

$$\lim_{\varepsilon\to 0^+}\left(\frac{X_{\alpha,\psi}(x+\varepsilon^{E(x)^t}u)-X_{\alpha,\psi}(x)}{\varepsilon}\right)_{u\in\mathbb{R}^d} \overset{(fdd)}{=} \big(X_{\psi_x}(u)\big)_{u\in\mathbb{R}^d}.$$

□

6. Proofs

6.1. **Polar coordinates.**

Proof of Lemma 3.3. Let $\delta > 0$ and $r_0 > 0$. Let us recall that $t^{\lambda_j(x)}$, $1 \le j \le d$ are eigenvalues of $t^{E(x)}$. Then, for every $j = 1, \dots, d$,

$$\big|t^{\lambda_j(x)}\big| = t^{\mathrm{Re}\,(\lambda_j(x))} \le \big\|t^{E(x)}\big\|,$$

which leads to the lower bounds since $1/\underline{H}(x) = \max_{1\le j\le d}\mathrm{Re}\,(\lambda_j(x))$ and $1/\overline{H}(x) = \min_{1\le j\le d}\mathrm{Re}\,(\lambda_j(x))$. Let us now prove the upper bounds.

(*i*) Since $\delta > 0$ and since E is continuous on T and satisfies Assumption 6, the map $x \mapsto E(x) - \big(1/\overline{H}(x) - \delta\big)I_d$ is also continuous on T, satisfies Assumption 6 and takes values in $\mathcal{M}^{>0}\big(\mathbb{R}^d\big)$. Then, by Lemma 3.2, the function

$$(t,x) \mapsto t^{E(x)-\big(1/\overline{H}(x)-\delta\big)I_d}$$

is continuous on $[0,+\infty)\times T$ and thus bounded on the compact set $[0,r_0]\times T$. Therefore, there exists a finite constant $c_{3,1} = c_{3,1}(T,\delta,r_0) > 0$ which only depends on δ, T and r_0 such that

$$\forall (t,x) \in [0,r_0]\times T,\ \left\|t^{E(x)-(1/\overline{H}(x)-\delta)I_d}\right\| \le c_{3,1}.$$

Since for $t > 0$, $t^{E(x)-(1/\overline{H}(x)-\delta)I_d} = t^{\delta-1/\overline{H}(x)}t^{E(x)}$, the last inequality leads to

$$\forall (t,x) \in]0,r_0]\times T,\ \big\|t^{E(x)}\big\| \le c_{3,1}t^{1/\overline{H}(x)-\delta}.$$

This inequality is obviously fulfilled for $t = 0$ since $0^{E(x)} = 0$ by convention.

(*ii*) Since $\delta > 0$ and since E is continuous on T and satisfies Assumption 6, the map $x \mapsto -E(x) + (1/\underline{H}(x) + \delta)I_d$ is also continuous on T, satisfies Assumption 6 and takes values in $\mathcal{M}^{>0}\big(\mathbb{R}^d\big)$. Then, using the same arguments as in the proof of assertion (*i*), there exists a finite constant $c_{3,2} = c_{3,2}(T,\delta,r_0) > 0$ which only depends on δ, T and r_0 such that

$$\forall (u,x) \in [0,1/r_0]\times T,\ \big\|u^{-E(x)+(1/\underline{H}(x)+\delta)I_d}\big\| \le c_{3,2}.$$

Hence,

$$\forall (t,x) \in [r_0,+\infty) \times T,\ \left\| t^{E(x)-(1/\underline{H}(x)+\delta)I_d} \right\| \leq c_{3,2},$$

that is

$$\forall (t,x) \in [r_0,+\infty) \times T,\ \left\| t^{E(x)} \right\| \leq c_{3,2} t^{1/\underline{H}(x)+\delta}.$$

The proof is then complete.

□

Proof of Lemma 3.4. Since E is continuous on T and satisfies Assumption 6, one can easily see that the map

$$\begin{array}{rccc} \mathcal{N}\ : & T \times \mathbb{R}^d & \longrightarrow & [0,+\infty) \\ & (x,\xi) & \longmapsto & \|\xi\|_{E(x)}, \end{array}$$

where $\|\cdot\|_M$ is defined by (8), is continuous using Lemma 3.2, Lemma 3.3 and dominated convergence. Furthermore,

$$\forall x \in T,\ \forall \xi \in \mathbb{R}^d \backslash \{0\},\ \|\xi\|_{E(x)} = \|\xi\| \left\| \frac{\xi}{\|\xi\|} \right\|_{E(x)} = \|\xi\| \mathcal{N}\left(x, \frac{\xi}{\|\xi\|} \right).$$

Since $\mathcal{N}$ is continuous and positive on the compact set $T \times \mathcal{S}^{d-1}$, we have

$$0 < m_T = \inf_{T \times \mathcal{S}^{d-1}} \mathcal{N}(y,\theta) \leq M_T = \sup_{T \times \mathcal{S}^{d-1}} \mathcal{N}(y,\theta) < +\infty.$$

Hence for every $x \in T$ and every $\xi \in \mathbb{R}^d \backslash \{0\}$,

$$\frac{\|\xi\|_{E(x)}}{M_T} \leq \|\xi\| \leq \frac{\|\xi\|_{E(x)}}{m_T}. \tag{29}$$

This inequality is obviously fulfilled for $\xi = 0$ since $\|0\| = \|0\|_{E(x)} = 0$.

Let us now focus on $\sigma_{E(x)}\big(S_{E(x)}\big)$. Applying Proposition 3.1, one obtains that

$$\forall x \in T,\ \sigma_{E(x)}(S_{E(x)}) = \int_{S_{E(x)}} \sigma_{E(x)}(d\theta) = q(x) \int_{\mathbb{R}^d} \mathbf{1}_{\{\tau_{E(x)}(\xi) \leq 1\}} d\xi$$

where q is defined by (6). By definition of $\|\cdot\|_{E(y)}$ (see (8)), for any $y \in T$ and $\xi \in \mathbb{R}^d$,

$$\|\xi\|_{E(y)} \leq 1 \text{ if and only if } \tau_{E(y)}(\xi) \leq 1,$$

which leads to

$$\forall x \in T,\ \sigma_{E(x)}(S_{E(x)}) = q(x) \int_{\mathbb{R}^d} \mathbf{1}_{\|\xi\|_{E(x)} \leq 1} d\xi.$$

Then, using (29) and the continuity of the positive function q on the compact set T, one easily finds two positive finite constants c, C such that

$$\forall x \in T,\ c \leq \int_{S_{E(x)}} \sigma_{E(x)}(d\theta) = \sigma_{E(x)}\left(S_{E(x)}\right) \leq C.$$

Therefore Lemma 3.4 holds with $c_{3,3} = \min(1/M_T, c)$ and $c_{3,4} = \max(1/m_T, C)$. □

Proof of Proposition 3.5. Let $r_0 = \inf_{\|\xi\|\geq 1} \inf_{x\in T} \tau_{E(x)}(\xi)$. First, let us prove that $\tau_{E(x)}(\xi)$ is uniformly bounded below for $x \in T$ and $\|\xi\| \geq 1$, that is $r_0 > 0$. Otherwise, for any $\varepsilon \in (0,1)$, one could find $x \in T$ and $\xi \in \mathbb{R}^d$ such that $\|\xi\| \geq 1$ and

$$\tau_{E(x)}(\xi) \leq \varepsilon < 1.$$

Since E is continuous, $\overline{H}$ is also continuous and we can choose $\eta > 0$ such that $2\eta < \min_{x\in T} \frac{1}{\overline{H}(x)}$. Then, according to Lemma 3.3 there would exist $c_{3,1}(T,\delta,1)$ such that

$$\|\tau_{E(x)}(\xi)^{E(y)}\| \leq c_{3,1}\tau_{E(x)}(\xi)^{\eta} \leq c_{3,1}\varepsilon^{\eta}.$$

However, this would imply that

$$1 \leq \|\xi\| = \|\tau_{E(y)}(\xi)^{E(y)}\ell_{E(y)}(\xi)\| \leq c_{3,1}c_{3,4}\varepsilon^{\eta}$$

according to Lemma 3.4, which is impossible for ε small enough. Hence $r_0 > 0$.

Let $x \in T$ and $\xi \in \mathbb{R}^d$ such that $\|\xi\| \geq 1$. Let $\delta \in (0, \min_{x\in T} \underline{H}(x))$ and $\delta_1 = \min_{x\in T} \frac{\delta}{\underline{H}(x)(\underline{H}(x)-\delta)}$. Using again Lemma 3.3, there exists a finite positive constant $c_{3,2} = c_{3,2}(T,\delta_1,r_0)$ which only depends on T, δ and r_0 such that

$$\|\xi\| = \|\tau_{E(x)}(\xi)^{E(x)}\ell_{E(x)}(\xi)\| \leq c_{3,2}c_{3,4}\tau_{E(x)}(\xi)^{1/\underline{H}(x)+\delta_1}$$

where the finite positive constant $c_{3,4}$ is given by Lemma 3.4 and only depends on T. Therefore, the lower bound of equation (12) holds with $c_{3,5} = \min_{x\in T} \left(\frac{1}{c_{3,2}c_{3,4}}\right)^{1/(1/\underline{H}(x)+\delta_1)}$.

Moreover, by Lemma 3.3, for $\delta_2 = \min_{x\in T} \frac{\delta}{\overline{H}(x)(\overline{H}(x)+\delta)}$ there exists $c_{3,1} = c_{3,1}(K,\delta_2,r_0^{-1})$ such that

$$c_{3,3} \leq \|\ell_{E(x)}(\xi)\| = \|\tau_{E(x)}(\xi)^{-E(x)}\xi\| \leq c_{3,1}\tau_{E(x)}(\xi)^{-1/\overline{H}(x)+\delta_2}\|\xi\|$$

where the finite positive constant $c_{3,3}$ is given by Lemma 3.4 and only depends on T. Therefore, the upper bound of equation (12) holds with $c_{3,6} = \max_{x\in T} \left(\frac{c_{3,1}}{c_{3,3}}\right)^{1/(1/\overline{H}(x)-\delta_2)}$.
The proof of equation (11), that is the case where $\|\xi\| \leq 1$, is similar. □

Proof of Proposition 3.6. Let $\varepsilon \in (0,1)$. Then, since E is continuous on T, there exists $\gamma = \gamma(\varepsilon) > 0$ such that

$$\|E(x) - E(y)\| \leq \varepsilon \tag{30}$$

for $x, y \in T$ with $\|x - y\| \leq \gamma$.
Let $x, y \in T$ such that $\|x - y\| \leq \gamma$ and let $\xi \in \mathbb{R}^d$ such that $0 < \|\xi\| \leq 1$. Let us write $\xi = \tau_{E(y)}(\xi)^{E(y)}\ell_{E(y)}(\xi)$. Then, Assumption 6 implies that

$$\xi = \tau_{E(y)}(\xi)^{E(x)}\tau_{E(y)}(\xi)^{E(y)-E(x)}\ell_{E(y)}(\xi),$$

which leads to

$$\tau_{E(x)}(\xi) = \tau_{E(y)}(\xi)\tau_{E(x)}\Big(\tau_{E(y)}(\xi)^{E(y)-E(x)}\ell_{E(y)}(\xi)\Big) \tag{31}$$

by definition of τ_M (see (10)).
Let us first assume that $\tau_{E(y)}(\xi) \leq 1$. Then

$$\|\tau_{E(y)}(\xi)^{E(y)-E(x)}\ell_{E(y)}(\xi)\| \leq \tau_{E(y)}(\xi)^{-\|E(y)-E(x)\|}\Big\|\ell_{E(x)}(\xi)\Big\|.$$

Hence, by (30) and by Lemma 3.4,

$$\|\tau_{E(y)}(\xi)^{E(y)-E(x)}\ell_{E(y)}(\xi)\| \leq c_{3,4}\tau_{E(y)}(\xi)^{-\varepsilon}$$

where $c_{3,4}$ is a finite positive constant which only depends on T. Note that we can assume that $c_{3,4} \geq 1$. Let us now choose $\delta \in (0, \min_{v\in T} \underline{H}(v))$ such that

$$\max_{v\in T} \overline{H}(v) + \delta < 1.$$

Since $c_{3,4}\tau_{E(y)}(\xi)^{-\varepsilon} \geq 1$ and since $\underline{H}(x) - \delta \leq \overline{H}(x) + \delta$, using Proposition 3.5, we obtain that

$$\tau_{E(x)}(\tau_{E(y)}(\xi)^{E(y)-E(x)}\ell_{E(y)}(\xi)) \leq c_{3,6}\Big(c_{3,4}\tau_{E(y)}(\xi)^{-\varepsilon}\Big)^{\overline{H}(x)+\delta}$$

where $c_{3,6}$ is a finite positive constant which only depends on T and δ. Then, since $\overline{H}(x) + \delta < 1$ and since $c_{3,4}\tau_{E(y)}(\xi)^{-\varepsilon} \geq 1$,

$$\tau_{E(x)}(\tau_{E(y)}(\xi)^{E(y)-E(x)}\ell_{E(y)}(\xi)) \leq c_{3,6}c_{3,4}\tau_{E(y)}(\xi)^{-\varepsilon}.$$

Hence, by Equation (31),

$$\tau_{E(x)}(\xi) \leq c_{3,6}c_{3,4}\tau_{E(y)}(\xi)^{1-\varepsilon}.$$

Let us now assume that $\tau_{E(y)}(\xi) \geq 1$. Since $\|\xi\| \leq 1$ and $\tau_{E(y)}(\xi)^{1-\varepsilon} \geq 1$,

$$\tau_{E(x)}(\xi) \leq c_{3,6} \leq c_{3,6}\tau_{E(y)}(\xi)^{1-\varepsilon}.$$

Therefore, for any $x, y \in T$ such that $\|x - y\| \leq \gamma$, for any $\xi \in \mathbb{R}^d$ such that $0 < \|\xi\| \leq 1$,

$$\tau_{E(x)}(\xi) \leq c_{3,8}\tau_{E(y)}(\xi)^{1-\varepsilon}$$

where the finite positive constant $c_{3,8} = \max(c_{3,6}c_{3,4}, c_{3,6}) = c_{3,6}c_{3,4}$ does not depend on $x, y \in T$, nor on ε, γ. Note that the last inequality also holds for $\xi = 0$ since $\tau_{E(x)}(0) = \tau_{E(y)}(0) = 0$.
Moreover, by symmetry in $\tau_{E(x)}(\xi)$ and $\tau_{E(y)}(\xi)$, one can easily find a finite positive constant $c_{3,7}$ which only depends on T such that

$$c_{3,7}\tau_{E(y)}(\xi)^{1+\varepsilon} \leq \tau_{E(x)}(\xi)$$

for any $x, y \in T$ such that $\|x - y\| \leq \gamma$, for any $\xi \in \mathbb{R}^d$ such that $\|\xi\| \leq 1$. The proof of (13) is then complete. The proof of (14) is similar. □

6.2. **Results on the scale parameter.** This section is devoted to the proof of Lemmas 4.1, 4.2 and 4.3. We begin with two auxiliary lemmas:

Lemma 6.1. *Let $T = \prod_{i=1}^{d}[b_i, d_i]$ with $b_i < d_i$ for $1 \leq i \leq d$. Assume that $E : T \to \mathcal{M}^{>0}(\mathbb{R}^d)$ is continuous on T and satisfies Assumptions 2 and 6. Let $\alpha \in (0, 2]$. Then, for all $\varepsilon \in$*

$(0, \min_{w\in T}(1/\overline{H}(w) - 1)$ *there exist two finite positive constants* $\gamma_1 = \gamma_1(\varepsilon)$ *and* $c_{6,1} = c_{6,1}(T, \varepsilon)$ *such that*

$$\int_{\tau_{E(u)}(\xi)\leq\eta} \min\left(\|\xi\|^\alpha, 1\right)\tau_{E(u)}(\xi)^{-\alpha\beta_\alpha(u)-\alpha\varepsilon}d\xi \leq c_{6,1}\,\eta^{\alpha\gamma_1}$$

for every $\eta \in (0, 1]$ *and* $u \in T$.

Proof. Let $\varepsilon \in (0, \min_{w\in T}(1/\overline{H}(w) - 1)$, $u \in T$ and $\eta \in (0, 1]$. We set

$$I_\varepsilon(\eta, u) = \int_{\tau_{E(u)}(\xi)\leq\eta} \min\left(\|\xi\|^\alpha, 1\right)\tau_{E(u)}(\xi)^{-\alpha\beta_\alpha(u)-\alpha\varepsilon}d\xi.$$

By definition of $\tau_{E(u)}$ (see (10)) and of β_α (see (6)), Proposition 3.1 (applied with $M = E(u)$) leads to

$$I_\varepsilon(\eta, u) \leq \int_0^\eta \int_{S_{E(u)}} \left\|r^{E(u)}\theta\right\|^\alpha r^{-\alpha(1+\varepsilon)-1}\sigma_{E(u)}(d\theta)dr$$

Let $\delta \in (0, \min_{w\in T}(1/\overline{H}(w) - 1) - \varepsilon)$. Applying Lemma 3.3 (with $r_0 = 1$) and Lemma 3.4, one obtains that for any $w \in T$, any $r \in (0, \eta]$ and any $\theta \in S_{E(w)}$,

$$\left\|r^{E(w)}\theta\right\| \leq \left\|r^{E(w)}\right\|\|\theta\| \leq c_{3,1}c_{3,4}\|\theta\|_{E(u)}r^{1/\overline{H}(w)-\delta} = c_{3,1}c_{3,4}r^{1/\overline{H}(w)-\delta}$$

where the finite positive constants $c_{3,1} = c_{3,1}(T, \delta)$ and $c_{3,4} = c_{3,4}(T)$ do not depend on (w, r, θ). Therefore,

$$I_\varepsilon(\eta, u) \leq \left(c_{3,1}c_{3,4}\right)^\alpha\sigma_{E(u)}\left(S_{E(u)}\right)\int_0^\eta r^{\alpha(1/\overline{H}(u)-1-\varepsilon-\delta)-1}dr.$$

Since $\delta < \min_{w\in T}(1/\overline{H}(w) - 1) - \varepsilon$ and since $u \in T$, we get $1/\overline{H}(u) - 1 - \varepsilon - \delta > 0$. Then, applying again Lemma 3.4, one easily sees that

$$I_\varepsilon(\eta, u) \leq \frac{c_{3,1}^\alpha c_{3,4}^{\alpha+1}\eta^{\alpha(1/\overline{H}(u)-1-\varepsilon-\delta)}}{\alpha(1/\overline{H}(u) - 1 - \varepsilon - \delta)}.$$

Since $\eta \in (0, 1]$,

$$I_\varepsilon(\eta, u) \leq c_{6,1}\eta^{\alpha\gamma_1}$$

with

$$c_{6,1} = \frac{c_{3,1}^\alpha c_{3,4}^{\alpha+1}}{\alpha\min_{w\in T}(1/\overline{H}(w) - 1 - \varepsilon - \delta)} \in (0, +\infty)$$

and

$$\gamma_1 = \min_{w\in T}(1/\overline{H}(w) - 1 - \varepsilon - \delta) \in (0, +\infty).$$

Note that $c_{6,1}$ and γ_1 are well-defined by continuity of $\overline{H}$ on the compact set T. □

Lemma 6.2. *Let* $T = \prod_{i=1}^d [b_i, d_i]$ *with* $b_i < d_i$ *for* $1 \leq i \leq d$. *Assume that* $E : T \to \mathcal{M}^{>0}(\mathbb{R}^d)$ *is continuous on* T *and satisfies Assumptions 2 and 6. Let* $\alpha \in (0, 2]$. *Then, for all* $\varepsilon \in (0, 1)$ *there exist two finite positive constants* $\gamma_2 = \gamma_2(\varepsilon)$ *and* $c_{6,2} = c_{6,2}(T, \varepsilon)$ *such that*

$$\int_{\tau_{E(u)}(\xi)>A} \min\left(\|\xi\|^\alpha, 1\right)\tau_{E(u)}(\xi)^{-\alpha\beta_\alpha(u)+\alpha\varepsilon}d\xi \leq c_{6,2}\,A^{-\alpha\gamma_2}$$

for every $A \geq 1$ *and* $u \in T$.

Proof. Let $A \in [1,\infty)$, $u \in T$, $\varepsilon \in (0,1)$ and

$$\widetilde{I}_\varepsilon(A,u) = \int_{\tau_{E(u)}(\xi)>A} \min\left(\|\xi\|^\alpha, 1\right) \tau_{E(u)}(\xi)^{-\alpha\beta_\alpha(u)+\alpha\varepsilon} d\xi.$$

Let us first observe that

$$\widetilde{I}_\varepsilon(A,u) \leq \int_{\tau_{E(u)}(\xi)>A} \tau_{E(u)}(\xi)^{-\alpha\beta_\alpha(u)+\alpha\varepsilon} d\xi.$$

Then, applying as in the proof of Lemma 6.1 Proposition 3.1 with $M = E(u)$ and Lemma 3.4, one obtains that

$$\widetilde{I}_\varepsilon(A,u) \leq c_{3,4} \int_A^\infty r^{-\alpha(1-\varepsilon)-1} dr$$

with $c_{3,4} = c_{3,4}(T)$ a finite positive constant which only depends on T. Then since $\varepsilon < 1$,

$$\widetilde{I}_\varepsilon(A,u) \leq \frac{c_{3,4}}{\alpha(1-\varepsilon)} A^{-\alpha(1-\varepsilon)},$$

which concludes the proof. □

Proof of Lemma 4.1. Since Assumption 5 is fulfilled, q and $\overline{H}$ are uniformly continuous on the compact set T. Then we can consider $\varepsilon \in \left(0, \min\left(\min_{w\in T} 1/\overline{H}(w) - 1, 1\right)\right)$ and there exists $\gamma = \gamma(\varepsilon) \in (0,1)$ such that

$$|q(u) - q(v)| < \alpha\varepsilon,$$

for any $u, v \in T$ with $\|u - v\| \leq \gamma$. Henceforth, by continuity of $\overline{H}$ on the compact set T, for any $u, v \in T$ with $\|u - v\| \leq \gamma$, (15) holds and then $Y_{1,\alpha,x}(u,v)$ is well-defined for any $x \in \mathbb{R}^d$.

Let us now consider $x \in K$ and $u, v \in T$ such that $\|u - v\| \leq \gamma$. Then,

$$Y_{1,\alpha,x}(u,v) = \mathrm{Re} \int_{\mathbb{R}^d} f_{1,\alpha,x}(u,v,\xi) W_\alpha(d\xi)$$

where

$$f_{1,\alpha,x}(u,v,\xi) = \left(\mathrm{e}^{i\langle x,\xi\rangle} - 1\right)\left(\psi_u(\xi)^{-\beta_\alpha(u)} - \psi_u(\xi)^{-\beta_\alpha(v)}\right). \tag{32}$$

Therefore, by definition of $\|\cdot\|_\alpha$,

$$\|Y_{1,\alpha,x}(u,v)\|_\alpha^\alpha = \int_{\mathbb{R}^d} |f_{1,\alpha,x}(u,v,\xi)|^\alpha d\xi. \tag{33}$$

Moreover, for any $\xi \in \mathbb{R}^d \backslash \{0\}$, by Assumption 3, $\psi_u(\xi) \neq 0$ and then by the Mean Value Theorem,

$$\left|\psi_u(\xi)^{-\beta_\alpha(u)} - \psi_u(\xi)^{-\beta_\alpha(v)}\right| = \psi_u(\xi)^{-\beta_\alpha(u)} |\beta_\alpha(u) - \beta_\alpha(v)| \psi_u(\xi)^{-\beta_{\xi,u,v}} |\ln|\psi_u(\xi)|| \tag{34}$$

for some $|\beta_{\xi,u,v}| \in [0, |\beta_\alpha(v) - \beta_\alpha(u)|]$. Furthermore, since $\beta_\alpha = 1 + q/\alpha$,

$$|\beta_\alpha(w) - \beta_\alpha(w')| < \varepsilon$$

for any $w, w' \in T$ with $\|w - w'\| \leq \gamma$. Then, since T is a compact set, one can easily find a finite positive constant $c_1 = c_1(T, \gamma(\varepsilon))$ such that

$$\left|\psi_w(\xi)^{-\beta_{\xi,w,w'}} \ln|\psi_w(\xi)|\right| \leq c_1 \max\left(\psi_w(\xi)^{-1}, \psi_w(\xi)\right)^\varepsilon \tag{35}$$

for any $\xi \in \mathbb{R}^d \backslash \{0\}$ and any $w, w' \in T$ with $\|w - w'\| \leq \gamma$.
Moreover, for any $w \in T$, since ψ_w is $E(w)$-homogeneous,

$$\forall \xi \in \mathbb{R}^d \backslash \{0\},\ \psi_w(\xi) = \tau_{E(w)}(\xi)\psi_w\big(l_{E(w)}(\xi)\big).$$

By Assumptions 3 and 4, the function ψ is positive and continuous on the compact set

$$\Big\{(w,\theta) \in T \times \mathbb{R}^d;\ \|\theta\|_{E(w)} = 1\Big\}.$$

Then, there exists two finite positive constants $c_2 = c_2(T), c_3 = c_3(T)$ such that

$$\forall w \in T,\ \forall \xi \in \mathbb{R}^d \backslash \{0\},\ c_2 \tau_{E(w)}(\xi) \leq \psi_w(\xi) \leq c_3 \tau_{E(w)}(\xi). \tag{36}$$

Let us also remark that since K is a compact set, there exists a finite positive constant $c_4 = c_4(K)$ such that

$$\forall y \in K,\ \forall \xi \in \mathbb{R}^d,\ \left|\mathrm{e}^{i\langle y,\xi\rangle} - 1\right| \leq c_4 \min(\|\xi\|, 1), \tag{37}$$

Therefore, by (32), (34), (35), (36) and (37), for any $\xi \in \mathbb{R}^d \backslash \{0\}$

$$|f_{1,\alpha,x}(u,v,\xi)| \leq c_5 |\beta_\alpha(u) - \beta_\alpha(v)| \min(\|\xi\|, 1) \tau_{E(u)}(\xi)^{-\beta_\alpha(u)} \max(\tau_{E(u)}(\xi)^{-1}, \tau_{E(u)}(\xi))^{\varepsilon} \tag{38}$$

where the finite positive constant c_5 does not depend on (x,u,v,ξ).
Then, by (33),

$$\|Y_{1,\alpha,x}(u,v)\|_\alpha^\alpha \leq c_5^\alpha |\beta_\alpha(u) - \beta_\alpha(v)|^\alpha \int_{\mathbb{R}^d} \min(\|\xi\|^\alpha, 1) \tau_{E(u)}(\xi)^{-\alpha\beta_\alpha(u)} \max(\tau_{E(u)}(\xi)^{-1}, \tau_{E(u)}(\xi))^{\alpha\varepsilon} d\xi.$$

Since $\varepsilon < \min(\min_{w\in T} 1/\overline{H}(w) - 1, 1)$, Lemma 6.1 applied with $\eta = 1$ and Lemma 6.2 applied with $A = 1$ lead to

$$\|Y_{1,\alpha,x}(u,v)\|_\alpha^\alpha \leq c_5^\alpha \big(c_{6,1} + c_{6,2}\big) |\beta_\alpha(u) - \beta_\alpha(v)|^\alpha$$

where $c_{6,1}$ and $c_{6,2}$ does not depend on (x,u,v). One easily concludes the proof since by Assumption 5, q and then $\beta_\alpha = 1 + q/\alpha$ is Lipschitz on the compact set T. □

Proof of Lemma 4.2. From the proof of Lemma 4.1, we can choose γ small enough such that (15) holds for any $u, v \in T$ with $\|u - v\| \leq \gamma$. Hence, $Y_{2,\alpha,x}(u,v)$ is well-defined for any $x \in \mathbb{R}^d$, and $u, v \in T$ with $\|u - v\| \leq \gamma$.

Let us now consider $x \in K$ and $u, v \in T$ with $\|u - v\| \leq \gamma$. Then,

$$Y_{2,\alpha,x}(u,v) = \mathrm{Re} \int_{\mathbb{R}^d} f_{2,\alpha,x}(u,v,\xi) W_\alpha(d\xi),$$

where

$$f_{2,\alpha,x}(u,v,\xi) = \big(\mathrm{e}^{i\langle x,\xi\rangle} - 1\big)\Big(\psi_u(\xi)^{-\beta_\alpha(v)} - \psi_v(\xi)^{-\beta_\alpha(v)}\Big). \tag{39}$$

Therefore, by definition of $\|\cdot\|_\alpha$,

$$\|Y_{2,\alpha,x}(u,v)\|_\alpha^\alpha = \int_{\mathbb{R}^d} |f_{2,\alpha,x}(u,v,\xi)|^\alpha d\xi.$$

Let $\xi \neq 0$ and let us split

$$\begin{aligned} g_\alpha(u,v,\xi) &= \Big|\psi_u(\xi)^{-\beta_\alpha(v)} - \psi_v(\xi)^{-\beta_\alpha(v)}\Big| \\ &= g_{1,\alpha,\eta}(u,v,\xi) + g_{2,\alpha,\eta}(u,v,\xi) \end{aligned}$$

with

$$g_{1,\alpha,\eta}(u,v,\xi) = \left(\mathbf{1}_{\tau_{E(v)}(\xi)<\eta} + \mathbf{1}_{\tau_{E(v)}(\xi)>1/\eta}\right) g_\alpha(u,v,\xi)$$

and

$$g_{2,\alpha,\eta}(u,v,\xi) = \mathbf{1}_{\eta\leq\tau_{E(v)}(\xi)\leq 1/\eta} g_\alpha(u,v,\xi),$$

where $\eta \in (0,1)$.

First Step: Study of $g_{1,\alpha,\eta}$ and choice of η.

By Assumption 5, $\beta_\alpha = 1 + q/\alpha$ is continuous on T and we can consider

$$\overline{\beta_\alpha} = \max_{w\in T} \beta_\alpha(w) \in (1,+\infty).$$

Let us choose $\varepsilon = \varepsilon(\alpha,T) > 0$ such that $\varepsilon < \overline{\beta_\alpha}\min(\min_{w\in T} 1/\overline{H}(w) - 1, 1)$. Then, according to Proposition 3.6, up to change γ, we can assume $\gamma = \gamma(\varepsilon) > 0$ and for all $\xi \neq 0$ and $w, w' \in T$ such that $\|w - w'\| \leq \gamma$,

$$\tau_{E(w)}(\xi) \geq c_{3,7}\tau_{E(w')}(\xi)\min(\tau_{E(w')}(\xi)^{-1}, \tau_{E(w')}(\xi))^{\varepsilon/\overline{\beta_\alpha}},$$

where the finite positive constant $c_{3,8} = c_{3,8}(T,\varepsilon)$ does not depend on w, w' and ξ. Then, by (36) (see the proof of Lemma 4.1) and continuity of β_α, there exists a finite positive constant $C_1 = C_1(T,\varepsilon)$, which does not depend on (x,u,v,ξ), such that

$$g_{1,\alpha,\eta}(u,v,\xi) \leq C_1\left(\mathbf{1}_{\tau_{E(v)}(\xi)<\eta} + \mathbf{1}_{\tau_{E(v)}(\xi)>1/\eta}\right)\tau_{E(v)}(\xi)^{-\beta_\alpha(v)}\max(\tau_{E(v)}(\xi)^{-1}, \tau_{E(v)}(\xi))^{\varepsilon}. \qquad (40)$$

Then, combining Equations (40) and (37), according to Lemma 6.1 and Lemma 6.2, there exist two finite positive constants $\nu = \nu(\varepsilon)$ and $C_2 = C_2(K,T,\varepsilon)$, which do not depend on (u,v,x), such that for all $\eta \in (0,1]$ one has

$$I_{1,\alpha,\eta}(x,u,v) \;=\; \int_{\mathbb{R}^d} \left|e^{i\langle x,\xi\rangle} - 1\right|^\alpha g_{1,\alpha,\eta}(u,v,\xi)^\alpha d\xi \leq C_2\eta^{\alpha\nu}.$$

Choosing $\eta = \eta(\varepsilon,u,v) = \|u-v\|^{1/\nu}$, one gets that $I_{1,\alpha,\eta}(x,u,v) \leq C_2\|u-v\|^\alpha$.

Second Step: Study of $g_{2,\alpha,\eta}$.

Now let us focus on $g_{2,\alpha,\eta}$ for this particular choice of η. By homogeneity of ψ_u and ψ_v,

$$g_{2,\alpha,\eta}(u,v,\xi) \;=\; \mathbf{1}_{\eta\leq\tau_{E(v)}(\xi)\leq 1/\eta}\tau_{E(v)}(\xi)^{-\beta_\alpha(v)}\Big|\psi_u\Big(\tau_{E(v)}(\xi)^{-E(u)}\tau_{E(v)}(\xi)^{E(v)}\ell_{E(v)}(\xi)\Big)^{-\beta_\alpha(v)} - \psi_v\Big(\ell_{E(v)}(\xi)\Big)^{-\beta_\alpha(v)}\Big|.$$

By Lemma 3.4, there exist two finite positive constants $c_{3,3} = c_{3,3}(T)$ and $c_{3,4} = c_{3,4}(T)$ such that

$$\forall w \in T,\ c_{3,3} \leq \left\|\ell_{E(w)}(\xi)\right\| \leq c_{3,4}.$$

Then, since $\xi \neq 0$ and $v \in T$,

$$\left\|\ell_{E(v)}(\xi) - \tau_{E(v)}(\xi)^{-E(u)}\tau_{E(v)}(\xi)^{E(v)}\ell_{E(v)}(\xi)\right\| \leq c_{3,4}\left\|I - \tau_{E(v)}(\xi)^{-E(u)}\tau_{E(v)}(\xi)^{E(v)}\right\|.$$

By Assumption 6, $E(u)E(v) = E(v)E(u)$ and then

$$\left\|I - \tau_{E(v)}(\xi)^{-E(u)}\tau_{E(v)}(\xi)^{E(v)}\right\| = \left\|I - \tau_{E(v)}(\xi)^{E(v)-E(u)}\right\|.$$

Therefore,

$$\left\|I-\tau_{E(v)}(\xi)^{-E(u)}\tau_{E(v)}(\xi)^{E(v)}\right\| \leq \|E(v)-E(u)\|\left|\ln \tau_{E(v)}(\xi)\right| \max\left(\tau_{E(v)}(\xi),\tau_{E(v)}(\xi)^{-1}\right)^{\|E(v)-E(u)\|},$$

since $\left\|e^{M}-e^{M'}\right\| \leq \|M-M'\|e^{\|M\|+\|M'\|}$, for any $M, M' \in \mathcal{M}(\mathbb{R}^d)$ such that $MM' = M'M$. Then, since $\eta \leq \tau_{E(v)}(\xi) \leq 1/\eta$,

$$\left\|I-\tau_{E(v)}(\xi)^{-E(u)}\tau_{E(v)}(\xi)^{E(v)}\right\| \leq \|E(v)-E(u)\||\ln \eta|\eta^{-\|E(v)-E(u)\|}.$$

Hence, according to Assumption 5, there exists $c_{2,2} = c_{2,2}(T)$ such that

$$\left\|I-\tau_{E(v)}(\xi)^{-E(u)}\tau_{E(v)}(\xi)^{E(v)}\right\| \leq c_{2,2}\|u-v\||\ln \eta|\eta^{-c_{2,2}\|u-v\|}$$

since $\eta \leq 1$. Finally, since $\eta = \|u-v\|^{1/\nu}$, one can choose γ small enough such that

$$\left\|I-\tau_{E(v)}(\xi)^{-E(u)}\tau_{E(v)}(\xi)^{E(v)}\right\| \leq \frac{c_{3,3}}{2c_{3,4}} \leq \frac{1}{2},$$

which implies that

$$\left\|\ell_{E(v)}(\xi)-\tau_{E(v)}(\xi)^{-E(u)}\tau_{E(v)}(\xi)^{E(v)}\ell_{E(v)}(\xi)\right\| \leq \frac{c_{3,3}}{2} \leq \frac{c_{3,4}}{2}.$$

Then,

$$\frac{c_{3,3}}{2} \leq \left\|\tau_{E(v)}(\xi)^{-E(u)}\tau_{E(v)}(\xi)^{E(v)}\ell_{E(v)}(\xi)\right\| \leq \frac{3c_{3,4}}{2}.$$

Using the Mean Value Theorem for $t \mapsto t^{-\beta_\alpha(v)}$, the continuity of β_α and Assumption 4 with

$$\mathcal{K} = T \times \left\{y \in \mathbb{R}^d; \frac{c_{3,3}}{2} \leq \|y\| \leq \frac{3c_{3,4}}{2}\right\},$$

one can find two finite positive constants C_3 and C_4 that only depend on T and γ such that

$$\begin{aligned}
&\left|\psi_u\left(\tau_{E(v)}(\xi)^{E(v)-E(u)}\ell_{E(v)}(\xi)\right)^{-\beta_\alpha(v)} - \psi_v\left(\ell_{E(v)}(\xi)\right)^{-\beta_\alpha(v)}\right| \\
\leq\ & C_3\left|\psi_u\left(\tau_{E(v)}(\xi)^{E(v)-E(u)}\ell_{E(v)}(\xi)\right) - \psi_v\left(\ell_{E(v)}(\xi)\right)\right| \\
\leq\ & C_4\|u-v\|\left(1+\left|\ln \tau_{E(v)}(\xi)\right| \max(\tau_{E(v)}(\xi)^{-1},\tau_{E(v)}(\xi))^{c_{2,2}\|u-v\|}\right).
\end{aligned}$$

To conclude let us recall that we have chosen $\varepsilon \in (0, \min(\min_{w\in T} 1/\overline{H}(w)-1, 1))$. Up to choose γ smaller we may assume that $c_{2,2}\gamma < \varepsilon$. Then, one can find a finite positive constant $C_5 = C_5(T,\varepsilon)$ such that

$$g_{2,\alpha,\eta}(u,v,\xi) \leq C_5\|u-v\|\tau_{E(v)}(\xi)^{-\beta_\alpha(v)} \max(\tau_{E(v)}(\xi)^{-1},\tau_{E(v)}(\xi))^{\varepsilon},$$

for all $\xi \neq 0$ and $u, v \in T$ such that $\|u-v\| \leq \gamma$. Then by Lemmas 6.1 and 6.2 and (37), there exists a finite positive constant $C_6 = C_6(T,K,\varepsilon)$ such that

$$I_{2,\alpha,\eta}(x,u,v) \ = \ \int_{\mathbb{R}^d} \left|\mathrm{e}^{i\langle x,\xi\rangle}-1\right|^{\alpha} g_{2,\alpha,\eta}(u,v,\xi)^{\alpha} d\xi \leq C_6\|u-v\|^{\alpha},$$

for all $x \in K$ and all $u, v \in T$ such that $\|u-v\| \leq \gamma$. The conclusion follows from

$$\|Y_{2,\alpha,x}(u,v)\|_{\alpha}^{\alpha} = I_{1,\alpha,\eta}(x,u,v) + I_{2,\alpha,\eta}(x,u,v).$$

□

Proof of Lemma 4.3. Let $x \in \mathbb{R}^d$. Then, the random field

$$\big(X_{\psi_x}(v)\big)_{v\in\mathbb{R}^d} = \big(Y_{\alpha,\psi}(v,x,x)\big)_{v\in\mathbb{R}^d}$$

is well-defined and is a harmonizable operator scaling α-stable random field in the sense of Theorem 4.1 of [7] with respect to $E(x)^t$. Moreover,

$$Y_{3,\alpha,x}(u,v) = X_{\psi_x}(u) - X_{\psi_x}(v).$$

Then, by stationarity of increments of X_{ψ_x} and the operator scaling property (see Corollary 4.2 of [7]), when $u \neq v$

$$\|Y_{3,\alpha,x}(u,v)\|_\alpha^\alpha = \tau_{_{E(x)^t}}(u-v)^\alpha J_\alpha\big(x, \ell_{E(x)^t}(u-v)\big)$$

where

$$\forall\theta \in S_{_{E(x)^t}},\ J_\alpha(x,\theta) = \int_{\mathbb{R}^d} \big|\mathrm{e}^{i\langle\theta,\xi\rangle} - 1\big|^\alpha \psi_x^{-\alpha\beta(x)}(\xi)d\xi.$$

Since J_α is positive and continuous on the compact set

$$\Big\{(y,\theta) \in \mathbb{R}^d \times \mathbb{R}^d; y \in K, \text{ and } \theta \in S_{_{E(y)^t}}\Big\},$$

there exist $c_{_{4,3}} = c_{_{4,3}}(\alpha, K)$ and $c_{_{4,4}} = c_{_{4,4}}(\alpha, K)$ two finite positive constants such that

$$\forall y \in K,\ \forall\theta \in S_{_{E(y)^t}},\ c_{_{4,3}} \leq J_\alpha(x,\theta) \leq c_{_{4,4}},$$

which concludes the proof. □

6.3. **Modulus of continuity.**

Proof of Lemma 4.7. If $\alpha = 2$, assertion (2) is a direct consequence of Theorem 4.4. Let us now assume that $\alpha \in (0,2)$. Then, according to (17)

$$v_\alpha^2((u,v) \mid (T_n,\xi_n)_n) = \frac{d_\alpha^2}{2}\,\mathbb{E}\big(|g_1|^2\big)\sum_{n=1}^{+\infty} T_n^{-2/\alpha} m_{x_0}(\xi_n)^{-2/\alpha} |f_\alpha(u,\xi_n) - f_\alpha(v,\xi_n)|^2$$

where f_α is defined by (16). Similarly to the proof of Theorem 4.4 we write

$$f_\alpha(u,\xi_n) - f_\alpha(v,\xi_n) = f_{1,\alpha,u}(u,v,\xi_n) + f_{2,\alpha,u}(u,v,\xi_n) + f_{3,\alpha,v}(u,v,\xi_n)$$

where $f_{1,\alpha,u}$ is defined by (32), $f_{2,\alpha,u}$ by (39) and

$$f_{3,\alpha,y}(u,v,\xi) = \big(\mathrm{e}^{i\langle u,\xi\rangle} - \mathrm{e}^{i\langle v,\xi\rangle}\big)\psi_y(\xi)^{-\beta_\alpha(y)}.$$

We then denote for $j \in \{1,2\}$,

$$v_{j,\alpha}^2((u,v) \mid (T_n,\xi_n)_n) = \frac{d_\alpha^2}{2}\,\mathbb{E}\big(|g_1|^2\big)\sum_{n=1}^{+\infty} T_n^{-2/\alpha} m_{x_0}(\xi_n)^{-2/\alpha} |f_{j,\alpha,u}(u,v,\xi_n)|^2$$

and

$$v_{3,\alpha}^2((u,v) \mid (T_n,\xi_n)_n) = \frac{d_\alpha^2}{2}\,\mathbb{E}\big(|g_1|^2\big)\sum_{n=1}^{+\infty} T_n^{-2/\alpha} m_{x_0}(\xi_n)^{-2/\alpha} |f_{3,\alpha,v}(u,v,\xi_n)|^2.$$

Hence,

$$v_\alpha^2((u,v) \mid (T_n,\xi_n)_n) \leq 4\sum_{j=1}^{3} v_{j,\alpha}^2((u,v) \mid (T_n,\xi_n)_n).$$

Let $\varepsilon_1 \in (0, \min\big(\min_{w\in T} 1/\overline{H}(w) - 1, 1\big))$.

First Step: Study of $v_{1,\alpha}$

Using (38), Proposition 3.6 and the Lipschitz property of β_α on T, one can find $\gamma = \gamma(\varepsilon_1) \in (0,1)$ and a finite positive constant $c_1 = c_1(T, \varepsilon_1)$ such that

$$|f_{1,\alpha,u}(u,v,\xi)| \leq c_1 \|u-v\| \min\left(1, \|\xi\|\right) \tau_{E(x_0)}(\xi)^{-\beta_\alpha(x_0)} \max(\tau_{E(x_0)}(\xi)^{-1}, \tau_{E(x_0)}(\xi))^{\varepsilon_1/3}$$

for any $\xi \in \mathbb{R}^d\backslash\{0\}$ and any $u, v \in T$ such that $\|u - x_0\| \leq \gamma$ and $\|v - x_0\| \leq \gamma$. Hence, almost surely

$$v_{1,\alpha}^2((u,v) \mid (T_n, \xi_n)_n) \leq \|u-v\|^2 W$$

where

$$W = c_1 \sum_{n=1}^{+\infty} T_n^{-2/\alpha} \zeta_n \tag{41}$$

with $\zeta_n = m_{x_0}(\xi_n)^{-2/\alpha} \min\left(1, \|\xi_n\|^2\right) \tau_{E(x_0)}(\xi_n)^{-2\beta_\alpha(x_0)} \max(\tau_{E(x_0)}(\xi_n)^{-1}, \tau_{E(x_0)}(\xi_n))^{2\varepsilon_1/3}$.
One easily checks that ζ_n, $n \in \mathbb{N}\backslash\{0\}$ are i.i.d. integrable random variables and then that $W < \infty$ almost surely (since $T_n/n \to 1$ almost surely and $2/\alpha > 1$).

Second Step: Study of $v_{2,\alpha}$ Following the proof of Lemma 4.2, one can choose two finite positive constants $\nu = \nu(\varepsilon_1)$ and $c_2 = c_2(T, \varepsilon_1)$ such that for η small enough,

$$\int_{\mathbb{R}^d} \Big(\|\xi\|^2 \mathbf{1}_{\tau_{E(x_0)}(\xi)<\eta} + \mathbf{1}_{\tau_{E(x_0)}(\xi)>1/\eta}\Big) \tau_{E(x_0)}(\xi)^{-2\beta_2(x_0)} \max\Big(\tau_{E(x_0)}(\xi), \tau_{E(x_0)}(\xi)^{-1}\Big)^{\varepsilon_1} \leq c_2 \eta^{2\nu}. \tag{42}$$

Moreover, following the proof of Lemma 4.2 and using Proposition 3.6, choosing $\gamma = \gamma(\varepsilon_1)$ smaller if necessary, one can also find a finite positive constant $c_3 = c_3(T, \varepsilon_1)$ such that for $\|u - x_0\| \leq \gamma/2$ and $\|v - x_0\| \leq \gamma/2$,

$$v_{2,\alpha}^2((u,v) \mid (T_n, \xi_n)_n) \leq c_3\big(\|u-v\|^2 W + \sigma_2^2(\|u-v\|)\big),$$

where W is defined by (41) and for all $h \geq 0$,

$$\sigma_2^2(h) = \sum_{n=1}^{+\infty} T_n^{-2/\alpha} m_{x_0}(\xi_n)^{-2/\alpha} \min\left(\|\xi_n\|^2, 1\right) \Big(\mathbf{1}_{\tau_{E(x_0)}(\xi_n)<h^{1/\nu}} + \mathbf{1}_{\tau_{E(x_0)}(\xi_n)>h^{-1/\nu}}\Big) \tau_{E(x_0)}(\xi_n)^{-2\beta_\alpha(x_0)}$$
$$\times \max(\tau_{E(x_0)}(\xi)^{-1}, \tau_{E(x_0)}(\xi))^{2\varepsilon_1/3}.$$

Let us recall that the density function of ξ_n is m_{x_0}. Then, using the definition of m_{x_0}, of β_α and β_2, one can easily find a finite positive constant $c_4 = c_4(T, \varepsilon_1)$ such that for $h \geq 0$ small enough,

$$\mathbb{E}\left(\sigma_2^2(h) | (T_n)_n\right) \leq c_4 \varsigma_2(h)^2 \sum_{n=1}^{+\infty} T_n^{-2/\alpha}$$

with

$$\varsigma_2(h)^2 = \int_{\mathbb{R}^d} \min\left(\|\xi\|^2, 1\right) \Big(\mathbf{1}_{\tau_{E(x_0)}(\xi)<h^{1/\nu}} + \mathbf{1}_{\tau_{E(x_0)}(\xi)>h^{-1/\nu}}\Big) \tau_{E(x_0)}(\xi)^{-2\beta_2(x_0)} \max(\tau_{E(x_0)}(\xi)^{-1}, \tau_{E(x_0)}(\xi))^{\varepsilon_1}.$$

Then, (42) leads to the existence of a finite positive constant $c_5 = c_5(\varepsilon_1)$ such that almost surely for $h \geq 0$ small enough,

$$\mathbb{E}\left(\sigma_2^2(h)|(T_n)_n\right) \leq c_5 h^2 \sum_{n=1}^{+\infty} T_n^{-2/\alpha}.$$

Then, since $h \mapsto \sigma_2^2(h)$ is monotone, almost surely

$$\lim_{h\to 0} \frac{\sigma_2^2(h)}{h^{2-\varepsilon}} = 0$$

for any $\varepsilon \in (0,1)$ (see for instance [6]).

Third Step: Study of $v_{3,\alpha}$

Using Proposition 3.6, there exist $\gamma = \gamma(\varepsilon_1) \in (0,1)$ and a finite positive constant $c_6 = c_6(T, \varepsilon_1)$ such that for any $\|u - x_0\| \leq \gamma/2$ and $\|v - x_0\| \leq \gamma/2$,

$$v_{3,\alpha}^2((u,v) \mid (T_n, \xi_n)_n) \leq c_6 \sigma_3^2(\tau_{E(x_0)^t}(u-v)),$$

where, for all $h \geq 0$,

$$\sigma_3^2(h) = \sum_{n=1}^{+\infty} T_n^{-2/\alpha} m_{x_0}(\xi_n)^{-2/\alpha} \min\left(\left\|h^{E(x_0)}\xi_n\right\|^2, 1\right) \tau_{E(x_0)}(\xi_n)^{-2\beta_\alpha(x_0)} \max(\tau_{E(x_0)}(\xi), \tau_{E(x_0)}(\xi)^{-1})^{2\varepsilon_1/3}.$$

Following the proof of Lemma 5.2 of [6], one obtains that

$$\mathbb{E}\left(\sigma_3^2(h)|(T_n)_n\right) \leq c_7 h^{2-\varepsilon_1} \sum_{n=1}^{+\infty} T_n^{-2/\alpha},$$

where the finite positive constant $c_7 = c_7(\varepsilon_1)$ does not depend on h. Therefore, almost surely

$$\lim_{h\to 0} \frac{\sigma_3^2(h)}{h^{2-\varepsilon}} = 0$$

for all $\varepsilon \in (0, \varepsilon_1)$.

Proposition 3.5, Step 1, 2 and 3 lead to the conclusion. □

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