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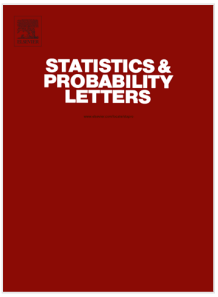

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Tail dependence of skewed grouped t-distributions

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Abstract

We derive the upper tail dependence coefficient (TDC) for a random vector following a grouped skewed t-distribution. We also analyze the impact of parameter changes on the TDC.

Key words: Multivariate distribution, tail dependence

1 Introduction

The *upper tail dependence coefficient* of a random vector (X_1, X_2) with marginal distribution functions F_1 and F_2 is defined as the limit (if it exists)

$$\lambda := \lim_{u \uparrow 1} \mathrm{P}\big(F_1(X_1) \geq u \big| F_2(X_2) \geq u\big).$$

We are interested in this quantity for (X_1, X_2) possessing a *skewed, grouped t-distribution*, which can be defined structurally by the equations

$$X_1 = \frac{\gamma}{G_\mu^{-1}(U)/\mu} + \frac{Z_1}{\sqrt{G_\mu^{-1}(U)/\mu}}, \qquad X_2 = \frac{\delta}{G_\nu^{-1}(U)/\nu} + \frac{Z_2}{\sqrt{G_\nu^{-1}(U)/\nu}}, \quad (1)$$

for numbers $\gamma, \delta, \mu > 0, \nu > 0$, a bivariate normal vector (Z_1, Z_2) with mean 0, variances 1 and correlation ρ, a uniform random variable U independent of (Z_1, Z_2), and for G_μ the cumulative distribution function of a chi-square distribution with μ degrees of freedom.

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Multivariate distributions of this type have been found useful in financial risk analysis, where increasing globalization of financial markets and relaxation of regulations have lead to increased dependency between financial assets. Quantification of such dependencies is important in risk management, in particular of dependencies between upper extremal values. As argued in e.g. **?** and **?**, the linear (Pearson) correlation coefficient, which is still commonly used together with a Gaussian copula, is inadequate in many practical situations. The upper tail dependence coefficient provides an alternative, and is particularly sensitive to dependencies among extremes. It is zero for Gaussian distributions, but traverses a full range of values for the family of multivariate t-distributions. Because this family is based on a small set of parameters, it can be easily fitted to (financial) data, where it often gives a reasonable fit. The Gaussian copula is contained in the t-family as the extreme case with degrees of freedom equal to infinity. Reviews of other dependency measures are given by **?**, **?** and **?**.

The ordinary multivariate t-distribution is defined as the distribution of the vector $\sqrt{W}\mathbf{Z}$ for $\mathbf{Z}$ a multivariate normal vector and $1/W$ an independent chisquare variable. For increased flexibility this family has been extended in two ways. The *grouped t-distribution*, put forward in **?**, allows for nonidentical marginal distributions by replacing the scalar product by a coordinate-wise product $\sqrt{\mathbf{W}}\mathbf{Z}$ of vectors $\sqrt{\mathbf{W}}$ and $\mathbf{Z}$, where the vector $1/\mathbf{W}$ has chisquare marginals of different degrees of freedom that are perfectly dependent (in the sense of a trivial copula). Second the *skewed t-copula*, introduced in **?**, allows for asymmetric marginal distributions by considering the distribution of the vector $\alpha W + \sqrt{W}\mathbf{Z}$. These definitions are valid in any dimension, but since the tail dependence coefficient is a bivariate measure, in our situation it suffices to consider bivariate vectors. Then definition (1) encompasses both the grouped copula and the skewed t-copula as special cases, to which it reduces by choosing $\delta = \gamma = 0$ or $\mu = \nu$, respectively. Further discussion of so-called *meta-t*-distributions can be found in **?**.

Tail dependence in the ordinary t-distribution was considered by **?**, among others. In this paper we extend these results to the full family of grouped, skewed t-distributions. Tail dependence in the symmetric grouped t-distributions depends in a complicated manner on the three parameters (μ, ν, ρ), as shown in Theorem 2.1 and illustrated in Figures A.3 and A.4. Tail dependence in the skewed distributions is strongly dependent on the signs of the skewness parameters. For positive γ and δ the upper tails of the variables X_1 and X_2 are determined by the skewing variables, and tail dependence is complete ($\lambda = 1$). For negative skewing parameters the symmetric parts of the variables are responsible for the upper tails (which we show to be exponential) and the tail dependence is 0. A nontrivial tail dependence parameter is obtained when one marginal is positively skewed and the other symmetric.

That skewing leads to trivial values of the tail dependence coefficient seems

disappointing from a modelling perspective. Skewing was introduced to improve the fit of the t-distribution, but turns out to change the dependence in the tails completely.

2 Main result

The main result of the paper is the following theorem characterizing the extreme properties of the grouped skewed-t distribution. Because the upper tail coefficient is symmetric in X_1 and X_2, to compute it in the case that $\mu \neq \nu$ we may assume without loss of generality that $\mu > \nu$.

Let Φ be the standard normal cumulative distribution function and let g_ν be the density of the chisquare distribution with ν degrees of freedom. Set

$$l_{\mu,\nu} = \left(\frac{2^{\mu/2}\Gamma\left((\mu+1)/2\right)}{2^{\nu/2}\Gamma\left((\nu+1)/2\right)} \right)^{1/\mu}, \quad k_{\mu,\nu} = \left(\frac{2^{\mu/2}\Gamma\left((\mu+1)/2\right)}{2\sqrt{\pi}} \right)^{1/\mu}. \tag{2}$$

Theorem 2.1 *The upper tail coefficient of the vector (X_1, X_2) defined in (1) is given by:*

(1) If $\mu > \nu$ and $\gamma = \delta = 0$, then

$$\lambda = 1 - \int_0^\infty \Phi\left(\frac{l_{\mu,\nu} t^{\nu/(2\mu)} - \rho\sqrt{t}}{\sqrt{1-\rho^2}} \right) g_{\nu+1}(t)\, dt.$$

(2) If $\mu = \nu$ and $\gamma = \delta = 0$, then it is the same expression, but multiplied by 2.
(3) If $\gamma > 0$ and $\delta > 0$, then $\lambda = 1$.
(4) If $\gamma < 0$ and $\delta < 0$, then $\lambda = 0$.
(5) If $\gamma < 0$ and $\delta > 0$, then $\lambda = 0$.
(6) If $\gamma = 0$ and $\delta < 0$, then $\lambda = 0$.
(7) If $\gamma = 0$ and $\delta > 0$, then

$$\lambda = \int_0^1 \left(1 - \Phi\left(k_{\mu,\nu} u^{1/\mu}\right)\right) du.$$

Figures A.3 and A.4 show the numerical value of the tail dependence coefficient in cases (1) and (2). Each plot gives the value of λ, computed by numerical integration, as a function of two of the three parameters μ, ν, ρ with the third parameter fixed at the value as indicated. Scatterplots illustrating the cases (3)-(5) are shown in Figures A.1 to A.2.

3 Proofs

We use the notation $f(x) \asymp g(x)$ for $x \to a$ if $f(x)/g(x) \to 1$ as $x \to a$, and $f(x) \sim g(x)$ for $x \to a$ if $f(x)/g(x)$ tends to a finite positive limit as $x \to a$.

Recall that F_1 and F_2 are the marginal distribution functions of X_1 and X_2, and G_ν is the distribution function of the chisquare distribution with ν degrees of freedom. Let f_1, f_2 and g_ν be the corresponding density functions. It will be convenient to use the abbreviations

$$f(x) = F_1^{-1} \circ F_2(x), \qquad g(x) = \sqrt{G_\mu^{-1} \circ G_\nu(x^2)}, \qquad S^2 = G_\nu^{-1}(U).$$

The representation (1) can be written as

$$X_1 = \frac{\gamma}{g(S)^2/\mu} + \frac{Z_1}{g(S)/\sqrt{\mu}}, \qquad X_2 = \frac{\delta}{S^2/\nu} + \frac{Z_2}{S/\sqrt{\nu}}. \tag{3}$$

The upper tail dependence coefficient can be reexpressed as

$$\begin{aligned}
\lambda :&= \lim_{x\to\infty} \mathrm{P}\Big(X_1 \geq F_1^{-1} \circ F_2(x) \big|\, X_2 \geq x\Big) \\
&= \lim_{x\to\infty} \frac{\mathrm{P}\Big(X_1 \geq f(x), X_2 \geq x\Big)}{\mathrm{P}(X_2 \geq x)} \\
&= \lim_{x\to\infty} \frac{\int_{f(x)}^\infty \int_x^\infty p^{X_1,X_2}(s,t)\, dt\, ds}{\mathrm{P}(X_2 \geq x)} \\
&= \lim_{x\to\infty} \frac{-\int_{f(x)}^\infty p^{X_1,X_2}(s,x)\, ds - \int_x^\infty p^{X_1,X_2}\Big(f(x),t\Big)\, dt\ f'(x)}{-p^{X_2}(x)} \\
&= \lim_{x\to\infty} \left[\mathrm{P}\Big(X_1 \geq f(x) \big|\, X_2 = x\Big) + \frac{f_1(x)}{f_2(x)} \mathrm{P}\Big(X_2 \geq x \big|\, X_1 = f(x)\Big) f'(x)\right].
\end{aligned} \tag{4}$$

Here we have used de l'Hôpital's rule to derive the third equality, and p^{X_1,X_2} is the density of the vector (X_1, X_2).

For easy reference we recall the behaviour of the tails of the chisquare and t-distributions. The marginal distribution of S^2 is $\chi^2_\nu = \Gamma(\frac{\nu}{2}, \frac{1}{2})$. It follows that, as $x \downarrow 0$,

$$G_\nu(x) = \int_0^x d_\nu e^{-t/2} t^{\nu/2-1}\, dt \asymp x^{\nu/2} d_\nu \frac{2}{\nu}, \qquad d_\nu = (\tfrac{1}{2})^{\nu/2} \frac{1}{\Gamma(\frac{\nu}{2})}. \tag{5}$$

Using Lemma 3.1 (below) we can invert this into the statement, for $u \downarrow 0$,

$$G_\nu^{-1}(u) \asymp \left(\frac{u}{2d_\nu/\nu}\right)^{2/\nu} = 2u^{2/\nu} \Gamma\left(\frac{\nu}{2} + 1\right)^{2/\nu}. \tag{6}$$

Taken together this yields, for $x \downarrow 0$,

$$g(x) \asymp d_{\mu,\nu}\, x^{\nu/\mu}, \qquad d_{\mu,\nu} = \sqrt{2}\frac{\Gamma(\mu/2+1)^{1/\mu}}{\Gamma(\nu/2+1)^{1/\mu}} \left(\tfrac{1}{2}\right)^{\nu/(2\mu)}. \tag{7}$$

The density f_ν of the t-distribution with ν degrees of freedom is given by

$$f_\nu(x) = c_\nu\Big(1+\frac{x^2}{\nu}\Big)^{-(\nu+1)/2}, \qquad c_\nu = \frac{\Gamma\Big((\nu+1)/2\Big)}{\Gamma(\nu/2)\sqrt{\pi\nu}}.$$

It follows that the corresponding distribution function F_ν satisfies, as $x \to \infty$,

$$1 - F_\nu(x) \asymp \int_x^\infty c_\nu \left(\frac{s^2}{\nu}\right)^{-(\nu+1)/2} ds = c_\nu \nu^{(\nu-1)/2} x^{-\nu}. \tag{8}$$

Using Lemma 3.1 we can invert this to see that, for $u \uparrow 1$,

$$F_\nu^{-1}(u) \asymp c_\nu^{1/\nu} \nu^{(\nu-1)/(2\nu)} (1-u)^{-1/\nu}.$$

Lemma 3.1 *Let F be a cumulative distribution function of a distribution on $(0,\infty)$ and F^{-1} its quantile function.*

(1) If $1 - F(x) \asymp cx^{-k}$ as $x \to \infty$, then $F^{-1}(1-u) \asymp (c/u)^{1/k}$ as $u \downarrow 0$.
(2) If $F(x) \asymp cx^k$ as $x \downarrow 0$ for some $k > 0$, then $F^{-1}(u) \asymp (u/c)^{1/k}$ as $u \downarrow 0$.
(3) If $1 - F(x) \asymp cx^k e^{-\gamma x}$ as $x \to \infty$ for some $\gamma > 0$, then $F^{-1}(1-u) \asymp -(\log u)/\gamma$ as $u \downarrow 0$.

Proof If the assumption of (1) holds, then, for any $r > 0$,

$$\lim_{u\downarrow 0} \frac{1 - F\Big(r(c/u)^{1/k}\Big)}{u} = \frac{1}{r^k}.$$

Monotonicity shows that if $r < 1$, then $r(c/u)^{1/k} \le F^{-1}(1-u)$ eventually, whereas if $r > 1$, then $r(c/u)^{1/k} \ge F^{-1}(1-u)$ eventually. This proves (1). Assertion (2) follows similarly. Under assumption (3),

$$\lim_{u\downarrow 0} \frac{1 - F\Big(-r(\log u)/\gamma\Big)}{u}$$

is ∞ for $r < 1$ and 0 for $r > 1$, and assertion (3) follows. ■

3.1 The symmetric case (1) and (2)

Assume that $\gamma = \delta = 0$. The result is different in the cases that $\mu = \nu$ or $\mu > \nu$. If $\mu = \nu$, then the marginal densities coincide, $f'(x) = 1$, and the two

terms in the sum in the last line of (4) are identical. We shall show that in the case that $\mu > \nu$ the second term in this sum gives zero contribution to the limit. It is then clear that the tail independence coefficient in the case that $\mu = \nu$ is twice the expression we obtain in the case $\mu > \nu$.

The marginal distributions F_1 and F_2 are t-distributions with μ and ν degrees of freedom, respectively. It follows that the strictly increasing function $f = F_1^{-1} \circ F_2$ satisfies, as $x \to \infty$,

$$f(x) \asymp c_{\mu,\nu} x^{\nu/\mu}, \qquad c_{\mu,\nu} = \Big(\frac{c_\mu}{c_\nu}\Big)^{1/\mu} \frac{\mu^{(\mu-1)/(2\mu)}}{\nu^{(\nu-1)/(2\mu)}}. \tag{9}$$

The derivative of f is equal to $f' = f_2/(f_1 \circ f)$, and hence, as $x \to \infty$,

$$\frac{f_1(x)}{f_2(x)} f'(x) = \frac{f_1(x)}{f_1 \circ f(x)} \sim x^{(\mu+1)(\nu/\mu-1)}.$$

We conclude that this factor tends to zero for $\mu > \nu$. Because the probability $\mathrm{P}\big(X_2 \geq x | \, X_1 = f(x)\big)$ is bounded, the second term on the right in (4) is negligible in this case.

The conditional density of S^2 given X_2 satisfies, by Bayes' rule,

$$\begin{aligned} p_{S^2|X_2=x_2}(t) &\propto p^{X_2|S^2=t}(x_2) p^{S^2}(t) \propto \sqrt{\frac{t}{\nu}} e^{-\frac{1}{2}x_2^2 t/\nu} \; e^{-t/2} t^{\nu/2-1} \\ &\propto e^{-\frac{1}{2}(x_2^2/\nu+1)t} \; t^{(\nu+1)/2-1} \quad . \end{aligned}$$

It follows that given X_2 the variable S^2 possesses a $\Gamma\big(\nu+1)/2, (X_2^2/\nu+1)/2\big)$-distribution, whence the variable $(X_2^2/\nu + 1)S^2$ is chisquare distributed with $\nu + 1$ degrees of freedom and is independent of X_2.

In view of the bivariate normal distribution of (Z_1, Z_2), the conditional distribution of Z_1 given $Z_2 = z_2$ is normal with mean ρz_2 and variance $1 - \rho^2$. We can write this structurally as the distribution of $\rho z_2 + \sqrt{1-\rho^2} W$ for a standard normal variable W, independent of Z_2 and S.

With g_ν the density of the chisquare distribution with ν degrees of freedom,

$$
\begin{aligned}
&\mathrm{P}(X_1 \geq x_1 |\, X_2 = x_2) \\
&= \int \mathrm{P}\big(X_1 \geq x_1 |\, X_2 = x_2, (X_2^2/\nu + 1)S^2 = t\big)\, p^{(X_2^2/\nu+1)S^2 | X_2 = x_2}(t)\, dt \\
&= \int \mathrm{P}\left(\frac{Z_1\sqrt{\mu}}{g(S)} \geq x_1 \mid Z_2 = \sqrt{\frac{t}{x_2^2/\nu + 1}}\frac{x_2}{\sqrt{\nu}}, \left(\frac{x_2^2}{\nu} + 1\right) S^2 = t\right) g_{\nu+1}(t)\, dt \\
&= \int \mathrm{P}\left(\frac{\rho\sqrt{t/(x_2^2+\nu)}\, x_2 + \sqrt{1-\rho^2}W}{g\Big(\sqrt{t\nu/(x_2^2+\nu)}\Big)}\sqrt{\mu} \geq x_1\right) g_{\nu+1}(t)\, dt.
\end{aligned}
$$

In the last two steps we have first rewritten the conditioning event in terms of Z_2 and S^2 and next used the conditional distribution of Z_1 found in the preceding paragraph. We can now reexpress the right side of (4) as

$$
\lim_{x\to\infty} \int \mathrm{P}\left(\sqrt{1-\rho^2}W \geq \frac{f(x)}{\sqrt{\mu}} g\left(\sqrt{\frac{t\nu}{x^2+\nu}}\right) - \frac{\rho\sqrt{t}x}{\sqrt{x^2+\nu}}\right) g_{\nu+1}(t)\, dt.
$$

By the dominated convergence theorem, we can exchange the order of limit and integral, and by continuity of the Gaussian distribution function, we see that it suffices to compute

$$
\lim_{x\to\infty} \left[\frac{f(x)}{\sqrt{\mu}} g\left(\sqrt{\frac{t\nu}{x^2+\nu}}\right) - \frac{\rho\sqrt{t}x}{\sqrt{x^2+\nu}}\right].
$$

The second term obviously tends to $\rho\sqrt{t}$. For the first term we use (7) and (9). The final result is that, for $\mu > \nu$,

$$
\lambda = 1 - \int_0^\infty \Phi\left(\frac{c_{\mu,\nu} d_{\mu,\nu} \nu^{\nu/2\mu} \mu^{-1/2} t^{\nu/(2\mu)} - \rho\sqrt{t}}{\sqrt{1-\rho^2}}\right) g_{\nu+1}(t)\, dt. \qquad (10)
$$

This can be reduced to the expression given in the theorem. For $\mu = \nu$ this should be multiplied by 2.

3.2 The positively skewed case (3)

Assume that $\gamma > 0$ and $\delta > 0$. We shall show that the upper tails of the variables X_1 and X_2 are in this case dominated by the "skewing" variables $\gamma/g(S)^2$ and δ/S^2. Because these possess a deterministic increasing relationship, the result is an upper tail coefficient equal to 1. For simplicity of notation we replace the constants μ and ν in the algebraic definitions (3) of X_1 and X_2 by 1 (meanwhile keeping these constants in the definitions of S and g the same).

To make this precise it suffices to show that

$$\lim_{x\to\infty} \frac{\mathrm{P}\Big(\delta/S^2 + Z_2/S \geq x\Big)}{\mathrm{P}\Big(\delta/S^2 \geq x\Big)} = 1, \tag{11}$$

and, for $f = F_1^{-1} \circ F_2$, as $x \to \infty$,

$$\lim_{x\to\infty} \frac{\mathrm{P}\Big(\gamma/g(S)^2 + Z_1/g(S) \geq f(x), \delta/S^2 + Z_2/S \geq x\Big)}{\mathrm{P}\Big(\gamma/g(S)^2 \geq f(x), \delta/S^2 \geq x\Big)} = 1, \tag{12}$$

and, for $\bar{f} = \bar{F}_1^{-1} \circ \bar{F}_2$, with $\bar{F}_1$ and $\bar{F}_2$ the marginal distribution functions of $\bar{X}_1 = \gamma/g(S)^2$ and $\bar{X}_2 = \delta/S^2$,

$$\lim_{x\to\infty} \frac{\mathrm{P}\Big(\gamma/g(S)^2 \geq f(x), \delta/S^2 \geq x\Big)}{\mathrm{P}\Big(\gamma/g(S)^2 \geq \bar{f}(x), \delta/S^2 \geq x\Big)} = 1. \tag{13}$$

The first two displays show that the central t-variables in the probabilities given in the denominator and numerator in the second line of (4) can be deleted without changing the limit. The third display shows that in the resulting quotient the function f, which is based on the true marginals F_1 and F_2, can be replaced by the corresponding function $\bar{f}$ defined from the marginals of $\bar{X}_1$ and $\bar{X}_2$. Because $\bar{X}_1 = \gamma/g(S)^2$ can be written as a monotone, continuous function of $\bar{X}_2 = \delta/S^2$, the corresponding copula, the distribution of $\big(\bar{F}_1(\bar{X}_1), \bar{F}_2(\bar{X}_2)\big)$, is equal to the distribution of (U, U) for a uniform variable U. Hence given (11), (12) and (13) it follows without further calculations that the upper tail dependence coefficient is 1.

For any function $r\colon \mathbb{R}^+ \to \mathbb{R}^+$,

$$\begin{aligned}\mathrm{P}\Big(\frac{\delta}{S^2} + \frac{Z_2}{S} \geq x\Big) &\leq \mathrm{P}\Big(\frac{\delta}{S^2} \geq x - r(x)\Big) + \mathrm{P}\Big(\frac{Z_2}{S} \geq r(x)\Big) \\ &\asymp d_\nu \frac{2}{\nu}\left(\frac{\delta}{x - r(x)}\right)^{\nu/2} + O\left(\frac{1}{r(x)^\nu}\right),\end{aligned}$$

in view of (5) and (8), if $x - r(x) \to \infty$ and $r(x) \to \infty$. We choose $r(x) = x^{3/4}$, so that $r(x)/x \to 0$ and $r^2(x)/x \to \infty$ as $x \to \infty$. The second term on the right is negligible relative to the first, and the first is equivalent to the upper tail of the variable δ/S^2. It follows that the limit in (11) is smaller than 1. To prove that it is bigger than 1 we argue similarly that, for any function $t\colon \mathbb{R}^+ \to \mathbb{R}^+$ and y such that $t(y) > y > 0$,

$$\mathrm{P}\Big(\frac{\delta}{S^2}+\frac{Z_2}{S}\geq y-t(y)\Big)\geq \mathrm{P}\Big(\frac{\delta}{S^2}\geq y, Z_2\geq -\sqrt{\frac{\delta}{y}}\,t(y)\Big)$$
$$\geq \mathrm{P}\Big(\frac{\delta}{S^2}\geq y\Big)\Big(1-e^{-\frac{1}{2}(\delta/y)t^2(y)}\Big),$$

by the independence of S and Z_2 and a bound on the left tail of the normal distribution. We can choose $t(y)=y^{3/4}$ and $y-t(y)=x$ (for $x>1$), so that $y\asymp x$ for $x\to\infty$ to conclude the proof of (11).

We have now also proved that $1-F_2(x)$ is equivalent to $d_\nu(2/\nu)(\delta/x)^{\nu/2}$ as $x\to\infty$, whence by symmetry $1-F_1(x)$ is equivalent to the same expression, but with (μ,γ) instead of (ν,δ). Then we find that $f=F_1^{-1}\circ F_2$ satisfies, as $x\to\infty$,

$$f(x)\asymp \frac{\frac{1}{2}\gamma}{(\frac{1}{2}\delta)^{\nu/\mu}}\Big(\frac{\Gamma(\nu/2+1)}{\Gamma(\mu/2+1)}\Big)^{2/\mu}x^{\nu/\mu}. \tag{14}$$

For $r\colon\mathbb{R}^+\to\mathbb{R}^+$ as before and $s=f^{3/4}\colon\mathbb{R}^+\to\mathbb{R}^+$, so that $s(x)/f(x)\to 0$ and $s(x)^2/f(x)\to\infty$, the numerator of (12) is bounded above by

$$\mathrm{P}\left(\frac{\gamma}{g(S)^2}\geq f(x)-s(x),\frac{\delta}{S^2}\geq x-r(x)\right)+\mathrm{P}\left(\frac{Z_1}{g(S)}\geq s(x)\right)$$
$$+\mathrm{P}\left(\frac{Z_2}{S}\geq r(x)\right)$$
$$\leq \mathrm{P}\left(S\leq g^{-1}\Big(\sqrt{\frac{\gamma}{f(x)-s(x)}}\Big)\wedge\sqrt{\frac{\delta}{x-r(x)}}\right)+O\Big(\frac{1}{s(x)^\mu}\Big)+O\Big(\frac{1}{r(x)^\nu}\Big).$$

The behaviour of the function g is given in (7), from which it follows that $g^{-1}(u)\sim u^{\mu/\nu}$ as $u\downarrow 0$. Consequently the functions $x\mapsto g^{-1}\circ\sqrt{(\gamma/f)}(x)$ and $x\mapsto\sqrt{\delta/x}$ have the same order as $x\to\infty$. We conclude that the first term on the right is of the order $x^{-\nu/2}$ and the last two terms on the far right are negligible relative to the first, as $x\to\infty$. This shows that the limit in (12) is bounded above by 1. To conclude the proof of (12) we bound, for given functions t and u,

$$\mathrm{P}\Big(\frac{\gamma}{g(S)^2}+\frac{Z_1}{g(S)}\geq f(y)-u(y),\frac{\delta}{S^2}+\frac{Z_2}{S}\geq y-t(y)\Big)$$
$$\geq \mathrm{P}\Big(\frac{\gamma}{g(S)^2}\geq f(y),\frac{\delta}{S^2}\geq y\Big)\mathrm{P}\Big(Z_1\geq -\sqrt{\frac{\gamma}{f(y)}}\,u(y),Z_2\geq -\sqrt{\frac{\delta}{y}}\,t(y)\Big).$$

The second probability on the right can be seen to converge to 1 as $y\to\infty$ if u^2 and t^2 are chosen to increase faster to infinity than f and the identity. We can achieve this with functions such that $f(y)-u(y)=f(x)$, $y-t(y)=x$ and $f(y)\asymp f(x)$ and $y\asymp x$. This completes the proof of (12).

Finally, to prove (13) we write both probabilities as left tail probabilities for S and use (5). Thus it suffices to show that $\bar{f}(x) \asymp f(x)$, as $x \to \infty$. This is implicit in the proof of (14).

For later reference we state the exact order of the tails of the marginal distribution of a positively skewed t-variable. Interestingly, its order is the root of the order of the tail of the symmetric t-distribution with the same number of degrees of freedom. The following lemma follows from from (11) and (5).

Lemma 3.2 *For $\delta > 0$ and $\nu > 0$, as $x \to \infty$,*

$$\mathrm{P}\Big(\frac{\delta}{S^2/\nu} + \frac{Z_2}{S/\sqrt{\nu}} \geq x\Big) \asymp \frac{(\delta\nu/2)^{\nu/2}}{\Gamma(\nu/2+1)}\Big(\frac{1}{x}\Big)^{\nu/2}.$$

3.3 The negatively skewed case (4)

Consider the case that $\gamma < 0$ and $\delta < 0$. Negative skewing decreases the right tails. The (proof of the) following lemma shows that the tails at x arise predominantly from values of the Gaussian variables Z_i near $4|\delta|\sqrt{x}$ combined with values of S of the order $1/\sqrt{x}$.

Lemma 3.3 *For $\delta < 0$ and $\nu > 0$, as $x \to \infty$,*

$$\mathrm{P}\left(\frac{\delta}{S^2/\nu} + \frac{Z_2}{S/\sqrt{\nu}} \geq x\right) \asymp \frac{\nu(-\delta\nu/2)^{\nu/2-1}}{4\Gamma(\nu/2)}\left(\frac{1}{x}\right)^{\nu/2+1} e^{2x\delta}.$$

Proof. The event $\delta/S^2 + Z_2/S \geq x$ is empty if $D := Z_2^2 + 4x\delta < 0$ and can otherwise be rewritten as $Z_2 - \sqrt{D} \leq 2xS \leq Z_2 + \sqrt{D}$. Because $D \leq Z_2^2$ this interval for $2xS$ is contained in $(0,\infty)$ and necessarily $Z_2 > 0$. Writing G_ν for the distribution function of S^2, we find that

$$\begin{aligned}
&\mathrm{P}\left(\frac{\delta}{S^2} + \frac{Z_2}{S} \geq x\right) = \\
&= \mathrm{E}1_{Z_2^2 \geq 4x|\delta|, Z_2 \geq 0}\left[G_\nu\left(\left(\frac{Z_2+\sqrt{D}}{2x}\right)^2\right) - G_\nu\left(\left(\frac{Z_2-\sqrt{D}}{2x}\right)^2\right)\right] \\
&\asymp e_\nu \mathrm{E}1_{Z_2^2 \geq 4x|\delta|, Z_2 \geq 0}\left[\left(\frac{Z_2+\sqrt{D}}{2x}\right)^\nu - \left(\frac{Z_2-\sqrt{D}}{2x}\right)^\nu\right], \qquad (15)
\end{aligned}$$

for $e_\nu = (2/\nu)d_\nu$. The last step is explained by (5) and can be argued rigorously as follows. We note first that, for $0 < a < b$,

$$\left|\frac{G_\nu(b) - G_\nu(a)}{e_\nu(b^{\nu/2} - a^{\nu/2})} - 1\right| = \left|\frac{\int_a^b (\nu/2)t^{\nu/2-1}(e^{-t/2}-1)\,dt}{(b^{\nu/2}-a^{\nu/2})}\right| \leq \max_{a<t<b}|e^{-t/2}-1| \lesssim b.$$

For $r(x) \to \infty$ the usual bounds on the normal tail give $\mathrm{P}(Z_2 \geq r(x)) \leq e^{-r(x)^2/2}$ and $\mathrm{E}Z_2^\nu 1_{Z_2 \geq r(x)} \lesssim r(x)^{\nu-1} e^{-r(x)^2/2}$. For $r(x) = x^{3/4}$ and $x \to \infty$ both terms are smaller than e^{-x^k} for some $k > 1$ and hence of smaller order than the order of the upper tail claimed in the lemma. The random variable in the expectation on the right of (15) is bounded by a multiple of $|Z_2|^\nu$, because $D \leq Z_2^2$. Therefore, if we restrict the expectations in the middle and right side of (15) to the event $Z_2 \leq r(x)$ this makes a difference of $O(\exp(-x^k))$, which is of smaller order than the order claimed in the lemma. It therefore suffices to prove the equivalence (15) for the two expectations restricted to the event $Z_2 \leq r(x)$. On this event we have $b := (Z_2 + \sqrt{D})^2/(2x)^2 \leq r(x)^2/x^2$, and hence the difference between the two expectations in (15) restricted to the event $Z_2 \leq r(x)$ is bounded above by

$$e_\nu \mathrm{E} 1_{Z_2^2 \geq 4x|\delta|, Z_2 \geq 0} \Big(\frac{r(x)}{x}\Big)^2 \Big[\Big(\frac{Z_2 + \sqrt{D}}{2x}\Big)^\nu - \Big(\frac{Z_2 - \sqrt{D}}{2x}\Big)^\nu\Big].$$

Because $r(x)/x \to 0$ this is negligible relative to the right side of (15). Thus (15) has been justified.

The right side of (15) can be expressed as an integral relative to the χ_1^2-density of Z_2^2, as

$$\begin{aligned}
&\frac{e_\nu}{2(2x)^\nu} \int_{4x|\delta|}^\infty \Big[\Big(\sqrt{s} + \sqrt{s + 4x\delta}\Big)^\nu - \Big(\sqrt{s} - \sqrt{s + 4x\delta}\Big)^\nu\Big] \frac{(\frac{1}{2})^{1/2}}{\Gamma(1/2)} s^{-1/2} e^{-s/2}\, ds \\
&\quad = \frac{e_\nu}{(2x)^\nu} \frac{(\frac{1}{2})^{3/2} e^{-y/2}}{\Gamma(1/2)} \int_0^\infty \Big[\Big(\sqrt{s+y} + \sqrt{s}\Big)^\nu - \Big(\sqrt{s+y} - \sqrt{s}\Big)^\nu\Big] \frac{e^{-s/2}}{\sqrt{s+y}}\, ds,
\end{aligned}$$

where $y = 4x|\delta| \to \infty$. Because $(a+b)^\nu - (a-b)^\nu \asymp 2\nu a^{\nu-1} b$ for $a \to \infty$ and fixed b, the last display is asymptotically equivalent to

$$\begin{aligned}
&\frac{e_\nu}{(2x)^\nu} \frac{(\frac{1}{2})^{3/2} e^{-y/2}}{\Gamma(1/2)} \int_0^\infty \Big[2\nu \Big(\sqrt{s+y}\Big)^{\nu-1} \sqrt{s}\Big] \frac{1}{\sqrt{s+y}} e^{-s/2}\, ds \\
&\quad = \frac{e_\nu}{(2x)^\nu} \frac{(\frac{1}{2})^{3/2} e^{-y/2}}{\Gamma(1/2)} (\sqrt{y})^{\nu-2} 2\nu \int_0^\infty \Big(\sqrt{\frac{s}{y} + 1}\Big)^{\nu-2} \sqrt{s} e^{-s/2}\, ds.
\end{aligned}$$

The integral tends to $\Gamma(3/2)2^{3/2}$, as $y \to \infty$. The lemma follows upon replacing δ and x by $\delta\sqrt{\nu}$ and $x/\sqrt{\nu}$ and simplifying this expression. ■

For the proof of (4) of the main theorem we again simplify notation by taking $\mu = \nu = 1$ in the algebraic definitions of X_1 and X_2. It follows from the preceding lemma that the marginal distribution functions in the negatively skewed case satisfy $1 - F_1(x) \asymp c_1 e^{2\gamma x} x^{-k_1}$ and $1 - F_2(x) \asymp c_2 e^{2\delta x} x^{-k_2}$, for some constants c_1, c_2, k_1, k_2, for $x \to \infty$. Inverting the first relationship yields

$F_1^{-1}(1-u) \asymp (2\gamma)^{-1}\log u$, for $u \to 0$, and hence, for $x \to \infty$,

$$f(x) \asymp \frac{\delta}{\gamma}\, x.$$

The conditional distribution of Z_1 given (S, Z_2) is normal with mean ρZ_2 and variance $1-\rho^2$. Because the event $\gamma/g(S)^2 + Z_1/g(S) \geq C(\delta/\gamma)x$ is empty if $Z_1^2 + 4Cx\delta < 0$, we have for $|\rho| < 1$ and any $C > 0$,

$$\begin{aligned}\mathrm{P}\left(\frac{\gamma}{g(S)^2} + \frac{Z_1}{g(S)} \geq \frac{C\delta x}{\gamma}, \frac{\delta}{S^2} + \frac{Z_2}{S} \geq x\right) &\leq \mathrm{P}\left(Z_1^2 + 4Cx\delta > 0, \frac{\delta}{S^2} + \frac{Z_2}{S} \geq x\right) \\ &= 2\mathrm{E}1_{\frac{\delta}{S^2}+\frac{Z_2}{S}\geq x}\left[1 - \Phi\left(\frac{\sqrt{4C|\delta|x} - \rho Z_2}{\sqrt{1-\rho^2}}\right)\right].\end{aligned}$$

Choose constants c and C such that $\rho^2 < c^2 < C < 1$, so that $c^2/\rho^2 > 1$ and $C\delta x/\gamma < f(x)$ for large x. The left side of the preceding display can be bounded above by

$$2\mathrm{P}\left(\rho Z_2 > c\sqrt{4|\delta|x}\right) + 2\mathrm{E}1_{\frac{\delta}{S^2}+\frac{Z_2}{S}\geq x}\left[1 - \Phi\left(\frac{(\sqrt{C}-c)\sqrt{4|\delta|x}}{\sqrt{1-\rho^2}}\right)\right].$$

The first term is of order $o(e^{2\delta x c^2/\rho^2})$ and the second term is $\big(1-F_2(x)\big)\big[o(1)\big]$. Thus both terms are of smaller order than $1-F_2(x)$, as $x \to \infty$. Together with (4) this proves Theorem 2.1 in the case that $\gamma < 0$ and $\delta < 0$.

3.4 The combined positively and negatively skewed case (5)

Consider the case that $\gamma < 0$ and $\delta > 0$. For any function $r\colon \mathbb{R}^+ \to \mathbb{R}^+$,

$$\begin{aligned}&\mathrm{P}\Big(\frac{\gamma}{g(S)^2} + \frac{Z_1}{g(S)} \geq f(x), \frac{\delta}{S^2} + \frac{Z_2}{S} \geq x\Big) \\ &\quad\leq \mathrm{P}\Big(Z_1 \geq f(x)g(S) + \frac{-\gamma}{g(S)}, \frac{\delta}{S^2} \geq x - r(x)\Big) + \mathrm{P}\Big(\frac{Z_2}{S} \geq r(x)\Big) \\ &\quad\leq \mathrm{E}1_{\delta/S^2 \geq x - r(x)}\Big(1 - \Phi\Big(f(x)g(S) + \frac{-\gamma}{g(S)}\Big)\Big) + \mathrm{P}\Big(\frac{Z_2}{S} \geq r(x)\Big).\end{aligned}$$

The second term on the right is $O(r(x)^{-\nu})$ if $r(x) \to \infty$, by (8). The function $s \mapsto f(x)s - \gamma/s$ is bounded below by $2\sqrt{-\gamma f(x)}$ on $(0,\infty)$. Therefore the argument in the normal distribution function in the first term tends to infinity uniformly in $g(S)$. It follows that the preceding display is

$$\mathrm{E}1_{\delta/S^2 \geq x - r(x)}o(1) + O\big(r(x)^{-\nu}\big).$$

Here for $r(x)/x \to 0$, the expectation in the first term is equivalent to $1-F_2(x)$, as $x \to \infty$, by (11), which is of bigger order than the second term if $r^2(x)/x \to \infty$ by Lemma 3.2. A reference to (4) completes the proof.

3.5 The combined symmetric and negatively skewed case (6)

In the case that $\gamma = 0$ and $\delta < 0$ the variable X_1 is t_μ-distributed and hence possesses a polynomial upper tail, while the upper tail of X_2 is exponentially small, by Lemma 3.3. It follows that, as $x \to \infty$,

$$f(x) \sim e^{2x|\delta|/\mu} x^{(\nu/2+1)/\mu}.$$

If $S \le 1/x$, then $\delta/S^2 + Z_2/S > x$ implies that $Z_2 > xS - \delta/S > |\delta|x$. Hence

$$\mathrm{P}\Big(\frac{Z_1}{g(S)} > f(x), S \le \frac{1}{x}, \frac{\delta}{S^2} + \frac{Z_2}{S} > x\Big) \le \mathrm{P}\Big(Z_2 > |\delta|x\Big)$$

is subGaussian and hence of smaller order than the tail of X_2. It follows that

$$\begin{aligned}\mathrm{P}\Big(\frac{Z_1}{g(S)} > f(x), \frac{\delta}{S^2} + \frac{Z_2}{S} > x\Big) &\le \mathrm{P}\Big(Z_1 > f(x)g(S), S > 1/x\Big) + o\Big(1 - F_2(x)\Big) \\ &\le \mathrm{P}\Big(Z_1 > f(x)g(1/x)\Big) + o\Big(1 - F_2(x)\Big).\end{aligned}$$

Because $f(x)g(1/x)$ is exponentially large as $x \to \infty$, the first term is sub-Gumbel and also $o\Big(1 - F_2(x)\Big)$. Thus the limit in the second line of (4) is 0.

3.6 The combined symmetric and positively skewed case (7)

Consider the case that $\gamma = 0$ and $\delta > 0$. The tails of X_2 are in this case determined by the skewing variable. For the marginal distribution this follows from (11). An argument similar to the one given in Section 3.2 also gives

$$\lim_{x\to\infty} \frac{\mathrm{P}\Big(Z_1/g(S) \ge f(x), \delta/S^2 + Z_2/S \ge x\Big)}{\mathrm{P}\Big(Z_1/g(S) \ge f(x), \delta/S^2 \ge x\Big)} = 1. \tag{16}$$

In view of (4) the tail index can therefore be computed for the variables $X_1 = \sqrt{\mu}Z_1/g(S)$ and $X_2' = \delta\nu/S^2$. The function f can be seen to satisfy, as $x \to \infty$,

$$f(x) \asymp c_\mu^{1/\mu} \mu^{(\mu-1)/(2\mu)} \Big(\frac{\Gamma(\nu/2+1)2^{\nu/2}}{(\nu\delta)^{\nu/2}}\Big)^{1/\mu} x^{\nu/(2\mu)}.$$

With g_ν the density of S^2 we have

$$\mathrm{P}\Big(X_1 > f(x), X_2' > x\Big) = \int_0^{\delta\nu} \Big(1 - \Phi\Big(\frac{f(x)g\Big(\sqrt{u/x}\Big)}{\sqrt{\mu}}\Big)\, g_\nu\Big(\frac{u}{x}\Big)\, \frac{du}{x}.$$

Here, as $x \to \infty$,

$$\frac{g_\nu(u/x)/x}{G_\nu(\delta\nu/x)} \asymp \frac{(u/x)^{\nu/2-1}e^{-u/(2x)}d_\nu/x}{(\delta\nu/x)^{\nu/2}d_\nu 2/\nu} \to \frac{\nu u^{\nu/2-1}}{2(\delta\nu)^{\nu/2}},$$

where the convergence is pointwise in u and the functions on the left are uniformly bounded in $u \le \delta\nu$. Furthermore,

$$f(x)g\Big(\sqrt{u/x}\Big) \to \frac{\Gamma\Big((\mu+1)/2\Big)^{1/\mu}\sqrt{\mu}2^{1/2-1/\mu}}{\pi^{1/(2\mu)}(\nu\delta)^{\nu/(2\mu)}}u^{\nu/(2\mu)},$$

also pointwise in u, as $x \to \infty$. An application of the dominated convergence theorem and subsequent algebraic simplication shows that the quotient $\mathrm{P}\Big(X_1 > f(x), X_2' > x\Big)/G_\nu(\delta\nu/x)$ tends to the expression given in (7) of the theorem.

A Figures

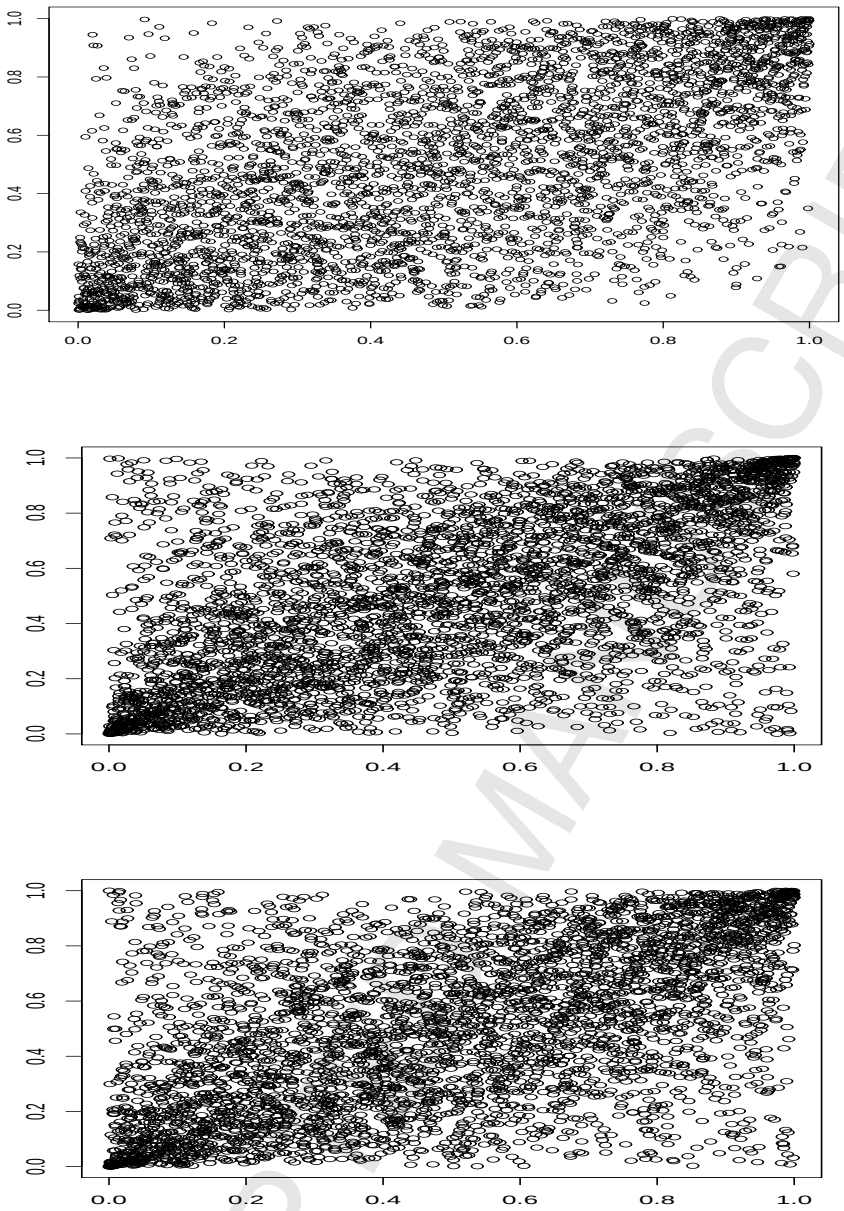

Fig. A.1. Scatter plots of random samples from symmetric, grouped t-distribution ($\delta = \gamma = 0$). Top to bottom: Gaussian ($\mu = \nu = \infty$), regular t ($\nu = \mu = 4$) and grouped t ($\nu = 4$, $\mu = 12$) copula with correlation $\rho = 0.5$

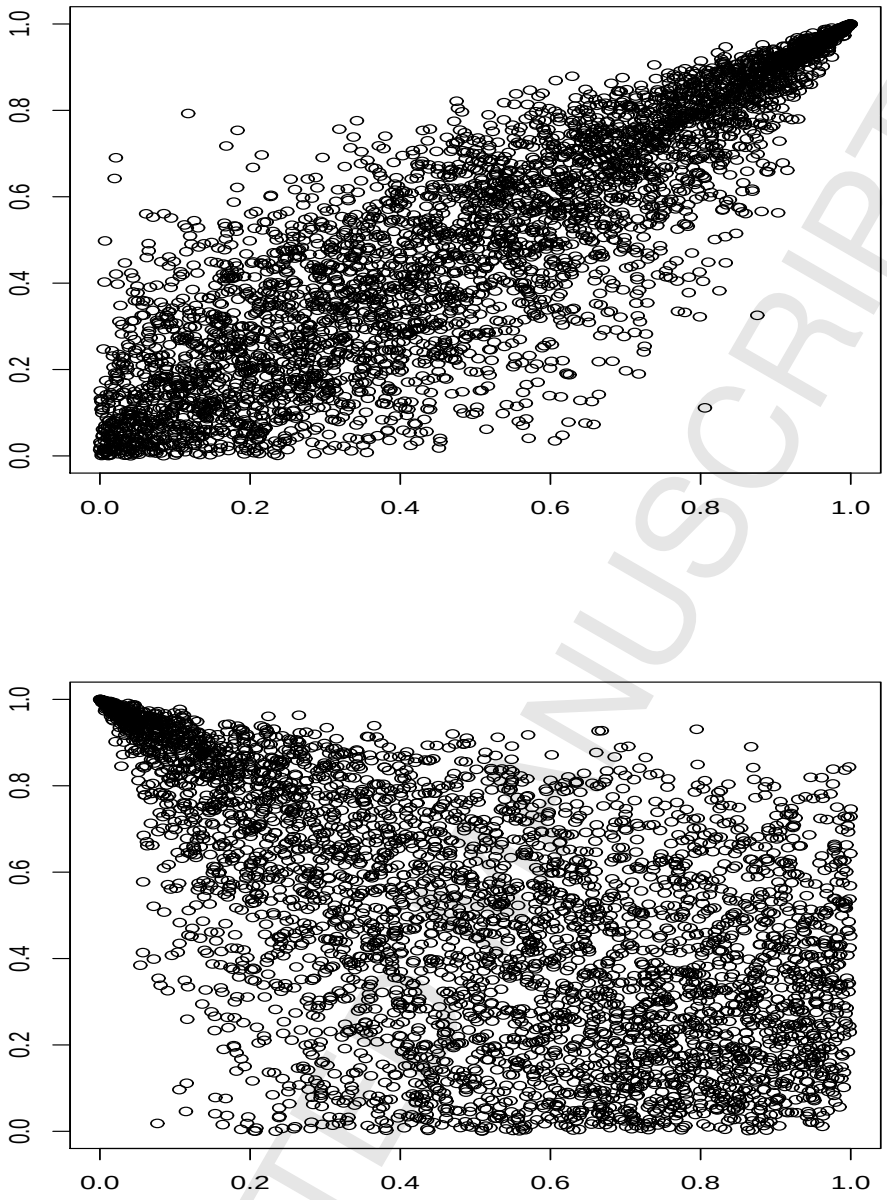

Fig. A.2. Scatter plots of random samples from grouped skewed t-distribution ($\mu = 4$, $\nu = 12$, $\rho = 0.5$) with different combinations of skewing parameters. Top to bottom: $\{\gamma > 0,\ \delta > 0\}$ and $\{\gamma < 0,\ \delta > 0\}$

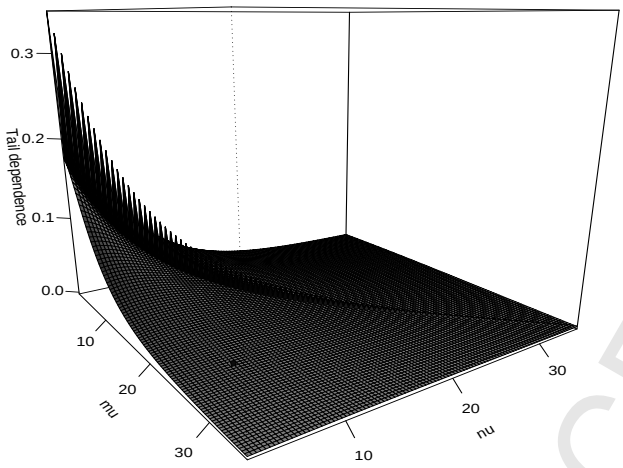

Fig. A.3. Tail dependence coefficient λ as a function of μ and ν with fixed correlation $\rho = 0.5$ and $\delta = \gamma = 0$.

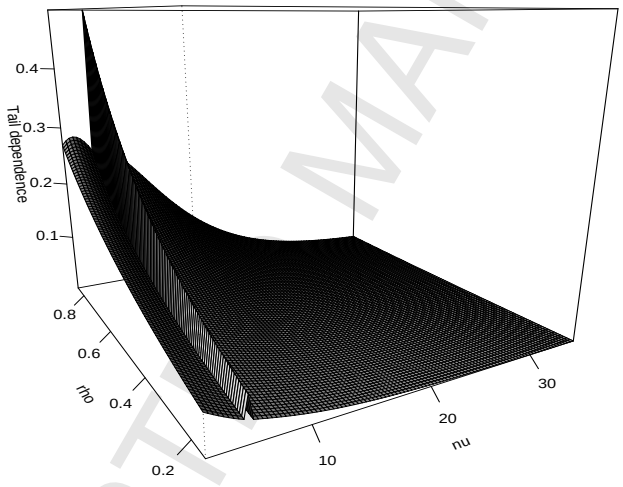

Fig. A.4. Tail dependence coefficient λ as a function of μ and ρ with fixed degrees of freedom on the first coordinate $\mu = 6$ and $\delta = \gamma = 0$. The ridge is the discontinuity at $\mu = \nu$.