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L-FUNCTIONS OF EXPONENTIAL SUMS ON CURVES OVER RINGS.

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ABSTRACT. Let C be a smooth curve over $\mathcal{O}/p^l\mathcal{O}$, $\mathcal{O}$ being the valuation ring of an unramified extension of the field $\mathbb{Q}_p$ of p-adic numbers, with residue field $k = \mathbb{F}_q$. Let f be a function over C, and Ψ be an additive character of order p^l over R; in this paper we study the exponential sums associated to f and Ψ over C, and their L-functions. We show the rationality of the L-functions in a more general setting, then in the case of curves we express them as products of L-functions associated to polynomials over the affine line, each factor coming from a singularity of f. Finally we show that in the case of Morse functions (i.e. having only simple singularities), the degree of the L-functions are, up to sign, the same as in the case of finite fields, yielding very similar bounds for exponential sums.

0. INTRODUCTION

The aim of this paper is to study exponential sums over p-adic curves, and their associated L-functions. Let K_m denote the unramified extension of degree m of $\mathbb{Q}_p$; we denote by $\mathcal{O}_m$ its valuation ring. This is a local ring with maximal ideal generated by p and residue field $k := \mathbb{F}_q$, $q := p^m$. Let $l \geq 2$ be an integer; we set $R := \mathcal{O}_m/p^l\mathcal{O}_m$, and, for any integer $r \geq 1$, $R_r := \mathcal{O}_{mr}/p^l\mathcal{O}_{mr}$ its "unramified extension" of degree r. We fix an additive character Ψ of R of order p^l, and denote by $\Psi^{(r)} := \Psi \circ \mathrm{Tr}_{R_r/R}$ the character induced on R_r.

If C is a smooth projective curve over R, and f is a function over C, we define the sum

$$S_r(C, f) := \sum_{\Pi \in C_f(R_r)} \Psi^{(r)}(f(\Pi)),$$

where $C_f(R_r)$ is the set of R_r-points of C at which f is defined. We associate to the sums $(S_r(C, f))_{r\geq 1}$ the L-function

$$L(C, f; T) := \exp\left(\sum_{r\geq 1} S_r(C, f)\frac{T^r}{r}\right).$$

In the case $l = 1$, these sums and their L-functions have been thoroughly studied; it follows from Weil's Theorem on the Riemann hypothesis over finite fields that the L-function is a polynomial whose degree depends on the polar divisor of the function f, and all of whose reciprocal roots are q-Weil numbers of weight 1, i.e. algebraic integers all of whose conjugates have complex absolute value $q^{\frac{1}{2}}$. Precisely, if the polar divisor of f is $(f)_\infty = \sum n_i P_i$, the degree of the L-function is at most (exactly if p does not divide any of the n_i)

$$2g - 2 + \sum (n_i + 1)\deg(P_i),$$

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where g denotes the genus of C (*cf.* [5] Section 3, [9]).

When $l \geq 2$, little is known when the genus of $C \otimes k$ is positive. A bound is given in [14] in the case of plane nonsingular curves and polynomial functions (note that the assumption of nonsingularity there is weaker that our assumption of smoothness). The author gives a bound for sums similar to $S_r(C, f)$, and l big enough, in terms of the intersection multiplicities of the equation of the curve and the polynomial to be summed. Some work has also been done when the summation set is not the whole set of R_r-points of the curve, but the image of a section of reduction modulo p: $C(R_r) \to C(k_r)$ (*cf.* [1], [10], [12], [13]).

When $l \geq 2$ and C is rational (i.e. when $C \otimes k$ has genus 0), the sums associated to one-variable polynomials have drawn much attention (*cf.* [3], [11]), and recently the case of rational functions has been studied (*cf.* [4]). In these works, the authors give bounds for such sums when the residue field is the prime field $\mathbb{F}_p$. The method is based upon some kind of stationnary phase formula: the sum over the fiber of reduction modulo p above $\alpha \in \mathbb{F}_p$ is zero unless the reduction modulo p of the function f has a singularity at α. It only remains to study locally the behaviour of the function at its singularities using a Taylor expansion. We shall extend this point of view to the case of curves: precisely, we show that the sum over the fiber of reduction modulo p above $P \in C(k)$ is zero unless the reduction modulo p of the function f, say $\rho(f)$ has a singularity at P (that is: the differential $d\rho(f)$, has a zero at P). We use local parameters to obtain a Taylor expansion for the function, and compare it with a polynomial.

We adopt the langage of L-functions here, since it gives a unified point of view between the case of finite fields and the case of rings. Moreover the evaluation of the L-function allows bounds over any of the extensions; these bounds are reduced to the estimation of the degree of the corresponding L-function, and the weights of its reciprocal roots as q-Weil numbers. We begin by showing that the L-functions are rational in the more general setting of schemes (not necessarily one-dimensional) over R. Then we show a stationary phase formula, which allows us to write the L-function as a product of local factors. Each of these local factors comes from a singularity, and can be written as the L-function associated to a polynomial (coming from the expansion of the function in terms of a local parameter) over the affine line. Thus we are able to reduce the general case of curves to the well known case of polynomials.

We emphasize on the case of Morse functions, i.e. functions whose reduction modulo p has only simple singularities. In this case, we show that the situation is very similar to the one in finite fields. Precisely, the degree of the L-function can be estimated in terms of the polar divisor of the reduction modulo p of the function, and in most cases, it remains the same as for finite fields. Only the weight of the reciprocal roots as q-Weil numbers change, and for instance we get the following bound (compare with [5] 3.5.2, [9] Corollary 3.5)

$$|S(C,f)| \leq \left(2g - 2 + \sum_{i=1}^{k}(n_i+1)\deg(P_i)\right) q^{\frac{l}{2}},$$

where the polar divisor of $\rho(f)$ is $(\rho(f))_\infty = \sum_{i=1}^{k}(n_i+1)P_i$.

The paper is organized as follows: we begin by showing the rationality of the L-functions we consider in the general setting of schemes over the ring R, using ℓ-adic cohomology and Greenberg's functor to reduce to the case of varieties over finite fields. In section 2, we use once again Greenberg's functor to describe precisely the set $C(R_r)$: we write this set as a disjoint union of fibers of reduction modulo p,

then we show that over each fiber, the image of a local parameter describes tbe maximal ideal pR_r of R_r. This allows us to reduce the sum over $C(R)$ to sums attached to polynomials over the affine line in section 3; moreover we show that the sum over the fiber above $P \in C(k)$ vanishes unless the reduction modulo p of the function has a singularity at P. Thus we can write the L-function as a product of local factors. Finally we consider the case of Morse functions in section 4 and we generalize in this case known results over finite fields.

Notations

Let $l \geq 2$ be an integer ; we set $R := \mathcal{O}_m/p^l\mathcal{O}_m$. Sometimes this ring is called a Galois ring, and denoted $GR(p^l, m)$. Recall that there is a canonical isomorphism between this ring and $W_l(k)$, the ring of Witt vectors of length l with coefficients in k.

It is well known that $R^\times$ has a cyclic subgroup $\mathcal{T}^\times$ of order $q-1$. We set $\mathcal{T} := \mathcal{T}^\times \cup \{0\}$, this is the *Teichmüller* of R, and every element in R can be written uniquely as

$$x = x_0 + px_1 + \cdots + p^{l-1}x_{l-1},\ x_0, \ldots, x_{l-1} \in \mathcal{T}.$$

One can consider R as the (unramified) extension of $\mathbb{Z}/p^l\mathbb{Z}$ of degree m. The Galois group of this extension is isomorphic to $\mathbb{Z}/m\mathbb{Z}$, generated by the *Frobenius* F, whose action is described (when we write $x \in R$ as above) by

$$F(x) = x_0^p + px_1^p + \cdots + p^{l-1}x_{l-1}^p.$$

Recall that the additive characters of R are the morphims from $(R, +)$ to $(\mathbb{C}^*, \times)$; they can be defined in the following way

$$\Psi_{l,y}(x) = \exp\left(\frac{2i\pi}{p^l}\mathrm{Tr}(xy)\right),\ y \in R,$$

where $\mathrm{Tr} := \mathrm{Tr}_{R/\mathbb{Z}/p^l\mathbb{Z}} = Id + F + \cdots + F^{m-1}$ is the trace. In the following we set $\Psi := \Psi_{l,1}$.

If $r \geq 1$ is an integer, let $R_r := \mathcal{O}_{mr}/p^l\mathcal{O}_{mr}$ be the unramified extension of degree r of R, and $\Psi^{(r)} := \Psi \circ \mathrm{Tr}_{R_r/R}$ an additive character.

Recall that for any $r \geq 1$ there is a canonical isomorphism $w : R_r \to W_l(k_r)$ between R_r and the ring of Witt vectors of length l with coefficients in k_r.

In the following, we shall need to consider exponential sums and L-functions both over the rings R_r and over the rings $R'_r := R_r \otimes \mathbb{Z}/p^{l-1}\mathbb{Z}$. In order to simplify notations, we shall denote by $\Psi' := \Psi_{l-1,1}$ and $\Psi^{(r)'}$ the corresponding characters, by $S'_r(C, f) := S_r(C \otimes R', f)$ the sum associated to $\Psi^{(r)'}$, and by

$$L'(C, f; T) := L'(C \otimes R', \Psi', f; T)$$

the L-function attached to this situation.

1. Rationality of the L-functions.

Let X be a scheme of finite type over R, and $f \in \Gamma(X, \mathcal{O}_X)$ a regular function over X. Denote by $X(R_r)$ the set of R_r-points of X, i.e. the set of morphisms from Spec R_r to X.We consider the following family of exponential sums

$$S_r(X, f) := \sum_{\Pi \in X(R_r)} \Psi^{(r)}(f(\Pi)),$$

and the associated L-function

$$L(X, f; T) := \exp\left(\sum_{r \geq 1} S_r(X, f)\frac{T^r}{r}\right).$$

The aim of this section is to show that such a function is rational.

We begin by recalling some facts about Greenberg's functor: the proofs can be found in [6]. To any k-scheme Y, we associate functorially a scheme WY over Spec R, with the same underlying topological space, and structure sheaf $\mathcal{O}_{WY} = W_l(\mathcal{O}_Y)$, defined for every U open in Y by

$$\Gamma(U, \mathcal{O}_{WY}) = W_l(\Gamma(U, \mathcal{O}_Y)).$$

If X is a R-scheme, the functor associating $\mathrm{Hom}_R(WY, X)$ to every k-scheme Y is representable by a k-scheme $\mathcal{F}X$: we have a functorial isomorphism

$$\mathrm{Hom}_R(WY, X) = \mathrm{Hom}_k(Y, \mathcal{F}X).$$

Now Greenberg's functor is the functor $X \mapsto \mathcal{F}X$ from the category of R-schemes to the one of k-schemes. Note that we get a morphism $\lambda_X : W\mathcal{F}X \to X$ corresponding to the identity of $\mathcal{F}X$ by the adjunction formula; let $\gamma_X : \Gamma(X, \mathcal{O}_X) \to W_l(\Gamma(\mathcal{F}X, \mathcal{O}_{\mathcal{F}X}))$ be the morphism induced by λ_X on global sections.

Since we have $W\mathrm{Spec}\ k_r = \mathrm{Spec}\ R_r$ for any $r \geq 1$, the adjunction formula gives a bijection, we shall call *Greenberg's bijection* in the sequel

$$X(R_r) = \mathcal{F}X(k_r),$$

between the set of R_r-points of X and the set of k_r-points of $\mathcal{F}X$. Moreover, if $\Pi \in X(R_r)$, and $P \in \mathcal{F}X(k_r)$ is its image by the above bijection, the following diagram commutes

$$\begin{array}{ccc} \Gamma(X, \mathcal{O}_X) & \xrightarrow{\gamma_X} & W_l(\Gamma(\mathcal{F}X, \mathcal{O}_{\mathcal{F}X})) \\ {\scriptstyle \Gamma(\Pi)}\downarrow & & \downarrow{\scriptstyle W_l(\Gamma(P))} \\ R & \xrightarrow{w} & W_l(k) \end{array}$$

that is, for every $f \in \Gamma(X, \mathcal{O}_X)$ with image $\gamma_X(f) := (f_0, \ldots, f_{l-1}) \in W_l(\Gamma(\mathcal{F}X, \mathcal{O}_{\mathcal{F}X}))$, and for every R_r-point Π of X, corresponding to the k_r-point P of $\mathcal{F}X$, we have $f(\Pi) = w^{-1}(\gamma_X(f)(P)) = w^{-1}(f_0(P), \ldots, f_{l-1}(P))$. Thus from Greenberg's bijection, we can rewrite the sum S_r in the following way

$$S_r(X, f) = S_r(\mathcal{F}X, \gamma_X f) := \sum_{P \in \mathcal{F}X(k_r)} \Psi^{(r)}(w^{-1}\gamma_X f(P)),$$

thus the function $L(X, f; t)$ can be rewritten

$$L(X, f; T) = \exp\left(\sum_{r \geq 1} S_r(\mathcal{F}X, \gamma_X f)\frac{T^r}{r}\right) = L(\mathcal{F}X, \gamma_X f; T) \tag{1}$$

and it remains to show the rationality of this last function.

We denote by $\mathbb{W}_l$ the affine group variety of Witt vectors of length l, defined over k (note that we have the isomorphism of algebraic varieties $\mathbb{W}_l = \mathbb{A}^l$), and by F the Frobenius morphism on $\mathbb{W}_l$ relative to k, defined by $(X_0, \ldots, X_{l-1}) \mapsto (X_0^q, \ldots, X_{l-1}^q)$. Let L be Lang's isogeny on $\mathbb{W}_l$, i.e $L = F - Id_{\mathbb{W}_l}$; it is surjective, and we get the following sequence of group varieties over k

$$0 \longrightarrow \mathbb{W}_l(k) \longrightarrow \mathbb{W}_l \xrightarrow{L} \mathbb{W}_l \longrightarrow 0.$$

We denote by $\mathcal{L}$ the $\mathbb{W}_l(k) = W_l(k)$-torsor over $\mathbb{W}_l$ defined by this exact sequence (cf. [5], 1.7).

Now let ℓ be a prime, different from p, and ψ_l be the character of order p^l from $W_l(\mathbb{F}_q)$ to $\overline{\mathbb{Q}}_\ell^\times$ corresponding to $\Psi \circ w^{-1}$ via a fixed embedding of $\overline{\mathbb{Q}}_\ell$ into $\mathbb{C}$. To the torsor $\mathcal{L}$, we associate a lisse rank 1 $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{L}_{\psi_l} := \psi_l(\mathcal{L})$ over $\mathbb{W}_l$.

Let $g : Y \to \mathbb{W}_l$ be a morphism of varieties over k. Consider the sheaf $g^*\mathcal{L}_{\psi_l}$ on Y. If y is a closed point of Y, say $y \in Y(k_r)$, let $\overline{y}$ be a geometric point over y. The fiber $g^*\mathcal{L}_{\psi_l,\overline{y}}$ is a $\overline{\mathbb{Q}}_\ell$-vector space of dimension 1, on which $\mathrm{Gal}(k(\overline{y})/k(y))$ acts. If $F_{\overline{y}}$ is the geometric Frobenius relative to $k(y)$, it acts as multiplication by $\psi_l(\mathrm{Tr}_{W_l(k_r)/W_l(k)}(g(y)))$; thus the trace of the action of $F_{\overline{y}}$ on $g^*\mathcal{L}_{\psi_l,\overline{y}}$ is this root of unity.

Let $H^i_c(Y, g^*\mathcal{L}_{\psi_l})$, $0 \le i \le 2\dim Y$, denote the groups (in fact $\overline{\mathbb{Q}}_\ell$-vector spaces of finite dimension) of cohomology with compact supports attached to Y and the ℓ-adic sheaf $g^*\mathcal{L}_{\psi_l}$. The Frobenius morphism F_Y of Y acts linearly on these spaces, and the Lefschetz trace formula [8] ensures us that for every $n \ge 1$, we have

$$\sum_{y \in Y(k_r)} \psi_l(\mathrm{Tr}_{W_l(k_r)/W_l(k)} g(y)) = \sum_{i=1}^{2\dim Y} (-1)^i \mathrm{Tr}(F_Y^r | H^i_c(Y, g^*\mathcal{L}_{\psi_l})).$$

In terms of L-functions, we get

$$L(Y, g; T) = \prod_{i=1}^{2\dim Y} \det(1 - TF_Y | H^i_c(Y, g^*\mathcal{L}_{\psi_l}))^{(-1)^{i+1}}. \tag{2}$$

Finally, applying this formula to $Y = \mathcal{F}X$ and $g = \gamma_X f$, we get from (1)

Theorem 1.1. *Let X be a scheme of finite type over R, and f a regular function on X; then the L-function $L(X, f; T)$ is a rational function.*

2. The geometry of curves over local rings.

Let C be an irreducible, projective and smooth of relative dimension 1 scheme over Spec (R); in the following we simply say "C is a smooth curve over R". The aim of this section is to study the sets $C(R_r)$ for $r \ge 1$ an integer. We first write them as an union of fibers of reduction modulo p in Lemma 2.1, each fiber corresponding to a point in $C(k_r)$, then we study the geometry of the fibers: we show that they can be "realized" as affine spaces using Greenberg's functor. Finally, we study local parameters for the R_r-points of C, and we show in Theorem 2.4 that they send bijectively a fiber on the set of elements in the maximal ideal of R_r.

Recall that $C_k := C \times_{\mathrm{Spec}\ R} \mathrm{Spec}\ k$ is a smooth irreducible projective curve over Spec k, and for any $r \ge 1$ there is a bijection between the sets $C(k_r)$ and $C_k(k_r)$; we shall identify these sets in the sequel. In the following we denote by C_r (*resp.* C_{k_r}) the curve $C \otimes R_r$ (*resp.* $C_k \otimes k_r$); once again we get a bijection between the sets $C_r(k_r)$ and $C_{k_r}(k_r)$.

The morphism $\varrho : \mathrm{Spec}\ k_r \to \mathrm{Spec}\ R_r$ induced by reduction modulo p associates to every R_r-point Π a unique point $P \in C_r(k_r)$ via $P = \Pi \circ \varrho$. Conversely, since the structural morphism $C_r \to \mathrm{Spec}\ (R_r)$ is smooth, the map

$$\mathrm{Hom}_{\mathrm{Spec}\ R_r}(\mathrm{Spec}\ R_r, C_r) \to \mathrm{Hom}_{\mathrm{Spec}\ R_r}(\mathrm{Spec}\ k_r, C_r)$$

induced by ϱ is surjective ([7] 17.1.1), and every point in $C_r(k_r)$ comes from a R_r-point in this way. Let P be a point in $C_r(k_r)$, and $\Pi \in C_r(R_r)$ such that $P = \Pi \circ \varrho$. If U is any affine open in C_r containing P, and $\Gamma(U, \mathcal{O}_{C_r})$ is the ring of regular functions over U, the morphism $\widetilde{P} = \widetilde{\Pi \circ \varrho} = \widetilde{\varrho} \circ \widetilde{\Pi}$ from $\Gamma(U, \mathcal{O}_{C_r})$ to k_r corresponding to P factors via $\mathcal{O}_{C_r,P}$. Thus the images via $\widetilde{\Pi}$ of the elements of

$\Gamma(U, \mathcal{O}_{C_r})$ which are invertible in $\mathcal{O}_{C_r,P}$ are in $R_r^\times = \widetilde{\rho}^{-1}(k_r^\times)$, and $\widetilde{\Pi}$ factors to a morphism from $\mathcal{O}_{C_r,P}$ to R_r. Finally, Π is a R_r-point of the R_r-scheme Spec $\mathcal{O}_{C_r,P}$. We deduce the following

Lemma 2.1. *For any $r \geq 1$, we have the disjoint union:*

$$C_r(R_r) = \coprod_{P \in C_r(k_r)} (\text{Spec } \mathcal{O}_{C_r,P})(R_r),$$

where (Spec $\mathcal{O}_{C_r,P}$)(R_r) *is the set of R_r-points of the R-scheme* Spec $\mathcal{O}_{C_r,P}$.

Let P be a point in $C_r(k_r)$; we now study precisely the set of R_r-points of the affine scheme Spec $\mathcal{O}_{C_r,P}$. We shall use Greenberg's functor to describe the k_r-scheme $\mathcal{F}$Spec $\mathcal{O}_{C_r,P}$. In order to simplify notations, we assume $r = 1$ in the rest of this section.

Let us describe the image of an affine R-scheme $U \subset \mathbb{A}_R^n$ under the functor $\mathcal{F}$, and explicit the morphism λ_U in this case. If $U = \mathbb{A}_R^n$ is affine n-space over R, we have $\mathcal{F}U = \mathbb{A}_k^{nl}$, the map $\lambda_U := \lambda$ is the morphism corresponding to the morphism of R-algebras

$$\begin{array}{rccc} \gamma & : & R[x_1, \ldots, x_n] & \to & W_l\left(k[x_{ij}]_{\substack{0 \leq j \leq l-1 \\ 1 \leq i \leq n}}\right) \\ & & \alpha \in R & \mapsto & w(\alpha) \\ & & x_i & \mapsto & (x_{i0}, \ldots, x_{i,l-1}). \end{array}$$

and Greenberg's bijection is just the map

$$\begin{array}{rccc} w \times \cdots \times w & : & \mathbb{A}_R^n(R) = R^n & \to & \mathbb{A}_k^{nl}(k) = k^{nl} \\ & & (\alpha_1 \ldots, \alpha_n) & \mapsto & (w(\alpha_1), \ldots, w(\alpha_n)). \end{array}$$

In the following, in order to simplify notations, and since no confusion can occur, we set: $R[x_i] := R[x_1, \ldots, x_n]$, and $k[x_{ij}] := k[x_{ij}]_{1 \leq i \leq n, 0 \leq j \leq l-1}$. Let us write the image $\gamma(f)$ of a polynomial $f \in R[x_i]$.

We first fix some notations. Let δ be the morphism

$$\begin{array}{rccc} \delta & : & R[x_i] & \to & W_l(k[x_{i0}]) \\ & & x_i & \mapsto & (x_{i0}, 0, \ldots, 0) \end{array}$$

$\delta_{|R} = w$, and $U_i = (0, x_{i1}, \ldots, x_{i,l-1})$. If $M = \{m_1, \ldots, m_t\}$ is a t-uple of positive integers, we set $|M| := m_1 + \cdots + m_t$, $\#M := t$, and $M! := m_1! \ldots m_t!$. If $J = \{j_1, \ldots, j_t\}$ is a t-uple of indexes in $\{1, \ldots, n\}$, let JM be the set of indexes j_i in J, each counted with the multiplicity m_i (note that $\#JM = |M|$) ; we set

$$\frac{\partial^{|M|} f}{\partial x_J^M} := \frac{\partial^{|M|} f}{\partial x_{j_1}^{m_1} \ldots \partial x_{j_t}^{m_t}} \; ; \; U_J^M := \prod_{j_i \in JM} U_{j_i} = U_{j_1}^{m_1} \ldots U_{j_t}^{m_t}.$$

Then we can write, for any $f \in R[x_i]$:

$$\gamma(f) = \delta(f) + \sum_{h=1}^{l-1} \sum_{M, |M|=h} \sum_{J, \#J=\#M} \delta\left(\frac{1}{M!} \frac{\partial^h f}{\partial x_J^M}\right) U_J^M.$$

Note that $\frac{1}{M!} \frac{\partial^h f}{\partial x_J^M}$ is again a polynomial in $R[x_i]$. Developing, we obtain another description for γ: if $\gamma(f) = (f_0, \ldots, f_{l-1})$, we have

$$f_0 = \bar{f}(x_{i0}), \qquad f_u = f^{(u)}(x_{ij}) + \sum_{i=1}^{n} \left(\frac{\partial \bar{f}}{\partial x_i}(x_{i0})\right)^{p^u} x_{iu}, \; 1 \leq u \leq l-1,$$

with $\bar{f}$ the reduction modulo p of f, and $f^{(u)}$ in $k[x_{ij}]_{1\leq i\leq n,\ 0\leq j\leq u-1}$; in other words, the polynomial f_u depends only on the variables x_{ij}, $1\leq i\leq n$, $0\leq j\leq u$, and it is linear in the variables x_{iu}, $1\leq i\leq n$.

Now let U be an affine R-scheme $U=\text{Spec } R[x_i]/I$, $I=(f_1,\dots,f_m)$. For every $1\leq t\leq m$, set $\gamma(f_t)=(f_{t0},\dots,f_{t,l-1})$; let I' be the ideal in $k[x_{ij}]$ generated by $\{f_{tu}\}_{1\leq t\leq m,0\leq u\leq l-1}$, and $\pi_{I'}$ the surjection $k[x_{ij}]\to k[x_{ij}]/I'$. From its construction, the morphism

$$W_l(\pi_{I'})\circ\gamma : R[x_i]\to W_l(k[x_{ij}])\to W_l(k[x_{ij}]/I')$$

vanishes on I; it induces a morphism $\gamma_U : R[x_i]/I\to W_l(k[x_{ij}]/I')$. Now the k-scheme Spec $k[x_{ij}]/I'$ is $\mathcal{F}U$, and λ_U is the morphism of affine R-schemes corresponding to γ_U.

We come back to the study of (Spec $\mathcal{O}_{C,P}$)(R). Let U be an affine open of C containing P, $U=\text{Spec } R[x_i]/I$, $I=(f_1,\dots,f_m)$. Let m_P be the maximal ideal of P in $R[x_i]/I$, and m_{P_k} the ideal of P in $(R[x_i]/I)\otimes k$. Then we have $\mathcal{O}_{C,P}=(R[x_i]/I)_{m_P}$, and since the elements in $(R[x_i]/I)\backslash m_P$ are precisely those whose images by γ_U have first coordinate invertible in $k[x_{i0}]_{m_{P_k}}$, we can extend γ_U to a morphism

$$\gamma_{C,P}:\mathcal{O}_{C,P}\to k[x_{i0}]_{m_{P_k}}[x_{ut}]/I'_{m_{P_k}}$$

with $I'_{m_{P_k}}$ the ideal in $k[x_{i0}]_{m_{P_k}}[x_{ut}]$ generated by the images of

$$\left(\bar{f}_j,\sum_{i=1}^{n}\left(\frac{\partial\bar{f}_j}{\partial x^{0i}}\right)^{p^k}x_{ik}+f_j^{(k)}\right)_{\substack{1\leq j\leq m\\ 1\leq k\leq l-1}}.$$

Finally, the image of Spec $\mathcal{O}_{C,P}$ by Greenberg's functor is

$$\begin{aligned}\mathcal{F}\text{Spec } \mathcal{O}_{C,P} &= \text{Spec } k[x_{i0}]_{m_{P_k}}[x_{ut}]/I'_{m_{P_k}}\\ &\simeq \text{Spec } \mathcal{O}_{C_k,P}[x_{ut}]/\left(\sum_{i=1}^{n}\left(\frac{\partial\bar{f}_j}{\partial x_{i0}}\right)^{p^k}x_{ik}+f_j^{(k)}\right)_{\substack{1\leq j\leq m\\ 1\leq k\leq l-1}}.\end{aligned}$$

Now since C is smooth of relative dimension 1 over Spec R, the Jacobian criterion says that there are $n-1$ elements in I, say $f_1,\dots,f_{n-1}$, whose images in $I_{m_P}/I_{m_P}^2$ generate this $R[x_1,\dots,x_n]_{m_P}$-module, and $n-1$ indices (say $2,\dots,n$) such that the function

$$\det\left(\left(\frac{\partial f_j}{\partial x_i}\right)_{\substack{1\leq j\leq n-1\\ 2\leq i\leq n}}\right)$$

is not in m_P. Since $R[x_1,\dots,x_n]_{m_P}$ is a local ring, Nakayama's lemma tells us that $f_1,\dots,f_{n-1}$ generate I_{m_P}. Thus the ideal of $\mathcal{F}$Spec $\mathcal{O}_{C,P}$ in $\mathcal{O}_{C_k,P}[x_{ut}]$ is generated by the

$$\left(\sum_{i=1}^{n}\left(\frac{\partial\bar{f}_j}{\partial x^{i0}}\right)^{p^k}x_{ik}+f_j^{(k)}\right)_{\substack{1\leq j\leq n-1\\ 1\leq k\leq l-1}}.$$

Now we apply the second part of the Jacobian criterion; assume for a while that $l=2$; then the ideal of $\mathcal{F}$Spec $\mathcal{O}_{C,P}$ in $\mathcal{O}_{C_k,P}[x_{tu}]$ is generated by the $n-1$ functions above with $k=1$. We can rewrite this system as

$$\left(\left(\frac{\partial\bar{f}_j}{\partial x_i}\right)^{p}\right)_{\substack{1\leq j\leq n-1\\ 2\leq i\leq n}}(x_{i1})_{2\leq i\leq n}=\left(-f_j^{(1)}-\left(\frac{\partial\bar{f}_j}{\partial x_1}\right)^{p}x_{11}\right)_{1\leq j\leq n-1}.$$

The matrix in the left hand side is invertible in $\mathcal{O}_{C_k,P}$, and we get in this case that $\mathcal{F}\text{Spec } \mathcal{O}_{C,P} = \text{Spec } \mathcal{O}_{C_k,P}[x_{11}]$. The general case can be treated by induction on l, considering the systems

$$\left(\left(\frac{\partial \bar{f}_j}{\partial x_i}\right)^{p^{l-1}}\right)_{\substack{1\leq j\leq n-1\\ 2\leq i\leq n}} (x_{i,l-1})_{2\leq i\leq n} = \left(-f_j^{(l-1)} - \left(\frac{\partial \bar{f}_j}{\partial x_1}\right)^{p^{l-1}} x_{1,l-1}\right)_{1\leq j\leq n-1}$$

since the description of γ, joint with the induction hypothesis, ensures that $f_j^{(l-1)} \in \mathcal{O}_{C_k,P}[x_{11},\dots,x_{1,l-2}]$. Finally, we get

Proposition 2.2. *If C is a smooth curve over R, and P a point of $C(k)$, we have:*

$$\mathcal{F}\text{Spec } \mathcal{O}_{C,P} = \text{Spec } \mathcal{O}_{C_k,P}[X_1,\dots,X_{l-1}].$$

Thus $\mathcal{F}\text{Spec } \mathcal{O}_{C,P}(k) = \mathbb{A}_k^{l-1}(k)$, and Greenberg's bijection gives

Corollary 2.3. *If C is a smooth curve over R, and P a point of $C(k)$, we have* $\#\text{Spec } \mathcal{O}_{C,P}(R) = p^{m(l-1)}$.

We now recall the definition of a local parameter for a R-point of C. The ideal sheaf associated to a R-point Π of C (considered as a subscheme of C) is locally principal thus there exists an element $t \in \mathcal{O}_{C,P}$ such that the kernel of the morphism $\Pi : \mathcal{O}_{C,P} \to R$ is the principal ideal $(t) \subset \mathcal{O}_{C,P}$. We call such an element a *local parameter* for C at Π. If m_P denotes the maximal ideal of $\mathcal{O}_{C,P}$, we have the following properties for a local parameter t

i) t is not a zero divisor in $\mathcal{O}_{C,P}$;
ii) $t \in m_P \backslash m_P^2$;
iii) $\mathcal{O}_{C,P}/(t^n)$ is a free R-module with basis $1, t, t^2, \dots, t^{n-1}$;
iv) the quotient ring of $\mathcal{O}_{C,P}$, K_C, is an $\mathcal{O}_{C,P}$-module generated by $1, t^{-1}, \dots$.

Note that to every R-point Π of C, corresponding to P in $C(k)$, one can associate a Cartier divisor (Π), defined as $((U,t),(V,1))$, where U is a suitable open in C containing Π, t is a local parameter at Π, and $V = C\backslash\{P\}$.

We are ready to show the main result of this section

Theorem 2.4. *Let C be a smooth curve over R, P a point of $C(k)$, and t a local parameter for an R-point of* Spec $\mathcal{O}_{C,P}(R)$. *Then the function $t \mapsto t(\Pi)$ from* Spec $\mathcal{O}_{C,P}(R)$ *to R, is a bijection onto pR.*

Proof. This is a local question, so we restrict our attention to an affine open subset containing P, say $U = \text{Spec } R[x_1,\dots,x_n]/I$, $I = (f_1,\dots,f_m)$. Applying the Jacobian criterion of smoothness to U at P, we get $n-1$ functions among the f_i, say $f_1,\dots,f_{n-1}$, and $n-1$ indexes in $\{1,\dots,n\}$. Let i be the remaining element of this set, and a_i be the i-th coordinate of P seen as a point in $\mathbb{A}^n(k)$. Let $\alpha_i \in R$ be any element reducing to a_i modulo p. If S is the closed subscheme of $\mathbb{A}_R^n$ defined by the ideal $(g_1,\dots,g_{n-1},x_i-\alpha_i)$, then P is in $S(k)$, and the Jacobian criterion says that S is smooth over Spec R at P, of relative dimension 0. Now $\mathcal{O}_{S,P}$ is a local ring of dimension 0, and a flat R-module with maximal ideal (p) and residue field k. Therefore we have $\mathcal{O}_{S,P} \simeq \mathcal{O}_{C,P}/(x_i-\alpha_i)_P \simeq R$, and $x_i - \alpha_i$ is the local parameter for an R-point above P. Moreover, when α varies in the class of α_i modulo p, we get q^{l-1} distinct morphisms from $\mathcal{O}_{C,P}$ to R, and these are the q^{l-1} different R-points of Spec $\mathcal{O}_{C,P}$.

Thus each of the functions $x_i - \alpha_i$, $\overline{\alpha}_i = a_i$, is a local parameter for an R-point of Spec $\mathcal{O}_{C,P}(R)$. Conversely any of these R-points has a local parameter in this family, say $x_i - \alpha_i$, and its image by $x_i - \alpha_i'$ is $\alpha_i - \alpha_i'$. We have shown the theorem for any one of these local parameters.

If t is a local parameter for a point Π in Spec $\mathcal{O}_{C,P}(R)$, let t_0 be the local parameter for Π in the family above. Then t and t_0 generate the same ideal in $\mathcal{O}_{C,P}$, and we have $t = ut_0$ for some $u \in \mathcal{O}_{C,P}^{\times}$. Reducing this equality modulo t_0^l, and since $\mathcal{O}_{C,P}/(t_0^l)$ is a free R-module with basis $1, \dots, t_0^{l-1}$, isomorphic to $R[t_0]/(t_0^l)$, we get that $t = c_1 t_0 + ...$, with $c_1 \in R^{\times}$. Now the function $x \mapsto c_1 x + \dots$ is a bijection over pR, and we get the result.

□

3. The L-function as a product of local factors.

This section is the heart of the paper: we express the L-function associated to the function f over C as a product of local factors, corresponding to the singularities of the reduction modulo p of the function f. Each of these factors is the L-function associated to an additive exponential sum and a polynomial over the affine line $\mathbb{A}^1$.

3.1. **Rewriting the sums $S_r(C, f)$.** As above, C is a smooth curve over R, and f is a function over C. For any point $P \in C(\overline{k})$, we say that f *is defined at* P when $f \in \mathcal{O}_{C,P}$. We denote by $\mathbb{P}_f$ the set of points P in $C(\overline{k})$ such that $f \notin \mathcal{O}_{C,P}$, i.e. the set of poles of f. For any $r \geq 1$, we can define the sums

$$S_r(C, f) := \sum_{\Pi \in C_f(R_r)} \Psi^{(r)}(f(\Pi)),$$

where $C_f(R_r)$ is the set of R_r-points of C at which f is regular. Note that from Lemma 2.1, we have $C_f(R_r) := \coprod_{P \in C(k_r) \setminus \mathbb{P}_f(k_r)} (\text{Spec } \mathcal{O}_{C,P})(R_r)$; thus we write the sum as

$$S_r(C, f) := \sum_{P \in C(k_r) \setminus \mathbb{P}_f(k_r)} S_P^{(r)}(C, f),$$

$$S_P^{(r)}(f) := \sum_{\Pi \in \text{Spec } \mathcal{O}_{C \otimes R_r, P}(R_r)} \Psi^{(r)}(f(\Pi)).$$

We begin by rewriting the sum $S_P^{(r)}(f)$ with the help of Theorem 2.4. First we need a definition.

Definition 3.1. Let C be a curve as above, P a point in $C(R_r)$, f a function over C defined at P; assume that the image of f in $\mathcal{O}_{C \otimes R_r, P}/(t^l)$ is $f_\Pi(t) = \alpha_0 + \alpha_1 t + \dots + \alpha_{l-1} t^{l-1}$, t being a local parameter at $\Pi_0 \in \text{Spec } \mathcal{O}_{C \otimes R_r, P}(R_r)$. We define the *local polynomial associated to* f *at* Π_0 as the polynomial defined in $R_r[T]$ by

$$f_{\Pi_0}(T) = \alpha_1 T + p\alpha_2 T^2 \cdots + p^{l-2} \alpha_{l-1} T^{l-1}.$$

Now we have

Proposition 3.2. *Let P be a point in $C(k_r)$, f a function on C, defined at P. If Π_0 is a point in $C(R_r)$ above P, and f_{Π_0} is the local polynomial associated to f at Π_0, we have:*

$$S_P^{(r)}(f) = \Psi^{(r)}(f(\Pi_0)) \sum_{x \in R_r'} \Psi^{(r)}(f_{\Pi_0}(x)).$$

Proof. Let $f \in t^l \mathcal{O}_{C,P}$; it follows from Theorem 2.4 that we have $f(\Pi) = 0$ for all Π in Spec $\mathcal{O}_{C,P}(R)$. Thus the map from Spec $\mathcal{O}_{C,P}(R)$ to R sending Π to $f(\Pi)$ only depends on the image of f in $\mathcal{O}_{C,P}/(t^l)$. Now the corollary in an easy consequence of the preceding theorem applied to the curve $C \otimes R_d$, once we have remarked that in the definition above we have $f(\Pi_0) = \alpha_0$. □

As in the preceding section, let C_k be the reduction modulo p of C. We denote by K_C the sheaf of total quotient rings of $\mathcal{O}_C$; this is a constant sheaf, whose associated ring we also denote by K_C. If $U := \text{Spec } B$ is any affine open subset of C, then $K_C = i(U)^{-1}\Gamma(U, \mathcal{O}_C) = i(B)^{-1}B$, where $i(B)$ is the set of elements of B which are not zero divisors in any of the local rings B_m. These are just the elements in $B \backslash pB$, thus $K_C = B_{(p)}$. The morphism ρ_U induced by reduction modulo p from B to $B_0 := B/pB$ extends to a morphism from K_C to the fraction field of B_0 which is $K(C_k)$, the function field of C_k. It is easily seen that this last morphism does not depend on the choice of U, and we denote it by ρ. Note that if Π is a R_d-point of C, corresponding to P in $C(k_d)$, and t is a local parameter at Π, then $t_0 := \rho(t)$ is a local parameter for C_k at P.

Definition 3.3. Let $g \in K(C_k)$ be a function on C_k; we say that g has a *singularity at* $P \in C_k(\overline{k})$ if the divisor of dg has a zero at P. The *multiplicity of the singularity of* g *at* P is the multiplicity of the divisor $(dg)_0$ at P.

In the following, we denote by Σ_g the support of $(dg)_0$, i.e. the set of singularities of g.

Now we show some kind of a stationnary phase formula: if P is a point in $C(k_r)$, and f is a function over C defined at P, the sum associated to f on the fiber (Spec $\mathcal{O}_{C_r,P}$)(R_r) is zero unless the reduction modulo p of f has a singularity at P.

Proposition 3.4. *Let P be a point of $C(k_r)$, and $f \in O_{C,P}$. For any integer $n \geq 1$, let $S_P^{(nr)}(f)$ be as above, i.e. the sum associated to f over the set of R_{nr}-points of the fiber over P. If $\rho(f)$ has no singularity at P, we have*

$$S_P^{(nr)}(f) = 0.$$

Proof. Let $f = \alpha_0 + \alpha_1 t + \cdots + \alpha_{l-1} t^{l-1} + t^l g$, $g \in \mathcal{O}_{C_r,P}$ be the development of f in terms of a local parameter $t \in \mathcal{O}_{C_r,P}$ at a R_r-point Π above P. Then the reduction modulo p of t, $t_0 := \rho(t)$, is a local parameter at P for C_{k_r}, and we have

$$\rho(f) = \overline{\alpha}_0 + \overline{\alpha}_1 t_0 + \cdots + \overline{\alpha}_{l-1} t_0^{l-1} + t_0^l \rho(g),$$

and $d\rho(f) = \overline{\alpha}_1 + 2\overline{\alpha}_2 t_0 + \cdots + (l-1)\overline{\alpha}_{l-1} t_0^{l-2} + t_0^{l-1} h$. Thus $\rho(f)$ has no singularity at P iff $\overline{\alpha}_1 \neq 0$, and in this case Lemma 2.10 in [2] gives the desired result. □

This result, joint with lemma 2.1, allows us to rewrite the sum over $C(R_r)$ associated to f in the following way

$$S_r(C, f) = \sum_{\Pi \in C_f(R_r)} \Psi^{(r)}(f(\Pi)) = \sum_{P \in \Sigma_{\rho(f)}(k_r) \backslash \mathbb{P}_f(k_r)} S_P^{(r)}(f), \tag{3}$$

From the equation above and Proposition 3.2, we can rewrite the sum $S_r(C, f)$ as a sum of sums over the affine line, attached to the local polynomials at the singularities of f.

Proposition 3.5. *Let r be a positive integer. Assume that*

$$\Sigma_{\rho(f)}(k_r)\backslash\mathbb{P}_f(k_r) = \{P_1, \dots, P_t\},$$

and choose for any $1 \leq i \leq t$ a R_r-point Π_i of $C \otimes R_r$ above P_i. Let f_{Π_i} denote the local polynomial associated to f at Π_i; then we have

$$S_r(C, f) = \sum_{i=1}^{t} \Psi^{(r)}(f(\Pi_i)) S'_r(\mathbb{A}^1, f_{\Pi_i}).$$

3.2. **Rewriting the L-function $L(C, f; T)$.** In terms of L-functions, we get from equation (3):

$$L(C, f; T) = \prod_{P \in \Sigma_{\rho(f)}\backslash\mathbb{P}_f} L_P(f; T^{d_P}), \quad L_P(f; T^{d_P}) := \exp\left(\sum_{r\geq 1} S_P^{(rd_P)}(f) \frac{T^{rd_P}}{rd_P}\right),$$

where for each P in $\Sigma_{\rho(f)}\backslash\mathbb{P}_f$, d_P is the degree of the extension $[k(P) : k]$, with $k(P)$ the residue field of P.

We now consider more closely these local terms; since f is defined over R, the set $\Sigma_{\rho(f)}\backslash\mathbb{P}_f(\overline{k})$ is stable under the action of $\mathrm{Gal}(\overline{k}/k)$, and rather than considering the points of this set, we consider the orbits of the action of the Galois group. Thus it is natural to use places of the function field $K(C_k)$, rather than points in $C_k(\overline{k})$. We will group together terms in equation (3) coming from the same place since they are very similar, in order to write the L-function as a product of local factors, each coming from a place in the support of $\Sigma_{\rho(f)}\backslash\mathbb{P}_f$.

Assume that $P \in \Sigma_{\rho(f)}(k_d)$ is a point above a place $\mathcal{P}$ of degree d of $K(C_k)$. It is clear that the sum $S_P^{(r)}(f)$ is zero unless d divides r. In this case, $\mathcal{P}$ corresponds to d points $P = P_1, \dots, P_d$ of $C(k_r)$, which are conjugate under the action of $\mathrm{Gal}(k_d/k)$. Moreover, since C is defined over R, the fibers are conjugate under the action of $\mathrm{Gal}(R_d/R)$. We define $S_{\mathcal{P}}^{(r)}(f) := \sum_{i=1}^{d} S_{P_i}^{(r)}(f)$, and we get

$$\begin{aligned}
S_{\mathcal{P}}^{(dr)}(f) &= \textstyle\sum_{\Pi \in (\mathrm{Spec}\ \mathcal{O}_{C\otimes R_{dr}, P})(R_{dr})} \Psi^{(dr)}(f(\Pi)) \\
&= \textstyle\sum_{\sigma \in \mathrm{Gal}(R_d/R)} \sum_{\Pi \in (\mathrm{Spec}\ \mathcal{O}_{C,P_1})(R_{dr})} \Psi^{(dr)}(f(\Pi^\sigma)) \\
&= \textstyle\sum_{\sigma \in \mathrm{Gal}(R_d/R)} \sum_{\Pi \in (\mathrm{Spec}\ \mathcal{O}_{C,P_1})(R_{dr})} \Psi^{(dr)}(f(\Pi)^\sigma) \\
&= d \textstyle\sum_{\Pi \in (\mathrm{Spec}\ \mathcal{O}_{C,P_1})(R_{dr})} \Psi^{(dr)}(f(\Pi)) \\
&= d S_{P_1}^{(dr)}
\end{aligned}$$

Thus if Π_1 is an R_r-point above P_1, we get

$$S_{\mathcal{P}}^{(dr)}(f) = d\Psi^{(dr)}(f(\Pi_1)) S'_{dr}(\mathbb{A}^1, f_{\Pi_1}),$$

and the local factor at $\mathcal{P}$ can be rewritten

$$\begin{aligned}
L_{\mathcal{P}}(f, T) &= \exp\left(\textstyle\sum_{r\geq 1} S_{\mathcal{P}}^{(di)}(f) \frac{T^{dr}}{dr}\right) \\
&= \exp\left(\textstyle\sum_{r\geq 1} \Psi^{(dr)}(f(\Pi_1)) S'_{dr}(\mathbb{A}^1, f_{\Pi_1}) \frac{(T^d)^r}{r}\right) \\
&= L'(\mathbb{A}^1, f_{\Pi_1}; \Psi^{(d)}(f(\Pi_1))T^d),
\end{aligned}$$

the last equality coming from the fact that since $f(\Pi_1)$ is in R_d, we have $\Psi^{(dr)}(f(\Pi_1)) = \Psi^{(d)}(f(\Pi_1))^r$ for any $r \geq 1$.

We end this section putting together the results above: we are able to write the L-function $L(C, f; T)$ as a product of local factors. Each of these factors corresponds to an orbit of the action of $\mathrm{Gal}(\overline{k}/k)$ on $\Sigma_{\rho(f)}\backslash\mathbb{P}_f(\overline{k})$ (i.e. to a place in the support

of $\Sigma_{\rho(f)}\backslash\mathbb{P}_f$), and can be written as the L-function associated to a polynomial over the affine line.

Theorem 3.6. *Let f be a function on C, and $\mathcal{P}_1,\dots,\mathcal{P}_t$ a set of representatives for the orbits in $\Sigma_{\rho(f)}\backslash\mathbb{P}_f(\overline{k})$ under the action of $\mathrm{Gal}(\overline{k}/k)$. If $\deg(\mathcal{P}_i)=d_i$, let Π_i be an R_{d_i}-point above one of the points in $C(k_i)$ dividing $\mathcal{P}_i$; then we have the following decomposition in local factors for the L-function associated to f*

$$L(C,f;T)=\prod_{i=1}^{t} L'(\mathbb{A}^1_{R'_{d_i}},f_{\Pi_i};\Psi^{(d_i)}(f(\Pi_i))T^{d_i})$$

where for each $1\leq i\leq t$, f_{Π_i} is the local polynomial associated to f at Π_i.

4. The case of Morse functions.

In this section we assume $p\geq 3$ is an odd prime number, and we study the degree of the L-functions $L(C,f;T)$ and the weights as q-Weil numbers of their reciprocal roots and poles when the function $\rho(f)$ has only simple singularities, i.e. when all the zeroes of $d\rho(f)$ occur with multiplicity one. We shall find results generalizing closely those of Weil in the case of finite fields.

We begin by recalling results from [2]. Assume that $f(T)=\alpha_0+\alpha_2T^2+\cdots+\alpha_nT^n\in R'[T]$ is a polynomial with leading monomial of degree 2, i.e. that we have $v_p(\alpha_d)>v_p(\alpha_2)$ for any $d>2$, where v_p is the usual p-adic valuation. We denote by $e:=v_p(2\alpha_2)=v_p(\alpha_2)$ the *arithmetic multiplicity* of f and let $n:=\lfloor\frac{l-e-3}{2}\rfloor$, where as usual $\lfloor r\rfloor$ denotes the greatest integer less than r. Then we have the following for the L-function $L'(\mathbb{A}^1_{R'},f;T)$ (*cf.* [2] Corollary 4.11):

- if $l-e-1$ is even, then we have
$$L'(\mathbb{A}^1_{R'},f;T)=\frac{1}{1-q^{\frac{l+e-1}{2}}\Psi'(f(0))T};$$
- if $l-e-1$ is odd, we have the following expression for the L-function
$$L'(\mathbb{A}^1_{R'},f;T)=1-q^{\frac{l+e-2}{2}}\chi_2(p^{-e}\alpha_2)\mathcal{G}_q\Psi'(f(0))T,$$
where χ_2 is the multiplicative character of order 2 of $R^\times$ (note that $p^{-e}\alpha_2$ is in $R^\times$), and $\mathcal{G}_q:=\sum_{x\in\mathbb{F}_q}\Psi_1(x^2)$ is the quadratic Gauss sum over $\mathbb{F}_q$.

Note that in any case, the L-function is a rational function of degree $(-1)^{l-1}$, and that its reciprocal root or pole is a Weil number of weight $l+e-1$.

We come back to the case of curves; let C be a curve of genus g, and $f\in K(C)$ be a function on C.

Definition 4.1. We say that f is a *Morse function* on C when the differential $d\rho(f)$ has only simple zeroes (i.e. when all the points in $(d\rho(f))_0$ occur with multiplicity one), and when f is defined at all the points in the support of $(d\rho(f))_0$.

Lemma 4.2. *Assume f is a Morse function, and let $(d\rho(f))_0:=\sum_{i=1}^t\mathcal{P}_i$, $\mathcal{P}_i$ being a place of degree d_i on C_k. Let P_i be a place in $C_{k_{d_i}}$ above $\mathcal{P}_i$. Then there exists Π_i, a R_{d_i}-point above P_i, such that the local polynomial associated to f at Π_i has leading monomial of degree 2 and arithmetic multiplicity one.*

Proof. Since $d\rho(f)$ has a simple zero at P_i, and $p \neq 2$, we can write $\rho(f) = a_0 + a_2 t_{0i}^2 + \cdots + a_{l-1} t_{0i}^{l-1}$ in $\mathcal{O}_{C_{k_{d_i}},P_i}/(t_{0i}^l)$, where t_{0i} is a local parameter for $C_{k_{d_i}}$ at P_i. Fix Π, a R_{d_i}-point above P_i, and t a local parameter for C_{d_i} at Π as in of Theorem 2.4; we can write $f = \alpha_0 + \alpha_1 t + \cdots + \alpha_{l-1} t^{l-1}$ in $\mathcal{O}_{C_{d_i},P_i}/(t^l)$, for some α_i in R_{d_i} reducing to a_i modulo p (and actually $\alpha_1 = p\beta_1$). Let $P(T) := \alpha_0 + p\beta_1 T + \alpha_2 T^2 + \cdots + \alpha_{l-1} T^{l-1}$ in $R[T]$; since the reduction modulo p of $P'(T)$ has a simple zero at 0, we can apply Hensel's lemma and there exists a θ_i in R_{d_i} such that $P'(p\theta_i) = 0$. Thus $P(T + p\theta_i) = P(p\theta_i) + \alpha'_2 T^2 + \alpha'_3 T^3 + \cdots + \alpha'_{l-1} T^{l-1}$, with $\overline{\alpha}'_i = a_i$.

Now from the proof of Theorem 2.4, we see that there exists an R_{d_i}-point above P_i, Π_i, such that $t_i := t + p\theta_i$ is a local parameter at Π_i; moreover the image of f in $\mathcal{O}_{C_{d_i},P_i}/(t_i^l)$ is $f(\Pi_i) + \alpha'_2 t_i^2 + \alpha'_3 t_i^3 + \cdots + \alpha'_{l-1} t_i^{l-1}$, thus the local polynomial associated to f at Π_i is $p\alpha'_2 T^2 + p^2 \alpha'_3 T^3 \cdots + p^{l-2} \alpha'_{l-1} T^{l-1}$, which clearly has leading monomial of degree 2 and arithmetic multiplicity one since $\overline{\alpha}'_2 = a_2 \in \mathbb{F}_{q^{d_i}}^{\times}$. □

From Lemma 4.2 and the discussion preceding it, we get the following (notations are as in the Lemma and its proof)

- if l is even, then we have

$$L'(\mathbb{A}^1_{R'_{d_i}}, f_{\Pi_i}; \Psi^{(d_i)}(f(\Pi_i))T^{d_i}) = \frac{1}{1 - q^{\frac{ld_i}{2}} \Psi^{(d_i)}(f(\Pi_i))T^{d_i}};$$

- if l is odd, then

$$L'(\mathbb{A}^1_{R'_{d_i}}, f_{\Pi_i}; \Psi^{(d_i)}(f(\Pi_i))T^{d_i}) = 1 - q^{d_i \frac{l-1}{2}} \chi_2(\alpha'_2) \mathcal{G}_{q^{d_i}} \Psi^{(d_i)}(f(\Pi_i))T^{d_i},$$

thus we see that the (local) function $L_P^{(-1)^{l-1}}$ has degree d_i, and its reciprocal roots are Weil numbers of weight l.

From Theorem 3.6, the function $L(C, f; T)^{(-1)^{l-1}}$ is a polynomial of degree $\sum_{i=1}^t d_i$; since we have $\sum_{i=1}^t d_i = \deg(d\rho(f))_0$, we get the following

Theorem 4.3. *Let f be a Morse function over C; then the function $L(C, f; t)^{(-1)^{l-1}}$ is a polynomial of degree $\deg(d\rho(f))_0$, and all of its reciprocal roots are Weil numbers of weight l.*

Now we evaluate the number $\deg(d\rho(f))_0$ in terms of the genus of C_k and the pole orders of the function $\rho(f)$. Let $(\rho(f))_\infty := \sum_{i=1}^k n_i P_i^\infty \in \mathrm{Div}(C_k)(k)$ be this divisor of poles. Assume that $n_1 \cdots n_k$ is prime to p; then the divisor of poles of the differential $d\rho(f)$ is $(d\rho(f))_\infty = \sum_{i=1}^k (n_i + 1) P_i^\infty$, and its degree is $\sum_{i=1}^k (n_i + 1) \deg(P_i^\infty)$. The degree of the divisor $(d\rho(f))$ is $2g - 2$, where g denotes the genus of C_k; thus the degree of $\deg(d\rho(f))_0$ is $2g - 2 + \sum_{i=1}^k (n_i + 1) \deg(P_i^\infty)$, and we get the following generalisation of Proposition 3.4 in [9]

Corollary 4.4. *Assume that f is Morse, and that the pole orders of $\rho(f)$ are prime to the characteristic. Then if the polar divisor of $\rho(f)$ is $\sum_{i=1}^k n_i P_i^\infty$, the function $L(C, f; t)^{(-1)^{l-1}}$ is a polynomial with degree*

$$2g - 2 + \sum_{i=1}^k (n_i + 1) \deg(P_i^\infty),$$

and all of its reciprocal roots are Weil numbers of weight l.

Finally, since in any case we have the inequality

$$\deg\left((d\rho(f))_0\right) \leq 2g-2+\sum_{i=1}^{k}(n_i+1)\deg(P_i^\infty),$$

we deduce the following bound for the sums $S_r(C,f)$ in the case of simple singularities, generalizing Corollary 3.5 in [9]

Corollary 4.5. *Let f be a Morse function over C with*

$$(\rho(f))_\infty := \sum_{i=1}^{k} n_i P_i^\infty \in \mathrm{Div}C_k(k);$$

then we have the bound

$$|S_r(C,f)| \leq \left(2g-2+\sum_{i=1}^{k}(n_i+1)\deg(P_i^\infty)\right) q^{\frac{lr}{2}}.$$

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