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# Dynamic eigenvector problem in thermoelectroelasticity of dissipative ionic crystals

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## Abstract

A two dimensional problem in the Stroh formalism is derived for the continuum theory of thermoelectroelasticity with polarization gradients. Dissipative effects are accounted for, according to a constitutive model outlined in previous works. The eigenvector problem is studied in the frequency domain to obtain a representation of the solution in terms of two classes of modes corresponding to opposite signs of imaginary part of the eigenvalues. Impedance and admittance tensors are exploited to express the energy flux of the thermoelectroelastic transformed field across an interface  $\mathcal{S}$ . The compatibility conditions at  $\mathcal{S}$  are also derived. The eigenvector equations are then rewritten in the time domain to obtain two convolution-type integral equations for the Hilbert transforms of the real fields corresponding to each mode.

*Keywords:* continuum theory of electroelastic solids; electroelasticity with polarization gradients; Stroh formalism.

## 1 Introduction

The electromechanical continuum theory of ionic crystals has received a noticeable improvement by the works of Mindlin in the late sixties (Mindlin 1968, 1969). The main feature of the original approach consists in accounting for the polarization gradient in the constitutive equations in order to describe electroelastic couplings, just at a linear level, also in polarizable crystals which do not allow for the piezoelectric effect. A lattice's dynamic derivation of electroelastic coupling in alkali halide has supported this point of view showing that, in the long-wave approximation, contributions due to polarization gradient arise in the linearized model as a consequence of shell-shell and core-core interactions between the lattice's constituents (Askar et al. 1970, Askar and Lee, 1974).

Some application of this theory have been developed in the past, concerning both static and dynamic specific problems (see Mindlin 1969, Maugin 1988 and references therein, and, more recently, Nowacki 2004). It is worth remarking that theories of electroelastic media which include strain gradients in the constitutive equations have been recently investigated to account for the so called "flexoelectric effect" which consists in the converse effect, where polarization arises due to a strain gradient (Maranganti et al. 2006, Majdoub et al. 2008). The interest into both direct and converse effects is motivated by their suitability to account for a noticeable electromechanical coupling in thin structures and at interfaces or surfaces.

A comprehensive theoretical approach of the continuum theory of polarizable crystals was given by Maugin (1988). He derived a non-linear theory including electromagnetic and thermal coupling, also accounting for polarization gradient and polarization inertia. Some improvements of the general theory have been recently suggested introducing the point of view of internal variables to model dissipation in accordance with the second law of thermodynamics (Romeo 2007).

In this paper we formulate a two-dimensional vector problem in the Stroh formalism (Stroh 1962), rewriting the governing equations of the linear theory of continuum thermoelasticity of polarizable dielectric solids in the form of an eigenvector equation for a sixteen components field. We allow for non stationary solutions and use the Fourier transform to state an eigenvector problem in the frequency domain. The set of balance and constitutive equations is given in section 2 according to the Maugin's approach. Here, looking at the Stroh-type formulation, we write the constitutive equation for the shell-shell interaction tensor as a purely homogeneous equation, accounting for the polarization effects at the free surface within the boundary conditions. The dissipative effects are modeled via internal variables according to Romeo (2007). These variables are then eliminated in writing the governing equations for the transformed field, as shown in section 3. In section 4 we

show how the integral formalism (Lothe and Barnett, 1976a,1976b) can be applied in the frequency domain, and give a detailed discussion about the splitting of the solution to the eigenvector problem in two classes of modes corresponding to different signs of the imaginary part of the eigenvalues. The impedance and admittance tensors are used to express the energy flux at interfaces and the specific role of the polarization constraint at the free surface is outlined in section 5. Owing to its relevance in the theory of surface and interfacial wave propagation, in section 6 we give the compatibility conditions at an interface  $\mathcal{S}$  separating two different polarizable crystals. These conditions are directly obtained by the continuity of the pertinent physical fields at  $\mathcal{S}$ . Finally, in section 7 we rewrite the integrated eigenvector problem in the time domain, thus arriving at a couple of convolution-type integral equations for the Hilbert transform of the real physical field.

## 2 Thermoelastic continuum model for ionic crystals

Here we summarize the governing equations of the continuum model of thermoelectroelastic polarizable anisotropic media (Maugin 1988, ch. 7). They are characterized by the dependence of the constitutive functions from the polarization gradient and the presence of the polarization inertia in the balance equation for polarization.

We denote by  $\rho$  the mass density in the current configuration  $\mathcal{B}_t$  of an arbitrary portion of the continuum and by  $\mathbf{u}$ ,  $\mathbf{p}$ ,  $\varepsilon$  and  $\mathbf{q}$ , respectively, the mechanical displacement, the polarization vector, the energy density and the heat flux. Adopting the quasi-static hypothesis, the electric field will be given by  $-\nabla\phi$ , where  $\phi$  is the electric potential. Looking at the linearized model of the general non-linear theory, in absence of external mechanical body forces and heat supplies, we write the balance equations in  $\mathcal{B}_t$  as (Romeo 2007)

$$\rho\ddot{\mathbf{u}} = \nabla \cdot \mathbf{T}, \quad (2.1)$$

$$\delta\ddot{\mathbf{p}} = -\nabla\phi + \boldsymbol{\varepsilon} + \frac{1}{\rho}\nabla \cdot \boldsymbol{\mathfrak{E}}, \quad (2.2)$$

$$\rho\dot{\varepsilon} = -\nabla \cdot \mathbf{q}, \quad (2.3)$$

where  $\delta$  is the polarization inertia per unit mass. The quantities  $\mathbf{T}$ ,  $\boldsymbol{\varepsilon}$  and  $\boldsymbol{\mathfrak{E}}$  represent, respectively, the (Cauchy) stress tensor, the local electric field and the shell-shell or core-core interaction tensor due to the effects of polarization and its gradient. According to the divergence free condition for the electric displacement

$$\mathbf{d} = -\nabla\phi + \mathbf{p}, \quad (2.4)$$

we also have

$$\nabla \cdot \mathbf{p} - \Delta\phi = 0, \quad (2.5)$$

Denoting by  $\mathbf{n}$  the outward normal to the boundary  $\partial\mathcal{B}_t$  we have

$$\mathbf{n}\mathbf{T} = \mathbf{t}^{(\mathbf{n})}, \quad (2.6)$$

$$\frac{1}{\rho}\mathbf{n}\mathfrak{E} = \boldsymbol{\pi}^{(s)} - \mathbf{b}^{(0)}, \quad (2.7)$$

$$\mathbf{n} \cdot \mathbf{q} = q_n, \quad (2.8)$$

$$\mathbf{n} \cdot \mathbf{p} - \mathbf{n} \cdot \nabla\phi = d_n, \quad (2.9)$$

at any point in  $\partial\mathcal{B}_t$ .  $\mathbf{t}^{(\mathbf{n})}$ ,  $q_n$  and  $d_n$  represent, respectively, the mechanical traction acting at the boundary, the heat flux across  $\partial\mathcal{B}_t$  and the normal component of the electric displacement at  $\partial\mathcal{B}_t$ , while  $\boldsymbol{\pi}^{(s)}$ ,  $-\mathbf{b}^{(0)}$  are, respectively, the surface density of (possible) electric dipoles and the intrinsic "polarization traction" on  $\partial\mathcal{B}_t$  (cf. Romeo 2008). The last quantity is a constitutive parameter of the polarizable continuum, which depends on the microscopic structure of the crystal lattice (Askar and Lee 1974). According to the previous equations, the energy flux vector of the thermoelectroelastic field can be written as

$$\mathbf{J} = -\mathbf{T}\dot{\mathbf{u}} - \frac{1}{\rho}\mathfrak{E}\dot{\mathbf{p}} + \phi\dot{\mathbf{d}} + \mathbf{q}. \quad (2.10)$$

This expression is a consequence of the energy balance for the model at hand (Maugin 1988). It reduces to the usual energy flux vector of linear electroelasticity if the effects of polarization gradient and heat conduction are discarded (see for example Auld 1990).

In Romeo (2007) it has been shown that an effective description of dissipative effects in the thermoelectroelastic model with polarization gradients can be obtained by introducing a symmetric second order tensor  $\boldsymbol{\Omega}$  and a vector  $\boldsymbol{\chi}$  which play the role of internal variables obeying a couple of supplementary suitable evolution equations. The constitutive functions for  $\mathbf{T}$ ,  $\boldsymbol{\epsilon}$  and  $\mathfrak{E}$  also depends on  $\boldsymbol{\Omega}$  and  $\boldsymbol{\chi}$ . In the linear case we have

$$\mathbf{T} = \mathbb{U}^{\eta\eta}\boldsymbol{\eta} + \mathbb{U}^{\eta\mathbf{p}}\mathbf{p} + \mathbb{U}^{\eta\Pi}\nabla\mathbf{p} + \mathbb{U}^{\Omega\Omega}\boldsymbol{\Omega} + \mathbb{U}^{\Omega\boldsymbol{\chi}}\boldsymbol{\chi}, \quad (2.11)$$

$$\boldsymbol{\epsilon} = -(\mathbb{U}^{\mathbf{p}\eta}\boldsymbol{\eta} + \mathbb{U}^{\mathbf{p}\mathbf{p}}\mathbf{p} + \mathbb{U}^{\mathbf{p}\Pi}\nabla\mathbf{p}), \quad (2.12)$$

$$\frac{1}{\rho}\mathfrak{E} = \mathbb{U}^{\Pi\eta}\boldsymbol{\eta} + \mathbb{U}^{\Pi\mathbf{p}}\mathbf{p} + \mathbb{U}^{\Pi\Pi}\nabla\mathbf{p} + \mathbb{N}^t(\mathbb{U}^{\Omega\Omega}\boldsymbol{\Omega} + \mathbb{U}^{\Omega\boldsymbol{\chi}}\boldsymbol{\chi}) \quad (2.13)$$

$$\mathbf{q} = -(\mathbb{U}^{\boldsymbol{\chi}\Omega}\boldsymbol{\Omega} + \mathbb{U}^{\boldsymbol{\chi}\boldsymbol{\chi}}\boldsymbol{\chi})\mathbb{R}, \quad (2.14)$$

where  $\boldsymbol{\eta} = \frac{1}{2}[\nabla\mathbf{u} + (\nabla\mathbf{u})^T]$  is the infinitesimal strain tensor and where  $\mathbb{R}$  and  $\mathbb{N}$  are non singular, respectively, second and fourth order tensors. The tensor coefficients  $\mathbb{U}$  are taken to be constant and comply with the properties

$$\mathbb{U}^{\mathbf{ab}} = (\mathbb{U}^{\mathbf{ba}})^t,$$

where the superimposed  $t$  denotes transposition with respect to the sets of tensorial order of  $\mathbf{a}$  and  $\mathbf{b}$ . For instance

$$\mathbb{U}_{ijk}^{\Pi\mathbf{p}} = \mathbb{U}_{kij}^{\mathbf{p}\Pi}.$$

Within the same approximation, the evolution equations for the internal variables take the form

$$\dot{\boldsymbol{\Omega}} = \boldsymbol{\eta} + \mathbb{N}\nabla\dot{\mathbf{p}} + \gamma_{\boldsymbol{\Omega}}\boldsymbol{\Omega}, \quad (2.15)$$

$$\dot{\boldsymbol{\chi}} = \alpha\mathbb{R}\nabla\varepsilon + \gamma_{\boldsymbol{\chi}}\boldsymbol{\chi}, \quad (2.16)$$

where  $\gamma_{\boldsymbol{\Omega}}, \gamma_{\boldsymbol{\chi}}$  are real negative parameters and  $\alpha = -d^2\psi/d\theta^2|_0\theta_0$ . Here  $\psi$  is the free energy density and  $\theta_0$  is the thermodynamic temperature of the unperturbed continuum. Since  $-\psi$  is a convex function of  $\theta$ ,  $\alpha$  turns out to be a real positive parameter. Also, as a consequence of the second law of thermodynamics, the block matrix

$$\begin{pmatrix} \mathbb{U}^{\boldsymbol{\Omega}\boldsymbol{\Omega}} & \mathbb{U}^{\boldsymbol{\Omega}\boldsymbol{\chi}} \\ \mathbb{U}^{\boldsymbol{\chi}\boldsymbol{\Omega}} & \mathbb{U}^{\boldsymbol{\chi}\boldsymbol{\chi}} \end{pmatrix},$$

turns out to be positive definite. The same property is supposed to hold for the block matrix (cf. Romeo 2008)

$$\begin{pmatrix} \mathbb{U}^{\boldsymbol{\eta}\boldsymbol{\eta}} & \mathbb{U}^{\boldsymbol{\eta}\mathbf{p}} & \mathbb{U}^{\boldsymbol{\eta}\Pi} \\ \mathbb{U}^{\mathbf{p}\boldsymbol{\eta}} & \mathbb{U}^{\mathbf{p}\mathbf{p}} & \mathbb{U}^{\mathbf{p}\Pi} \\ \mathbb{U}^{\Pi\boldsymbol{\eta}} & \mathbb{U}^{\Pi\mathbf{p}} & \mathbb{U}^{\Pi\Pi} \end{pmatrix}.$$

### 3 The derivation of the eigenvector problem

We consider a two-dimensional problem assuming that the thermoelectroelastic field depends on the spatial coordinates in an arbitrary plane  $\pi$  in  $\mathcal{B}_t$ . Denoting by  $\mathbf{m}$  and  $\mathbf{n}$  a pair of orthogonal unit vectors in  $\pi$ , we pose  $\xi = \mathbf{m} \cdot \mathbf{x}$ ,  $\nu = \mathbf{n} \cdot \mathbf{x}$  and write the spatial and time dependence of the field  $(\mathbf{u}, \mathbf{p}, \varepsilon, \phi, \boldsymbol{\Omega}, \boldsymbol{\chi})$ , in the following form

$$(\mathbf{u}, \mathbf{p}, \varepsilon, \phi, \boldsymbol{\Omega}, \boldsymbol{\chi})(\mathbf{x}, t) = [a(\xi, \cdot) * b(\nu, \cdot) * (\bar{\mathbf{u}}, \bar{\mathbf{p}}, \bar{\varepsilon}, \bar{\phi}, \bar{\boldsymbol{\Omega}}, \bar{\boldsymbol{\chi}})(\cdot)](t), \quad (3.1)$$

where  $a$  and  $b$  are, respectively, functions of  $(\xi, t)$  and  $(\nu, t)$ . The symbol  $(*)$  denotes time convolution according to

$$(a * b)(t) = \int_0^t a(\tau) b(t - \tau) d\tau.$$

In the following we shall consider fields which vanish for  $t < 0$ . Hence the previous integral can be thought extended to  $(-\infty, t)$ .

By Fourier transforming equation (3.1) we get

$$(\hat{\mathbf{u}}, \hat{\mathbf{p}}, \hat{\varepsilon}, \hat{\phi}, \hat{\boldsymbol{\Omega}}, \hat{\boldsymbol{\chi}})(\xi, \nu; \omega) = \hat{a}(\xi, \omega)\hat{b}(\nu, \omega)(\hat{\mathbf{u}}, \hat{\mathbf{p}}, \hat{\varepsilon}, \hat{\phi}, \hat{\boldsymbol{\Omega}}, \hat{\boldsymbol{\chi}})(\omega),$$

Then, posing

$$\frac{1}{\hat{a}} \frac{\partial \hat{a}}{\partial \xi} = q, \quad \frac{1}{\hat{b}} \frac{\partial \hat{b}}{\partial \nu} = p, \quad (3.2)$$

where  $q$  and  $p$  are assumed to be independent on  $\xi$  and  $\nu$ , equations (2.1)-(2.3), (2.5) become

$$-\rho\omega^2 \hat{\mathbf{u}} = q\mathbf{m}\hat{\mathbf{T}} + p\mathbf{n}\hat{\mathbf{T}}, \quad (3.3)$$

$$-\mu\omega^2 \hat{\mathbf{p}} = -q\mathbf{m}\hat{\phi} - p\mathbf{n}\hat{\phi} + \hat{\mathbf{e}} + \frac{1}{\rho}q\mathbf{m}\hat{\mathbf{E}} + \frac{1}{\rho}p\mathbf{n}\hat{\mathbf{E}}, \quad (3.4)$$

$$\rho i\omega \hat{\mathbf{e}} = -q\mathbf{m} \cdot \hat{\mathbf{q}} - p\mathbf{n} \cdot \hat{\mathbf{q}}, \quad (3.5)$$

$$\mathbf{m} \cdot \hat{\mathbf{p}} - q\hat{\phi} + \frac{p}{q}\mathbf{n} \cdot \hat{\mathbf{d}} = 0. \quad (3.6)$$

In writing equation (3.6) we have exploited equation (2.4). The dependence on  $\hat{\mathbf{\Omega}}$  and  $\hat{\mathbf{\chi}}$  can be removed by exploiting their evolution equations. By the Fourier transform in equations (2.15) and (2.16) we express  $\hat{\mathbf{\Omega}}$  and  $\hat{\mathbf{\chi}}$  in terms of the other transformed fields, as

$$\hat{\mathbf{\Omega}} = \frac{i\omega}{i\omega - \gamma_{\Omega}} (\hat{\boldsymbol{\eta}} + \mathbf{N}\nabla\hat{\mathbf{p}}), \quad (3.7)$$

$$\hat{\mathbf{\chi}} = \frac{\alpha}{i\omega - \gamma_{\chi}} \mathbf{R}\nabla\hat{\mathbf{e}}, \quad (3.8)$$

We observe that  $(\mathbf{u}, \mathbf{p}, \varepsilon, \phi)$  are real valued fields. Hence, denoting by a superimposed asterisk the complex conjugate of a given quantity, we have

$$\hat{\mathbf{u}}^*(\xi, \nu; \omega) = \hat{\mathbf{u}}(\xi, \nu; -\omega),$$

and analogous equalities for the other transformed fields. This, in particular, implies that

$$\hat{a}^*(\xi, \omega) = \hat{a}(\xi, -\omega), \quad \hat{b}^*(\nu, \omega) = \hat{b}(\nu, -\omega), \quad \forall (\xi, \nu) \in \mathbb{R}^2, \quad \forall \omega \in \mathbb{R}.$$

In view of equations (3.2), we get

$$q^*(\omega) = q(-\omega), \quad p^*(\omega) = p(-\omega). \quad (3.9)$$

for any  $\omega$ . In the following it will be sufficient to assume that  $q$  and  $p$  are complex-valued piecewise continuous functions of  $\omega$ .

In the following it will be convenient to pose

$$\hat{\vartheta} = \frac{\alpha}{i\omega} \frac{i\omega - \gamma_{\Omega}}{i\omega - \gamma_{\chi}} \hat{\mathbf{e}},$$

and define a complex wavelength vector  $\mathbf{l}$  as

$$\mathbf{l} = \frac{1}{q}\mathbf{m}.$$



The quantity  $\vartheta$  has the dimension of the time antiderivative of a thermodynamic temperature. Introducing the following quantities,

$$\begin{aligned} \mathbf{A} &= \mathbf{U}^{\eta\eta} + \frac{i\omega}{i\omega - \gamma_{\Omega}} \mathbf{U}^{\Omega\Omega} + \rho\omega^2 \mathbf{1} \otimes \mathbf{I} \otimes \mathbf{1}, \\ \mathbf{B} &= \mathbf{U}^{\text{III}} + \frac{i\omega}{i\omega - \gamma_{\Omega}} \mathbf{N}^t \mathbf{U}^{\Omega\Omega} \mathbf{N} - (\mathbf{1} \otimes \mathbf{U}^{\text{PI}} - \mathbf{U}^{\text{IP}} \otimes \mathbf{1}) + \delta\omega^2 \mathbf{1} \otimes \mathbf{I} \otimes \mathbf{1} - \mathbf{1} \otimes \mathbf{U}^{\text{PP}} \otimes \mathbf{1}, \\ \mathbf{C} &= \mathbf{U}^{\eta\text{II}} + \frac{i\omega}{i\omega - \gamma_{\Omega}} \mathbf{U}^{\Omega\Omega} \mathbf{N} + \mathbf{U}^{\eta\text{P}} \otimes \mathbf{1}, \\ \mathbf{D} &= \frac{\rho\omega^2}{\alpha} \frac{i\omega - \gamma_{\Omega}}{i\omega - \gamma_{\Omega}} \mathbf{1} \otimes \mathbf{1} + \frac{i\omega}{i\omega - \gamma_{\Omega}} \mathbf{R}^T \mathbf{U}^{\mathbf{x}\mathbf{x}} \mathbf{R}, \\ \mathbf{E} &= -\frac{i\omega}{i\omega - \gamma_{\Omega}} \mathbf{U}^{\Omega\mathbf{x}} \mathbf{R}, \\ \mathbf{F} &= -\frac{i\omega}{i\omega - \gamma_{\Omega}} \mathbf{N}^t \mathbf{U}^{\Omega\mathbf{x}} \mathbf{R}, \\ \mathbf{H} &= -\mathbf{1} \otimes \mathbf{I}, \end{aligned}$$

equations (3.3)-(3.6) can be rewritten in the following form

$$\mathbf{m}\mathbf{A}\mathbf{m}\hat{\mathbf{u}} + \mathbf{m}\mathbf{C}\mathbf{m}\hat{\mathbf{p}} + \mathbf{m}\mathbf{E}\mathbf{m}\hat{\vartheta} = -\frac{p}{q} [\mathbf{m}\mathbf{A}\mathbf{n} + \mathbf{m}\mathbf{C}\mathbf{n}\hat{\mathbf{p}} + \mathbf{m}\mathbf{E}\mathbf{n}\hat{\vartheta} + q^{-1} \mathbf{n}\hat{\mathbf{T}}], \quad (3.10)$$

$$\mathbf{m}\mathbf{C}^\dagger \mathbf{m}\hat{\mathbf{u}} + \mathbf{m}\mathbf{B}\mathbf{m}\hat{\mathbf{p}} + \mathbf{m}\mathbf{F}\mathbf{m}\hat{\vartheta} + \mathbf{m}\mathbf{H}\mathbf{m}\hat{\phi} = -\frac{p}{q} [\mathbf{m}\mathbf{C}^\dagger \mathbf{n}\hat{\mathbf{u}} + \mathbf{m}\mathbf{B}\mathbf{n}\hat{\mathbf{p}} + \mathbf{m}\mathbf{F}\mathbf{n}\hat{\vartheta} + \mathbf{m}\mathbf{H}\mathbf{n}\hat{\phi} + (q\rho)^{-1} \mathbf{n}\hat{\mathbf{E}}] \quad (3.11)$$

$$\mathbf{m}\mathbf{E}^\dagger \mathbf{m}\hat{\mathbf{u}} + \mathbf{m}\mathbf{F}^\dagger \mathbf{m}\hat{\mathbf{p}} + \mathbf{m}\mathbf{D}\mathbf{m}\hat{\vartheta} = -\frac{p}{q} [\mathbf{m}\mathbf{E}^\dagger \mathbf{n}\hat{\mathbf{u}} + \mathbf{m}\mathbf{F}^\dagger \mathbf{n}\hat{\mathbf{p}} + \mathbf{m}\mathbf{D}\mathbf{n}\hat{\vartheta} + q^{-1} \mathbf{n} \cdot \hat{\mathbf{q}}], \quad (3.12)$$

$$\mathbf{m}\mathbf{H}^\dagger \mathbf{m}\hat{\mathbf{p}} - \hat{\phi} = -\frac{p}{q^2} \mathbf{n} \cdot \hat{\mathbf{d}}, \quad (3.13)$$

where, for any quantity  $\mathbf{Q}$  in the set  $\{\mathbf{A}, \dots, \mathbf{H}\}$ , viewed as functions of  $\mathbf{l}$ , we have posed

$$\mathbf{Q}^\dagger(\mathbf{l}) = \mathbf{Q}^t(-\mathbf{l}).$$

With a similar reasoning, after multiplication by  $\mathbf{n}$  of equations (2.11), (2.13), (2.14) and (2.4), we obtain

$$\mathbf{n}\mathbf{A}\mathbf{m}\hat{\mathbf{u}} + \mathbf{n}\mathbf{C}\mathbf{m}\hat{\mathbf{p}} + \mathbf{n}\mathbf{E}\mathbf{m}\hat{\vartheta} - q^{-1} \mathbf{n}\hat{\mathbf{T}} = -\frac{p}{q} [\mathbf{n}\mathbf{A}\mathbf{n} + \mathbf{n}\mathbf{C}\mathbf{n}\hat{\mathbf{p}} + \mathbf{n}\mathbf{E}\mathbf{n}\hat{\vartheta}], \quad (3.14)$$

$$\mathbf{n}\mathbf{C}^\dagger \mathbf{m}\hat{\mathbf{u}} + \mathbf{n}\mathbf{B}\mathbf{m}\hat{\mathbf{p}} + \mathbf{n}\mathbf{F}\mathbf{m}\hat{\vartheta} + \mathbf{n}\mathbf{H}\mathbf{m}\hat{\phi} - (q\rho)^{-1} \mathbf{n}\hat{\mathbf{E}} = -\frac{p}{q} [\mathbf{n}\mathbf{C}^\dagger \mathbf{n}\hat{\mathbf{u}} + \mathbf{n}\mathbf{B}\mathbf{n}\hat{\mathbf{p}} + \mathbf{n}\mathbf{F}\mathbf{n}\hat{\vartheta} + \mathbf{n}\mathbf{H}\mathbf{n}\hat{\phi}] \quad (3.15)$$

$$\mathbf{n}\mathbf{E}^\dagger \mathbf{m}\hat{\mathbf{u}} + \mathbf{n}\mathbf{F}^\dagger \mathbf{m}\hat{\mathbf{p}} + \mathbf{n}\mathbf{D}\mathbf{m}\hat{\vartheta} - q^{-1} \mathbf{n} \cdot \hat{\mathbf{q}} = -\frac{p}{q} [\mathbf{n}\mathbf{E}^\dagger \mathbf{n}\hat{\mathbf{u}} + \mathbf{n}\mathbf{F}^\dagger \mathbf{n}\hat{\mathbf{p}} + \mathbf{n}\mathbf{D}\mathbf{n}\hat{\vartheta}], \quad (3.16)$$

$$\mathbf{n}\mathbf{H}^\dagger \mathbf{m}\hat{\mathbf{p}} - q^{-1} \mathbf{n} \cdot \hat{\mathbf{d}} = -\frac{p}{q} [\mathbf{n}\mathbf{H}^\dagger \mathbf{n}\hat{\mathbf{p}} - \hat{\phi}]. \quad (3.17)$$

More synthetically, equations (3.10)-(3.17) can be rewritten as

$$\mathcal{P}\hat{\mathbf{X}} = -\frac{p}{q}\mathcal{R}\hat{\mathbf{X}}, \quad (3.18)$$

where

$$\hat{\mathbf{X}} = \left( \hat{\mathbf{u}}, \hat{\mathbf{p}}, \hat{\vartheta}, \hat{\phi}, q^{-1}\mathbf{n}\hat{\mathbf{T}}, (q\rho)^{-1}\mathbf{n}\hat{\mathbf{e}}, q^{-1}\mathbf{n} \cdot \hat{\mathbf{q}}, q^{-1}\mathbf{n} \cdot \hat{\mathbf{d}} \right)^T$$

and

$$\mathcal{P} = \begin{pmatrix} (\mathbf{mm}) & \mathbf{0} \\ (\mathbf{nm}) & -\mathbb{I} \end{pmatrix}, \quad \mathcal{R} = \begin{pmatrix} (\mathbf{mn}) & \mathbb{I} \\ (\mathbf{nn}) & \mathbf{0} \end{pmatrix}.$$

In the previous block matrices we have posed  $\mathbb{I} = \text{diag}(\mathbf{I}, \mathbf{I}, 1, 1)$  and we have used the following notation which holds for  $\mathbf{a} = \mathbf{m}, \mathbf{n}$  and  $\mathbf{b} = \mathbf{m}, \mathbf{n}$ ,

$$(\mathbf{ab}) = \begin{pmatrix} \mathbf{aAb} & \mathbf{aCb} & \mathbf{aEb} & \mathbf{0} \\ \mathbf{aC}^\dagger\mathbf{b} & \mathbf{aBb} & \mathbf{aFb} & \mathbf{aHb} \\ \mathbf{aE}^\dagger\mathbf{b} & \mathbf{aF}^\dagger\mathbf{b} & \mathbf{aDb} & \mathbf{0} \\ \mathbf{0} & \mathbf{aH}^\dagger\mathbf{b} & \mathbf{0} & -\mathbf{a} \cdot \mathbf{b} \end{pmatrix}.$$

We observe that, in view of the definition of tensors  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}, \mathbf{F}, \mathbf{H}$ , the block matrices  $(\mathbf{ab})$  satisfy the property

$$(\mathbf{ab})^\dagger = (\mathbf{ba}). \quad (3.19)$$

Moreover, accounting for (3.9)<sub>1</sub> we also have

$$(\mathbf{ab})^*(\omega) = (\mathbf{ab})(-\omega). \quad (3.20)$$

Owing to the Shur factorization of determinants,

$$\det\mathcal{R} = \det(\mathbf{nn}),$$

hence, if  $(\mathbf{nn})$  is not singular,  $\mathcal{R}$  turns out to be invertible and we get

$$\mathcal{R}^{-1} = \begin{pmatrix} \mathbf{0} & (\mathbf{nn})^{-1} \\ \mathbb{I} & -(\mathbf{mn})(\mathbf{nn})^{-1} \end{pmatrix}.$$

In this case equation (3.18) reads

$$\mathcal{N}\hat{\mathbf{X}} = -\mu\hat{\mathbf{X}}, \quad (3.21)$$

where  $\mu = p/q$  and

$$\mathcal{N} = \mathcal{R}^{-1}\mathcal{P} = \begin{pmatrix} (\mathbf{nn})^{-1}(\mathbf{nm}) & -(\mathbf{nn})^{-1} \\ (\mathbf{mm}) - (\mathbf{mn})(\mathbf{nn})^{-1}(\mathbf{nm}) & (\mathbf{mn})(\mathbf{nn})^{-1} \end{pmatrix}. \quad (3.22)$$

We observe that  $(\mathbf{nn})$  turns out to be effectively non singular in view of the constitutive assumptions on the block matrices  $\mathbf{U}$ .

## 4 Integral formalism in the frequency domain

An integrated eigenvalue problem can be derived in order to obtain a dispersion equation for the dynamic transformed problem. The procedure, which traces back to the works by Lothe and Barnett (1976a, 1976b) concerning elastic and piezoelectric media (see also Tanuma 2007), can be straightforwardly extended to the present case.

We suppose that the eigenvalue problem (3.21) admits a complete set of complex-valued eigenvectors,

$$\hat{\mathbf{X}}^{(\alpha)}, \quad \alpha = 1, \dots, 16,$$

corresponding to the eigenvalues (not necessarily distinct)  $\mu_\alpha$ . We let  $\mathbf{m}$  and  $\mathbf{n}$  to rotate in the sagittal plane by an angle  $\theta$ , while keeping fixed the direction of  $\mathbf{l}$  which we shall denote by  $\mathbf{m}_0$ . As a consequence, the eigenvalue problem (3.21) holds for those orientations  $\mathbf{n}$  such that  $(\mathbf{nn})$  is not singular. In this respect we note that in the limit  $\omega \rightarrow \pm\infty$ , there exists a set of values  $q$  such that  $(\mathbf{nn})$  is not singular for any orientation  $\mathbf{n}$ . Actually, we have

$$(\mathbf{nn})_\infty := \lim_{\omega \rightarrow \pm\infty} (\mathbf{nn}) = \mathbf{n}\mathbb{P}\mathbf{n} - \frac{1}{q_0^2}(\mathbf{n} \cdot \mathbf{m}_0)^2 \bar{\mathbf{I}},$$

where

$$\mathbb{P} = \begin{pmatrix} \frac{1}{\rho}(\mathbf{U}\boldsymbol{\eta}\boldsymbol{\eta} + \mathbf{U}\boldsymbol{\Omega}\boldsymbol{\Omega}) & \frac{1}{\sqrt{\rho\delta}}(\mathbf{U}\boldsymbol{\eta}\boldsymbol{\Pi} + \mathbf{U}\boldsymbol{\Omega}\boldsymbol{\Omega}\mathbf{N}) & -\frac{\sqrt{\alpha}}{\rho}(\mathbf{U}\boldsymbol{\Omega}\boldsymbol{\chi}\mathbf{R}) \\ \frac{1}{\sqrt{\rho\delta}}(\mathbf{U}\boldsymbol{\eta}\boldsymbol{\Pi} + \mathbf{U}\boldsymbol{\Omega}\boldsymbol{\Omega}\mathbf{N})^T & \frac{1}{\delta}(\mathbf{U}\boldsymbol{\Pi}\boldsymbol{\Pi} + \mathbf{N}^t\mathbf{U}\boldsymbol{\Omega}\boldsymbol{\Omega}\mathbf{N}) & -\sqrt{\frac{\alpha}{\rho\delta}}(\mathbf{N}^t\mathbf{U}\boldsymbol{\Omega}\boldsymbol{\chi}\mathbf{R}) \\ -\frac{\sqrt{\alpha}}{\rho}(\mathbf{U}\boldsymbol{\Omega}\boldsymbol{\chi}\mathbf{R})^T & -\sqrt{\frac{\alpha}{\rho\delta}}(\mathbf{N}^t\mathbf{U}\boldsymbol{\Omega}\boldsymbol{\chi}\mathbf{R})^T & \frac{\alpha}{\rho}(\mathbf{R}^T\mathbf{U}\boldsymbol{\chi}\boldsymbol{\chi}\mathbf{R}) \end{pmatrix},$$

$$q_0 = \lim_{\omega \rightarrow \pm\infty} \left| \frac{q}{\omega} \right|, \quad \bar{\mathbf{I}} = \text{diag}(\mathbf{I}, \mathbf{I}, 1).$$

Owing to the properties of the block matrices  $\mathbf{U}$ , the block matrix  $\mathbb{P}$  turns out to be positive definite and, for  $\omega \rightarrow \pm\infty$ ,  $(\mathbf{nn})$  is not singular under the restriction

$$\frac{1}{q_0^2} < \lambda_L^2 = \min_{\mathbf{n} \in \pi} \left\{ \text{eigenvalues of } \frac{\mathbf{n}\mathbb{P}\mathbf{n}}{(\mathbf{n} \cdot \mathbf{m}_0)^2} \right\}.$$

The quantity  $\lambda_L$  is the smallest real wavelength at which there exists an orientation of  $\mathbf{n}$  such that  $(\mathbf{nn})_\infty$  become singular. In the next we shall consider  $q$  such that  $(\mathbf{nn})$  be non singular for any choice of  $\mathbf{n}$ .

It can be shown (cf. Lothe and Barnett 1976a, 1976b, Tanuma 2007) that the eigenvectors  $\hat{\mathbf{X}}^{(\alpha)}$  are independent on  $\theta$  and the eigenvalues can be written as

$$\mu_\alpha = -\tan(\theta + c_\alpha),$$

where  $c_\alpha$  is a complex-valued constant. Then we take the mean value over  $[0, 2\pi]$  in both sides of the eigenvalue problem

$$\mathcal{N}\hat{\mathbf{X}}^{(\alpha)} = -\mu_\alpha\hat{\mathbf{X}}^{(\alpha)}, \tag{4.1}$$

to obtain

$$\bar{\mathcal{N}}\hat{\mathbf{X}}^{(\alpha)} = -is_\alpha\hat{\mathbf{X}}^{(\alpha)}, \quad (4.2)$$

where  $s_\alpha = \text{sgn}(\Im\mu_\alpha)$  and

$$\bar{\mathcal{N}} = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{N}(\theta) d\theta.$$

Equation (4.2) also holds for degenerate eigenvalue problems (4.1) where  $\hat{\mathbf{X}}^{(\alpha)}$  are generalized eigenvectors. In view of equations (3.22) and (3.19) we have

$$\bar{\mathcal{N}} = \begin{pmatrix} \mathbf{S} & \mathbf{Q} \\ \mathbf{B} & \mathbf{S}^\dagger \end{pmatrix}$$

where  $\mathbf{S}, \mathbf{B}, \mathbf{Q}$  are obtained by taking the mean value over  $[0, 2\pi]$  of the matrix blocks in equation (3.22). Then, equation (4.2) implies that for any set of ordinary or generalized eigenvectors  $\hat{\mathbf{X}}^{(\alpha)}$ , ( $\alpha = 1, \dots, 16$ ),

$$\bar{\mathcal{N}}^2\hat{\mathbf{X}}^{(\alpha)} = -\hat{\mathbf{X}}^{(\alpha)},$$

and, consequently,

$$\bar{\mathcal{N}}^2 = -\mathbb{I}. \quad (4.3)$$

In particular, equation (4.3) implies

$$\mathbf{B}\mathbf{S} + \mathbf{S}^\dagger\mathbf{B} = \mathbf{0}, \quad \mathbf{S}\mathbf{Q} + \mathbf{Q}\mathbf{S}^\dagger = \mathbf{0}. \quad (4.4)$$

We remark that the dispersion equation

$$\det(\bar{\mathcal{N}} + is_\alpha\mathbb{I}) = 0, \quad (4.5)$$

and the secular equation

$$\det(\mathcal{N} + \mu_\alpha\mathbb{I}) = 0, \quad (4.6)$$

must be solved simultaneously to obtain  $q(\omega)$  and the corresponding eigenvalues  $\mu_\alpha(\omega)$ , ( $\alpha = 1, \dots, 16$ ). In particular, for a given solution  $q(\omega)$  satisfying the condition  $s_\alpha = 1$  at  $\omega = 0^+$ , each of the corresponding eigenvalues  $\mu_\alpha$  can change the sign of its imaginary part as  $\omega$  increases. Hence, possible successive turning points of  $\Im\mu_\alpha$  can occur, say  $\omega_1^{(\alpha)}, \omega_2^{(\alpha)}, \dots$ . Since  $\Im\mu_\alpha$  is odd, opposite turning points  $-\omega_1^{(\alpha)}, -\omega_2^{(\alpha)}, \dots$ , occur for  $\omega < 0$ .

In order to describe the dynamics of each mode we proceed as follows.

Posing  $s_0^{(\alpha)} = s_\alpha(\omega = 0^+)$  we consider the modes  $\alpha \in A^+$  where  $A^+ = \{\alpha : s_0^{(\alpha)} = 1\}$ , denoting by  $\Omega_\alpha^+$  the subset of  $\mathbb{R}^+$  where  $s_\alpha = 1$  and assume that  $s_\alpha = -1$  in  $\mathbb{R}^+ \setminus \Omega_\alpha^+$ . Since  $s_\alpha$  is an odd function of  $\omega$ , we obtain that  $s_\alpha = -1$  in  $\bar{\Omega}_\alpha^+ = \{\omega \in \mathbb{R}^- : -\omega \in \Omega_\alpha^+\}$ .

We also denote by  $\hat{\mathbf{X}}_a^{(\alpha)}$  and  $\hat{\mathbf{X}}_b^{(\alpha)}$  the restrictions of the field mode  $\hat{\mathbf{X}}^{(\alpha)}$ , respectively to  $\Omega_\alpha^+$  and to its complement in  $\mathbb{R}^+$ , i.e.,

$$\hat{\mathbf{X}}_a^{(\alpha)} = \hat{\mathbf{X}}^{(\alpha)}|_{\Omega_\alpha^+}, \quad \hat{\mathbf{X}}_b^{(\alpha)} = \hat{\mathbf{X}}^{(\alpha)}|_{\mathbb{R}^+ \setminus \Omega_\alpha^+}.$$

Hence, for such modes equation (4.2) can be rewritten as

$$\begin{aligned} \bar{\mathcal{N}}\hat{\mathbf{X}}_a^{(\alpha)} &= -i \operatorname{sgn}(\omega)\hat{\mathbf{X}}_a^{(\alpha)}, & |\omega| \in \Omega_\alpha^+, & \alpha \in A^+, \\ \bar{\mathcal{N}}\hat{\mathbf{X}}_b^{(\alpha)} &= i \operatorname{sgn}(\omega)\hat{\mathbf{X}}_b^{(\alpha)}, & |\omega| \in \mathbb{R}^+ \setminus \Omega_\alpha^+, & \alpha \in A^+, \end{aligned} \quad (4.7)$$

Similarly, we consider the modes  $\alpha \in A^-$  where  $A^- = \{\alpha : s_0^{(\alpha)} = -1\}$  and denote by  $\Omega_\alpha^-$  the subset of  $\mathbb{R}^+$  where  $s_\alpha = -1$ . It follows that, for these modes,  $s_\alpha = 1$  in  $\bar{\Omega}_\alpha^- = \{\omega \in \mathbb{R}^- : -\omega \in \Omega_\alpha^-\}$ . Posing again

$$\hat{\mathbf{X}}_a^{(\alpha)} = \hat{\mathbf{X}}^{(\alpha)}|_{\Omega_\alpha^-}, \quad \hat{\mathbf{X}}_b^{(\alpha)} = \hat{\mathbf{X}}^{(\alpha)}|_{\mathbb{R}^+ \setminus \Omega_\alpha^-},$$

for such modes equation (4.2) reads

$$\begin{aligned} \bar{\mathcal{N}}\hat{\mathbf{X}}_a^{(\alpha)} &= i \operatorname{sgn}(\omega)\hat{\mathbf{X}}_a^{(\alpha)}, & |\omega| \in \Omega_\alpha^-, & \alpha \in A^-, \\ \bar{\mathcal{N}}\hat{\mathbf{X}}_b^{(\alpha)} &= -i \operatorname{sgn}(\omega)\hat{\mathbf{X}}_b^{(\alpha)}, & |\omega| \in \mathbb{R}^+ \setminus \Omega_\alpha^-, & \alpha \in A^-. \end{aligned} \quad (4.8)$$

From equations (4.7) and (4.8) we obtain the following dispersion equations

$$\det[\bar{\mathcal{N}} \pm i \operatorname{sgn}(\omega)\mathbb{I}] = 0, \quad (4.9)$$

which allows us to find two sets of functions  $q = q_+(\omega)$  and  $q = q_-(\omega)$ . If these functions are substituted into the expression of  $\mathcal{N}$ , the eigenvalues  $\mu_\alpha$  and the sets  $\Omega_\alpha^+$ ,  $\Omega_\alpha^-$  can be worked out simultaneously from equation (4.6). Owing to (3.2) and (4.7), (4.8), we can represent the superpositions  $\hat{\mathbf{X}}^+$  and  $\hat{\mathbf{X}}^-$  of modes with  $s_\alpha = 1$  and  $s_\alpha = -1$ , respectively, as

$$\hat{\mathbf{X}}^+(\omega; \xi, \nu) = \exp[q_+\xi] \left[ \sum_{\alpha \in A^+} c^{(\alpha)}(\omega) \exp[q_+\mu_\alpha \nu] \hat{\mathbf{X}}_a^{(\alpha)}(\omega) + \sum_{\alpha \in A^-} c^{(\alpha)}(\omega) \exp[q_+\mu_\alpha \nu] \hat{\mathbf{X}}_b^{(\alpha)}(\omega) \right], \quad (4.10)$$

$$\hat{\mathbf{X}}^-(\omega; \xi, \nu) = \exp[q_-\xi] \left[ \sum_{\alpha \in A^-} c^{(\alpha)}(\omega) \exp[q_-\mu_\alpha \nu] \hat{\mathbf{X}}_a^{(\alpha)}(\omega) + \sum_{\alpha \in A^+} c^{(\alpha)}(\omega) \exp[q_-\mu_\alpha \nu] \hat{\mathbf{X}}_b^{(\alpha)}(\omega) \right]. \quad (4.11)$$

where, for any  $\alpha$ ,  $c^{(\alpha)*}(\omega) = c^{(\alpha)}(-\omega)$ . From equations (4.7)-(4.11) we obtain the following eigenvector problems for the fields  $\hat{\mathbf{X}}^+$  and  $\hat{\mathbf{X}}^-$

$$\mathcal{N}\hat{\mathbf{X}}^+ = -i \operatorname{sgn}(\omega)\hat{\mathbf{X}}^+, \quad (4.12)$$

$$\mathcal{N}\hat{\mathbf{X}}^- = i \operatorname{sgn}(\omega)\hat{\mathbf{X}}^-, \quad (4.13)$$

## 5 Impedance and admittance tensors and energy flux at interfaces

Let us decompose the eigenvectors  $\hat{\mathbf{X}}$  as

$$\hat{\mathbf{X}} = \begin{pmatrix} \hat{\mathbf{U}} \\ \frac{1}{q}\hat{\mathbf{V}} \end{pmatrix},$$

where

$$\hat{\mathbf{U}} = (\hat{\mathbf{u}}, \hat{\mathbf{p}}, \hat{\vartheta}, \hat{\phi})^T, \quad \hat{\mathbf{V}} = (\mathbf{n}\hat{\mathbf{T}}, \rho^{-1}\mathbf{n}\hat{\mathbf{e}}, \mathbf{n} \cdot \hat{\mathbf{q}}, \mathbf{n} \cdot \hat{\mathbf{d}})^T. \quad (5.1)$$

Then, for invertible tensors  $\mathbf{Q}$  and  $\mathbf{B}$ , from equations (4.12) and (4.13) we obtain

$$\hat{\mathbf{V}}^+ = iq\mathbf{Z}_+\hat{\mathbf{U}}^+, \quad \hat{\mathbf{V}}^- = iq\mathbf{Z}_-\hat{\mathbf{U}}^-, \quad (5.2)$$

$$\hat{\mathbf{U}}^+ = \frac{1}{iq}\mathbf{Y}_+\hat{\mathbf{V}}^+, \quad \hat{\mathbf{U}}^- = \frac{1}{iq}\mathbf{Y}_-\hat{\mathbf{V}}^-, \quad (5.3)$$

where we have posed

$$\begin{aligned} \mathbf{Z}_\pm &= \mp \operatorname{sgn}(\omega)\mathbf{Q}^{-1} + i\mathbf{Q}^{-1}\mathbf{S}, \\ \mathbf{Y}_\pm &= \pm \operatorname{sgn}(\omega)\mathbf{B}^{-1} - i\mathbf{B}^{-1}\mathbf{S}^\dagger. \end{aligned}$$

The quantities  $\mathbf{Z}_\pm$  and  $\mathbf{Y}_\pm$  are called, respectively, impedance and admittance tensors. In view of equations (4.4) we have

$$\mathbf{Z}_+^\dagger = -\mathbf{Z}_-, \quad \mathbf{Y}_+^\dagger = -\mathbf{Y}_-. \quad (5.4)$$

The tensors  $\mathbf{Z}_+$  and  $\mathbf{Z}_-$  represent a generalization of the impedance tensor introduced in the Stroh formalism to deal with surface constraints. In the case of monochromatic plane waves in non dissipative electroelastic media, the quantities  $s_\alpha$  are independent on  $\omega$  and can be chosen according to a given sign of the real wave number  $k$  ( $q = ik$ ). For example  $s_\alpha = 1$  for modes with  $k > 0$  in order to ensure the field decay away from the surface with inward normal  $\mathbf{n}$ . Accordingly, equation (3.19) reduces to

$$(\mathbf{ab})^{T*} = (\mathbf{ba}),$$

and the second of equations (4.4) becomes

$$\mathbf{S}\mathbf{Q} + \mathbf{Q}\mathbf{S}^T = \mathbf{0}.$$

Thus we pose  $\mathbf{Z}_+ = \mathbf{Z}$  and obtain

$$\mathbf{Z}^{T*} = \mathbf{Z},$$

i.e., the impedance tensor is hermitian (Lothe and Barnett 1976a).

Equations (5.2) and (5.3) are useful in connection with the existence of surface and interfacial waves at a given surface  $\mathcal{S}$  with normal  $\mathbf{n}$  (Lothe and Barnett 1976a, 1976b). Once suitable constraints are given at  $\mathcal{S}$ , the compatibility conditions of the algebraic systems (5.2) and (5.3) can be exploited to determine  $q$ . Here we exploit the impedance and admittance tensors to evaluate the energy flux across a given surface with normal  $\mathbf{n}$ .

Taking the Fourier transform of equation (2.10), after scalar multiplication by  $\mathbf{n}$  we get

$$\hat{\mathbf{J}} \cdot \mathbf{n} = -\frac{i}{2\pi} \left[ \mathbf{n}\hat{\mathbf{T}} * (\omega\hat{\mathbf{u}}) + \frac{1}{\rho}\mathbf{n}\hat{\mathbf{C}} * (\omega\hat{\mathbf{p}}) - \hat{\phi} * (\omega\hat{\mathbf{d}} \cdot \mathbf{n}) \right] + \hat{\mathbf{q}} \cdot \mathbf{n}. \quad (5.5)$$

This result can be expressed in terms of the single field  $\hat{\mathbf{U}}$  or the single field  $\hat{\mathbf{V}}$  by exploiting equations (5.2) and (5.3). Using the notations

$$\mathbf{Z} = \begin{pmatrix} \mathbf{Z}_{11} & \mathbf{Z}_{12} & \mathbf{z}_{13} & \mathbf{z}_{14} \\ \mathbf{Z}_{21} & \mathbf{Z}_{22} & \mathbf{z}_{23} & \mathbf{z}_{24} \\ \mathbf{z}_{31} & \mathbf{z}_{32} & \zeta_{33} & \zeta_{34} \\ \mathbf{z}_{41} & \mathbf{z}_{42} & \zeta_{43} & \zeta_{44} \end{pmatrix}, \quad \mathbf{Y} = \begin{pmatrix} \mathbf{Y}_{11} & \mathbf{Y}_{12} & \mathbf{y}_{13} & \mathbf{y}_{14} \\ \mathbf{Y}_{21} & \mathbf{Y}_{22} & \mathbf{y}_{23} & \mathbf{y}_{24} \\ \mathbf{y}_{31} & \mathbf{y}_{32} & \nu_{33} & \nu_{34} \\ \mathbf{y}_{41} & \mathbf{y}_{42} & \nu_{43} & \nu_{44} \end{pmatrix},$$

and omitting to specify the mode signs "+" or "-", from (5.5) we obtain the two alternative representations,

$$\begin{aligned} \hat{\mathbf{J}} \cdot \mathbf{n} = & \frac{1}{2\pi} \left\{ \omega(\hat{\mathbf{u}}, \hat{\mathbf{p}}) * \left[ q \begin{pmatrix} \mathbf{Z}_{11} & \mathbf{Z}_{12} \\ \mathbf{Z}_{21} & \mathbf{Z}_{22} \end{pmatrix} \right] \begin{pmatrix} \hat{\mathbf{u}} \\ \hat{\mathbf{p}} \end{pmatrix} + \omega(\hat{\mathbf{u}}, \hat{\mathbf{p}}) * \left[ q \begin{pmatrix} \mathbf{z}_{13} & \mathbf{z}_{14} \\ \mathbf{z}_{23} & \mathbf{z}_{24} \end{pmatrix} \right] \begin{pmatrix} \hat{\vartheta} \\ \hat{\phi} \end{pmatrix} \right. \\ & \left. - \omega\hat{\phi} * [q(\mathbf{z}_{41}, \mathbf{z}_{42}, \zeta_{43}, \zeta_{44})] \begin{pmatrix} \hat{\mathbf{u}} \\ \hat{\mathbf{p}} \\ \hat{\vartheta} \\ \hat{\phi} \end{pmatrix} \right\} + iq(\mathbf{z}_{31}, \mathbf{z}_{32}, \zeta_{33}, \zeta_{34}) \begin{pmatrix} \hat{\mathbf{u}} \\ \hat{\mathbf{p}} \\ \hat{\vartheta} \\ \hat{\phi} \end{pmatrix}, \end{aligned} \quad (5.6)$$

$$\begin{aligned} \hat{\mathbf{J}} \cdot \mathbf{n} = & -\frac{1}{2\pi} \left\{ (\mathbf{n}\hat{\mathbf{T}}, \frac{1}{\rho}\mathbf{n}\hat{\mathbf{C}}) * \left[ \frac{\omega}{q} \begin{pmatrix} \mathbf{Y}_{11} & \mathbf{Y}_{12} \\ \mathbf{Y}_{21} & \mathbf{Y}_{22} \end{pmatrix} \right] \begin{pmatrix} \mathbf{n}\hat{\mathbf{T}} \\ \frac{1}{\rho}\mathbf{n}\hat{\mathbf{C}} \end{pmatrix} \right. \\ & + (\mathbf{n}\hat{\mathbf{T}}, \frac{1}{\rho}\mathbf{n}\hat{\mathbf{C}}) * \left[ \frac{\omega}{q} \begin{pmatrix} \mathbf{y}_{13} & \mathbf{y}_{14} \\ \mathbf{y}_{23} & \mathbf{y}_{24} \end{pmatrix} \right] \begin{pmatrix} \hat{\mathbf{q}} \cdot \mathbf{n} \\ \hat{\mathbf{d}} \cdot \mathbf{n} \end{pmatrix} \\ & \left. - (\hat{\mathbf{d}} \cdot \mathbf{n}) * \left[ \frac{\omega}{q}(\mathbf{y}_{41}, \mathbf{y}_{42}, \nu_{43}, \nu_{44}) \right] \begin{pmatrix} \mathbf{n}\hat{\mathbf{T}} \\ \frac{1}{\rho}\mathbf{n}\hat{\mathbf{C}} \\ \hat{\mathbf{q}} \cdot \mathbf{n} \\ \hat{\mathbf{d}} \cdot \mathbf{n} \end{pmatrix} \right\} + \hat{\mathbf{q}} \cdot \mathbf{n}. \end{aligned} \quad (5.7)$$

Equation (5.7) can be exploited to evaluate the energy flux at the boundary surface  $\mathcal{S}$ . Owing to equations (2.6)-(2.9), for a mechanically free surface,  $\mathbf{t}^{(\mathbf{n})} = \mathbf{0}$  and we get

$$\begin{aligned} \hat{\mathbf{J}} \cdot \mathbf{n}|_{\mathcal{S}} = & -\frac{1}{2\pi} \left\{ \mathbf{b}^{(\mathbf{n})} * \left( \frac{\omega}{q} \mathbf{Y}_{22} \right) \mathbf{b}^{(\mathbf{n})} + \mathbf{b}^{(\mathbf{n})} * \left( \frac{\omega}{q} \mathbf{y}_{23} \right) \hat{q}_n + \mathbf{b}^{(\mathbf{n})} * \left[ \frac{\omega}{q} (\mathbf{y}_{24} - \mathbf{y}_{42}^T) \right] \hat{d}_n \right. \\ & \left. - \hat{d}_n * \left( \frac{\omega}{q} v_{43} \right) \hat{q}_n - \hat{d}_n * \left( \frac{\omega}{q} v_{44} \right) \hat{d}_n \right\} + \hat{q}_n, \end{aligned} \quad (5.8)$$

where

$$\mathbf{b}^{(\mathbf{n})} = \boldsymbol{\pi}^{(s)} - \mathbf{b}^{(0)}.$$

It is worth remarking that the first term in the right hand side of (5.8) is independent on the field amplitudes. It is a quadratic term in  $\mathbf{b}^{(\mathbf{n})}$  and essentially depends on the wavelength function  $1/q$ . This result is coherent with the static problem of the linear theory where the density of surface energy, for  $\boldsymbol{\pi}^{(s)} = \mathbf{0}$ , is quadratic in  $\mathbf{b}^{(0)}$  (Maugin 1988). In absence of polarization effects,  $\mathbf{b}^{(\mathbf{n})} = \mathbf{0}$  and the energy flux at  $\mathcal{S}$  reduces to the sole thermo-electromagnetic flux.

We also observe that, in view of equations (5.4), the coupling coefficient of the third term in the right hand side of equation (5.8) reads

$$\frac{\omega}{q_+} [\mathbf{y}_{24}^+(q_+) + \mathbf{y}_{24}^-(-q_+)], \quad \text{or} \quad \frac{\omega}{q_-} [\mathbf{y}_{24}^-(q_-) + \mathbf{y}_{24}^+(-q_-)],$$

respectively for (+) modes or (-) modes. In the case of monochromatic plane waves in non dissipative electroelastic media, we have  $q_- = -q_+$  and the previous coupling coefficients become opposite. Hence, couples of opposite modes with the same amplitudes carry a corresponding null contribution to the energy flux.

## 6 Compatibility conditions at interfaces

Now we derive a compatibility condition for the solution to the eigenvector problems (4.12) and (4.13) at a plane interface  $\mathcal{S}$  which separates two regions  $\mathcal{B}_1$  and  $\mathcal{B}_2$  occupied by two different crystals. We are concerned with the transformed fields  $\hat{\mathbf{U}}$  and  $\hat{\mathbf{V}}$  which are considered for a given mode "+" or "-". From equation (5.2), valued at  $\mathcal{S}$ , we have

$$\begin{pmatrix} \hat{\mathbf{t}}^{(\mathbf{n})} \\ \mathbf{b}^{(\mathbf{n})} \\ \hat{q}_n \\ \hat{d}_n \end{pmatrix} = iq \begin{pmatrix} \mathbf{Z}_{11} & \mathbf{Z}_{12} & \mathbf{z}_{13} & \mathbf{z}_{14} \\ \mathbf{Z}_{21} & \mathbf{Z}_{22} & \mathbf{z}_{23} & \mathbf{z}_{24} \\ \mathbf{z}_{31} & \mathbf{z}_{32} & \zeta_{33} & \zeta_{34} \\ \mathbf{z}_{41} & \mathbf{z}_{42} & \zeta_{43} & \zeta_{44} \end{pmatrix} \begin{pmatrix} \hat{\mathbf{u}} \\ \hat{\mathbf{p}} \\ \hat{\vartheta} \\ \hat{\phi} \end{pmatrix}. \quad (6.1)$$

The continuity requirements at  $\mathcal{S}$  for the thermoelectroelastic field imply

$$[[\hat{\mathbf{u}}]] = \mathbf{0}, \quad [[\hat{\mathbf{t}}^{(\mathbf{n})}]] = \mathbf{0}, \quad [[\hat{\mathbf{J}} \cdot \mathbf{n}]] = 0, \quad [[\hat{d}_n]] = 0, \quad (6.2)$$



where, for any field  $\varphi$ , we have denoted as  $[[\varphi]] = \varphi_2|_{\mathcal{S}} - \varphi_1|_{\mathcal{S}}$  its jump across the interface. Concerning to the electric potential  $\hat{\phi}$ , we can assume one of the following conditions.  $\hat{\phi}$  is continuous across  $\mathcal{S}$  or, alternatively,  $\hat{\phi}$  vanishes at  $\mathcal{S}$ . The first assumption applies to an ideal welding contact between the two dielectrics in  $\mathcal{B}_1$  and  $\mathcal{B}_2$ . The second models an electrically grounded interface. In both cases, in view of equation (5.5) and posing

$$\hat{f}_n = -\frac{i\omega}{2\pi} \mathbf{b}^{(\mathbf{n})} \cdot \hat{\mathbf{p}} + \hat{q}_n,$$

the continuity of the transformed energy flux at  $\mathcal{S}$  amounts to

$$[[\hat{f}_n]] = 0. \quad (6.3)$$

Equations (6.2) suggest to introduce the quantities

$$\hat{\xi} = (\hat{\mathbf{u}}, \hat{\mathbf{t}}^{(\mathbf{n})}, \hat{f}_n, \hat{d}_n)^T, \quad \hat{\sigma} = (\hat{\mathbf{p}}, \hat{\vartheta}, \hat{\phi})^T,$$

and rearrange equation (6.1) in the form

$$\hat{\xi} = \hat{\mathfrak{A}} \hat{\sigma} + \hat{\mathbf{b}} \quad (6.4)$$

where

$$\hat{\mathfrak{A}} = iq \begin{pmatrix} \frac{i}{q} \mathbf{Z}_{21}^{-1} \mathbf{Z}_{22} & \frac{i}{q} \mathbf{Z}_{21}^{-1} \mathbf{z}_{23} & \frac{i}{q} \mathbf{Z}_{21}^{-1} \mathbf{z}_{24} \\ \mathbf{Z}_{12} - \mathbf{Z}_{11} \mathbf{Z}_{21}^{-1} \mathbf{Z}_{22} & \mathbf{z}_{13} - \mathbf{Z}_{11} \mathbf{Z}_{21}^{-1} \mathbf{z}_{23} & \mathbf{z}_{14} - \mathbf{Z}_{11} \mathbf{Z}_{21}^{-1} \mathbf{z}_{24} \\ -\frac{\omega}{2\pi q} \mathbf{b}^{(\mathbf{n})} + \mathbf{z}_{32} - \mathbf{z}_{31} \mathbf{Z}_{21}^{-1} \mathbf{Z}_{22} & \zeta_{33} - \mathbf{z}_{31} \mathbf{Z}_{21}^{-1} \mathbf{z}_{23} & \zeta_{34} - \mathbf{z}_{31} \mathbf{Z}_{21}^{-1} \mathbf{z}_{24} \\ \mathbf{z}_{42} - \mathbf{z}_{41} \mathbf{Z}_{21}^{-1} \mathbf{Z}_{22} & \zeta_{43} - \mathbf{z}_{41} \mathbf{Z}_{21}^{-1} \mathbf{z}_{23} & \zeta_{44} - \mathbf{z}_{41} \mathbf{Z}_{21}^{-1} \mathbf{z}_{24} \end{pmatrix}$$

$$\hat{\mathbf{b}} = \begin{pmatrix} -\frac{i}{q} \mathbf{Z}_{21}^{-1} \mathbf{b}^{(\mathbf{n})} \\ \mathbf{Z}_{11} \mathbf{Z}_{21}^{-1} \mathbf{b}^{(\mathbf{n})} \\ \mathbf{z}_{31} \mathbf{Z}_{21}^{-1} \mathbf{b}^{(\mathbf{n})} \\ \mathbf{z}_{41} \mathbf{Z}_{21}^{-1} \mathbf{b}^{(\mathbf{n})} \end{pmatrix}$$

According to the discussion of section 4 and assuming  $\mathbf{n}$  as the outward normal with respect to  $\mathcal{B}_1$  (and inward with respect to  $\mathcal{B}_2$ ), we require that the field amplitude vanishes at large distances from the interface. As a consequence we shall consider the field  $\hat{\mathbf{X}}^+$  in  $\mathcal{B}_1$  and  $\hat{\mathbf{X}}^-$  in  $\mathcal{B}_2$ . The continuity requirements (6.2), (6.3) imply

$$[[\hat{\xi}]] = \mathbf{0},$$

whence, from (6.4) we obtain

$$\hat{\mathfrak{A}}_1 \hat{\sigma}^+ - \hat{\mathfrak{A}}_2 \hat{\sigma}^- = -\hat{\mathbf{b}}_1 + \hat{\mathbf{b}}_2. \quad (6.5)$$

This is an algebraic system for the field  $(\hat{\mathbf{p}}_1, \hat{\vartheta}_1, \hat{\phi}_1, \hat{\mathbf{p}}_2, \hat{\vartheta}_2, \hat{\phi}_2)$ . It consists of eight scalar non homogeneous equations for ten scalar unknowns. Once the solutions of equations (4.9)

have been substituted respectively in  $\hat{\mathfrak{A}}_1, \hat{\mathfrak{b}}_1$  and  $\hat{\mathfrak{A}}_2, \hat{\mathfrak{b}}_2$ , we can check the compatibility of system (6.5).

A simpler analysis applies to the case of a free boundary of a region  $\mathcal{B}$  occupied by a single crystal. Here, according to our assumptions, we must consider only the field  $\hat{\sigma}^+$  with the following constraints at  $\mathcal{S}$ ,

$$(\mathbf{n}\hat{\mathbf{T}}, \hat{f}_n, \hat{d}_n) = (\hat{\mathbf{t}}^{(\mathbf{n})}, \hat{q}_n^{\text{ext}}, \hat{d}_n^{\text{ext}}) =: \hat{\xi}^{\text{ext}}.$$

where, eventually,  $\hat{\mathbf{t}}^{(\mathbf{n})} = \mathbf{0}$ . Accordingly we obtain

$$\hat{\mathfrak{A}}_{\mathcal{B}}\hat{\sigma}^+ = \hat{\mathfrak{b}}_{\mathcal{B}} + \hat{\xi}^{\text{ext}}, \quad (6.6)$$

where

$$\hat{\mathfrak{A}}_{\mathcal{B}} = iq \begin{pmatrix} \mathbf{Z}_{12} - \mathbf{Z}_{11}\mathbf{Z}_{21}^{-1}\mathbf{Z}_{22} & \mathbf{z}_{13} - \mathbf{Z}_{11}\mathbf{Z}_{21}^{-1}\mathbf{z}_{23} & \mathbf{z}_{14} - \mathbf{Z}_{11}\mathbf{Z}_{21}^{-1}\mathbf{z}_{24} \\ \mathbf{z}_{32} - \mathbf{z}_{31}\mathbf{Z}_{21}^{-1}\mathbf{Z}_{22} & \zeta_{33} - \mathbf{z}_{31}\mathbf{Z}_{21}^{-1}\mathbf{z}_{23} & \zeta_{34} - \mathbf{z}_{31}\mathbf{Z}_{21}^{-1}\mathbf{z}_{24} \\ \mathbf{z}_{42} - \mathbf{z}_{41}\mathbf{Z}_{21}^{-1}\mathbf{Z}_{22} & \zeta_{43} - \mathbf{z}_{41}\mathbf{Z}_{21}^{-1}\mathbf{z}_{23} & \zeta_{44} - \mathbf{z}_{41}\mathbf{Z}_{21}^{-1}\mathbf{z}_{24} \end{pmatrix} \quad \hat{\mathfrak{b}}_{\mathcal{B}} = \begin{pmatrix} \mathbf{Z}_{11}\mathbf{Z}_{21}^{-1}\mathbf{b}^{(\mathbf{n})} \\ \mathbf{z}_{31}\mathbf{Z}_{21}^{-1}\mathbf{b}^{(\mathbf{n})} \\ \mathbf{z}_{41}\mathbf{Z}_{21}^{-1}\mathbf{b}^{(\mathbf{n})} \end{pmatrix}$$

Equation (6.6) represents an algebraic non homogeneous system of five scalar equations in the five components of the field  $\hat{\sigma}^+$ . Also in this case the pertinent solution of equation (4.9) must be substituted in the expressions of  $\hat{\mathfrak{A}}_{\mathcal{B}}$  and  $\hat{\mathfrak{b}}_{\mathcal{B}}$  in order to check the compatibility of system (6.6).

## 7 Eigenvector problem in the time domain

The problems (4.12) and (4.13) can be restated in the time domain. To this end we observe that, in view of equation (3.10) the inverse Fourier transform of  $\hat{\mathbf{X}}^+$  turns out to be a real field, which will be denoted by  $\mathbf{X}^+$ . Taking the inverse Fourier transform ( $\mathcal{F}^{-1}$ ) in both sides of equation (4.12) we obtain

$$[\mathcal{F}^{-1}(\bar{\mathcal{N}}) * \mathbf{X}^+](t) = -(\mathcal{H}\mathbf{X}^+)(t), \quad (7.1)$$

where  $\mathcal{H}$  denotes Hilbert transform according to

$$(\mathcal{H}\mathbf{X})(t) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\mathbf{X}(\tau)}{\tau - t} d\tau.$$

Since for any real function  $f$ ,

$$\mathcal{H}(\mathcal{H}f) = -f,$$

in view of the associative property of multiple convolution, we can rewrite equation (7.1) in the following form

$$[\mathcal{H}(\mathcal{F}^{-1}\bar{\mathcal{N}}) * \mathfrak{x}^+](t) = \mathfrak{x}^+(t), \quad (7.2)$$

where  $\mathfrak{X}^+ = \mathcal{H}\mathbf{X}^+$ . Accounting for the identity  $\mathcal{H}(\mathcal{F}^{-1}\bar{\mathcal{N}}) = \mathcal{F}^{-1}(-i\text{sgn}(\omega)\bar{\mathcal{N}})$  and in view of equation (3.21), we can rewrite (7.2) in the form

$$[\mathcal{M} * \mathfrak{X}^+](t) = \mathfrak{X}^+(t), \quad (7.3)$$

where

$$\mathcal{M}(t) = 2\Im[(\mathcal{F}_h^{-1}\bar{\mathcal{N}})(t)].$$

$\mathcal{F}_h^{-1}$  denotes the half-range inverse Fourier transform. Analogously, from equation (4.13) we arrive at

$$[\mathcal{M} * \mathfrak{X}^-](t) = -\mathfrak{X}^-(t). \quad (7.4)$$

Equations (7.3) and (7.4) are homogeneous convolution-type integral equations. We observe that the same integral equations also hold for each single mode  $\alpha$  only if  $s_\alpha$  is constant in  $\mathbb{R}^+$ . The more general time domain version of the eigenvector problem (4.2) for single modes is derived in the Appendix.

## 8 Conclusions

In this paper we have shown how the linearized theory of thermoelectroelasticity for ionic crystals can be handle to state a two dimensional dynamic problem in the Stroh formalism. It is worth remarking that, to this aim, we have used a slight modification of the governing equations with respect to the usual linear theory (see Mindlin 1968, Maugin 1988), bringing the intrinsic "polarization traction" in a boundary datum, instead of considering it as an inhomogeneous term in the constitutive equation for  $\mathfrak{E}$ . A detailed comment on this point can be found in Romeo (2008). Another relevant feature of the present work is that we have taken into account dissipative effects via suitable internal variables subjected to evolution equations compatible with the second law of thermodynamics (see Romeo 2007). Instead of the usual assumption of monochromatic plane wave fields we have considered the more general non stationary problem by means of the Fourier transforms. In this context, the transformed eigenvector equation in the dissipative case contains constitutive coefficients which are frequency dependent. The field representation for  $\hat{\mathbf{X}}^+$  and  $\hat{\mathbf{X}}^-$  in section 4 can be viewed as a generalization of the couple of modes obtained in the non dissipative plane wave type solutions in the approach of Lothe and Barnett (1976a, 1976b). In the present case, the general equations (7.3) and (7.4) hold in the time domain for the Hilbert transforms of the field.

As a final remark we observe that the energy flux across the boundary  $\mathcal{S}$  of the thermoelectroelastic crystal contains a quadratic term in  $\mathbf{b}^{(0)}$  (see equation (5.8)). This is a peculiar surface effect of polarizable media, which does not depend on the amplitude but

only on the propagation properties of the field. In particular, in the case of monochromatic waves, the energy flow across  $\mathcal{S}$  turns out to be proportional to the phase speed.

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## A Appendix

### A.1 Evaluation of integral equations for single modes

In order to evaluate the inverse transform of the right hand side of (4.2) we derive the following inverse transform

$$\mathcal{F}^{-1}(s_\alpha) = \mathcal{F}^{-1}(\text{sgn}[\mathfrak{S}(\mu_\alpha)]). \quad (\text{A.1})$$

We have

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{sgn}[\mathfrak{S}(\mu_\alpha)](\omega) e^{i\omega t} d\omega &= \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{sgn}[\mathfrak{S}(\mu_\alpha)](\omega) e^{i\omega t - \epsilon|\omega|} d\omega = \\ \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi} &\left[ \int_{-\infty}^0 \text{sgn}[\mathfrak{S}(\mu_\alpha)](\omega) e^{i\omega t + \epsilon\omega} d\omega + \int_0^{+\infty} \text{sgn}[\mathfrak{S}(\mu_\alpha)](\omega) e^{i\omega t - \epsilon\omega} d\omega \right]. \end{aligned}$$

Since  $\mathfrak{S}[\mu_\alpha]$  is an odd function of  $\omega$ , if it changes sign at  $\bar{\omega}$ , the opposite change occurs at  $-\bar{\omega}$ . Hence, denoting by  $\pm\omega_k^{(\alpha)}$ , ( $k = 1, \dots, n$ ) ( $|\omega_1^{(\alpha)}| < |\omega_2^{(\alpha)}| < \dots < |\omega_n^{(\alpha)}|$ ) the (possible) reversing points of  $\mathfrak{S}[\mu_\alpha]$ , the sum in the square bracket of the previous equation can be rewritten as

$$\begin{aligned} &- \left[ \int_{-\omega_1^{(\alpha)}}^0 s_0^{(\alpha)} e^{(it+\epsilon)\omega} d\omega - \int_{-\omega_2^{(\alpha)}}^{-\omega_1^{(\alpha)}} s_0^{(\alpha)} e^{(it+\epsilon)\omega} d\omega + \dots \int_{-\infty}^{-\omega_n^{(\alpha)}} s_0^{(\alpha)} e^{(it+\epsilon)\omega} d\omega \right] \\ &+ \left[ \int_0^{\omega_1^{(\alpha)}} s_0^{(\alpha)} e^{(it-\epsilon)\omega} d\omega - \int_{\omega_1^{(\alpha)}}^{\omega_2^{(\alpha)}} s_0^{(\alpha)} e^{(it-\epsilon)\omega} d\omega + \dots \int_{\omega_n^{(\alpha)}}^{+\infty} s_0^{(\alpha)} e^{(it-\epsilon)\omega} d\omega \right] \end{aligned}$$

where  $s_0^{(\alpha)}$  is the sign of  $\mathfrak{S}[\mu_\alpha]$  for  $\omega = 0^+$ . After integrating and taking the limit for  $\epsilon \rightarrow 0$ , we get

$$\mathcal{F}^{-1}(s_\alpha) = -\frac{s_0^{(\alpha)}}{\pi it} [1 - \cos \omega_1^{(\alpha)} t + \cos \omega_2^{(\alpha)} t - \dots \cos \omega_n^{(\alpha)} t] \quad (\text{A.2})$$

The convolution of  $\mathcal{F}^{-1}(s_\alpha)$  with the field  $\mathbf{X}^{(\alpha)}$  is thus the sum of

$$\frac{s_0^{(\alpha)}}{i} \mathcal{H}\mathbf{X}^{(\alpha)}(t), \quad (\text{A.3})$$

with  $n$  terms of the form

$$I_k^{(\alpha)}(t) = -(-1)^k \frac{s_0^{(\alpha)}}{\pi i} \int_{-\infty}^{+\infty} \frac{\cos \omega_k^{(\alpha)} \tau}{\tau} \mathbf{X}^{(\alpha)}(t - \tau) d\tau, \quad k = 1, \dots, n.$$

If we substitute

$$\mathbf{X}^{(\alpha)}(t - \tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{\mathbf{X}}^{(\alpha)}(\omega) e^{i\omega(t-\tau)} d\omega,$$

in the previous integral, we obtain

$$I_k^{(\alpha)}(t) = -(-1)^k \frac{s_0^{(\alpha)}}{4\pi^2} \int_{-\infty}^{+\infty} \hat{\mathbf{X}}^{(\alpha)}(\omega) e^{i\omega t} \int_{-\infty}^{+\infty} \left[ \frac{\sin[(\omega + \omega_k^{(\alpha)})\tau]}{\tau} + \frac{\sin[(\omega - \omega_k^{(\alpha)})\tau]}{\tau} \right] d\tau d\omega,$$

with  $k = 1, \dots, n$ . In view of the result

$$\int_{-\infty}^{+\infty} \frac{\sin \beta \tau}{\tau} d\tau = \begin{cases} \pi & \beta > 0 \\ 0 & \beta = 0 \\ -\pi & \beta < 0 \end{cases},$$

we arrive at

$$I_k^{(\alpha)}(t) = (-1)^k \frac{s_0^{(\alpha)}}{\pi i} \int_{\omega_k^{(\alpha)}}^{+\infty} \Im[\hat{\mathbf{X}}^{(\alpha)}(\omega) e^{i\omega t}] d\omega, \quad k = 1, \dots, n, \quad (\text{A.4})$$

By means of the results (A.3) and (A.4), the Fourier inversion of both sides of equation (4.2) yields

$$[(\mathcal{F}^{-1}\bar{\mathcal{N}}) * \mathbf{X}^{(\alpha)}](t) = -s_0^{(\alpha)} \left[ \mathcal{H}\mathbf{X}^{(\alpha)} + \frac{i}{\pi} \sum_{k=1}^n (-1)^k \int_{\omega_k^{(\alpha)}}^{+\infty} \Im[\hat{\mathbf{X}}^{(\alpha)}(\omega) e^{i\omega t}] d\omega \right]. \quad (\text{A.5})$$

According to the analysis in section 7, the transformed fields  $\hat{\mathbf{X}}_a^{(\alpha)}$  ( $\alpha \in A^+$ ) and  $\hat{\mathbf{X}}_b^{(\alpha)}$  ( $\alpha \in A^-$ ) vanish in the sets

$$\Omega_\alpha = [\omega_1^{(\alpha)}, \omega_2^{(\alpha)}] \cup [\omega_3^{(\alpha)}, \omega_4^{(\alpha)}] \cup \dots,$$

and the sum of integrals in the right hand side of (A.5) disappears for these restrictions. This fact gives the connection between equation (A.5) and the time domain representation of the eigenvector problems (4.12) and (4.13).

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