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Potentials for the Modified Cam-Clay model

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Abstract

Energy and dissipation pseudo-potentials are employed to derive constitutive relationships, in the context of thermodynamic concepts, for the widely used Modified Cam-Clay (MCC) model for soil mechanics. A variational formulation of the MCC evolution equations is proposed in this paper. Since plastic collapse of MCC soils cannot be embedded in the classical limit analysis theory, finding the critical amplification of the load that produces plastic collapse is formulated in the form of a system of equations and inequalities. Then, a mixed minimization principle is proposed for the plastic collapse analysis of MCC soils. This principle is obtained by the application of the variational formulation for the flow law introduced in the first part of the article.

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1 Introduction

The Modified Cam-Clay (MCC) model by Roscoe and Burland (1968) is an important conceptual framework for constitutive behavior in soil mechanics (Wood, 1990; Houlsby and Puzrin, 2006; Ulm and Coussy, 2003; Borja and Lee, 1990; Vaunat et al., 2000; Einav et al., 2007). It has been widely used in its simpler, classical form and also in more sophisticated versions that extend its application to a broader variety of situations.

This paper is focused on the formal structure of the classical MCC model as derived from the energy and dissipation potentials based on thermodynamics.

The systematic application of thermodynamic concepts to the formulation of constitutive models of inelastic solids has reached a sophisticated stage due to pioneering work and many outstanding contributions. In a simplified description of this research field, thermodynamic principles are invoked to find energy and dissipation functionals that are used to derive state and evolution relations in the form of potential laws.

Among many other contributions to the thermodynamic formulation of constitutive equations, we cite first the work of Ziegler (1983). Halphen and Nguyen (1975) (see also Nguyen, 2000) proposed a framework, the Standard Generalized Materials (GSM), which is a widely referenced approach to material

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behavior modeling. G. T. Houlsby and coworkers (Collins and Houlsby, 1997; Houlsby and Puzrin, 2006) contributed with the concept of Hyperplasticity, which is a complete methodology to consistently develop constitutive equations based on thermodynamic principles. Rich and rigorous contributions to thermomechanics with internal variables are also found in Maugin (1992), Han and Reddy (1999) and Šilhavý (1997).

We use an extension of the concept of potential laws proposed by de Saxcé (1992), based on a special class of functions, named *bipotentials* by de Saxcé. These functions are, by definition, separately convex in two independent variables and bounded below by the scalar product of the variables. Generalized potential laws are obtained, in direct and inverse forms, by partial subdifferentiation (Rockafellar, 1970). This gives implicit constitutive equations, instead of the usual explicit direct and inverse relations. Accordingly, the class of material represented by this kind of constitutive equations is called Implicit Standard Materials (ISM) (de Saxcé, 1995; de Saxcé and Bousshine, 1998; Hjiaj, 1999; Bodovillé, 2001). It is a generalization of GSM.

This paper contains a unified derivation of energy and dissipation potentials for the MCC model, beginning with some known potentials.

We propose a mixed approach to the evolution relations of MCC materials. Throughout this paper, the expressions mixed formulation and mixed potential are used to emphasize that generalized stresses and plastic strains (or strain rates) are taken together as a pair of independent variables. In the framework of this mixed approach, we propose a minimization principle for the MCC flow law.

The mixed variational principles for MCC are applied to the analysis of plastic

collapse of soils. The expression *plastic collapse* identifies the phenomenon of incipient unbounded plastic deformation of a body under constant loading and constant stress distribution. This is a critical state in soil mechanics. In the present context, this concept must be carefully distinguished from the case when a limit load eventually exists. Indeed, an incremental analysis of a simple system, for instance the conventional drained triaxial compression shown by Wood (1990, p. 123) and de Borst and Groen (2000, p. 37), demonstrates that the plastic collapse of MCC soils can effectively occur, in a proportional loading program, for a loading factor lower than the maximum amplification of loads previously attained. In triaxial compression this happens when heavily overconsolidated soil is sheared; we also discuss this example in Section 5.1 (see Figure 1).

Plastic collapse of MCC soils is the subject of Section 5, where we consider: (i) a formulation of plastic collapse analysis in terms of a system of equations and inequalities and (ii) a mixed minimization principle whose solution is precisely related to the plastic collapse solution by a formal proposition, which is proven with the variational tools developed in previous sections.

2 Potentials for elastoplastic materials

In this section, we briefly introduce the basic notation used in the paper and some general potentials for the elastoplastic material constitutive equations.

2.1 State variables and energy potentials

In an isothermal process the second law of thermodynamics can be written as (Lubliner, 1972; Halphen and Nguyen, 1975; Houlsby and Puzrin, 2006; Han and Reddy, 1999)

$$D = \boldsymbol{\sigma} \cdot \boldsymbol{d} - \dot{W} \geqslant 0 \tag{1}$$

where D denotes specific dissipation, W is the Helmholtz free energy, $\boldsymbol{\sigma}$ is the stress tensor, $\boldsymbol{d} := \dot{\boldsymbol{\varepsilon}}$ is the strain rate tensor and superposed dot denotes time derivative.

Let $\boldsymbol{\varepsilon}^p$ denote the plastic deformation and $\boldsymbol{\beta}$ the strain-like internal variable associated with hardening. We use this simplified notation although it is often the case that a list of strain-like scalars or tensors $\{\boldsymbol{\beta}^i;\ i = 1 : n_h\}$ is needed in order to represent different hardening mechanisms (Maugin, 1992, pp. 276-280; Han and Reddy, 1999, p. 52; Nguyen, 2000, p. 73; Nguyen, 2003, p. 82). The plastic strain is a special internal variable that is identified later. The internal variable $\boldsymbol{\beta}$ is not present in ideal plasticity. The generalized internal variable is defined as

$$\boldsymbol{\alpha} := (\boldsymbol{\varepsilon}^p, \boldsymbol{\beta}) \tag{2}$$

Then, the constitutive equations read as follows

$$W = \tilde{W}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \boldsymbol{\beta}) \qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \boldsymbol{\beta}) \tag{3}$$

We assume that $\tilde{W}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \boldsymbol{\beta})$ is convex and differentiable in $(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \boldsymbol{\beta})$ and strictly convex in $\boldsymbol{\varepsilon}$ (for any fixed $\boldsymbol{\varepsilon}^p$ and $\boldsymbol{\beta}$). Its partial Legendre-Fenchel conjugate function (Rockafellar, 1970, p. 104; Hiriart-Urruty and Lemaréchal,

1993b, p. 37; Maugin, 1992, p. 288; Han and Reddy, 1999, p. 75)

$$\tilde{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}^p, \boldsymbol{\beta}) := \sup_{\boldsymbol{\varepsilon}} \left[\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} - \tilde{W}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \boldsymbol{\beta}) \right] \tag{4}$$

is the negative of conventional Gibbs free energy, with strain-like parameters. It is convex with respect to $\boldsymbol{\sigma}$ (as are all conjugate functions) and differentiable in $\boldsymbol{\sigma}$ because $\tilde{W}$ is strictly convex in $\boldsymbol{\varepsilon}$ (Hiriart-Urruty and Lemaréchal, 1993b, pp. 38, 79).

The following state law, in direct and inverse form, is deduced from thermodynamic principles (Han and Reddy, 1999, pp. 50-52)

$$\boldsymbol{\sigma} = \nabla_{\boldsymbol{\varepsilon}} \tilde{W}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \boldsymbol{\beta}) \quad \Leftrightarrow \quad \boldsymbol{\varepsilon} = \nabla_{\boldsymbol{\sigma}} \tilde{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}^p, \boldsymbol{\beta}) \tag{5}$$

with $\nabla_{\boldsymbol{\varepsilon}} \tilde{W}$ denoting the partial gradient of $\tilde{W}$ with respect to $\boldsymbol{\varepsilon}$ and $\nabla_{\boldsymbol{\sigma}} \tilde{G}$ being defined likewise.

We restrict ourselves to consider uncoupled materials, that is, materials whose tangent elastic compliance $\nabla_{\boldsymbol{\sigma}\boldsymbol{\sigma}} \tilde{G}$ (second partial gradient with respect to $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}$) is independent of the internal variables. As proven in Collins and Houlsby (1997, pp. 1979-1981) and Han and Reddy (1999, pp. 50-52) (cf. Ulm and Coussy, 2003) this hypothesis leads to additive decomposition of strain

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e(\boldsymbol{\sigma}) + \boldsymbol{\varepsilon}^p \tag{6}$$

and also to the following form for the Helmholtz free energy

$$\tilde{W}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \boldsymbol{\beta}) = W^e(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) + W^h(\boldsymbol{\varepsilon}^p, \boldsymbol{\beta}) \tag{7}$$

We suppose that the elastic potential W^e is strictly convex and differentiable and that the plastic potential W^h is convex and differentiable (cf. Nguyen, 2006, p. 83).

The state equation (5) now gives

$$\boldsymbol{\sigma} = \nabla W^e(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) \tag{8}$$

and the thermodynamic forces conjugate to $\boldsymbol{\varepsilon}^p$ and $\boldsymbol{\beta}$ are defined as

$$\boldsymbol{\kappa} := -\nabla_{\boldsymbol{\varepsilon}^p}\tilde{W}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \boldsymbol{\beta}) = \nabla W^e(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) - \nabla_{\boldsymbol{\varepsilon}^p} W^h(\boldsymbol{\varepsilon}^p, \boldsymbol{\beta}) \tag{9}$$

$$\boldsymbol{A} := -\nabla_{\boldsymbol{\beta}}\tilde{W}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \boldsymbol{\beta}) = -\nabla_{\boldsymbol{\beta}} W^h(\boldsymbol{\varepsilon}^p, \boldsymbol{\beta}) \tag{10}$$

According to the definitions (5), (9) and (10) it holds that $\dot{W} = \boldsymbol{\sigma}\cdot\boldsymbol{d} - \boldsymbol{\kappa}\cdot\boldsymbol{d}^p - \boldsymbol{A}\cdot\dot{\boldsymbol{\beta}}$. Therefore, the second law of thermodynamics (1) implies the following constraint

$$D = \boldsymbol{\mathcal{A}}\cdot\dot{\boldsymbol{\alpha}} := \boldsymbol{\kappa}\cdot\boldsymbol{d}^p + \boldsymbol{A}\cdot\dot{\boldsymbol{\beta}} \geqslant 0 \tag{11}$$

where the generalized internal force and flux are

$$\boldsymbol{\mathcal{A}} := (\boldsymbol{\kappa}, \boldsymbol{A}) \qquad \dot{\boldsymbol{\alpha}} := (\boldsymbol{d}^p, \dot{\boldsymbol{\beta}}) \tag{12}$$

2.1.1 The conjugate function of the free energy

Let us consider first the conjugate function of the elastic part, W^e, of the Helmholtz free energy, i.e.

$$G^e(\boldsymbol{\sigma}) := \sup_{\boldsymbol{\varepsilon}^e}\{\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}^e - W^e(\boldsymbol{\varepsilon}^e)\} \tag{13}$$

Then, we have

$$\boldsymbol{\sigma} = \nabla W^e(\boldsymbol{\varepsilon}^e) \quad \Leftrightarrow \quad \boldsymbol{\varepsilon}^e = \nabla G^e(\boldsymbol{\sigma}) \quad \Leftrightarrow \quad G^e(\boldsymbol{\sigma}) + W^e(\boldsymbol{\varepsilon}^e) = \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}^e \tag{14}$$

It follows from the above relations that the partial Gibbs free energy, through the duality transformation (4) to the energy potential given by (7), is

$$\tilde{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}^p, \boldsymbol{\beta}) = G^e(\boldsymbol{\sigma}) + \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}^p - W^h(\boldsymbol{\varepsilon}^p, \boldsymbol{\beta}) \tag{15}$$

The rest of this subsection is devoted to obtaining the complete dual function of the free energy $\tilde{W}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \boldsymbol{\beta})$.

First, we consider the conjugate function of the inelastic part, W^h, of the Helmholtz free energy

$$G^h(\boldsymbol{\rho}, -\boldsymbol{A}) := \sup_{\boldsymbol{\varepsilon}^p, \boldsymbol{\beta}} \{\boldsymbol{\rho} \cdot \boldsymbol{\varepsilon}^p - \boldsymbol{A} \cdot \boldsymbol{\beta} - W^h(\boldsymbol{\varepsilon}^p, \boldsymbol{\beta})\} \tag{16}$$

This transformation may result in a nondifferentiable function G^h under the hypothesis that W^h is convex. Indeed, differentiability of G^h corresponds to strict convexity of W^h, which is not true for the Cam-Clay model presented in the next sections. Accordingly, the subdifferential concept (which we use for the dissipation potential in the following sections) should be applied to G^h in order to obtain the inverse relations. These inverse relations will not be needed in our presentation of the MCC model. Thus, we write below the inverse relations, under the differentiability hypothesis, for completeness and also for the purpose of comparing to references.

According to (16), in the differentiable case,

$$\begin{aligned} (\boldsymbol{\rho}, -\boldsymbol{A}) = \nabla W^h(\boldsymbol{\varepsilon}^p, \boldsymbol{\beta}) \quad &\Leftrightarrow \quad (\boldsymbol{\varepsilon}^p, \boldsymbol{\beta}) = \nabla G^h(\boldsymbol{\rho}, -\boldsymbol{A}) \\ &\Leftrightarrow \quad G^h(\boldsymbol{\rho}, -\boldsymbol{A}) + W^h(\boldsymbol{\varepsilon}^p, \boldsymbol{\beta}) = \boldsymbol{\rho} \cdot \boldsymbol{\varepsilon}^p - \boldsymbol{A} \cdot \boldsymbol{\beta} \end{aligned} \tag{17}$$

The variable $\boldsymbol{\rho}$ was introduced above as the conjugate of $\boldsymbol{\varepsilon}^p$ in the duality transformation (16); however, taking into account (8), (9) and (17), it holds that (cf.: Collins and Houlsby, 1997, p. 1980; Nguyen, 2006, p. 83)

$$\boldsymbol{\kappa} = \boldsymbol{\sigma} - \boldsymbol{\rho} \qquad \boldsymbol{\rho} = \nabla_{\boldsymbol{\varepsilon}^p} W^h(\boldsymbol{\varepsilon}^p, \boldsymbol{\beta}) \tag{18}$$

Thus, $\boldsymbol{\rho}$ has the meaning of a back-stress.

We choose $(\boldsymbol{\varepsilon}^e, \boldsymbol{\varepsilon}^p, \boldsymbol{\beta})$ and $(\boldsymbol{\sigma}, \boldsymbol{\rho}, \boldsymbol{A})$ as the dual pair of variables. An alternative choice may be the pair $(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \boldsymbol{\beta})$ and $(\boldsymbol{\sigma}, \boldsymbol{\kappa}, \boldsymbol{A})$ as adopted, for instance, in Collins and Houlsby (1997).

Consequently, we consider the following definition for the Helmholtz potential

$$W(\boldsymbol{\varepsilon}^e, \boldsymbol{\varepsilon}^p, \boldsymbol{\beta}) := W^e(\boldsymbol{\varepsilon}^e) + W^h(\boldsymbol{\varepsilon}^p, \boldsymbol{\beta}) \tag{19}$$

The complete Gibbs free energy, which is conjugated to the energy potential given by (7), is now defined as

$$G(\boldsymbol{\sigma}, \boldsymbol{\rho}, \boldsymbol{A}) := \sup_{\boldsymbol{\varepsilon}^e, \boldsymbol{\varepsilon}^p, \boldsymbol{\beta}} [\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}^e + \boldsymbol{\rho} \cdot \boldsymbol{\varepsilon}^p - \boldsymbol{A} \cdot \boldsymbol{\beta} - W(\boldsymbol{\varepsilon}^e, \boldsymbol{\varepsilon}^p, \boldsymbol{\beta})] \tag{20}$$

Thus, taking into account (17) and (14)

$$G(\boldsymbol{\sigma}, \boldsymbol{\rho}, \boldsymbol{A}) = G^e(\boldsymbol{\sigma}) + G^h(\boldsymbol{\rho}, -\boldsymbol{A}) \tag{21}$$

and the corresponding state equations are

$$\boldsymbol{\varepsilon}^e = \nabla G^e(\boldsymbol{\sigma}) \qquad \boldsymbol{\varepsilon}^p = \nabla_{\boldsymbol{\rho}} G^h(\boldsymbol{\rho}, -\boldsymbol{A}) \qquad \boldsymbol{\beta} = \nabla_{(-\boldsymbol{A})} G^h(\boldsymbol{\rho}, -\boldsymbol{A}) \tag{22}$$

Then, $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p$ and $\boldsymbol{\kappa} = \boldsymbol{\sigma} - \boldsymbol{\rho}$ appear as subsidiary equations.

2.2 Evolution equations and dissipation potentials

Because there is no characteristic time in inviscid plasticity, an appropriate dissipation potential for this case

$$D = D(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}) \tag{23}$$

must be positively homogeneous with degree one, with respect to $\dot{\boldsymbol{\alpha}} = (\boldsymbol{d}^p, \dot{\boldsymbol{\beta}})$; that is, $D(c\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}) = cD(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}) \;\forall\; c > 0$. Thus, the value of the dissipation

function for $\dot{\boldsymbol{\alpha}} = \mathbf{0}$ and any $\boldsymbol{\alpha}$ can only be zero or infinity; we rule out the latter case.

In addition, the dissipation potential is assumed to be convex, lower semicontinuous and positive for all $\dot{\boldsymbol{\alpha}} \neq \mathbf{0}$.

For the material models addressed here, it is necessary to consider non-differentiable dissipation potentials, although they are supposed to be within the class of functions endowed with sub-differentiability (Rockafellar, 1970; Hiriart-Urruty and Lemaréchal, 1993a).

Then, $\boldsymbol{\mathcal{A}}$ and $\dot{\boldsymbol{\alpha}}$ are related by the constitutive evolution relation if and only if they fulfill the following inclusion

$$\boldsymbol{\mathcal{A}} \in \partial_{\dot{\boldsymbol{\alpha}}} D(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}) \tag{24}$$

where $\partial_{\dot{\boldsymbol{\alpha}}} D(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha})$ denotes the partial subdifferential of D with respect to $\dot{\boldsymbol{\alpha}}$. The above inclusion is equivalent, by definition, to the following variational inequality

$$D(\dot{\boldsymbol{\alpha}}^*, \boldsymbol{\alpha}) - D(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}) \geqslant \boldsymbol{\mathcal{A}} \cdot (\dot{\boldsymbol{\alpha}}^* - \dot{\boldsymbol{\alpha}}) \quad \forall\ \dot{\boldsymbol{\alpha}}^* \tag{25}$$

This choice for the evolution equation automatically satisfies the second law constraint (11).

We note that the generalized strain-like internal variable $\boldsymbol{\alpha}$ is a passive parameter in this relation.

There is another hypothesis, regarding the coincidence of the internal forces in (24), (9) and (10), that is implicitly adopted here. This issue is discussed by Houlsby and Puzrin (2006) and resolved by restricting the class of materials envisaged or, alternatively, by accepting Ziegler's principle.

In the following, we focus on the inverse of the evolution equation (24), which is classically known as the (direct) plastic flow law. We show that it has the following form

$$\dot{\boldsymbol{\alpha}} \in N_P(\boldsymbol{\mathcal{A}}, \boldsymbol{\alpha}) := \partial \mathcal{I}_{P(\boldsymbol{\alpha})}(\boldsymbol{\mathcal{A}}) \tag{26}$$

where $P(\boldsymbol{\alpha})$ is the domain of plastic admissibility of the generalized internal force $\boldsymbol{\mathcal{A}}$, $N_P(\boldsymbol{\mathcal{A}}, \boldsymbol{\alpha})$ is the cone of normals to $P(\boldsymbol{\alpha})$ at $\boldsymbol{\sigma}$, the indicator function of $P(\boldsymbol{\alpha})$ is defined as $\mathcal{I}_{P(\boldsymbol{\alpha})}(\boldsymbol{\mathcal{A}}) := 0$ if $\boldsymbol{\mathcal{A}} \in P(\boldsymbol{\alpha})$ and $+\infty$ otherwise, and $\partial \mathcal{I}_{P(\boldsymbol{\alpha})}(\boldsymbol{\mathcal{A}})$ is the subdifferential set defined by the following variational inequality

$$\mathcal{I}_{P(\boldsymbol{\alpha})}(\boldsymbol{\mathcal{A}}^*) - \mathcal{I}_{P(\boldsymbol{\alpha})}(\boldsymbol{\mathcal{A}}) \geqslant (\boldsymbol{\mathcal{A}}^* - \boldsymbol{\mathcal{A}}) \cdot \dot{\boldsymbol{\alpha}} \quad \forall\, \boldsymbol{\mathcal{A}}^* \tag{27}$$

This is equivalent to

$$\boldsymbol{\mathcal{A}} \in P(\boldsymbol{\alpha}) \tag{28}$$

$$\boldsymbol{\mathcal{A}} \cdot \dot{\boldsymbol{\alpha}} \leqslant \boldsymbol{\mathcal{A}}^* \cdot \dot{\boldsymbol{\alpha}} \quad \forall\, \boldsymbol{\mathcal{A}}^* \in P(\boldsymbol{\alpha}) \tag{29}$$

In fact, functions like $D(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha})$, that are convex, closed, non-negative and positively homogeneous of degree one in $\dot{\boldsymbol{\alpha}}$ (i.e. functions called gauges), are associated, through a bijective relation, with sets $P(\boldsymbol{\alpha})$ in the dual space of the variable $\boldsymbol{\mathcal{A}}$ that are closed, convex and contain the origin (Rockafellar, 1970, p. 128; Hiriart-Urruty and Lemaréchal, 1993a, p. 218). This is a well known duality relation in convex analysis. In the present constitutive relationship, this means that if we assume that inelastic processes have no characteristic time then an admissibility domain for the dissipative forces necessarily exists.

There are formulas implementing the bijective relation between $D(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha})$ and $P(\boldsymbol{\alpha})$. Initially, we know that the dissipation potential is the support function of the admissibility domain (Rockafellar, 1970, p. 112; Aubin and Ekeland,

1984, p. 204), that is

$$D(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}) = \sup_{\boldsymbol{\mathcal{A}}} \{\boldsymbol{\mathcal{A}} \cdot \dot{\boldsymbol{\alpha}} \ \mid \ \boldsymbol{\mathcal{A}} \in P(\boldsymbol{\alpha})\} \tag{30}$$

This allows the computation of the dissipation function if the plastic admissibility function is given.

Furthermore, the indicator function $\mathcal{I}_{P(\boldsymbol{\alpha})}(\boldsymbol{\mathcal{A}})$ is the conjugate, in a Fenchel transformation, of the dissipation function $D(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha})$. Then, a fundamental theorem in convex analysis (Rockafellar, 1970, p. 218; Aubin and Ekeland, 1984, p. 202) states that (24), (25), (26), (27) and the equation

$$D(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}) + \mathcal{I}_{P(\boldsymbol{\alpha})}(\boldsymbol{\mathcal{A}}) = \boldsymbol{\mathcal{A}} \cdot \dot{\boldsymbol{\alpha}} \tag{31}$$

are all equivalent expressions of the same relation between $\dot{\boldsymbol{\alpha}}$ and $\boldsymbol{\mathcal{A}}$, i.e. these are equivalent evolution equations.

Let us introduce the following representation for the plastic admissibility set

$$P(\boldsymbol{\alpha}) = \{\boldsymbol{\mathcal{A}} \ \mid \ f(\boldsymbol{\mathcal{A}}, \boldsymbol{\alpha}) \leqslant 0\} \tag{32}$$

where f is the plastic admissibility function that is assumed to be regular for the sake of simplicity. Then, the flow law is expressed in the usual form (Rockafellar, 1970, p. 222)

$$\dot{\boldsymbol{\alpha}} = \dot{\lambda} \nabla_{\boldsymbol{\mathcal{A}}} f(\boldsymbol{\mathcal{A}}, \boldsymbol{\alpha}) \tag{33}$$

$$\dot{\lambda}\, f(\boldsymbol{\mathcal{A}}, \boldsymbol{\alpha}) = 0 \quad f(\boldsymbol{\mathcal{A}}, \boldsymbol{\alpha}) \leqslant 0 \quad \dot{\lambda} \geqslant 0 \tag{34}$$

If the dissipation function is given, we may compute its polar function as follows:

$$\Phi(\boldsymbol{\mathcal{A}}, \boldsymbol{\alpha}) = \sup_{\dot{\boldsymbol{\alpha}} \neq \mathbf{0}} \frac{\boldsymbol{\mathcal{A}} \cdot \dot{\boldsymbol{\alpha}}}{D(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha})} \tag{35}$$

and therefore determine the plastic admissibility set using the following plastic admissibility function

$$f^0(\boldsymbol{\mathcal{A}}, \boldsymbol{\alpha}) = \Phi(\boldsymbol{\mathcal{A}}, \boldsymbol{\alpha}) - 1 \leqslant 0 \tag{36}$$

Notice that the same convex set $P(\boldsymbol{\alpha})$ can be represented by different plastic admissibility functions (some of them being non-convex). However, there is always a unique canonical representation, in the form (36), in terms of the gauge function $\Phi(\boldsymbol{\mathcal{A}}, \boldsymbol{\alpha})$ associated with $P(\boldsymbol{\alpha})$ (Rockafellar, 1970, p. 79; Hiriart-Urruty and Lemaréchal, 1993a, p. 203).

Alternatively, if we suppose that a prescribed set is given $P(\boldsymbol{\alpha})$, which is closed, convex and contains the origin, then its gauge function (Rockafellar, 1970, p. 28), also known as the Minkowski function (Kamenjarzh, 1996, p. 147), is obtained by the formula

$$\Phi(\boldsymbol{\mathcal{A}}, \boldsymbol{\alpha}) = \inf\{\zeta \geqslant 0 \;\mid\; \boldsymbol{\mathcal{A}} \in \zeta P(\boldsymbol{\alpha})\} \tag{37}$$

This results in a function that is non-negative, closed, convex, positively homogeneous of degree one and with value zero at the origin $\boldsymbol{\mathcal{A}} = \mathbf{0}$. If the origin of the generalized stress space is strictly interior to the set $P(\boldsymbol{\alpha})$ then $\Phi(\boldsymbol{\mathcal{A}}, \boldsymbol{\alpha})$ is always positive, except at the origin. In the case of the MCC model the origin is exactly on the boundary of the plastic admissible domain.

3 Modified Cam-Clay model

This section is concerned with the energy and dissipation potentials associated with the Modified Cam-Clay model of Roscoe and Burland (1968). We adopt,

in the first part of this section, the assumptions of Collins and Houlsby (1997) and Houlsby and Puzrin (2006).

In Subsection 3.3, we comment on the non-associativity of the MCC equations and justify a re-interpretation of the variables and parameters in the dissipation potential. This leads to the mixed approach proposed in Section 4.

3.1 MCC state equations

Let us introduce the notation

$$\sigma_m := \frac{1}{3}\mathrm{tr}\,\boldsymbol{\sigma} \qquad \boldsymbol{S} = \boldsymbol{\sigma}^{\mathrm{dev}} := \boldsymbol{\sigma} - \sigma_m \boldsymbol{1} \tag{38}$$

$$\varepsilon_v := \mathrm{tr}\,\boldsymbol{\varepsilon} \qquad \boldsymbol{\varepsilon}^{\mathrm{dev}} := \boldsymbol{\varepsilon} - \frac{1}{3}(\mathrm{tr}\,\boldsymbol{\varepsilon})\boldsymbol{1} \qquad d_v := \mathrm{tr}\,\boldsymbol{d} \tag{39}$$

for the following variables: the mean stress σ_m, the deviatoric part of stress $\boldsymbol{S}$, the volumetric strain ε_v, the deviatoric part of strain $\boldsymbol{\varepsilon}^{\mathrm{dev}}$ and the volumetric strain rate d_v.

We adopt in this paper the basic isotropic elastic potentials (40) and (41), as given by Houlsby and Puzrin (2006, p. 189). This allows us to present and discuss the essential features of the Modified Cam-Clay model without going through more cumbersome expressions. In this simplified elastic model the bulk modulus results proportional to pressure and the shear modulus is assumed constant (different from the more usual assumption of constant Poisson ratio). However, it is known that in order to improve the adequacy of the ideal model to actual soil behavior more sophisticated elastic equations should be adopted. A thorough discussion of this issue is given by Houlsby and Puzrin (2006) in Chapter 9, which also contains non-linear elastic relationships

more closely representing real soils. It is worth mentioning that some other elastic relations adopted for MCC in the literature are not derived from an energy potential and this may lead to non-reversible responses.

Let the Helmholtz free energy be given by

$$W^e(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) = \mathsf{G}\|\boldsymbol{\varepsilon}^{\mathrm{dev}} - (\boldsymbol{\varepsilon}^p)^{\mathrm{dev}}\|^2 + \tilde{\kappa} p_0 \exp\left(-\frac{\varepsilon_v - \varepsilon_v^p}{\tilde{\kappa}}\right) \tag{40}$$

$$W^h(\boldsymbol{\varepsilon}^p) = \frac{1}{2}(\tilde{\lambda} - \tilde{\kappa}) p_{x0} \exp\left(-\frac{\varepsilon_v^p}{\tilde{\lambda} - \tilde{\kappa}}\right) \tag{41}$$

where the shear modulus G is assumed to be constant, p_0 and p_{x0} are reference values of hydrostatic pressure, $\tilde{\lambda}$ is the virgin compression index and $\tilde{\kappa}$ is the swell-recompression index.

The corresponding state equations are (cf.: Hjiaj, 1999, p. 61; de Borst and Groen, 2000, p. 33; Vaunat et al., 2000, p. 126; Ulm and Coussy, 2003, p. 293, Houlsby and Puzrin, 2006, p. 189).

$$\boldsymbol{S} = \nabla_{\boldsymbol{\varepsilon}^{\mathrm{dev}}} W^e = 2\mathsf{G}\left[\boldsymbol{\varepsilon}^{\mathrm{dev}} - (\boldsymbol{\varepsilon}^p)^{\mathrm{dev}}\right] \tag{42}$$

$$\sigma_m = \nabla_{\varepsilon_v} W^e = -p_0 \exp\left(-\frac{\varepsilon_v - \varepsilon_v^p}{\tilde{\kappa}}\right) \tag{43}$$

$$\boldsymbol{\rho} = \nabla_{\boldsymbol{\varepsilon}^p} W^h = -\rho \boldsymbol{1} \tag{44}$$

with

$$\rho := \frac{1}{2} p_{x0} \exp\left(-\frac{\varepsilon_v^p}{\tilde{\lambda} - \tilde{\kappa}}\right) \tag{45}$$

Thus

$$\boldsymbol{\kappa} = \boldsymbol{\sigma} + \rho \boldsymbol{1} \tag{46}$$

3.2 MCC evolution equations

The dissipation function that characterizes MCC constitutive behavior, given by

$$D(\boldsymbol{d}^p, \varepsilon_v^p) = \rho(\varepsilon_v^p)\sqrt{\frac{2M^2}{3}\|\boldsymbol{d}^{p\,\mathrm{dev}}\|^2 + (d_v^p)^2} \tag{47}$$

was first found by Houlsby (1981). We take this potential as the starting point for the definition of the evolution equations. In fact, this dissipation function leads to the widely known MCC elastic domains in the form of ellipses in the stress space, as proven in the following.

It is worth identifying the main features in (47). The following analysis shows that the MCC dissipation potential is, in a way, a simple quadratic proposal. Indeed, according to (47):

(1) Shear plastic strain rate components are weighted by a nondimensional material parameter M, when compared to the volumetric plastic strain rate.

(2) The positively homogeneous function expressing the dependence of the dissipation with respect to the plastic strain rate $\boldsymbol{d}^p$ is the Euclidean norm of the mean and weighted shear strain rate components

$$\|\boldsymbol{d}^p\|_* := \sqrt{a^{-1}\|\boldsymbol{d}^{p\,\mathrm{dev}}\|^2 + (d_v^p)^2} \tag{48}$$

where

$$a := \frac{3}{2M^2} \tag{49}$$

The function (48) belongs to the class of gauge functions, therefore it is admissible as a dissipation potential. In particular, it is a norm for strain rate tensors, whose polar function is then its dual norm, computed in

Appendix A (cf. de Saxcé, 1995, p. 4)

$$\|\boldsymbol{\kappa}\|_{**} := \sqrt{a\|\boldsymbol{\kappa}^{\mathrm{dev}}\|^2 + (\kappa_m)^2} \tag{50}$$

(3) The dissipation is proportional to a strength parameter that only depends on the volumetric part of the kinematical internal variable. This is a crucial assumption of this model, which is founded on experimental evidence (Wood, 1990, p. 89). In the present framework, the role of the strength parameter is played by the (scalar) back stress ρ.

Obtaining the plastic admissibility domain is just a matter of applying the tools of convex analysis. Indeed, the polar function of the dissipation potential, computed in Appendix A, is

$$\hat{\Phi}(\boldsymbol{\kappa}, \varepsilon_v^p) = [\rho(\varepsilon_v^p)]^{-1}\|\boldsymbol{\kappa}\|_{**} = [\rho(\varepsilon_v^p)]^{-1}\sqrt{a\|\boldsymbol{\kappa}^{\mathrm{dev}}\|^2 + (\kappa_m)^2} \tag{51}$$

As a result, the corresponding canonical representation of the plastic admissibility domain $P(\varepsilon_v^p)$ is given by the condition

$$f^0(\boldsymbol{\kappa}, \varepsilon_v^p) := [\rho(\varepsilon_v^p)]^{-1}\sqrt{a\|\boldsymbol{\kappa}^{\mathrm{dev}}\|^2 + (\kappa_m)^2} - 1 \leqslant 0 \tag{52}$$

However, for the sake of convenience, we adopt the following equivalent plastic admissibility constraint

$$\tilde{f}(\boldsymbol{\kappa}, \varepsilon_v^p) := \|\boldsymbol{\kappa}\|_{**} - \rho(\varepsilon_v^p) = \sqrt{a\|\boldsymbol{\kappa}^{\mathrm{dev}}\|^2 + (\kappa_m)^2} - \rho(\varepsilon_v^p) \leqslant 0 \tag{53}$$

The evolution equations are obtained by computing derivatives of the plastic function. This gives

$$\boldsymbol{d}^{p\,\mathrm{dev}} = \dot{\lambda}\frac{a}{\|\boldsymbol{\kappa}\|_{**}}\boldsymbol{\kappa}^{\mathrm{dev}} \qquad d_v^p = \dot{\lambda}\frac{\kappa_m}{\|\boldsymbol{\kappa}\|_{**}} \tag{54}$$

We note that the above equations imply $\|\boldsymbol{d}^p\|_* = \dot{\lambda}$.

Finally, we rewrite the plastic admissibility constraint in terms of the true stress by substituting, in view of (46), $\boldsymbol{\kappa}^{\mathrm{dev}} = \boldsymbol{S}$ and $\kappa_m = \sigma_m + \rho$.

$$\hat{f}(\boldsymbol{\sigma}, \varepsilon_v^p) := \|\boldsymbol{\sigma} + \rho(\varepsilon_v^p)\boldsymbol{1}\|_{**} - \rho(\varepsilon_v^p) = \sqrt{a\|\boldsymbol{S}\|^2 + [\sigma_m + \rho(\varepsilon_v^p)]^2} - \rho(\varepsilon_v^p) \leqslant 0 \quad (55)$$

Accordingly, the flow equations are

$$\boldsymbol{d}^{p\,\mathrm{dev}} = \dot{\lambda}\frac{a}{\hat{R}}\boldsymbol{S} \qquad d_v^p = \dot{\lambda}\frac{\sigma_m + \rho}{\hat{R}} \quad (56)$$

$$\dot{\lambda}\hat{f}(\boldsymbol{\sigma}, \varepsilon_v^p) = 0 \qquad \hat{f}(\boldsymbol{\sigma}, \varepsilon_v^p) \leqslant 0 \qquad \dot{\lambda} \geqslant 0 \quad (57)$$

where

$$\hat{R} := \sqrt{a\|\boldsymbol{S}\|^2 + [\sigma_m + \rho(\varepsilon_v^p)]^2} \quad (58)$$

3.3 Non-associativity of the MCC model

In the previous presentation of the dissipation potential and corresponding evolution equations, we considered the statical internal variable ρ as a passive parameter, which is a function, given by a state law, of the kinematical hardening variable ε_v^p. However, ρ is first considered as an independent state variable and afterwards linked to ε_v^p by the gradients of the Helmholtz and Gibbs energy potentials. Therefore, there is no need, in the framework of the above development, to include a complementary evolution equation for the dual hardening variable because this is already taken into account by the second flow law (56).

We now change from the previous approach, discussed above, with respect to the role of ρ in the dissipation potential and the evolution equations. This

will allow some variational formulations for the constitutive equations of MCC materials to be obtained where the participation of the kinematical and statical variables is clearly understood. To this aim, it is convenient to formally identify the strain-like hardening internal variable by an independent symbol, β, which is the dual of ρ, and later introduce an additional constraint $\beta = \varepsilon_v^p$. Then, the dual pair of generalized stress and strain rate is denoted

$$\boldsymbol{\Sigma} := (\boldsymbol{\sigma}, \rho) \qquad \boldsymbol{D}^p := (\boldsymbol{d}^p, \dot{\beta}) \tag{59}$$

with scalar product $\boldsymbol{\Sigma} \cdot \boldsymbol{D}^p := \boldsymbol{\sigma} \cdot \boldsymbol{d}^p + \rho\dot{\boldsymbol{\beta}}$.

Accordingly, there is a fixed plastic admissibility domain P in the space of the generalized stress; in the present notation

$$\boldsymbol{\Sigma} = (\boldsymbol{\sigma}, \rho) \in P \qquad \Leftrightarrow$$
$$f(\boldsymbol{\sigma}, \rho) := \|\boldsymbol{\sigma} + \rho \boldsymbol{1}\|_{**} - \rho = \sqrt{a\|\boldsymbol{S}\|^2 + (\sigma_m + \rho)^2} - \rho \leqslant 0 \tag{60}$$

The associated flow law (56) cannot be extended as a complete associated flow law in terms of generalized stress, and at the same time, have the hardening strength ρ associated with a flux variable $\dot{\beta}$ that is identical to the volumetric plastic strain rate d_v^p. This is proven in the following.

Computing the gradients of the plastic function defined in (60), we get

$$\nabla_{\boldsymbol{S}} f = \frac{a}{R}\boldsymbol{S} \qquad \nabla_{\sigma_m} f = \frac{\sigma_m + \rho}{R} \qquad \nabla_{\rho} f = \frac{\sigma_m + \rho}{R} - 1 \tag{61}$$

where, for simplicity, it is denoted

$$R := \|\boldsymbol{\sigma} + \rho \boldsymbol{1}\|_{**} = \sqrt{a\|\boldsymbol{S}\|^2 + (\sigma_m + \rho)^2} \tag{62}$$

The first two components of the generalized gradient (61) give the correct

associated fluxes that correspond to true stress components, i.e.

$$\boldsymbol{d}^{p\,\mathrm{dev}} = \dot{\lambda}\frac{a}{R}\boldsymbol{S} \qquad d_v^p = \dot{\lambda}\frac{\sigma_m + \rho}{R} \tag{63}$$

$$\dot{\lambda} f(\boldsymbol{\sigma}, \rho) = 0 \qquad f(\boldsymbol{\sigma}, \rho) \leqslant 0 \qquad \dot{\lambda} \geqslant 0 \tag{64}$$

in accordance to (56) and (57). Equivalently

$$\boldsymbol{d}^p = \frac{\dot{\lambda}}{R}\left[a\boldsymbol{S} + \frac{\sigma_m + \rho}{3}\boldsymbol{1}\right] \tag{65}$$

We note, for future use, that the two equations in (63), or (65), imply that

$$\|\boldsymbol{d}^p\|_* = \dot{\lambda} \tag{66}$$

Now it is clear, from (61) and (63), that an associated flow for the strain-like hardening variable $\dot{\beta}$ cannot be adopted because $\dot{\lambda}\nabla_\rho f = \dot{\lambda}(\frac{\sigma_m+\rho}{R} - 1) \neq d_v^p$.

In summary, we propose the introduction of the dual pair of generalized stress and strain rate (59) and justify the necessity of developing a broader formal structure for the evolution equations in order to accommodate the non-associated hardening flow law of an MCC material. This structure is the subject of the next section.

4 A mixed approach to MCC

We introduce now the mixed dissipation function

$$\begin{aligned}\tilde{D}(\boldsymbol{\Sigma}, \boldsymbol{D}^p) &= \tilde{D}(\boldsymbol{\sigma}, \rho, \boldsymbol{d}^p, \dot{\beta}) \\ &:= \rho\|\boldsymbol{d}^p\|_* + \mathcal{I}_P(\boldsymbol{\sigma}, \rho) + \mathcal{I}_K(\boldsymbol{d}^p, \dot{\beta}) \\ &= \rho\sqrt{a^{-1}\|\boldsymbol{d}^{p\,\mathrm{dev}}\|^2 + (d_v^p)^2} + \mathcal{I}_P(\boldsymbol{\sigma}, \rho) + \mathcal{I}_K(\boldsymbol{d}^p, \dot{\beta})\end{aligned} \tag{67}$$

where the indicator function $\mathcal{I}_P(\boldsymbol{\sigma}, \rho)$ equals 0 if $\|\boldsymbol{\sigma} + \rho \mathbf{1}\|_{**} \leqslant \rho$ and $+\infty$ otherwise, and the indicator function $\mathcal{I}_K(\boldsymbol{d}^p, \dot{\boldsymbol{\beta}})$ equals 0 if $\dot{\beta} = d_v^p$ and $+\infty$ otherwise.

In other words, the functions $\mathcal{I}_P$ and $\mathcal{I}_K$ include, by exact penalty in the dissipation potential (47), respectively: the plastic admissibility constraint for the generalized stress and the kinematical admissibility constraint for the generalized flux.

We prove in the following proposition that the mixed dissipation (67) gives an implicit potential form of the MCC flow law. This result was first found by de Saxcé (1995, p. 4).

Proposition 1. The set of MCC evolution equations (63), together with $\dot{\beta} = d_v^p$, is equivalent to

$$(\boldsymbol{d}^p, \dot{\beta}) \in \partial_{(\boldsymbol{\sigma},\rho)} \tilde{D}(\boldsymbol{\sigma}, \rho, \boldsymbol{d}^p, \dot{\beta}) \tag{68}$$

Proof. The subdifferential of a sum can be computed by adding the two separate contributions if one of the terms is a differentiable function (Hiriart-Urruty and Lemaréchal, 1993a, p. 261); thus,

$$\begin{aligned} \partial_{(\boldsymbol{\sigma},\rho)} \tilde{D}(\boldsymbol{\sigma}, \rho, \boldsymbol{d}^p, \dot{\beta}) &= \nabla_{(\boldsymbol{\sigma},\rho)} \left(\rho \|\boldsymbol{d}^p\|_*\right) + \partial_{(\boldsymbol{\sigma},\rho)} \mathcal{I}_P(\boldsymbol{\sigma}, \rho) \\ &= (\mathbf{0}, \|\boldsymbol{d}^p\|_*) + \partial_{(\boldsymbol{\sigma},\rho)} \mathcal{I}_P(\boldsymbol{\sigma}, \rho) \end{aligned} \tag{69}$$

Then $(\boldsymbol{d}^p, \dot{\beta}) \in \partial_{(\boldsymbol{\sigma},\rho)} \tilde{D}(\boldsymbol{\sigma}, \rho, \boldsymbol{d}^p, \dot{\beta})$ is equivalent to

$$\boldsymbol{d}^{p\,\mathrm{dev}} = \dot{\lambda} \frac{a}{R} \boldsymbol{S} \qquad d_v^p = \dot{\lambda} \frac{\sigma_m + \rho}{R} \qquad \dot{\beta} = \|\boldsymbol{d}^p\|_* + \dot{\lambda} \left(\frac{\sigma_m + \rho}{R} - 1 \right) \tag{70}$$

$$\dot{\lambda} f(\boldsymbol{\sigma}, \rho) = 0 \qquad f(\boldsymbol{\sigma}, \rho) \leqslant 0 \qquad \dot{\lambda} \geqslant 0 \tag{71}$$

The first two equations in (70) are equivalent to (63). The third one, taking into account (66), gives

$$\dot{\beta} = \dot{\lambda} + \dot{\lambda}\left(\frac{\sigma_m + \rho}{R} - 1\right) = \dot{\lambda}\frac{\sigma_m + \rho}{R} = d_v^p \tag{72}$$

This completes the proof. □

In the following, we verify that the mixed dissipation potential (67) belongs to the class of functions called *bipotentials* by de Saxcé (1992). The function $\tilde{D}(\boldsymbol{\Sigma}, \boldsymbol{D}^p)$ complies with the definition of bipotentials if it is bi-convex, i.e. separately convex in $\boldsymbol{\Sigma}$ and $\boldsymbol{D}^p$, and satisfies the following condition

$$\tilde{D}(\boldsymbol{\Sigma}, \boldsymbol{D}^p) \geqslant \boldsymbol{\Sigma} \cdot \boldsymbol{D}^p \quad \forall\ (\boldsymbol{\Sigma}, \boldsymbol{D}^p) \tag{73}$$

In fact, considering (67), the non-strict convexity with respect to $\boldsymbol{\Sigma}$ holds because the mixed dissipation is the sum of a linear function in $\boldsymbol{\Sigma}$ with an indicator function, which is also non-strictly convex. It is also non-strictly convex with respect to $\boldsymbol{D}^p$ for similar reasons.

It only remains to prove the variational inequality below to verify that $\tilde{D}(\boldsymbol{\Sigma}, \boldsymbol{D}^p)$ is a bipotential.

$$\tilde{D}(\boldsymbol{\sigma}, \rho, \boldsymbol{d}^p, \dot{\beta}) \geqslant \boldsymbol{\sigma} \cdot \boldsymbol{d}^p + \rho\dot{\beta} \quad \forall\ (\boldsymbol{\sigma}, \rho, \boldsymbol{d}^p, \dot{\beta}) \tag{74}$$

Proof. The inequality is trivially valid whenever any indicator function in (67) equals $+\infty$. Thus, we have to consider the above condition when the admissibility constraints are fulfilled. Therefore, we must prove that

$$\rho\|\boldsymbol{d}^p\|_* \geqslant (\boldsymbol{\sigma} + \rho\boldsymbol{1}) \cdot \boldsymbol{d}^p \quad \forall\ (\boldsymbol{\sigma}, \rho, \boldsymbol{d}^p) \ \mid\ \|\boldsymbol{\sigma} + \rho\boldsymbol{1}\|_{**} \leqslant \rho \tag{75}$$

We now use the fundamental inequality (A.5) for the dual norms $\|\boldsymbol{\kappa}\|_{**}$ and

$\|\boldsymbol{d}^p\|_*$ and the constraint above to get

$$(\boldsymbol{\sigma} + \rho\boldsymbol{1}) \cdot \boldsymbol{d}^p \leqslant \|\boldsymbol{\sigma} + \rho\boldsymbol{1}\|_{**}\|\boldsymbol{d}^p\|_* \leqslant \rho\|\boldsymbol{d}^p\|_* \tag{76}$$

This is (75) and the proof is complete. □

The equivalence of the following relations holds true as a consequence of the verification that the function $\tilde{D}(\boldsymbol{\Sigma}, \boldsymbol{D}^p)$ is a bipotential (de Saxcé, 1992; see also Zouain et al., 2007, p. 4396).

$$\tilde{D}(\boldsymbol{\Sigma}, \boldsymbol{D}^p) = \boldsymbol{\Sigma} \cdot \boldsymbol{D}^p \quad \Leftrightarrow \boldsymbol{D}^p \in \partial_{\boldsymbol{\Sigma}} \tilde{D}(\boldsymbol{\Sigma}, \boldsymbol{D}^p) \quad \Leftrightarrow \quad \boldsymbol{\Sigma} \in \partial_{\boldsymbol{D}^p} \tilde{D}(\boldsymbol{\Sigma}, \boldsymbol{D}^p) \tag{77}$$

Moreover, since the second relation above is the flow law (68) of the MCC material, we conclude that (77) displays three equivalent forms of the constitutive relation identifying this flow law.

In summary, we use the mixed dissipation $\tilde{D}(\boldsymbol{\Sigma}, \boldsymbol{D}^p)$, defined in (67), to characterize the MCC flow law in the following manner: a generalized stress $\boldsymbol{\Sigma}$ and a generalized plastic strain rate $\boldsymbol{D}^p$ are related by the flow law, and we denote this by

$$(\boldsymbol{\Sigma}, \boldsymbol{D}^p) \in \mathfrak{F} \tag{78}$$

if and only if this pair is extremal for the bipotential, i.e. this pair satisfies one of the equivalent conditions (77).

In general, as discussed by de Saxcé and coworkers (de Saxcé, 1992, 1995; de Saxcé and Bousshine, 1998; see also Bodovillé, 2001), implicit standard materials (ISM) are characterized by evolution equations of this kind.

4.1 The dissipation gap potential

We introduce the concept of potential dissipation excess, which is represented by the dissipation gap function

$$\psi(\boldsymbol{\Sigma}, \boldsymbol{D}^p) := \tilde{D}(\boldsymbol{\Sigma}, \boldsymbol{D}^p) - \boldsymbol{\Sigma} \cdot \boldsymbol{D}^p \tag{79}$$

For an arbitrary choice of $(\boldsymbol{\Sigma}, \boldsymbol{D}^p)$ the dissipation gap function gives the difference between the available power for dissipation and actual dissipated power of the arguments.

The dissipation gap function of the MCC material is, in unabridged notation,

$$\psi(\boldsymbol{\sigma}, \rho, \boldsymbol{d}^p, \dot{\beta}) := \rho\sqrt{a^{-1}\|\boldsymbol{d}^{p\,\mathrm{dev}}\|^2 + (d_v^p)^2} - \boldsymbol{\sigma}\cdot\boldsymbol{d}^p - \rho\dot{\beta} + \mathcal{I}_P(\boldsymbol{\sigma}, \rho) + \mathcal{I}_K(\boldsymbol{d}^p, \dot{\beta}) \tag{80}$$

The following proposition gives the relation of this dissipation gap function to the flow law.

Proposition 2. Let ψ be given by (79) and $\mathfrak{F}$ defined in (78), or (77), then

$$\psi_0 := \inf_{\boldsymbol{\Sigma}, \boldsymbol{D}^p} \psi(\boldsymbol{\Sigma}, \boldsymbol{D}^p) \tag{81}$$

is finite and nonnegative. Further,

(1) If $\psi_0 = 0$ then

$$(\boldsymbol{\Sigma}, \boldsymbol{D}^p) \in \mathfrak{F} \qquad \Leftrightarrow \qquad (\boldsymbol{\Sigma}, \boldsymbol{D}^p) \in \arg\inf_{\boldsymbol{\Sigma}, \boldsymbol{D}^p} \psi(\boldsymbol{\Sigma}, \boldsymbol{D}^p) \tag{82}$$

(2) If $\psi_0 > 0$ then $\mathfrak{F}$ is empty.

The symbol arg inf above denotes the set of solutions to the minimization

problem.

Proof. The dissipation gap function $\psi(\boldsymbol{\Sigma}^*, \boldsymbol{D}^{p*})$ is proper (Rockafellar, 1970) and nonnegative for all $(\boldsymbol{\Sigma}^*, \boldsymbol{D}^{p*})$, as a consequence of the definition of the bipotential. Further, if $(\boldsymbol{\Sigma}, \boldsymbol{D}^p) \in \mathfrak{F}$ then using (77), it follows that $\psi(\boldsymbol{\Sigma}, \boldsymbol{D}^p) = 0$, which leads directly to the thesis. □

It is worth noting that we consider in Proposition 2 that $\mathfrak{F}$ is eventually empty because this is possible for functions that only comply with the hypothesis of the theorem. However, if $\mathfrak{F}$ is empty, the proposed model for the flow law is not applicable. This situation is detected by the presence of a positive minimum value ψ_0.

In the particular case of the MCC model, the set of stresses and fluxes related by the flow law is not empty because $(\boldsymbol{\Sigma}, \boldsymbol{D}^p) = (\mathbf{0}, \mathbf{0})$, at least, belongs to $\mathfrak{F}$. Thus, according to the proposition above, $\mathfrak{F}$ coincides, for MCC, with the set of solutions of the optimization problem (81).

We will show in the next sections that a proposition analogous to the one stated above, for a material point, describes the behavior of the whole body with respect to the occurrence of plastic collapse.

5 Plastic collapse of MCC soils

Plastic collapse is a phenomenon characterized by (incipient) unbounded plastic deformation in a process where loadings and the stress distribution remain constant in time. As an example, this concept is identified in the case of the drained triaxial compression test of a MCC soil.

Then, we consider a precise definition of this critical state, taking into account the particular features introduced when considering non-associated plasticity.

Afterwards, we present a non-standard mixed minimization principle and its relationship with plastic collapse solutions.

5.1 Triaxial compression test and plastic collapse

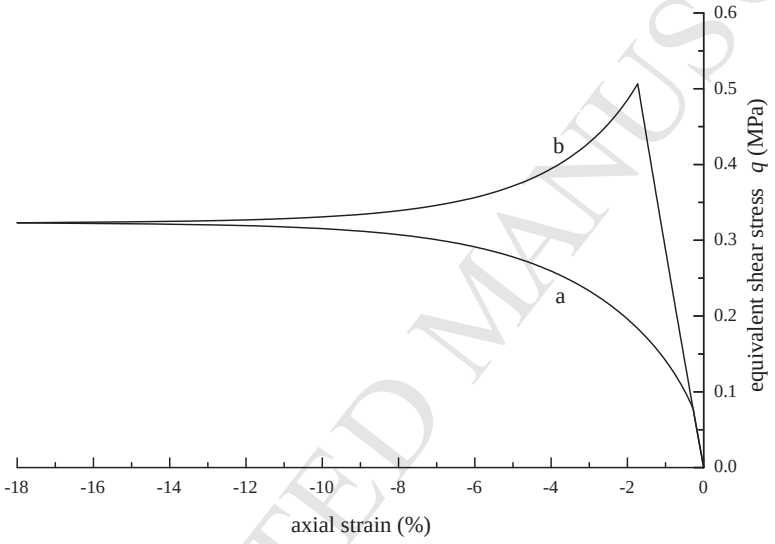

Fig. 1. Equivalent shear stress q versus axial strain ε_z in a simulated triaxial compression test with cell pressure $p_0 = 0.2\,\mathrm{MPa}$. Curve (a) corresponds to a lightly overconsolidated soil with OCR = 1.25 and (b) refers to a heavily overconsolidated soil with OCR = 5. Both are exact solutions for a MCC material with $G = 11.54\,\mathrm{MPa}$, $M = 1.05$, $\tilde{\lambda} = 0.032$ and $\tilde{\kappa} = 0.013$.

The influence of non-associated yield laws in the plastic collapse phenomenon can be recognized in the response of a MCC soil to a simulated triaxial compression test. For this purpose we consider a cylinder of MCC material under constant confining pressure $p_0 = 0.2\,\mathrm{MPa}$ and subjected to a displacement-

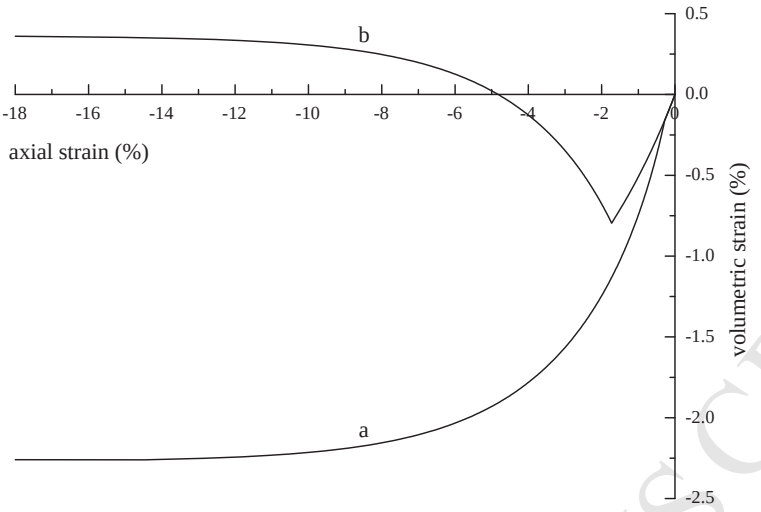

Fig. 2. Volumetric strain ε_v versus axial strain ε_z in a simulated triaxial compression test with cell pressure $p_0 = 0.2\,\mathrm{MPa}$ and same material as in Figure 1. Curve (a) corresponds to a lightly overconsolidated soil with OCR = 1.25 and (b) refers to a heavily overconsolidated soil with OCR = 5.

driven loading program imposing an additional variable axial compression q, i.e. $\sigma_z = -p_0 - q$ and $\sigma_r = \sigma_\theta = -p_0$ (r, θ and z are cylindrical coordinates). In Figures 1 and 2, we present exact solutions (computed from closed form solutions) for a MCC material with constant shear modulus $G = 11.54\,\mathrm{MPa}$, and with $M = 1.05$, $\tilde{\lambda} = 0.032$ and $\tilde{\kappa} = 0.013$. This example was solved by de Borst and Groen (2000, p. 34) for a MCC material model with constant Poisson ratio $\nu = 0.2$ and same values of M, $\tilde{\lambda}$ and $\tilde{\kappa}$ (see also Wood, 1990, p. 123).

We compare the incremental responses of two samples having different overconsolidation ratios OCR $:= 2\rho_0/p_0$ (ρ_0 is the initial value of the hardening variable): (a) a lightly overconsolidated soil with OCR = 1.25 and (b) a heavily overconsolidated soil, with OCR = 5. Two facts are worth noting:

(1) It is apparent in Figure 1 that, irrespective of the initial conditions and the loading path, there is a critical load $q_c = 3Mp_0/(3-M) = 0.323\,\text{MPa}$ (Zouain et al., 2007, p. 4393) producing plastic collapse, i.e. unbounded deformation under constant loadings and stress distribution. This is a critical state in soil mechanics. In this case the collapse load depends on the confining pressure p_0, not only on material constants.

(2) When the heavily overconsolidated soil (b) is sheared (see Figure 1) the load q is amplified up to a peak value of $q_{\text{peak}} = 0.05\,\text{MPa}$ and then decreases asymptotically to the collapse load q_c. Consequently, the collapse load cannot be called limit load (in the classical sense) for this non-associated elastoplastic material. Indeed, there is no load greater or equal than all loading values effectively sustained by the system in any loading path, under constant confining pressure, initiating at arbitrary maximum past isotropic consolidation pressure (cf. Wood, 1990, p. 188).

(3) Figure 2 shows that plastic collapse occurs under constant volume deformation. This is predicted since during the collapse phenomenon: (i) the volumetric elastic strain rate is zero because stresses are constant and (ii) hardening must remain frozen and therefore the volumetric plastic strain rate must be zero because it is associated to hardening evolution in the MCC material.

5.2 Additional notation

In the following the symbol $\mathcal{D}$ denotes the linear deformation operator mapping velocities (displacements) into compatible strain rates (respective strains). The internal power associated with a stress field $\boldsymbol{\sigma}$ and a velocity distribution

w is denoted

$$\langle \boldsymbol{\sigma}, \mathcal{D}\boldsymbol{w} \rangle := \int_{\mathcal{B}} \boldsymbol{\sigma} \cdot \mathcal{D}\boldsymbol{w} \, d\mathcal{B} \tag{83}$$

Likewise, the external power of a load system $\boldsymbol{F}$ is

$$\langle \boldsymbol{F}, \boldsymbol{w} \rangle := \int_{\mathcal{B}} \boldsymbol{b} \cdot \boldsymbol{w} \, d\mathcal{B} + \int_{\Gamma_{\boldsymbol{\tau}}} \boldsymbol{\tau} \cdot \boldsymbol{w} \, d\Gamma \tag{84}$$

where $\boldsymbol{b}$ and $\boldsymbol{\tau}$ are volume and surface load densities and $\Gamma_{\boldsymbol{\tau}}$ is part of the boundary Γ where traction is prescribed (null displacements are imposed in $\Gamma_{\boldsymbol{u}}$, with $\Gamma = \Gamma_{\boldsymbol{\tau}} \bigcup \Gamma_{\boldsymbol{u}}$ and $\Gamma_{\boldsymbol{\tau}} \bigcap \Gamma_{\boldsymbol{u}}$ empty). The set of stress fields in equilibrium with a given load system $\boldsymbol{F}$ is defined as

$$\mathcal{S}(\boldsymbol{F}) := \{\boldsymbol{\sigma} \;\mid\; \langle \boldsymbol{\sigma}, \mathcal{D}\boldsymbol{w} \rangle = \langle \boldsymbol{F}, \boldsymbol{w} \rangle \; \forall \; \boldsymbol{w}\} \tag{85}$$

where the virtual velocity $\boldsymbol{w}$ varies in the linear space of admissible velocities.

5.3 The plastic collapse equations

When the body undergoes plastic collapse the stress and the internal thermodynamic forces remain constant. In view of the elastic state equations, a constant stress field induces an elastic deformation distribution that is also constant in time. This, in turn, means that the strain rate field is at the same time compatible and purely plastic.

Likewise, according to the state equation relative to internal variables, internal hardening forces that are constant during plastic collapse imply that the kinematical internal variables are also constant, that is, $\dot{\boldsymbol{\beta}} = 0$ at all points in the body (cf. Polizzotto et al., 1991).

This description of plastic collapse is implemented in the following. The computation of the critical factor α that amplifies a prescribed load system $\boldsymbol{F}$ so

as to produce unbounded purely plastic deformation, when superposed to a fixed (non-amplified) load $\boldsymbol{F}^0$, can be formulated as

PC - *The plastic collapse problem.* Find $(\alpha, \boldsymbol{\sigma}, \rho, v)$ such that

$$\boldsymbol{\sigma} \in \mathcal{S}(\boldsymbol{F}^0 + \alpha \boldsymbol{F}) \tag{86}$$

$$\langle \boldsymbol{F}, \boldsymbol{v} \rangle = 1 \tag{87}$$

$$\tilde{D}(\boldsymbol{\sigma}, \rho, \mathcal{D}\boldsymbol{v}, 0) = \boldsymbol{\sigma} \cdot \mathcal{D}\boldsymbol{v} \quad \text{in } \mathcal{B} \tag{88}$$

$$\alpha \geqslant 0 \tag{89}$$

Condition (87) is only included in order to select one normalized velocity distribution, which rules out all its scalar multiples that would be solutions otherwise.

The system (86-89) represents general conditions for plastic collapse that are applicable (with minor changes in notation) to any material behavior derived from a bipotential. In addition to these conditions, MCC materials also obey some particular equations that we consider in the following.

For MCC materials, the fact that the internal variable rate $\dot{\boldsymbol{\beta}}$ is zero implies that

$$d_v^p = 0 \tag{90}$$

Thus, plastic collapse for MCC materials takes place under conditions of constant volume deformation. A condition of plastic collapse under constant volume deformation is known in soil mechanics as a *critical state* (Schofield and Wroth, 1968; Roscoe and Burland, 1968; Wood, 1990, p. 139; Houlsby and Puzrin, 2006, p. 187).

From the condition $\dot{\boldsymbol{\beta}} = d_v^p = 0$ and the evolution equation (70) we get

$$\rho = -\sigma_m \qquad \text{if} \qquad \dot{\lambda} > 0 \tag{91}$$

We note that this relation only applies to material points having effective plastic deformation during collapse.

Finally, we introduce the additional constraint (90) in the mixed dissipation potential, to conclude that for MCC materials undergoing plastic collapse

$$\tilde{D}(\boldsymbol{\sigma}, \rho, \mathcal{D}\boldsymbol{v}, 0) = \begin{cases} \sqrt{\frac{2}{3}} M \rho \|\mathcal{D}\boldsymbol{v}\| & \text{if} \quad \operatorname{tr}\mathcal{D}\boldsymbol{v} = 0, \ f(\boldsymbol{\sigma}, \rho) \leqslant 0 \\ +\infty & \text{otherwise} \end{cases} \tag{92}$$

5.4 Variational plastic collapse analysis

We propose a minimization principle and then explain how it is used to analyze the existence of collapse solutions $(\alpha, \boldsymbol{\sigma}, \rho, \boldsymbol{v})$ to the plastic collapse problem PC, defined by (86-89).

In order to obtain the variational formulation given below, we impose (88) in the form of a minimization principle as given in (79), but now we attach (86), (87) and (89) as constraints.

MP - *A minimization principle*

$$\Upsilon = \inf_{\alpha, \boldsymbol{\sigma}, \rho, \boldsymbol{v}} \left\{ \Psi(\boldsymbol{\sigma}, \rho, \mathcal{D}\boldsymbol{v}, 0) \ \mid \ \boldsymbol{\sigma} \in \mathcal{S}(\boldsymbol{F}^0 + \alpha \boldsymbol{F}); \ \langle \boldsymbol{F}, \boldsymbol{v} \rangle = 1; \ \alpha \geqslant 0 \right\} \tag{93}$$

where $\Psi := \int_{\mathcal{B}} \psi d\mathcal{B}$.

The first remark on this variational formulation is the following lemma, whose proof is straightforward.

Lemma. The infimum Υ of the minimization principle MP is finite and non-negative if the feasible set is nonempty, otherwise it is $+\infty$.

By using $D := \int_{\mathcal{B}} \tilde{D} d\mathcal{B}$ and the constraints in (93), we get

$$\Psi(\boldsymbol{\sigma}, \rho, \mathcal{D}\boldsymbol{v}, 0) = D(\boldsymbol{\sigma}, \rho, \mathcal{D}\boldsymbol{v}, 0) - \langle \boldsymbol{\sigma}, \mathcal{D}\boldsymbol{v} \rangle = D(\boldsymbol{\sigma}, \rho, \mathcal{D}\boldsymbol{v}, 0) - \langle \boldsymbol{F}^0, \boldsymbol{v} \rangle - \alpha \tag{94}$$

Thus, the minimization principle (93) can be rewritten as

$$\Upsilon = \inf_{\alpha, \boldsymbol{\sigma}, \rho, \boldsymbol{v}} \Big\{ D(\boldsymbol{\sigma}, \rho, \mathcal{D}\boldsymbol{v}, 0) - \langle \boldsymbol{F}^0, \boldsymbol{v} \rangle - \alpha \;\; | \;\; \boldsymbol{\sigma} \in \mathcal{S}(\boldsymbol{F}^0 + \alpha \boldsymbol{F});\; \langle \boldsymbol{F}, \boldsymbol{v} \rangle = 1;\; \alpha \geqslant 0 \Big\} \tag{95}$$

The main results in the present variational analysis of plastic collapse can be summarized as follows: (i) MP has always a solution for Υ, finite or $+\infty$, and (ii) finding this solution can give the answer to whether or not solutions to the PC exist. This is precisely stated below.

Proposition 3. The plastic collapse problem PC and the minimization principle MP are related by the following implications:

(1) If there exists a solution $(\alpha, \boldsymbol{\sigma}, \rho, \boldsymbol{v})$ of PC, then this set is a minimizer for MP and corresponds to $\Upsilon = 0$.
(2) If MP has a minimizer $(\alpha, \boldsymbol{\sigma}, \rho, \boldsymbol{v})$ such that $\Upsilon = 0$, then this set is a solution for PC.

Proof.

(1) Collapse solutions satisfy all constraints in MP and give $\Upsilon = 0$ since $\psi = 0$ in $\mathcal{B}$.

(2) $\Upsilon = 0$ implies $\psi = 0$ in $\mathcal{B}$ and this, together with the constraints of MP, is the set of equations in PC. □

In other words, if we solve the minimization problem MP, analytically or numerically, we can conclude that:

(1) If $\Upsilon = 0$ then the computed minimal solution describes the plastic collapse for the obtained amplification factor α.
(2) If $\Upsilon > 0$ then the body does not collapse for any loading of the form $\boldsymbol{F}^0 + \alpha \boldsymbol{F}$.

The proposition above applies to material models derived from a bipotential, including those with associated flow laws, and not only for the MCC.

The minimization problem MP, given by (93) or (95), has convex constraints but its objective function is only bi-convex, and not necessarily jointly convex in its arguments. It encompasses the complexities inherent to non-convex optimization.

The expression of the minimization principle MP for soils consisting of MCC material is obtained by introducing (92) in (95); this gives

$$\Upsilon = \inf_{\alpha,\boldsymbol{\sigma},\rho,\boldsymbol{v}} \left\{ \int_{\mathcal{B}} \sqrt{\tfrac{2}{3}} M \rho \|\mathcal{D}\boldsymbol{v}\| d\mathcal{B} - \langle \boldsymbol{F}^0, \boldsymbol{v} \rangle - \alpha \;\; | \right.$$
$$\boldsymbol{\sigma} \in \mathcal{S}(\boldsymbol{F}^0 + \alpha \boldsymbol{F}),\; \langle \boldsymbol{F}, \boldsymbol{v} \rangle = 1,\; \alpha \geqslant 0,$$
$$\left. \operatorname{tr} \mathcal{D}\boldsymbol{v} = 0 \text{ in } \mathcal{B},\; \sqrt{\tfrac{3}{2M^2} \|\boldsymbol{S}\|^2 + (\sigma_m + \rho)^2} \leqslant \rho \text{ in } \mathcal{B} \right\} \quad (96)$$

The minimization principle (96) was also obtained in Zouain et al. (2007, p. 4390), but using a different development. Here, the starting point is an extension of the classical dissipation function to a mixed dissipation potential.

Moreover, now we have a variational formulation for the evolution equations that allows an easy transition to the global formulation in plastic collapse analysis. An example of the application of (96) and Proposition 3 is studied in Zouain et al. (2007, pp. 4390-4394).

6 Conclusions

In this paper we proposed a mixed variational formulation (81) for the evolution relations of MCC materials that matches all features of the most commonly used Cam-Clay model in soil mechanics.

It was shown in Section 4 that interpreting the known dissipation potential given by Houlsby (1981) as a mixed function, and adding penalty terms to exactly enforce plastic and kinematic admissibility, leads to the bi-convex potential $\tilde{D}(\boldsymbol{\sigma}, \rho, \boldsymbol{d}^p, \dot{\beta})$, in (67), giving implicit potential flow laws like those proposed by de Saxcé (1995). In this way, the bipotential for the evolution equations is interpreted as a dissipation potential. Moreover, it is obtained from a classical dissipation function by the simple consideration of statical and kinematic constraints.

The concept of excess of dissipation was introduced by defining the gap potential $\psi(\boldsymbol{\sigma}, \rho, \boldsymbol{d}^p, \dot{\beta})$ in (79). It was proven, in Proposition 2, that the set $\mathfrak{F}$ of stress and plastic strain rates related by the flow law coincides with the set of minimizers of the dissipation gap function.

Since the hardening evolution relation of MCC is not associated, the plastic collapse analysis of a soil obeying this constitutive behavior cannot be embedded in classical limit analysis theory. For example, a maximum load amplifier

does not necessarily exist and the well known lower and upper bounding principles are not valid.

However, we have shown, in Proposition 3, that there is a mixed minimization principle associated with the analysis of the critical state of plastic collapse of an MCC soil. This global variational formulation was directly deduced from the constitutive variational formulation of the flow law.

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A The polar of the dissipation function

Let us first justify (50). The dual norm corresponding to the norm for strain rates defined in (48) is defined by (Rockafellar, 1970, p. 128; Horn and Johnson, 1985, p. 277)

$$\|\boldsymbol{\kappa}\|_{**} := \sup_{\boldsymbol{d}\neq 0} \frac{\boldsymbol{\kappa}\cdot\boldsymbol{d}}{\|\boldsymbol{d}\|_{*}} \tag{A.1}$$

The above problem can be solved by a change of variables:

$$\hat{\boldsymbol{\kappa}} := \sqrt{a}\boldsymbol{\kappa}^{\mathrm{dev}} + \frac{1}{3\sqrt{3}}(\mathrm{tr}\,\boldsymbol{\kappa})\boldsymbol{1} \tag{A.2}$$

$$\hat{\boldsymbol{d}} := \frac{1}{\sqrt{a}}\boldsymbol{d}^{\mathrm{dev}} + \frac{1}{\sqrt{3}}(\mathrm{tr}\,\boldsymbol{d})\boldsymbol{1} \tag{A.3}$$

Then $\boldsymbol{\kappa}\cdot\boldsymbol{d} = \hat{\boldsymbol{\kappa}}\cdot\hat{\boldsymbol{d}}$, $\|\boldsymbol{d}\|_{*} = \|\hat{\boldsymbol{d}}\|$ and

$$\|\boldsymbol{\kappa}\|_{**} = \sup_{\hat{\boldsymbol{d}}\neq 0} \frac{\hat{\boldsymbol{\kappa}}\cdot\hat{\boldsymbol{d}}}{\|\hat{\boldsymbol{d}}\|} = \|\hat{\boldsymbol{\kappa}}\| = \sqrt{a\|\boldsymbol{\kappa}^{\mathrm{dev}}\|^2 + (\tfrac{1}{3}\mathrm{tr}\,\boldsymbol{\kappa})^2} \tag{A.4}$$

where we applied the fact that the Euclidean norm is its own dual (Horn and Johnson, 1985, p. 277). This proves (50).

The fundamental inequality characterizing dual norms

$$\|\boldsymbol{\kappa}\|_{**}\,\|\boldsymbol{d}\|_{*} \geqslant \boldsymbol{\kappa}\cdot\boldsymbol{d} \qquad \forall\ (\boldsymbol{\kappa},\boldsymbol{d}) \tag{A.5}$$

is a consequence of the definition (A.1).

Finally

(1) The polar of

$$D(\boldsymbol{d}^p,\varepsilon_v^p) = \rho(\varepsilon_v^p)\|\boldsymbol{d}^p\|_{*} \tag{A.6}$$

with respect to $\boldsymbol{d}^p$, is

$$\hat{\Phi}(\boldsymbol{\kappa},\varepsilon_v^p) = [\rho(\varepsilon_v^p)]^{-1}\|\boldsymbol{\kappa}\|_{**} \tag{A.7}$$

This is proven by an additional change of variables.

(2) The polar of

$$\tilde{D}(\boldsymbol{\sigma},\rho,\boldsymbol{d}^p,\dot{\beta}) = \rho\|\boldsymbol{d}^p\|_{*} + \mathcal{I}_P(\boldsymbol{\sigma},\rho) + \mathcal{I}_K(\boldsymbol{d}^p,\dot{\beta}) \tag{A.8}$$

with respect to $(\boldsymbol{d}^p,\dot{\beta})$, is

$$\tilde{\Phi}(\boldsymbol{\sigma},\rho) = \rho^{-1}\|\boldsymbol{\sigma}+\rho\boldsymbol{1}\|_{**} \tag{A.9}$$

Proving this requires using the constraints imposed by exact penalty and the same changes of variables.

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