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Shear Stresses in Elastic Beams: an intrinsic approach

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Abstract

The theory of non-uniform flexure and torsion of Saint-Venant's beam with arbitrary multiply connected cross section is revisited in a coordinate-free form to provide a computationally convenient context. Numerical implementations, by Matlab, are performed to evaluate the maximum elastic shear stresses in beams with rectangular cross sections for different Poisson's ratios. The deviations between the maximum and mean stresses are then diagrammed to adjust the results provided by Jourawski's method.

Key words: Saint-Venant's beam, elastic shear stresses, rectangular cross section.

1. Introduction

In (Truesdell , 1976, page 124) it is stated that Saint-Venant's work, for penetration of concept, for assessment of a key problem, and for skill in solving it, stands as one of the supreme monuments of mechanics. Nevertheless, the exact solution of the problem of finding the elastic shear stress field in a Saint-Venant's beam with cross section of arbitrary shape is lacking. Saint-Venant himself studied in detail and solved torsion and flexure problems for circular, elliptic, equilater triangle and rectangular cross sections by providing numerical examples and graphs (De Saint Venant , 1856a,b). In the case of rectangular sections, Saint-Venant provided a solution in terms of a rapidly converging series. Grashof in 1878 discussed the case of a cross section whose boundary consists of two vertical sides and two hyperbolas. Further, the cases of cardioid, epitrochoidal, Booth's lemniscate, Bernoulli's lemniscate, confocal ellipses, eccentric circles cross sections may be dealt with as applications of conformal mappings (Muskhelishvili , 1949; Sokolnikoff ,

1956). For a triangle of arbitrary geometry SAINT-VENANT's solution, in terms of shear stresses, is not known. SETH dealt with the flexure of a beam whose cross section is a right-angled isosceles triangle in 1934, and then, in 1936, he discussed the flexure solutions only for three particular cases: for a shear force perpendicular to the axis of symmetry of an isosceles triangle, for a shear force parallel to the hypotenuse of a right-angled triangle and for any triangular cross section and for any direction of the shear force if POISSON's ratio is equal to 0.5. However SAINT-VENANT's solutions for cross sections occupied by multiply connected domains or of arbitrary shape are not available, but it occurs to make recourse to numerical or alternative calculus methods (see e.g. Di Paola et al., 2008). A simple strategy employed by engineers is the method invented by Jourawski (1856) and Rankine (1858), subsequently developed by Grashof (1878), based on equilibrium considerations. In the first part of the paper, we reformulate the problem of non-uniform flexure and torsion of SAINT-VENANT's beam with a arbitrary cross section in a coordinate-free form. We obtain the elastic shear stress field, at the points of the cross sections, elastically compatible with the linearized displacement field, assigned to within an unknown scalar warping function. By imposing that the elastic shear stresses, expressed as sum of a geometrical term plus the gradient of the warping, fulfill the differential and boundary CAUCHY's equations of equilibrium, a POISSON-NEUMANN's problem or equivalently a LAPLACE-NEUMANN's one for the warping is formulated. Two analogies with elastostatics show that these problems admit an unique solution, to within a constant, for any datum (Romano , 2002). For particular positions and geometries of the holes of multiply connected cross sections or in the case of pure torsion the evaluation of the elastic shear stresses may be alternatively carried out formulating a POISSON-DIRICHLET's problem. The second part of the paper concerns the evaluation, by MATLAB, of the elastic shear stresses, to within the shear modulus, on rectangular cross sections of nine SAINT-VENANT's beams subject to non-uniform flexure, shear force directed along the height and for seven POISSON's ratios. Moreover we evaluate, by polynomial approximation, the component of the elastic shear stresses along the height at the points of the middle line parallel to the base of each rectangle and, then, we compare the maximum value with the mean one provided by JOURAWSKI's formula. Subsequently, for chosen POISSON's ratios, we provide the corrective coefficient which must be multiplied by JOURAWSKI's mean value to get the maximum shear stress as a function of the ratio between base and height of the rectangle.

2. Formulation of the problems of flexure and torsion

Let Ω be a two-dimensional multiply connected domain, occupied by the cross section of Saint-Venant's beam, and assume that the boundary $\partial\Omega$ is made of several closed contours $\partial\Omega_o, \partial\Omega_1, \partial\Omega_2, \ldots, \partial\Omega_n$; the first of these contours contains all other (see fig. 1). Hereafter we denote by $\mathbf{r}$ the position vector of each point of the cross section Ω with respect to the centroid G, z the abscissa along the beam axis, $\mathbf{g}$ the euclidean metric tensor, $\mathbf{n}$ the unit outward normal to Ω, $\mathbf{k}$ the unit vector of the z-axis, $\{'\}$ the derivative with respect to z and $\mathbf{R}$ the orthogonal linear transformation which anticlockwise rotates a two-dimensional vector field of $\frac{\pi}{2}$. Let $\epsilon_z(\mathbf{r})$ be the scalar field of the extensions of the longitudinal fibers of the beam and $\mathbf{d}(z) := \nabla\epsilon_z$ be its gradient with respect to the position vector $\mathbf{r}$. The linear bending curvature vector $\mathbf{d}(z)$ is evaluated by imposing the rotation equilibrium condition about the centroid G:

$$\mathbf{M}_G(z) = \int_\Omega \mathbf{r} \times \sigma_z(\mathbf{r})\,\mathbf{k}\,da = \int_\Omega \mathbf{r} \times (E\,\mathbf{g}(\mathbf{r}, \mathbf{d}(z)))\,\mathbf{k}\,da \iff$$
$$\mathbf{R}\mathbf{M}_G(z) := \mathbf{k} \times \mathbf{M}_G(z) = \mathbf{J}_G(E)\,\mathbf{d}(z)$$

where $\mathbf{M}_G(z)$ is the bending resultant moment at the z abscissa about G, $\mathbf{J}_G(E) := E\int_\Omega \mathbf{r} \otimes_{\mathbf{g}} \mathbf{r}\,da$ is the elastic bending stiffness and $\sigma_z(\mathbf{r}) = E\,\epsilon_z(\mathbf{r})$ is the elastic normal stress field on Ω. By relation $\mathbf{R}\mathbf{M}_G(z) = \mathbf{J}_G(E)\,\mathbf{d}(z)$ being $\mathbf{M}_G(z)$ an affine function, i.e. $\mathbf{M}_G(z) = \mathbf{M}_G(0) - z\,\mathbf{R}\mathbf{S}$, it follows that the curvature vector $\mathbf{d}(z)$ is an affine function: $\mathbf{d}(z) = \mathbf{d}_0 + \mathbf{d}'\,z$, where $\mathbf{S}$ is the shear force. The vector $\mathbf{d}_0$ is the bending curvature of the cross section at $z = 0$ given by the relation $\mathbf{R}\mathbf{M}_G(0) = \mathbf{J}_G(E)\,\mathbf{d}_0$. By observing that $\mathbf{J}_G(E)\,\mathbf{d}(z) = \mathbf{R}\mathbf{M}_G(z) = \mathbf{R}\mathbf{M}_G(0) + z\,\mathbf{S} \implies \mathbf{J}_G(E)\,(\mathbf{d}(z))' = \mathbf{R}(\mathbf{M}_G(z))' = \mathbf{S}$ we get the formula $\mathbf{S} = \mathbf{J}_G(E)\,\mathbf{d}'$ to evaluate the vector $\mathbf{d}'$.

2.1. Displacement, strain and stress fields

The displacement field of Saint-Venant's beam subject to non-uniform flexure and torsion is given by:

$$\mathbf{u}(\mathbf{r}, z) = \frac{\nu}{2}\,(\mathbf{R}\mathbf{r} \otimes_{\mathbf{g}} \mathbf{R}\mathbf{r} - \mathbf{r} \otimes_{\mathbf{g}} \mathbf{r})\,\mathbf{d}(z) - \frac{z^2}{2}\,\mathbf{d}_0 - \frac{z^3}{6}\,\mathbf{d}' + \theta'\,z\,\mathbf{R}\mathbf{r}$$
$$+\big[\varphi(\mathbf{r}) + \mathbf{g}(\mathbf{d}_0, \mathbf{r})\,z + \mathbf{g}(\mathbf{d}', \mathbf{r})\,\frac{z^2}{2}\big]\,\mathbf{k}\,,$$

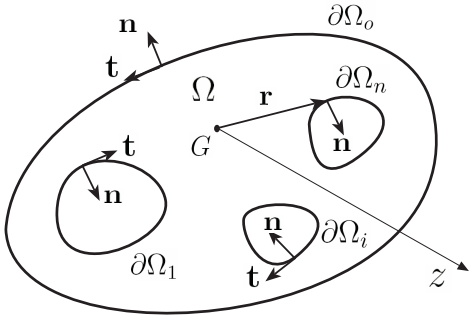

Figure 1: Multiply connected cross section of the beam

in which $\otimes_{\mathbf{g}}$ is the diadic product between vectors according to the metric tensor $\mathbf{g}$, θ' is the torsional curvature, ν is POISSON's ratio assumed to be constant with respect to the longitudinal fibers of the beam and $\varphi \in \mathrm{C}^2(\Omega\,;\mathcal{R})$ is an unknown scalar field to determine by equilibrium. Denoting by $BL(\,)$ a bounded linear map, V a three dimensional linear space and d the derivative, the compatible stretching $\mathbf{Du} \in BL(\Omega\,;BL(V\,;V))$ associated with the displacement field $\mathbf{u} \in \mathrm{C}^2(\Omega\,;V)$ is given by:

$$\mathbf{Du} := sym\, d\mathbf{u} = \frac{1}{2}\,(d\mathbf{u} + d\mathbf{u}^T)\,.$$

By assuming an isotropic linear elastic behavior of the material, the CAUCHY stress $\mathbf{T} \in BL(\Omega\,;BL(V\,;V))$ which is elastically compatible with the displacement field $\mathbf{u}$ writes as:

$$\mathbf{T} = 2\,G\,(\mathbf{Du}) + \frac{2\,G\,\nu}{1-2\,\nu}\,(I_1(\mathbf{Du}))\,\mathbf{I}\,,$$

where G is the shear modulus, $I_1()$ stands for the linear invariant of a linear operator and $\mathbf{I}$ is the identity operator. Accordingly the elastic normal and shear stress fields, evaluated at the points of the cross section of the beam, are given by:

$$\begin{cases} \sigma_z(\mathbf{r}) = E\,\mathbf{g}(\mathbf{r},\mathbf{d}(z))\,, \\ \boldsymbol{\tau}(\mathbf{r}) \;= \dfrac{G\,\nu}{2}\,(\mathbf{Rr}\otimes_{\mathbf{g}}\mathbf{Rr} - \mathbf{r}\otimes_{\mathbf{g}}\mathbf{r})\,\mathbf{d}' + G\,\theta'\,\mathbf{Rr} + G\,\nabla\varphi(\mathbf{r})\,, \end{cases}$$

where ∇ is the gradient operator. The determination of the elastic shear stresses is subordinated to the evaluation of the gradient of the unknown

function $\varphi \in \mathrm{C}^2(\Omega\,;\mathcal{R})$. To this end, the fulfillment of the differential and boundary CAUCHY's conditions of equilibrium

$$\begin{cases} \operatorname{div} \boldsymbol{\tau}(\mathbf{r}) = -\sigma'(\mathbf{r}) = -E\,\mathbf{g}(\mathbf{d}', \mathbf{r})\,, & \forall\, \mathbf{r} \in \Omega\,, \\ \displaystyle\int_{\mathbf{c}} \mathbf{g}(\mathbf{R}\boldsymbol{\tau}) = 0\,, & \forall\, \mathbf{c} \subset \partial\Omega\,, \end{cases}$$

leads to the formulation of the following POISSON-NEUMANN problem

$$\begin{cases} \Delta_2 \varphi = \dfrac{1}{G}(\operatorname{div} \boldsymbol{\tau}_\beta - \alpha)\,, & \text{in } \Omega\,, \\ \displaystyle\int_{\mathbf{c}} \mathbf{g}(\mathbf{R}(G\,\nabla\varphi - \boldsymbol{\tau}_\beta)) = 0\,, & \forall\, \mathbf{c} \subset \partial\Omega\,, \end{cases}$$

for the function $\varphi \in \mathrm{C}^2(\Omega\,;\mathcal{R})$, in which:

$$\begin{cases} \Delta_2 := \operatorname{div} \nabla \\ \boldsymbol{\tau}_\beta(\mathbf{r}) := -\dfrac{G\,\nu}{2}\,(\mathbf{R}\mathbf{r} \otimes_{\mathbf{g}} \mathbf{R}\mathbf{r} - \mathbf{r} \otimes_{\mathbf{g}} \mathbf{r})\,\mathbf{d}' - G\,\theta'\,\mathbf{R}\mathbf{r}\,, \\ \alpha(\mathbf{r}) := 2\,G\,(1+\nu)\,\mathbf{g}(\mathbf{d}', \mathbf{r})\,, \\ \operatorname{div}\left((\mathbf{R}\mathbf{r} \otimes_{\mathbf{g}} \mathbf{R}\mathbf{r})\,\mathbf{d}'\right) = -\mathbf{g}(\mathbf{r}, \mathbf{d}')\,, \\ \operatorname{div}\left((\mathbf{r} \otimes_{\mathbf{g}} \mathbf{r})\,\mathbf{d}'\right) = 3\,\mathbf{g}(\mathbf{r}, \mathbf{d}')\,, \\ \operatorname{div}\left(\mathbf{R}\mathbf{r}\right) = 0\,, \\ \operatorname{div} \boldsymbol{\tau}_\beta(\mathbf{r}) = 2\,G\,\nu\,\mathbf{g}(\mathbf{d}', \mathbf{r})\,, \\ \operatorname{div} \boldsymbol{\tau}_\beta(\mathbf{r}) - \alpha(\mathbf{r}) = -2\,G\,\mathbf{g}(\mathbf{d}', \mathbf{r})\,. \end{cases}$$

Remark 2.1. *Let us note that a metric tensor* $\mathbf{g} \in BL(V^2\,;\mathcal{R})$ *induces a linear isomorphism* $\mathbf{g}^\flat \in BL(V\,;V^*)$ *between the space* V *and its dual* V^*. *Indeed to any vector* $\mathbf{a} \in V$ *we may associate uniquely the covector* $\mathbf{g}^\flat\mathbf{a} \in V^*$ *defined by* $\langle \mathbf{g}^\flat\mathbf{a}, \mathbf{b}\rangle = \mathbf{g}(\mathbf{a}, \mathbf{b}) \quad \forall\, \mathbf{b} \in V$, *where* $\langle \cdot, \cdot \rangle$ *is the duality pairing between vectors and covectors. By performing the natural identification* $BL(V^2\,;\mathcal{R}) = BL(V\,;V^*)$ *we denote by the symbol* $\mathbf{ga}$ *the covector* $\mathbf{g}^\flat\mathbf{a} \in V^*$ *associated with the vector* $\mathbf{a} \in V$ *(see e.g. Abraham et al. , 2002). The boundary condition of equilibrium*

$$\int_{\mathbf{c}} \mathbf{g}(\mathbf{R}\boldsymbol{\tau}) := \int_{s_1}^{s_2} \mathbf{g}(\mathbf{R}\boldsymbol{\tau}, \mathbf{t})\,ds = \int_{s_1}^{s_2} \mathbf{g}(\mathbf{R}\boldsymbol{\tau}, \mathbf{R}\mathbf{n})\,ds = \int_{s_1}^{s_2} \mathbf{g}(\boldsymbol{\tau}, \mathbf{n})\,ds = 0$$

states that the integral of the covector $\mathbf{g}(\mathbf{R}\boldsymbol{\tau})$ *, over every curve* $\mathbf{c}$ *drawn on the boundary of the domain* Ω *, vanishes. This assertion is equivalent to require that the relation* $\mathbf{g}(\boldsymbol{\tau}(\mathbf{r}), \mathbf{n}(\mathbf{r})) = 0$ *, at each point of* $\partial\Omega$ *, holds. The reader is referenced to (Romano , 2009) for an exhaustive exposition about integration on manifolds.*

Remark 2.2. *The* Poisson-Neumann *problem formulated in subsection 2.1 may be transformed in the following* Laplace-Neumann *problem:*

$$\begin{cases} \Delta_2\psi = 0\,, \quad \textit{in } \Omega\,, \\ \displaystyle\int_{\mathbf{c}} \mathbf{g}(\mathbf{R}(\nabla\psi(\mathbf{r}) - \frac{G}{4}\,\nabla\big(\mathbf{g}(\mathbf{d}',\mathbf{r})\mathbf{g}(\mathbf{r},\mathbf{r})\big)) - \boldsymbol{\tau}_\beta(\mathbf{r})) = 0\,, \quad \forall\,\mathbf{c} \subset \partial\Omega\,, \end{cases}$$

where $\psi(\mathbf{r}) := G\,\varphi(\mathbf{r}) + \dfrac{G}{4}\,\mathbf{g}(\mathbf{d}',\mathbf{r})\,\mathbf{g}(\mathbf{r},\mathbf{r})\,, \quad \mathbf{r} \in \Omega$ *. Indeed, observing that*

$$\mathbf{g}(\mathbf{d}',\mathbf{r}) = \frac{1}{8}\,\Delta_2(\mathbf{g}(\mathbf{d}',\mathbf{r})\,\mathbf{g}(\mathbf{r},\mathbf{r}))\,,$$

the known term of Poisson*'s equation:* $\Delta_2\varphi = \dfrac{1}{G}(\operatorname{div}\boldsymbol{\tau}_\beta - \alpha)$ *, may be rewritten as:*

$$\frac{1}{G}(\operatorname{div}\boldsymbol{\tau}_\beta(\mathbf{r}) - \alpha(\mathbf{r})) = -2\,\mathbf{g}(\mathbf{d}',\mathbf{r}) = -\frac{1}{4}\,\Delta_2(\mathbf{g}(\mathbf{d}',\mathbf{r})\,\mathbf{g}(\mathbf{r},\mathbf{r}))\,,$$

so that the result follows.

Remark 2.3. *The condition for existence and uniqueness, to within a constant, of the solution* $\psi \in \mathrm{C}^2(\Omega\,;\mathcal{R})$ *of the* Laplace-Neumann *problem is that the integral of the normal derivative of the function* ψ *, over the boundary* $\partial\Omega$ *, vanishes. This result may be motivated by making recourse to an analogy with an elastostatic problem (Romano , 2002). This condition is satisfied in the case of the* Laplace-Neumann *problem:*

$$\begin{cases} \Delta_2\psi = 0\,, \quad \textit{in } \Omega\,, \\ \displaystyle\int_{\mathbf{c}} \mathbf{g}(\mathbf{R}(\nabla\psi(\mathbf{r}) - \frac{G}{4}\,\nabla\big(\mathbf{g}(\mathbf{d}',\mathbf{r})\mathbf{g}(\mathbf{r},\mathbf{r})\big)) - \boldsymbol{\tau}_\beta(\mathbf{r})) = 0\,, \quad \forall\,\mathbf{c} \subset \partial\Omega\,. \end{cases}$$

It suffices to show that: $\oint_{\partial\Omega} \mathbf{g}(\mathbf{R}\big(\frac{G}{4}\,\nabla(\mathbf{g}(\mathbf{d}',\mathbf{r})\,\mathbf{g}(\mathbf{r},\mathbf{r}))\big) + \boldsymbol{\tau}_\beta(\mathbf{r})) = 0$ *. The result can be obtained by making recourse to the divergence theorem, taking*

into account the formulas:

$$\begin{cases} \operatorname{div} \boldsymbol{\tau}_\beta(\mathbf{r}) = 2\,G\,\nu\,\mathbf{g}(\mathbf{d}',\mathbf{r})\,; \\ \nabla(\mathbf{g}(\mathbf{d}',\mathbf{r})\,\mathbf{g}(\mathbf{r},\mathbf{r}))) = (\mathbf{d}' \otimes_{\mathbf{g}} \mathbf{r})\mathbf{r} + 2\,(\mathbf{r} \otimes_{\mathbf{g}} \mathbf{r})\mathbf{d}'\,, \\ \Delta_2(\mathbf{g}(\mathbf{d}',\mathbf{r})\,\mathbf{g}(\mathbf{r},\mathbf{r})) = 8\,\mathbf{g}(\mathbf{d}',\mathbf{r})\,, \end{cases}$$

and observing that the first moment of area $\int_\Omega \mathbf{r}\,da$*, with respect to the centroid* G*, is equal to zero.*

2.2. The stress function

An alternative methodology for determining the elastic shear stresses consists in transforming the differential equation of equilibrium:

$$\operatorname{div} \boldsymbol{\tau}(\mathbf{r}) = -\alpha(\mathbf{r})\,,$$

with $\alpha(\mathbf{r}) := -2\,G\,(1+\nu)\,\mathbf{g}(\mathbf{d}',\mathbf{r})$ in the equivalent condition:

$$\operatorname{div}\,(\boldsymbol{\tau}(\mathbf{r}) + \boldsymbol{\tau}_\alpha(\mathbf{r})) = \operatorname{rot}\,(\mathbf{R}(\boldsymbol{\tau}(\mathbf{r}) + \boldsymbol{\tau}_\alpha(\mathbf{r}))) = 0\,,$$

where $\boldsymbol{\tau}_\alpha(\mathbf{r}) := \frac{1}{3}E\,(\mathbf{r} \otimes_{\mathbf{g}} \mathbf{r})\,\mathbf{d}'$ and $\operatorname{div} \boldsymbol{\tau}_\alpha(\mathbf{r}) = \operatorname{rot}\,(\mathbf{R}\boldsymbol{\tau}_\alpha(\mathbf{r})) = \alpha(\mathbf{r})\,.$

Two cases have to be distinguished and precisely whether the cross sections of the beam are simply or multiply connected. In the former case, the differential equation of equilibrium expressed by the vanishing of the curl of the vector field $\mathbf{R}(\boldsymbol{\tau}(\mathbf{r}) + \boldsymbol{\tau}_\alpha(\mathbf{r}))$ is an integrability condition, namely it implies the existence of a potential $F \in \mathrm{C}^2(\Omega\,;\mathcal{R})$:

$$\mathbf{R}(\boldsymbol{\tau}(\mathbf{r}) + \boldsymbol{\tau}_\alpha(\mathbf{r})) = \nabla F(\mathbf{r})\,,$$

or equivalently: $\boldsymbol{\tau}(\mathbf{r}) = \mathbf{R}\boldsymbol{\tau}_\alpha(\mathbf{r}) - \mathbf{R}\nabla F(\mathbf{r})\,.$ The evaluation of the function $F \in \mathrm{C}^2(\Omega\,;\mathcal{R})\,,$ to within a constant, may be obtained by imposing the differential condition of compatibility and the boundary one of equilibrium. To this end, to get the differential condition of compatibility

$$\operatorname{rot} \boldsymbol{\tau}(\mathbf{r}) = -\beta(\mathbf{r})\,,$$

we take into account the relation $\boldsymbol{\tau}(\mathbf{r}) + \boldsymbol{\tau}_\beta(\mathbf{r}) = G\,\nabla\varphi(\mathbf{r})$ (see subsection 2.1) and evaluate the curl, with $\beta(\mathbf{r}) := \operatorname{rot} \boldsymbol{\tau}_\beta(\mathbf{r}) = -2\,G\,\theta' + 2\,G\,\nu\,\mathbf{g}(\mathbf{R}\mathbf{d}',\mathbf{r})$ and being $\operatorname{rot}\nabla\varphi(\mathbf{r}) = 0\,.$ Moreover, the boundary condition of equilibrium $\int_{\mathbf{c}} \mathbf{g}(\mathbf{R}\boldsymbol{\tau}) = 0$ imposes that:

$$\int_{\mathbf{c}} \mathbf{g}(\nabla F - \boldsymbol{\tau}_\alpha) = 0\,, \quad \forall\,\mathbf{c} \subset \partial\Omega\,,$$

so that integrating on the contour of the section of the beam we get:

$$F(s) = F(s_0) + \int_{s_0}^{s} \mathbf{g}(\boldsymbol{\tau}_\alpha, \mathbf{t})\, ds\,.$$

Accordingly, the function $F \in \mathrm{C}^2(\Omega\,;\mathcal{R})$ may be obtained as a solution of Poisson-Dirichlet's problem:

$$\begin{cases} \Delta_2 F = \operatorname{div} \boldsymbol{\tau}_\alpha + \beta\,, \\ F(s) = \displaystyle\int_{s_0}^{s} \mathbf{g}(\boldsymbol{\tau}_\alpha, \mathbf{t})\, ds\,, \end{cases}$$

with the assumption that $F(s_0) = 0$, being $F \in \mathrm{C}^2(\Omega\,;\mathcal{R})$ defined to within a constant. The last problem admits solution for any datum. This result may be motivated by an analogy with an elastostatic problem (Romano , 2002). In the latter case, the integrability of the field $\mathbf{R}(\boldsymbol{\tau}(\mathbf{r}) + \boldsymbol{\tau}_\alpha(\mathbf{r}))$ is ensured if $\operatorname{rot}\left(\mathbf{R}(\boldsymbol{\tau}(\mathbf{r}) + \boldsymbol{\tau}_\alpha(\mathbf{r}))\right) = 0$ and on all internal contours of the cross section of the beam the following conditions

$$\begin{aligned} 0 = \oint_{\partial\Omega_i} \mathbf{g}(\mathbf{R}(\boldsymbol{\tau}_\alpha(\mathbf{r}))) &= \oint_{\partial\Omega_i} \mathbf{g}(\mathbf{R}(\boldsymbol{\tau}_\alpha(\mathbf{r})), \mathbf{t}(\mathbf{r}))\, ds = \oint_{\partial\Omega_i} \mathbf{g}(\boldsymbol{\tau}_\alpha(\mathbf{r}), \mathbf{n}(\mathbf{r}))\, ds \\ &= \int_{\Omega_i} \operatorname{div} \boldsymbol{\tau}_\alpha(\mathbf{r})\, da = \int_{\Omega_i} \alpha(\mathbf{r})\, da \end{aligned}$$

are fulfilled. A noteworthy example in which the previous integrability conditions are met is the case of torsion, where $\mathbf{d}' = 0$ and hence $\boldsymbol{\tau}_\alpha(\mathbf{r}) = 0$.

It follows that the expression of the elastic shear stress field in Saint-Venant's beam subject to torsion becomes:

$$\boldsymbol{\tau}(\mathbf{r}) = -\mathbf{R}\,\nabla F(\mathbf{r})\,.$$

In the case of torsion the potential $F \in \mathrm{C}^2(\Omega\,;\mathcal{R})$ was introduced by Prandtl (1903) and it is said to be Prandtl stress function. This function may be found as solution of the following Poisson-Dirichlet problem:

$$\begin{cases} \Delta_2 F(\mathbf{r}) = -2\,G\,\theta' \quad \text{on } \Omega\,, \\ F(\mathbf{r}) = 0 \quad \text{on } \partial\Omega_0\,, \\ F(\mathbf{r}) = k_i \quad \text{on } \partial\Omega_i\,, \quad k_i \in \mathcal{R}\,, \quad i = 0, 1, 2, \ldots, n\,. \end{cases}$$

Setting $\mathbf{S_G}(\Omega_i) := \int_{\Omega_i} \mathbf{r}\, da$ and $A(\Omega_i) := \int_{\Omega_i} da$, the evaluation of the n constants $k_i \in \mathcal{R}$ may be obtained by imposing the integrability conditions:

$$\oint_{\partial\Omega_i} \mathbf{g}(\boldsymbol{\tau}(\mathbf{r}), \mathbf{t}(\mathbf{r}))\, ds = -2\,G\,\nu\,\mathbf{g}(\mathbf{R}\,\mathbf{S_G}(\Omega_i), \mathbf{d}') - 2\,G\,\theta'\,A(\Omega_i) \quad i = 1, 2, \ldots, n\,,$$

deduced integrating along the internal contours of the holes, of the cross section, the compatibility condition (see subsection 2.1):

$$\boldsymbol{\tau}(\mathbf{r}) + \boldsymbol{\tau}_\beta(\mathbf{r}) = G\,\nabla\varphi(\mathbf{r})$$

and assuming that $\mathbf{d}' = 0$. Then, let us set: $F(\mathbf{r}) = F_o(\mathbf{r}) + \sum_{i=1}^{n} k_i\, F_i(\mathbf{r})$, where:

- $F_o(\mathbf{r})$ is the solution of Poisson-Dirichlet's problem with homogeneous boundary conditions:

$$\begin{cases} \Delta_2 F_o(\mathbf{r}) = -2\,G\,\theta' & \text{on } \Omega\,, \\ F_0(\mathbf{r}) = 0 & \text{on } \partial\Omega\,, \end{cases}$$

- $F_i(\mathbf{r})$, $i = 0, 1, 2, \ldots, n$ are the solutions of n Laplace-Dirichlet's problems:

$$\begin{cases} \Delta_2 F_i(\mathbf{r}) = 0 & \text{on } \Omega\,, \\ F_i(\mathbf{r}) = 0 & \text{on } \partial\Omega_0\,, \\ F_i(\mathbf{r}) = \delta_{ij} & \text{on } \partial\Omega_j\,, \quad j = 0, 1, 2, \ldots, n\,, \end{cases}$$

  where δ_{ij} is Kronecker's notation.

A straightforward computation shows that, being:

$$\begin{aligned} -\oint_{\partial\Omega_i} \mathbf{g}(\mathbf{R}\,\nabla F(\mathbf{r}), \mathbf{t}(\mathbf{r}))\, ds &= -\oint_{\partial\Omega_i} \mathbf{g}(\nabla F(\mathbf{r}), \mathbf{n}(\mathbf{r}))\, ds \\ &= -\oint_{\partial\Omega_i} \mathbf{g}(\nabla F_o(\mathbf{r}), \mathbf{n}(\mathbf{r}))\, ds \\ &\quad -\sum_{j=1}^{n} \left(\oint_{\partial\Omega_j} \mathbf{g}(\nabla F_j(\mathbf{r}), \mathbf{n}(\mathbf{r}))\, ds \right) k_j\,; \end{aligned}$$

the compatibility equations

$$\oint_{\partial\Omega_i} \mathbf{g}(\boldsymbol{\tau}(\mathbf{r}), \mathbf{t}(\mathbf{r}))\, ds = -2\, G\, \theta'\, A(\Omega_i)\,,\ i = 1, 2, \ldots, n$$

for the elastic shear stress field $\boldsymbol{\tau}(\mathbf{r}) = -\mathbf{R}\, \nabla F(\mathbf{r})$ provide a non homogeneous algebraic linear system in the unknowns $k_i\ ,\ i = 0, 1, 2, \ldots, n$:

$$\sum_{j=1}^{n} \left(\oint_{\partial\Omega_j} \mathbf{g}(\nabla F_j(\mathbf{r}), \mathbf{n}(\mathbf{r}))\, ds \right) k_j = 2\, G\, \theta'\, A(\Omega_i) - \oint_{\partial\Omega_i} \mathbf{g}(\nabla F_o(\mathbf{r}), \mathbf{n}(\mathbf{r}))\, ds\,.$$

Remark 2.4. Poisson-Dirichlet*'s problem:*

$$\begin{cases} \Delta_2 F(\mathbf{r}) = \dfrac{2\, G\, (1 + 4\, \nu)}{3}\, \mathbf{g}(\mathbf{R}\mathbf{d}', \mathbf{r}) - 2\, G\, \theta' \quad on\ \Omega\,, \\ F(\mathbf{r}) = 0 \quad on\ \partial\Omega_0\,, \\ F(\mathbf{r}) = k_i \quad on\ \partial\Omega_i\,, \quad k_i \in \mathcal{R}\,, \quad i = 0, 1, 2, \ldots, n \end{cases}$$

admits a unique solution $F \in \mathrm{C}^2(\Omega\,; \mathcal{R})$ *for any datum. This result may be motivated by an analogy with an elastostatic problem (Romano , 2002).*

3. Jourawski method

Let us consider a subdomain Ω^* of the cross section Ω of Saint-Venant's beam (fig. 2). The flux q^* of the shear stresses through the closed curve $\partial\Omega^*$ is defined by:

$$q^* := \int_{\partial\Omega^*} \mathbf{g}(\mathbf{R}\boldsymbol{\tau})\,.$$

By the equilibrium differential equation: $\operatorname{div}\boldsymbol{\tau}(\mathbf{r}) = -E\, \mathbf{g}(\mathbf{d}', \mathbf{r})$ and the divergence theorem we get the expression:

$$q^* = \int_{\partial\Omega^*} \mathbf{g}(\mathbf{R}\boldsymbol{\tau}) = \int_{\Omega^*} \operatorname{div}\boldsymbol{\tau}\, da = -E\, \mathbf{g}(\mathbf{d}', \mathbf{S}^*_G)\,,$$

in which $\mathbf{S}^*_G := \int_{\Omega^*} \mathbf{r}\, da$ is the first moment of area of Ω^* with respect to the centroid of the section Ω. The mean value of the component along the unit normal $\mathbf{n}$, outward to the domain Ω^*, of the elastic shear stresses $\boldsymbol{\tau}$ is given by:

$$\overline{\tau_{\mathbf{n}}} := \frac{q^*}{l(\partial\Omega^*)}\,,$$

where $l(\partial\Omega^*)$ is the length of the curve $\partial\Omega^*$. Accordingly the Jourawski formula writes as: $\overline{\tau_{\mathbf{n}}} = -E\, \dfrac{\mathbf{g}(\mathbf{d}', \mathbf{S}^*_G)}{l(\partial\Omega^*)} = -2\, G\, (1 + \nu)\, \dfrac{\mathbf{g}(\mathbf{d}', \mathbf{S}^*_G)}{l(\partial\Omega^*)}$.

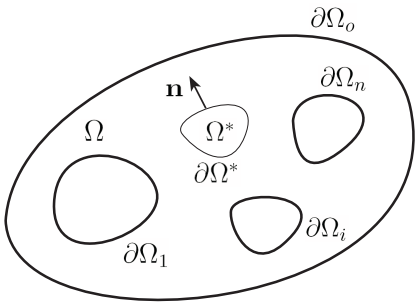

Figure 2: Cross section of the beam

4. Numerical results

Let us consider a rectangular cross section of Saint-Venant's beam, whose edges are B (base) and H (height), and assume that the derivative $\mathbf{d}'$ of the extension gradient $\mathbf{d}$ has components $\{\,0\,;1\,\}$ in a fixed set of $\mathbf{g}$-orthonormal cartesian axes $\{\,G, x, y\,\}$ (see fig. 3). By formula $\mathbf{S} = \mathbf{J}_G(E)\,\mathbf{d}'$ in section 2 it follows that the shear force is given by

$$\mathbf{S} = E\,\frac{B\,H^3}{12}\,\mathbf{d}'\,.$$

We evaluate the elastic shear stress field for rectangular cross sections, whose

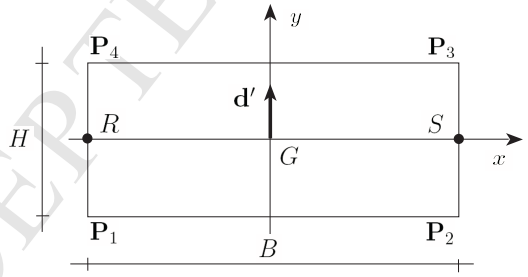

Figure 3: Rectangular cross section

ratio between base and height $\frac{B}{H} \in \{\,\frac{1}{8}\,;\frac{1}{6}\,;\frac{1}{4}\,;\frac{1}{2}\,;1\,;2\,;4\,;6\,;8\,\}$ and Poisson ratio $\nu \in \{\,0\,;0.1\,;0.2\,;0.25\,;0.3\,;0.4\,;0.4999\,\}$, by Matlab version 7.6 (R2008a). To this end, setting $u(\mathbf{r}) := -\dfrac{\varphi(\mathbf{r})}{2}$, the Poisson-Neumann

problem (see subsection 2.1) to solve is the following:

$$\begin{cases} \Delta_2 u(\mathbf{r}) = \mathbf{g}(\mathbf{r}, \mathbf{d}')\,, \\ d_{\mathbf{n}} u(\mathbf{r}) = \dfrac{1}{4}\,\nu\,[\mathbf{g}(\mathbf{Rr}, \mathbf{d}')\mathbf{g}(\mathbf{Rr}, \mathbf{n}) - \mathbf{g}(\mathbf{r}, \mathbf{d}')\mathbf{g}(\mathbf{r}, \mathbf{n})]\,, \end{cases}$$

and the expression of the elastic shear stress field writes as (subsection 2.1): $\boldsymbol{\tau}(\mathbf{r}) = -2\,G\,\nabla u(\mathbf{r}) - \boldsymbol{\tau}_\beta(\mathbf{r})$, with $\boldsymbol{\tau}_\beta(\mathbf{r}) = -\dfrac{G\,\nu}{2}\,(\mathbf{Rr} \otimes_{\mathbf{g}} \mathbf{Rr} - \mathbf{r} \otimes_{\mathbf{g}} \mathbf{r})\,\mathbf{d}'$, given that the torsional curvature θ' is equal to zero. In components, being $|\mathbf{r}| = \{\, x\,; y\,\}$ and $|\mathbf{Rr}| = \{\, -y\,; x\,\}$, the POISSON equation $\Delta_2 u(\mathbf{r}) = \mathbf{g}(\mathbf{r}, \mathbf{d}')$ becomes $\Delta_2 u(\mathbf{r}) = y$. Moreover, taking into account the components reported in table 1 of the normal unit vector outward to the rectangular domain and of the points located on its boundary, NEUMANN's condition

Table 1:

Side	$\lvert\mathbf{n}\rvert$		$\lvert\mathbf{r}\rvert$	
$\mathbf{P}_1\mathbf{P}_2$	0	-1	x	$-\frac{H}{2}$
$\mathbf{P}_2\mathbf{P}_3$	1	0	$\frac{B}{2}$	y
$\mathbf{P}_3\mathbf{P}_4$	0	1	x	$\frac{H}{2}$
$\mathbf{P}_4\mathbf{P}_1$	-1	0	$-\frac{B}{2}$	y

$d_{\mathbf{n}} u(\mathbf{r}) = \dfrac{1}{4}\,\nu\,[\mathbf{g}(\mathbf{Rr}, \mathbf{d}')\mathbf{g}(\mathbf{Rr}, \mathbf{n}) - \mathbf{g}(\mathbf{r}, \mathbf{d}')\mathbf{g}(\mathbf{r}, \mathbf{n})]$ specializes as:

- side $\mathbf{P}_1\mathbf{P}_2 \quad \Longrightarrow \quad d_{\mathbf{n}} u(\mathbf{r}) = -\dfrac{1}{4}\,\nu\,(x^2 - \dfrac{H^2}{4})\,,$
- sides $\mathbf{P}_2\mathbf{P}_3$ and $\mathbf{P}_4\mathbf{P}_1 \quad \Longrightarrow \quad d_{\mathbf{n}} u(\mathbf{r}) = -\dfrac{1}{4}\,\nu\,B\,y\,,$
- side $\mathbf{P}_3\mathbf{P}_4 \quad \Longrightarrow \quad d_{\mathbf{n}} u(\mathbf{r}) = \dfrac{1}{4}\,\nu\,(x^2 - \dfrac{H^2}{4})\,.$

Accordingly, being $|\mathbf{r} \otimes_{\mathbf{g}} \mathbf{r}| = \begin{vmatrix} x^2 & xy \\ xy & y^2 \end{vmatrix}$ and $|\mathbf{Rr} \otimes_{\mathbf{g}} \mathbf{Rr}| = \begin{vmatrix} y^2 & -xy \\ -xy & x^2 \end{vmatrix}$, the components of the elastic shear stresses, to within G, are given by:

$$\begin{cases} \dfrac{\tau_{xz}(x,y,z)}{G} = -\nu\,x\,y - 2\,\dfrac{\partial u}{\partial x}(x,y)\,, \\ \dfrac{\tau_{yz}(x,y,z)}{G} = \dfrac{\nu}{2}\,(x^2 - y^2) - 2\,\dfrac{\partial u}{\partial y}(x,y)\,. \end{cases}$$

We evaluate, by polynomial approximation, the component τ_y of the elastic shear stresses, to within the shear modulus, along the line RS (see fig. 3) and then we report in table 2 the ratios c between the obtained maximum values and JOURAWSKI's ones $\frac{\overline{\tau_y}}{G} = \frac{2\,(1+\nu)\,S^*_{Gy}}{B} = \frac{(1+\nu)}{4}\,H^2$ (see section 3), as a function of POISSON's ratio and $\frac{B}{H}$. Then, interpolating the numer-

Table 2: Coefficient c vs. $\frac{B}{H}$ and ν

$\frac{B}{H}$	$\nu = 0$	$\nu = 0.1$	$\nu = 0.2$	$\nu = 0.25$	$\nu = 0.3$	$\nu = 0.4$	$\nu = 0.4999$
$\frac{1}{8}$	1	1.0009	1.0017	1.0020	1.0025	1.0005	1.0034
$\frac{1}{6}$	1	1.0016	1.0030	1.0036	1.0042	1.0050	1.0059
$\frac{1}{4}$	1	1.0038	1.0069	1.0084	1.0096	1.0115	1.0136
$\frac{1}{2}$	1	1.0058	1.0272	1.0333	1.0373	1.0461	1.0537
1	1	1.0556	1.1020	1.1261	1.1412	1.1749	1.1899
2	1	1.1782	1.3273	1.3964	1.5600	1.7598	1.9323
4	1	1.4415	1.8093	1.9882	2.1205	2.3877	2.6179
6	1	1.7069	2.2964	2.5555	2.7948	3.2227	3.5932
8	1	1.9891	3.0704	3.1763	3.5079	4.1040	4.6187

ical results by the power law $c = \xi\,(\frac{B}{H})^{\eta} + \zeta$, with $\frac{B}{H} \in]0\,;8]$ and ξ, η and ζ provided in table 3, we get the diagrams in figure 4. The results in

Table 3: ξ, η and ζ vs. ν

	$\nu = 0$	$\nu = 0.1$	$\nu = 0.2$	$\nu = 0.25$	$\nu = 0.3$	$\nu = 0.4$	$\nu = 0.4999$
ξ	1	0.08781	0.11420	0.19240	0.24920	0.32830	0.39450
η	0	1.17500	1.38700	1.17700	1.11900	1.08800	1.07300
ζ	0	0.98240	0.99530	0.96550	0.95430	0.93740	0.92300

figure 4 are in agreement with ones reported in (Timoshenko and Goodier , 1951) where POISSON's ratio is equal to 0.25 and $\frac{B}{H} \in \{\,0.5\,;1\,;2\,;4\,\}$. In figures 5, 6 and 7 the elastic shear stress fields are plotted with reference to $\frac{B}{H} \in \{\,\frac{1}{8}\,;1\,;8\,\}$ and $\nu = 0.3$. In figures 8 and 9 the ratio $\frac{||\boldsymbol{\tau}(\mathbf{r})||}{G}$ and the warping function $\varphi(\mathbf{r}) = -2\,u(\mathbf{r})$ for a square cross section with POISSON's ratio $\nu = 0.3$ are respectively diagrammed. In figures 10, 11, 12, 13, 14, 15, 16 we plot $\frac{\overline{\tau_y}}{G}$ and $\frac{\tau_y}{G}$ along the line RS for $\frac{B}{H} = 8$ and $\nu \in \{\,0\,;0.1\,;0.2\,;0.25\,;0.3\,;0.4\,;0.4999\,\}$.

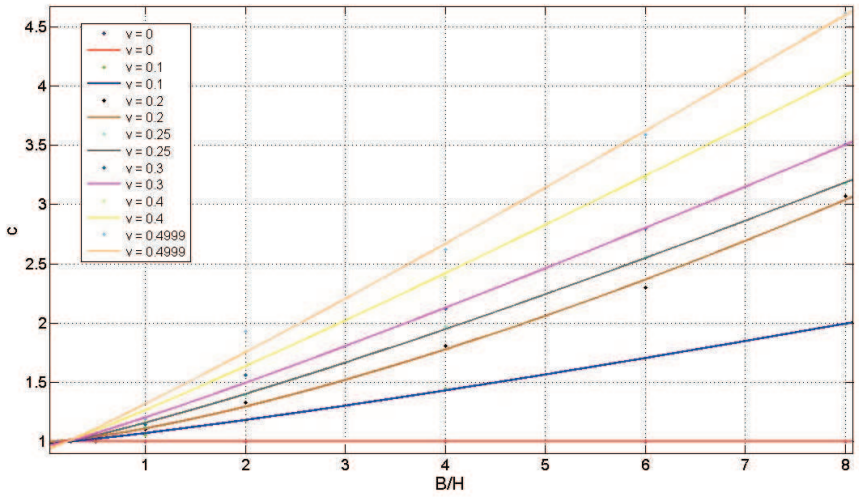

Figure 4: Corrective coefficient c

5. Conclusions

- The presentation concerning the evaluation of the elastic shear stresses in Saint-Venant's beam subject to non-uniform flexure and torsion, with arbitrary cross section, is nontraditional since it makes no reference to coordinate systems. The elastic shear stresses are expressed in terms of the derivative of the extension gradient $\mathbf{d}'$ of the longitudinal fibers of the beam, whose determination requires the solution of the two-dimensional algebraic linear problem: $\mathbf{S} = \mathbf{J}_G(E)\,\mathbf{d}'$, and of a scalar function fulfilling a boundary value problem of potential theory. This general treatment is particularly convenient whereas the form of the boundary of the cross section of Saint-Venant's beam suggests the use of curvilinear coordinates instead of Cartesian ones, e.g. the cases of circular and elliptic sections may be advantageously dealt with by introducing polar coordinate systems.

- It is shown that for arbitrary multiply connected cross sections the evaluation of the elastic shear stress of Saint-Venant requires the solution of a Poisson-Neumann problem or, equivalently, a Laplace-Neumann one. The Poisson-Neumann problem may be transformed in a Poisson-Dirichlet one if the domain occupied by the cross sec-

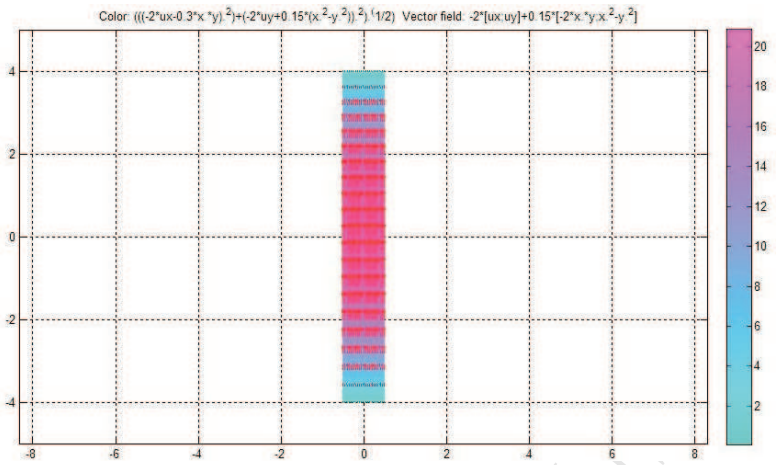

Figure 5: Shear stress field - B/H=1/8 and $\nu = 0.3$

tion is simply connected or in the general case of pure torsion. This is often more convenient to determine exact solutions. In the general case is not possible to perform such a transformation since the solution of a POISSON-DIRICHLET problem could be a multi-valued function. In the wake of PRANDTL the torsion and the flexure of SAINT-VENANT's beam are also treated in detail and the compatibility conditions are discussed. The existence of a stress function is proven in a tricky manner by rewriting the equilibrium differential equation of CAUCHY: $\operatorname{div} \boldsymbol{\tau} = -\alpha$ as the vanishing of the curl of a suitable vector field: $\operatorname{rot}(\mathbf{R}(\boldsymbol{\tau} + \boldsymbol{\tau}_\alpha)) = 0$.

- The intrinsic theory provides a computationally convenient context to perform numerical implementations in a program code whereas exact solutions are not available. With reference to beams with rectangular cross sections and shear force directed along the height, we have shown that the maximum norm of the elastic shear stress field is obtained at the end points of the middle line RS parallel to the base of each rectangle (see e.g. fig. 5, 6, 7 and 8). Then, by analyzing the results in table 2, we deduce that JOURAWSKI's method provides an exact estimate of the maximum elastic shear stress if POISSON's ratio is equal to zero and a subestimate of it as more significant as higher is the ratio between base and height of the rectangle and POISSON's ratio. We evaluate a corrective coefficient which must be multiplied by

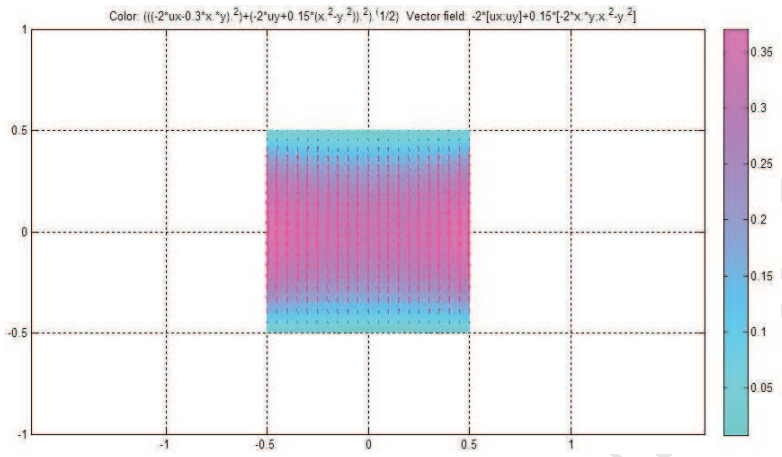

Figure 6: Shear stress field - B/H=1 and $\nu = 0.3$

Jourawski's mean stress, on the line RS, to get the maximum norm of Saint-Venant's elastic shear stress. The obtained numerical results are in agreement with ones available in the literature provided by De Saint Venant (1856a,b); Timoshenko and Goodier (1951).

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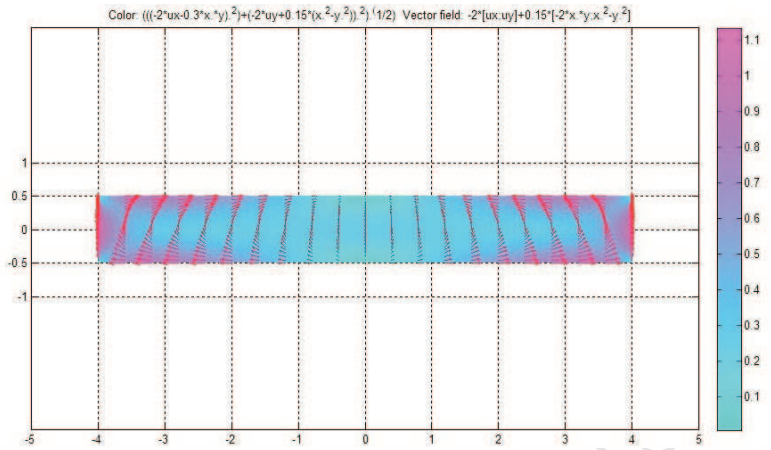

Figure 7: Shear stress field - B/H=8 and $\nu = 0.3$

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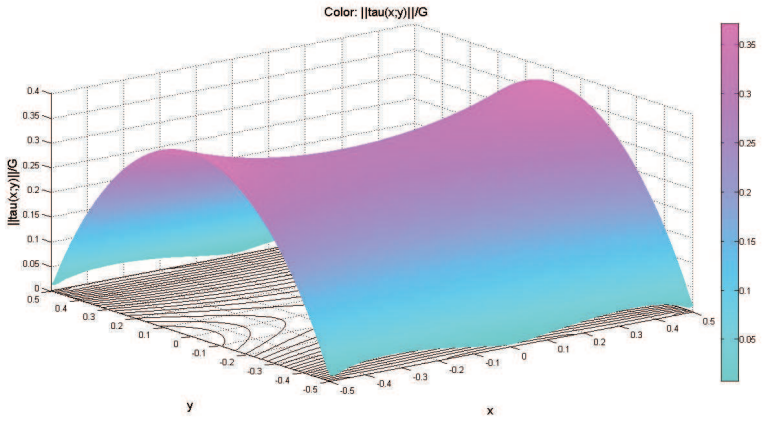

Figure 8: $\frac{||\boldsymbol{\tau}(\mathbf{r})||}{G}$ for a square cross section and $\nu = 0.3$

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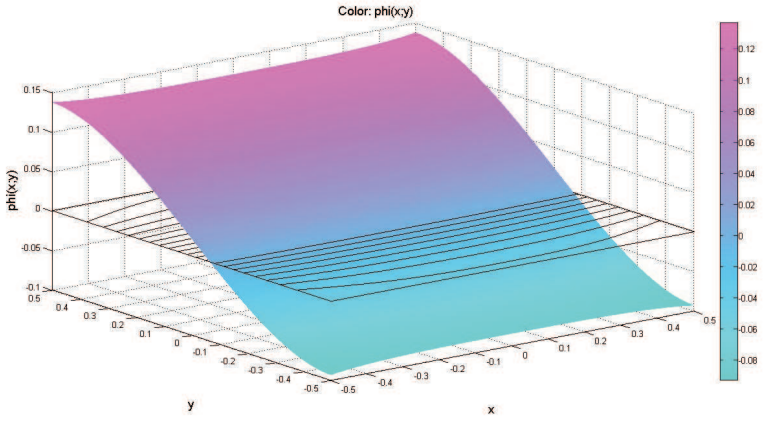

Figure 9: Warping function

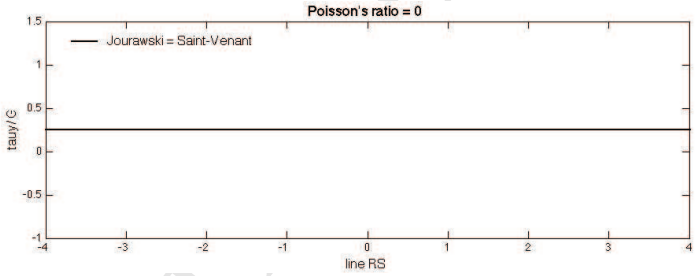

Figure 10: Saint-Venant vs. Jourawski - $\nu = 0$

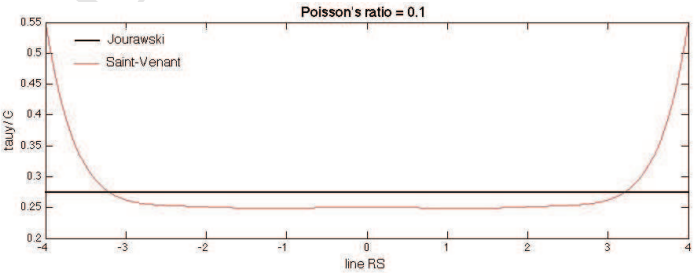

Figure 11: Saint-Venant vs. Jourawski - $\nu = 0.1$

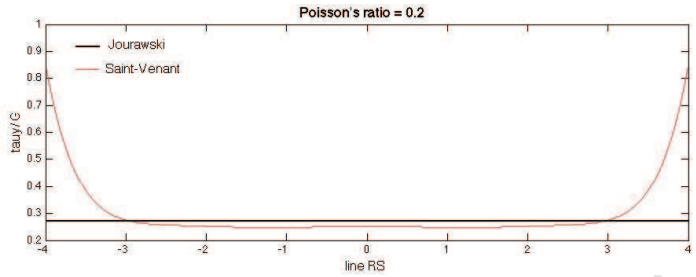

Figure 12: Saint-Venant vs. Jourawski - $\nu = 0.2$

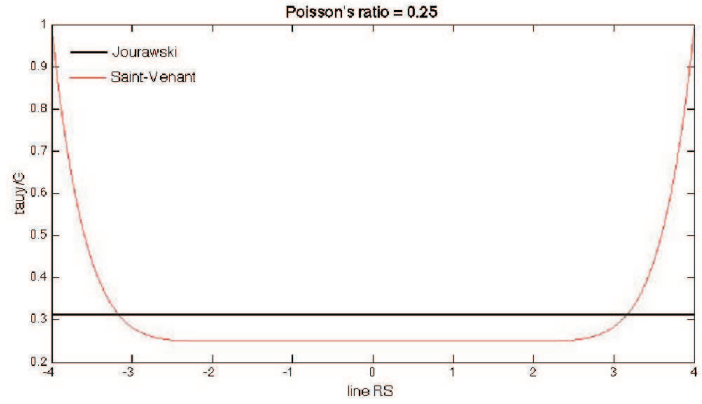

Figure 13: Saint-Venant vs. Jourawski - $\nu = 0.25$

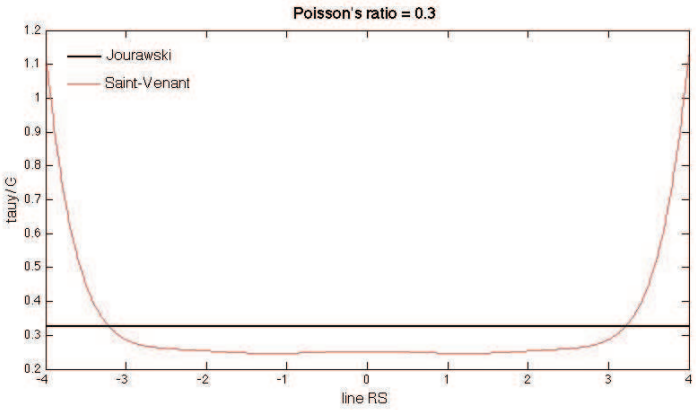

Figure 14: Saint-Venant vs. Jourawski - $\nu = 0.3$

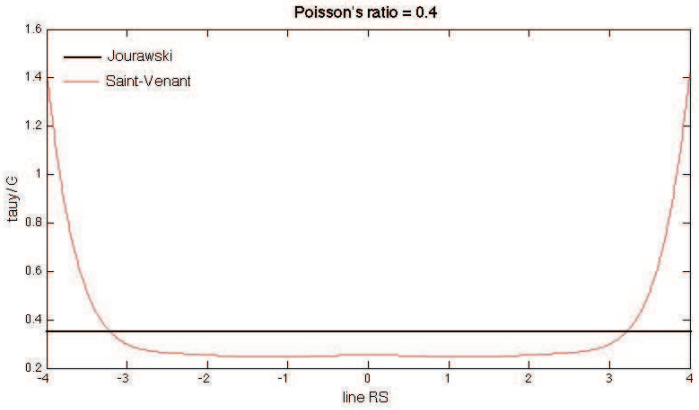

Figure 15: Saint-Venant vs. Jourawski - $\nu = 0.4$

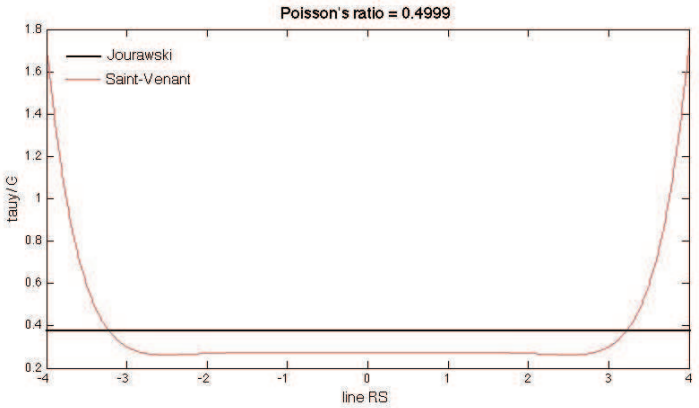

Figure 16: Saint-Venant vs. Jourawski - $\nu = 0.4999$