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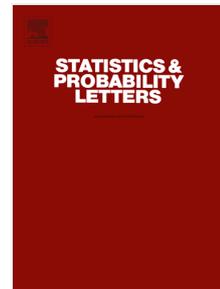
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Estimability of Parameters in a Linear Model and Related Characterizations

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Abstract

An alternative criterion is presented for a linear function of the unknown parameters, in a linear model, to be estimable. If there is a linear relationship between members of a subset of the columns of the design matrix \mathbf{X} , then the parameters associated with all the columns in that subset are not estimable. Also any part of their linear relationship is non estimable.

Key words and phrases. Linear space, projection matrix, generalized and ordinary least squares.

ORDINARY LEAST SQUARES

It is known that in a linear model with independent observations and constant variance, a linear function of the unknown parameters is estimable if the vector

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of coefficients of the parameters is a linear combination of the rows of the design matrix. The model is,

$$\mathbf{Y} = \mathbf{X}\mathbf{b} + \mathbf{e}, \quad \mathbf{Y} : nx1, \quad \mathbf{X} : nxp, \quad \mathbf{e} : nx1, \quad E(\mathbf{e}) = \mathbf{0}, \quad \text{var}(\mathbf{e}) = \sigma^2\mathbf{I}_n \quad (1)$$

$\mathbf{X} = (\mathbf{X}_1, \mathbf{x})$ is the design matrix, \mathbf{Y} is the vector of observations, \mathbf{e} is the error random variable, \mathbf{b} is the vector of unknown parameters. To each column of \mathbf{X} corresponds an unknown parameter. With $R(\mathbf{X})$ we denote the linear space of the columns of \mathbf{X} .

A linear function $\mathbf{q}^T\mathbf{b} = q_1b_1 + q_2b_2 + \dots + q_pb_p$ of the parameters is estimable if the vector of coefficients \mathbf{q}^T is a linear combination of the rows of \mathbf{X} , i.e., $\mathbf{q}^T = \mathbf{t}^T\mathbf{X}$, Searle (1997), here \mathbf{q}^T denotes the transpose of \mathbf{q} . Thus $b_p = (0, 0, \dots, 1)\mathbf{b}$ is estimable if $\mathbf{t}^T\mathbf{X}_1 = \mathbf{0}$ and $\mathbf{t}^T\mathbf{x} = 1$.

Here we give a different approach that leads to alternative equivalent criteria for a single parameter.

Theorem 1. If $\mathbf{X} = (\mathbf{X}_1, \mathbf{x})$ and (i) \mathbf{x} is a linear combination of the columns of \mathbf{X}_1 i.e. $\mathbf{x} \in R(\mathbf{X}_1)$, (ii) b_x is the parameter corresponding to column \mathbf{x} , then

(a) b_x is non estimable.

(b) The method of Ordinary Least Squares (OLS) gives $0\hat{b}_x = 0$.

Proof: (a) Let $\mathbf{c}^T\mathbf{Y} = c_1Y_1 + c_2Y_2 + \dots + c_nY_n$ be a linear function of the observations which is an unbiased estimator of b_x . If $\mathbf{b} = \begin{pmatrix} b_1 \\ b_x \end{pmatrix}$, then $E(\mathbf{c}^T\mathbf{Y}) = \mathbf{c}^T\mathbf{X}\mathbf{b} = \mathbf{c}^T(\mathbf{X}_1\mathbf{b}_1 + \mathbf{x}b_x) = b_x \forall \mathbf{b}_1, b_x$, which means that $\mathbf{c}^T\mathbf{X}_1 = \mathbf{0}$ $\mathbf{c}^T\mathbf{x} = 1$. Hence \mathbf{c} is orthogonal to the column space of \mathbf{X}_1 i.e. $\mathbf{c} \in R(\mathbf{I}_n - \mathbf{P}) \Leftrightarrow \mathbf{c} = (\mathbf{I}_n - \mathbf{P})\mathbf{z}$, for some $\mathbf{z} \in R^n$, where \mathbf{P} is the $n \times n$ matrix of orthogonal projection onto the column space of \mathbf{X}_1 .

So $\mathbf{c}^T\mathbf{x} = \mathbf{z}^T(\mathbf{I}_n - \mathbf{P})\mathbf{x} = \mathbf{z}^T\mathbf{0} = 0 \neq 1$ because $\mathbf{x} \in R(\mathbf{X}_1)$ and $\mathbf{P}\mathbf{x} = \mathbf{x}$, therefore b_x is non estimable.

(b) The method of Ordinary Least Squares (OLS) gives,

$$\mathbf{x}^T(\mathbf{I}_n - \mathbf{P})\mathbf{x}\hat{b}_x = \mathbf{x}^T(\mathbf{I}_n - \mathbf{P})\mathbf{Y}. \quad (2)$$

Hence (2) becomes $0\hat{b}_x = 0$, any linear function of the observations is a solution, none of these solutions is unbiased estimator of b_x , as it was proved before. ■

Theorem 2. If $\mathbf{X} = (\mathbf{X}_1, \mathbf{x})$ and \mathbf{x} is not a linear combination of the columns of \mathbf{X}_1 i.e., $\mathbf{x} \notin R(\mathbf{X}_1)$, then b_x is estimable and the method of OLS gives,

$$\mathbf{x}^T(\mathbf{I}_n - \mathbf{P})\mathbf{x}\hat{b}_x = \mathbf{x}^T(\mathbf{I}_n - \mathbf{P})\mathbf{Y} \quad \text{with} \quad \mathbf{x}^T(\mathbf{I}_n - \mathbf{P})\mathbf{x} > 0.$$

Proof: Since $\mathbf{x} \notin R(\mathbf{X}_1)$, then $\mathbf{P}\mathbf{x} \neq \mathbf{x}$ and

$$\mathbf{x}^T(\mathbf{I}_n - \mathbf{P})\mathbf{x} = ((\mathbf{I}_n - \mathbf{P})\mathbf{x})^T(\mathbf{I}_n - \mathbf{P})\mathbf{x} > 0.$$

Hence the OLS solution given in (2) has a unique solution. ■

Conclusion 1. If a column of the design matrix \mathbf{X} , in the linear model (1), is not a linear combination of other columns of \mathbf{X} , then the corresponding parameter is estimable, otherwise it is not estimable. If the rank of \mathbf{X} is p , i.e. $r(\mathbf{X}) = p$, then all the parameters and their linear combinations are estimable.

Conclusion 2. A necessary and sufficient condition for a parameter to be estimable is $\mathbf{P}\mathbf{x} \neq \mathbf{x}$, where $\mathbf{X} = (\mathbf{X}_1, \mathbf{x})$, \mathbf{x} is associated with that parameter and \mathbf{P} is the matrix of orthogonal projection onto the column space of \mathbf{X}_1 .

Conclusion 3. If some parameters in the linear model (1) are not estimable, then some of their linear combinations might be estimable.

Example: Let the linear model be $\mathbf{Y} = b_1\mathbf{x}_1 + b_2\mathbf{x}_2 + b_3\mathbf{x}_3 + \mathbf{e}$, where $\mathbf{x}_3 = 2\mathbf{x}_1 - 3\mathbf{x}_2$ and let $\mathbf{x}_1, \mathbf{x}_2$ be linearly independent. Then, Theorem 1, b_1, b_2, b_3 are not estimable but the linear combinations $b_1 + 2b_2, b_2 - 3b_3$ are estimable, because the model becomes $\mathbf{Y} = (b_1 + 2b_2)\mathbf{x}_1 + (b_2 - 3b_3)\mathbf{x}_2 + \mathbf{e}$ which is of full rank. So if there is a linear relationship between members of a subset of the columns of \mathbf{X} , then the b_i associated with all the columns in that subset are not estimable.

Conclusion 4. If in the model (1) a parameter of interest, say b_j , is not estimable, i.e., the corresponding column \mathbf{x}_j is linearly dependent on other columns of \mathbf{X} ,

$\mathbf{x}_j = \sum_{i \neq j} c_i \mathbf{x}_i$ with c_i known, then take the transformation $b_i^* = b_i + c_i b_j$, $i \neq j$, and the model becomes:

$$\mathbf{Y} = \sum_{\substack{i=1 \\ i \neq j}}^p b_i^* \mathbf{x}_i + \mathbf{e}. \quad (3)$$

If a parameter of interest is still not estimable, continue this process. This procedure was followed in the previous example.

Conclusion 5. If the covariance matrix of the errors in (1) is $\text{var}(\mathbf{e}) = \sigma^2 \mathbf{V}$, then follow the same process as above, because if $\mathbf{V} = \mathbf{R}\mathbf{R}^T$, the model (1) becomes:

$$\mathbf{Y}^* = \mathbf{X}^* \mathbf{b} + \mathbf{e}^*, \text{ where } \mathbf{Y}^* = \mathbf{R}^{-1} \mathbf{Y}, \mathbf{X}^* = \mathbf{R}^{-1} \mathbf{X}, \mathbf{e}^* = \mathbf{R}^{-1} \mathbf{e}, \text{ var}(\mathbf{e}^*) = \sigma^2 \mathbf{I}_n.$$

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