



# Identification of two independent coefficients with one observation for a nonlinear parabolic system

Michel Cristofol, Patricia Gaitan, Hichem Ramoul, Masahiro Yamamoto

## ► To cite this version:

Michel Cristofol, Patricia Gaitan, Hichem Ramoul, Masahiro Yamamoto. Identification of two independent coefficients with one observation for a nonlinear parabolic system. *Applicable Analysis*, 2012, 91 (11). hal-00550476

**HAL Id: hal-00550476**

**<https://hal.science/hal-00550476>**

Submitted on 28 Dec 2010

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Identification of two independent coefficients with one observation for a nonlinear parabolic system

Michel Cristofol <sup>\*</sup>    Patricia Gaitan <sup>†</sup>    Hichem Ramoul <sup>‡</sup>  
Masahiro Yamamoto <sup>§</sup>

December 14, 2010

## Abstract

This article is devoted to prove a stability result for two independent coefficients for a  $2 \times 2$  nonlinear parabolic system with only one observation. The main idea to obtain this result is to use a modified form of the Carleman estimate

*To cite this article: M. Cristofol, P. Gaitan, H. Ramoul and M. Yamamoto*

## 1 Introduction and main result

This paper is an improvement of the work [4] in the sense that we determine two independent coefficients with the observation of only one component in a nonlinear  $2 \times 2$  parabolic system. The system of coupled linear reaction-diffusion-convection equations had been recently widely studied in [1] and [2] with applications in inverse area but also in control domains.

Several works concern inverse problems associated with linear and nonlinear parabolic equations (see [13], [15], [3], [7], [8], [17], [5], ...) but few concern system of nonlinear parabolic equations. Such systems arise in biological,

---

<sup>\*</sup>Laboratoire d'analyse, topologie, probabilités CNRS UMR 6632, Marseille, France and Aix-Marseille Université, IUT de Marseille

<sup>†</sup>Laboratoire d'analyse, topologie, probabilités CNRS UMR 6632, Marseille, France and Aix-Marseille Université, IUT d'Aix-en-Provence

<sup>‡</sup>Laboratoire de Mathématiques Appliquées (LMA), Département de Mathématiques, Université de Badji Mokhtar, BP 12, 23000 Annaba, Algeria

<sup>§</sup>Department of Mathematical Sciences, University of Tokyo, Tokyo, Japan

ecological domain or combustion and chemical reactions (see [16], [18]).

Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain of  $\mathbb{R}^n$  with  $n \leq 3$  and  $\omega \subset \Omega$  a non empty subset. We denote by  $\nu$  the outward unit normal to  $\Omega$  on  $\Gamma = \partial\Omega$  assumed to be of class  $\mathcal{C}^1$ . Let  $T > 0$  and  $t_0 \in (0, T)$ . We shall use the following notations  $Q_0 = \Omega \times (0, T)$ ,  $Q = \Omega \times (t_0, T)$ ,  $Q_\omega = \omega \times (t_0, T)$ ,  $\Sigma = \Gamma \times (t_0, T)$  and  $\Sigma_0 = \Gamma \times (0, T)$ . We consider the following  $2 \times 2$  reaction-diffusion system:

$$\begin{cases} \partial_t U = \Delta U + a_{11}(x)U + a_{12}(x)V + a_{13}(x)f(U, V) & \text{in } Q_0, \\ \partial_t V = \Delta V + a_{21}(x)U + a_{22}(x)V & \text{in } Q_0, \\ U(x, t) = k_1(x, t), V(x, t) = k_2(x, t) & \text{on } \Sigma_0, \\ U(x, 0) = U_0 \text{ and } V(x, 0) = V_0 & \text{in } \Omega, \end{cases} \quad (1)$$

where, the function  $f$  is assumed to be Lipschitz with respect the two variables  $U$  and  $V$ .

Uniqueness and existence results for initial boundary value problem for such systems can be found in [14].

Throughout this paper, we consider the following set

$$\Lambda(R) = \{\Phi \in L^\infty(\Omega); \|\Phi\|_{L^\infty(\Omega)} \leq R\},$$

where  $R$  is a given positive constant.

For  $t_0 \in (0, T)$ , we denote  $T' = \frac{t_0+T}{2}$ . Let  $(U, V)$  (resp.  $(\tilde{U}, \tilde{V})$ ) be solution of (1) associated to  $(a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, k_1, k_2, U_0, V_0)$  (resp.  $(a_{11}, a_{12}, \tilde{a}_{13}, \tilde{a}_{21}, a_{22}, k_1, k_2, U_0, V_0)$ ) satisfying some regularity and positivity properties:

**Assumption 1.1.** 1. For  $i = 1, 2, j = 1, 2, 3$ ,  $a_{ij}$ ,  $\tilde{a}_{13}$  and  $\tilde{a}_{21} \in \Lambda(R)$ .

2. There exist constants  $r_1 > 0$  and  $a_0 > 0$  such that  
 $\tilde{U}_0 \geq r_1$ ,  $\tilde{V}_0 \geq 0$ ,  $a_{11}r_1 + a_{12}\tilde{V}_0 + \tilde{a}_{13}f(r_1, \tilde{V}_0) \geq 0$ ,  
 $a_{21} \geq a_0$ ,  $\tilde{a}_{21} \geq a_0$ ,  $k_1 \geq r_1$  and  $k_2 \geq 0$ .

Such assumptions allow us to state that the function  $\tilde{U}$  satisfies  $|\tilde{U}(x, T')| \geq r_1 > 0$  in  $\Omega$  (see [19]).

**Assumption 1.2.** 1. The function  $f$  checks a generalized Lipschitz property in the following sense :  $\exists C > 0$ , such that

$$|\partial_t f(U, V) - \partial_t f(\tilde{U}, \tilde{V})| \leq C \left( |U - \tilde{U}| + |V - \tilde{V}| + |(U - \tilde{U})_t| + |(V - \tilde{V})_t| \right).$$

2.  $\exists r_2 > 0$  such that  $f(\tilde{U}, \tilde{V})(T', x) \geq r_2 > 0$  in  $\Omega$ .

3.  $\partial_t f(U, V) \in L^2((0, T); H^2(\Omega))$ .

This set of functions is not empty and contains, in particular, a large class of semilinear terms (e.g.  $f(U, V) = U^\alpha V^\beta$  with  $\alpha$  and  $\beta$  non negative constants in chemical reactions).

The main result is the following Theorem:

**Theorem 1.3.** *Let  $\omega$  be a subdomain of an open set  $\Omega$  of  $\mathbb{R}^n$ . We suppose that Assumptions 1.1 and 1.2 are checked and  $(U, V)(\cdot, T') = (\tilde{U}, \tilde{V})(\cdot, T')$ . Furthermore, we assume that  $U_0, V_0$  in  $H^2(\Omega)$ . Then there exists a constant  $C = C(\Omega, \omega, a_0, t_0, T, r_1, r_2, R) > 0$  such that*

$$\|a_{21} - \tilde{a}_{21}\|_{L^2(\Omega)}^2 + \|a_{13} - \tilde{a}_{13}\|_{L^2(\Omega)}^2 \leq C \|\partial_t V - \partial_t \tilde{V}\|_{L^2(Q_\omega)}^2.$$

In [4], for a coupled linear reaction diffusion system, we prove a stability result for one coefficient with only one observation. Note that our method does not work if the system (1) contains a non linear term in each equation. The novelty in this paper is the identification of two coefficients with only one observation for a nonlinear system. The main tool is a Carleman estimate established in [4] which is adapted, using a "shift" of the large parameters, to recover two independent coefficients, one in each equation of (1).

The paper is organized as follows: In section 2, we give the modified Carleman estimate for a reaction-diffusion system with only one observation. Then using this modified Carleman estimate, we prove in section 3 a stability result for two coefficients with the observation of only one component.

## 2 Carleman estimate

At first, we recall the general form of the Carleman estimate associated to the operator  $\partial_t q - \Delta q$  (see [9], [11], [12]). Let  $\omega' \Subset \omega \Subset \Omega$  and let  $\tilde{\beta}$  be a  $C^2(\Omega)$  function such that

$$\tilde{\beta} > 0, \text{ in } \Omega, \quad \tilde{\beta} = 0 \text{ on } \partial\Omega, \quad \min\{|\nabla \tilde{\beta}(x)|, x \in \overline{\Omega \setminus \omega'}\} > 0 \quad \text{and} \quad \partial_\nu \tilde{\beta} < 0 \text{ on } \partial\Omega.$$

Then, we define  $\beta = \tilde{\beta} + K$  with  $K = m\|\tilde{\beta}\|_\infty$  and  $m > 1$ . For  $\lambda > 0$  and  $t \in (t_0, T)$ , we define the following weight functions (see [10])

$$\varphi(x, t) = \frac{e^{\lambda\beta(x)}}{(t - t_0)(T - t)}, \quad \eta(x, t) = \frac{e^{2\lambda K} - e^{\lambda\beta(x)}}{(t - t_0)(T - t)}.$$

We then have the following Carleman estimate:

**Theorem 2.1.** *Let  $\tau \in \mathbb{R}$ . Then there exist  $\lambda_0 = \lambda_0(\Omega, \omega) \geq 0$ ,  $s_0 = s_0(\lambda_0, T, \tau) > 0$  and a positive constant  $C_0 = C_0(\Omega, \omega, \tau)$  such that, for any  $\lambda \geq \lambda_0$  and any  $s \geq s_0$ , the following estimate holds:*

$$I(\tau, q) \leq C_0 \left[ \iint_{Q_\omega} e^{-2s\eta} \lambda^4 (s\varphi)^{\tau+3} |q|^2 \, dx \, dt + \iint_Q e^{-2s\eta} (s\varphi)^\tau |\partial_t q - \Delta q|^2 \, dx \, dt \right], \quad (2)$$

where

$$\begin{aligned} I(\tau, q) = & \iint_Q e^{-2s\eta} (s\varphi)^{\tau-1} (|\partial_t q|^2 + |\Delta q|^2) \, dx \, dt + \lambda^2 \iint_Q e^{-2s\eta} (s\varphi)^{\tau+1} |\nabla q|^2 \, dx \, dt \\ & + \lambda^4 \iint_Q e^{-2s\eta} (s\varphi)^{\tau+3} |q|^2 \, dx \, dt \end{aligned}$$

*Remark 1.* If we denote

$$M_1^{(\tau)} \psi = -\Delta \psi - s^2 \lambda^2 \varphi^2 |\nabla \beta|^2 \psi - \left( \frac{\tau}{2} - s \partial_t \eta \right) \psi,$$

and

$$M_2^{(\tau)} \psi = \partial_t \psi + 2s\lambda(\varphi + \frac{\tau}{2}) \nabla \beta \cdot \nabla \psi,$$

with  $\psi = e^{-s\eta} \varphi^{\frac{\tau}{2}} q$ , the Carleman estimate (2) also gives an upper bound of  $\|M_1^{(\tau)} \psi\|_{L^2(Q)}^2 + \|M_2^{(\tau)} \psi\|_{L^2(Q)}^2$  (see [9]).

We assume that  $a_{11}, a_{12}, a_{21}, a_{22} \in \Lambda(R)$ ,  $a_{21} \geq a_0 > 0$  and we consider the following system:

$$\begin{cases} \partial_t Y = \Delta Y + a_{11}(x)Y + a_{12}(x)Z + H_1, & \text{in } Q_0, \\ \partial_t Z = \Delta Z + a_{21}(x)Y + a_{22}(x)Z + H_2 & \text{in } Q_0, \\ Y(x, t) = Z(x, t) = 0 & \text{on } \Sigma_0, \\ Y(x, 0) = K_1(x), \quad Z(x, 0) = K_2(x) & \text{in } \Omega, \end{cases} \quad (3)$$

where  $H_1$  and  $H_2$  are arbitrary functions. Following [4], we can derive a modified Carleman estimate : "shifted Carleman", with a single observation acting on a subdomain  $\omega$  of  $\Omega$  for the system (3). Such estimate will be used in the next section to prove stability result for two coefficients with the observation of only one component. Then we can have the following theorem:

**Theorem 2.2.** *There exist  $\lambda_1 = \lambda_1(\Omega, \omega) \geq 1$ ,  $s_1 = s_1(\lambda_1, T) > 1$  and a positive constant  $C_1 = C_1(\Omega, \omega, R, T, a_0)$  such that, for any  $\lambda \geq \lambda_1$  and any*

$s \geq s_1$  and  $\epsilon > 0$  fixed, the following estimate holds:

$$\begin{aligned} \lambda^{-4+\epsilon} I(-3, Y) + I(0, Z) &\leq C_1 s^4 \lambda^{4+\epsilon} \iint_{Q_\omega} e^{-2s\eta} \varphi^4 |Z|^2 dx dt \quad (4) \\ + C_1 \left[ s^{-3} \lambda^{-4+\epsilon} \iint_Q e^{-2s\eta} \varphi^{-3} |H_1|^2 dx dt + \lambda^{2\epsilon} \iint_Q e^{-2s\eta} |H_2|^2 dx dt \right]. \end{aligned}$$

*Remark 2.* This "shifted Carleman" method allows us to estimate two coefficients, one in each equation, by the observation of only one component with  $L^2$ -norm in the observation term. This point is of interest in view to obtain controllability to trajectories with only one control force.

The proof derives from the proof given in [4, Theorem 2.3]. We only highlight the main points in the proof.

*Proof.* Using the Carleman estimate (2), the solution  $(Y, Z)$  of system (3) satisfies

$$\begin{aligned} \lambda^{-4+\epsilon} I(-3, Y) + I(0, Z) &\leq C_0 \left[ \lambda^\epsilon \iint_{Q_\omega} e^{-2s\eta} |Y|^2 dx dt \right. \\ &\quad \left. + s^3 \lambda^4 \iint_{Q_\omega} e^{-2s\eta} \varphi^3 |Z|^2 dx dt \right. \quad (5) \\ &\quad \left. + s^{-3} \lambda^{-4+\epsilon} \iint_Q e^{-2s\eta} \varphi^{-3} (|a_{11}Y|^2 + |a_{12}Z|^2 + |a_{13}H_1|^2) dx dt \right. \\ &\quad \left. + \iint_Q e^{-2s\eta} (|a_{21}Y|^2 + |a_{22}Z|^2 + |a_{23}H_2|^2) dx dt \right]. \end{aligned}$$

The main difficulty is in estimating the term  $I := \lambda^\epsilon \iint_{Q_\omega} e^{-2s\eta} |Y|^2 dx dt$  in function of the localized observation of  $Z$ . Thus, through the assumption  $a_{21}(x) \geq a_0$  with  $a_0$  is a positive constant, we can estimate the following integral:

$$I' := \lambda^\epsilon \iint_{Q_\omega} a_{21} \xi e^{-2s\eta} |Y|^2 dx dt,$$

where  $\xi$  is a smooth cut-function satisfying

$$\begin{cases} \xi(x) = 1 & \forall x \in \omega', \\ 0 < \xi(x) \leq 1 & \forall x \in \omega'', \\ \xi(x) = 0 & \forall x \in \mathbb{R}^n \setminus \omega'', \end{cases}$$

and  $\omega' \Subset \omega'' \Subset \omega \Subset \Omega$ .

Using the second equation of system (3), we have

$$\begin{aligned}
I' &= \lambda^\epsilon \iint_{Q_\omega} e^{-2s\eta} \xi (\partial_t Z - \Delta Z - a_{22}Z - a_{23}H_2) Y \, dx \, dt \\
&= \lambda^\epsilon \iint_{Q_\omega} e^{-2s\eta} \xi (\partial_t Z) Y \, dx \, dt - \lambda^\epsilon \iint_{Q_\omega} e^{-2s\eta} \xi (\Delta z) Y \, dx \, dt \\
&\quad \lambda^\epsilon \iint_{Q_\omega} a_{22} e^{-2s\eta} \xi Z Y \, dx \, dt - \lambda^\epsilon \iint_{Q_\omega} a_{23} e^{-2s\eta} \xi H_2 Y \, dx \, dt := I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Then, by integration by parts and Young inequalities (see [4] for more details), we obtain, for  $s$  sufficiently large,

$$|I_1| + |I_2| + |I_3| \leq C s^4 \lambda^{4+\epsilon} \iint_{Q_\omega} e^{-2s\eta} \varphi^4 |Z|^2 \, dx \, dt + \text{"absorbed terms"} ,$$

and

$$|I_4| \leq C \left( \lambda^{2\epsilon} \iint_{Q_\omega} e^{-2s\eta} |H_2|^2 \, dx \, dt + \iint_{Q_\omega} e^{-2s\eta} |Y|^2 \, dx \, dt \right),$$

$C$  being a generic constant which depends on  $\Omega$ ,  $\omega$ ,  $R$  and  $T$  and "absorbed terms" means integrals terms dominated by the left hand side of (5) for large  $s$ . Finally, we have, for  $s$  sufficiently large,

$$\begin{aligned}
|I| \leq \frac{1}{a_0} |I'| &\leq \frac{C}{a_0} s^4 \lambda^{4+\epsilon} \iint_{Q_\omega} e^{-2s\eta} \varphi^4 |Z|^2 \, dx \, dt + C \lambda^{2\epsilon} \iint_Q e^{-2s\eta} |H_2|^2 \, dx \, dt \\
&\quad + \text{"absorbed terms"} .
\end{aligned} \tag{6}$$

If we note

$$\begin{aligned}
J &:= s^{-3} \lambda^{-4+\epsilon} \iint_Q e^{-2s\eta} \varphi^{-3} (|a_{11}Y|^2 + |a_{12}Z|^2 + |a_{13}H_1|^2) \, dx \, dt \\
&\quad + \iint_Q e^{-2s\eta} (|a_{21}Y|^2 + |a_{22}Z|^2 + |a_{23}H_2|^2) \, dx \, dt,
\end{aligned}$$

we obtain, according (6), for  $s$  sufficiently large and  $\epsilon > 0$ ,

$$\begin{aligned}
|J| + |I| &\leq C \left( s^{-3} \lambda^{-4+\epsilon} \iint_Q e^{-2s\eta} \varphi^{-3} |H_1|^2 \, dx \, dt + \lambda^{2\epsilon} \iint_Q e^{-2s\eta} |H_2|^2 \, dx \, dt \right) \\
&\quad + \text{"absorbed terms"} .
\end{aligned}$$

Finally, the previous estimate achieves the proof of Theorem 2.2.  $\square$

### 3 Stability result

In this section we give the proof of Theorem 1.3.

Let  $(U, V)$  (resp.  $(\tilde{U}, \tilde{V})$ ) be solution of (1) associated to  $(a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, k_1, k_2, U_0, V_0)$  (resp.  $(a_{11}, a_{12}, \tilde{a}_{13}, \tilde{a}_{21}, a_{22}, k_1, k_2, U_0, V_0)$ ). Then, if we set  $u = U - \tilde{U}$ ,  $v = V - \tilde{V}$ ,  $Y = \partial_t u$  and  $Z = \partial_t v$ ,  $(Y, Z)$  is solution to the following problem

$$\begin{cases} \partial_t Y = \Delta Y + a_{11}(x)Y + a_{12}(x)Z + \gamma_1 \partial_t f(\tilde{U}, \tilde{V}) + a_{13}(x) \partial_t F(U, V, \tilde{U}, \tilde{V}), & \text{in } Q_0, \\ \partial_t Z = \Delta Z + a_{21}(x)Y + a_{22}(x)Z + \gamma_2 \partial_t \tilde{U} & \text{in } Q_0, \\ Y(x, t) = Z(x, t) = 0 & \text{on } \Sigma_0, \\ Y(x, 0) = \gamma_1 f(U_0, V_0), \quad Z(x, 0) = \gamma_2 U_0 & \text{in } \Omega, \end{cases} \quad (7)$$

where  $\gamma_1 = (a_{13} - \tilde{a}_{13})$ ,  $\gamma_2 = (a_{21} - \tilde{a}_{21})$  and  $F(U, V, \tilde{U}, \tilde{V}) = f(U, V) - f(\tilde{U}, \tilde{V})$ . If we apply the modified Carleman estimate (4) to the previous system (7), we have

$$\begin{aligned} \lambda^{-4+\epsilon} I(-3, Y) + I(0, Z) &\leq C_1 s^4 \lambda^{4+\epsilon} \iint_{Q_\omega} e^{-2s\eta} \varphi^4 |Z|^2 dx dt \\ &+ C_1 \left[ s^{-3} \lambda^{-4+\epsilon} \iint_Q e^{-2s\eta} \varphi^{-3} (|\gamma_1 \partial_t f(\tilde{U}, \tilde{V})|^2 + |\partial_t F|^2) dx dt \right. \\ &\quad \left. + \lambda^{2\epsilon} \iint_Q e^{-2s\eta} |\gamma_2 \partial_t \tilde{U}|^2 dx dt \right]. \end{aligned} \quad (8)$$

Now we shall "absorb" the term  $A = s^{-3} \lambda^{-4+\epsilon} \iint_Q e^{-2s\eta} \varphi^{-3} |\partial_t F|^2 dx dt$ . So, we need the following lemma (see [13]):

**Lemma 3.1.** *There exists a positive constant  $C > 0$  such that.*

$$\iint_Q \left| \int_{T'}^t q(x, \xi) d\xi \right|^2 e^{-2s\eta} dx dt \leq \frac{C}{s} \iint_Q |q(x, t)|^2 e^{-2s\eta} dx dt$$

for all large  $s > 0$  and  $q \in L^2(Q)$ .

Since  $\varphi^{-3} \leq C \frac{T^6}{4^3}$ ,  $\varphi^{-3} \leq C \frac{T^{12}}{4^6} \varphi^3$  and using Assumption 1.2-(1), the previous Lemma yields

$$A \leq C \lambda^{-4+\epsilon} s^{-3} (1 + s^{-1}) \iint_Q e^{-2s\eta} (|Y|^2 + \varphi^3 |Z|^2) dx dt. \quad (9)$$

Therefore, for  $s$  large enough and  $\epsilon > 0$ , the integral  $A$  can be "absorbed" by the left hand side of (8).



Then (8) can be written as follows

$$\begin{aligned} \lambda^{-4+\epsilon} I(-3, Y) + I(0, Z) &\leq C_1 s^4 \lambda^{4+\epsilon} \iint_{Q_\omega} e^{-2s\eta} \varphi^4 |Z|^2 dx dt \\ &+ C_1 \left[ s^{-3} \lambda^{-4+\epsilon} \iint_Q e^{-2s\eta} \varphi^{-3} |\gamma_1 \partial_t f(\tilde{U}, \tilde{V})|^2 dx dt \right. \\ &\quad \left. + \lambda^{2\epsilon} \iint_Q e^{-2s\eta} |\gamma_2 \partial_t \tilde{U}|^2 dx dt \right]. \end{aligned} \quad (10)$$

Let us introduce the following integral

$$\mathcal{I}_1 = \lambda^{-4+\epsilon} \int_{t_0}^{T'} \int_{\Omega} M_2^{(-3)} \psi_1 \cdot \psi_1 dx dt,$$

with  $\psi_1 = e^{-s\eta} Y \varphi^{-3/2}$  and  $T' = \frac{T+t_0}{2}$ .

We first estimate  $\mathcal{I}_1$  with the modified Carleman estimate (10):

$$\begin{aligned} \mathcal{I}_1 &\leq \frac{1}{2} \lambda^{-2} \left[ \lambda^{-4+\epsilon} \|M_2^{(-3)} \psi_1\|_{L^2(Q)}^2 + \lambda^\epsilon \int_{t_0}^{T'} \int_{\Omega} e^{-2s\eta} \varphi^{-3} |Y|^2 dx dt \right] \\ &\leq \frac{1}{2} \lambda^{-2} C_1 \left[ s^4 \lambda^{4+\epsilon} \iint_{Q_\omega} e^{-2s\eta} \varphi^4 |Z|^2 dx dt \right. \\ &\quad + s^{-3} \lambda^{-4+\epsilon} \iint_Q e^{-2s\eta} \varphi^{-3} |\gamma_1 \partial_t f(\tilde{U}, \tilde{V})|^2 dx dt \\ &\quad \left. + \lambda^{2\epsilon} \iint_Q e^{-2s\eta} |\gamma_2 \partial_t \tilde{U}|^2 dx dt \right]. \end{aligned} \quad (11)$$

By computing  $\mathcal{I}_1$ , we obtain

$$\frac{1}{2} \lambda^{-4+\epsilon} \int_{\Omega} |\psi_1(\cdot, T')|^2 dx \leq 2|\mathcal{I}_1| + C_1 \lambda^{-3+\epsilon} s(\lambda+1) \iint_Q e^{-2s\eta} \varphi^{-2} |Y|^2 dx dt. \quad (12)$$

Applying the modified Carleman estimate (10) to the last term in (12) with  $\varphi^{-2} \leq CT^4$  and using the estimate (11), we have

$$\begin{aligned} \frac{1}{2} \lambda^{-4+\epsilon} \int_{\Omega} |\psi_1(\cdot, T')|^2 dx &\leq \frac{1}{2} s \lambda^{-2} C_1 \left[ s^4 \lambda^{4+\epsilon} \iint_{Q_\omega} e^{-2s\eta} \varphi^4 |Z|^2 dx dt \right. \\ &\quad + s^{-3} \lambda^{-4+\epsilon} \iint_Q e^{-2s\eta} \varphi^{-3} |\gamma_1 \partial_t f(\tilde{U}, \tilde{V})|^2 dx dt \\ &\quad \left. + \lambda^{2\epsilon} \iint_Q e^{-2s\eta} |\gamma_2 \partial_t \tilde{U}|^2 dx dt \right]. \end{aligned} \quad (13)$$

Then, the estimate (13) yields

$$\begin{aligned} \lambda^{-4+\epsilon} \int_{\Omega} e^{-2s\eta(\cdot, T')} \varphi^{-3}(\cdot, T') |Y(\cdot, T')|^2 dx &\leq C \left[ s^5 \lambda^{2+\epsilon} \iint_{Q_\omega} e^{-2s\eta} \varphi^4 |Z|^2 dx dt \right. \\ &\quad + s^{-2} \lambda^{-6+\epsilon} \iint_Q e^{-2s\eta} \varphi^{-3} |\gamma_1 \partial_t f(\tilde{U}, \tilde{V})|^2 dx dt \\ &\quad \left. + s \lambda^{-2+2\epsilon} \iint_Q e^{-2s\eta} |\gamma_2 \partial_t \tilde{U}|^2 dx dt \right]. \end{aligned}$$

Since  $(U, V)(\cdot, T') = (\tilde{U}, \tilde{V})(\cdot, T')$ , we have  $Y(\cdot, T') = \gamma_1 f(\tilde{U}, \tilde{V})(\cdot, T')$ . Thus, we obtain

$$\begin{aligned} \lambda^{-4+\epsilon} \int_{\Omega} e^{-2s\eta(\cdot, T')} \varphi^{-3}(\cdot, T') |\gamma_1 f(\tilde{U}, \tilde{V})(\cdot, T')|^2 dx \\ \leq C \left[ s^5 \lambda^{2+\epsilon} \iint_{Q_\omega} e^{-2s\eta} \varphi^4 |Z|^2 dx dt \right. \\ + s^{-2} \lambda^{-6+\epsilon} \iint_Q e^{-2s\eta} \varphi^{-3} |\gamma_1 \partial_t f(\tilde{U}, \tilde{V})|^2 dx dt \\ \left. + s \lambda^{-2+2\epsilon} \iint_Q e^{-2s\eta} |\gamma_2 \partial_t \tilde{U}|^2 dx dt \right]. \end{aligned} \quad (14)$$

In a similar way, we introduce

$$\mathcal{I}_2 = \int_{t_0}^{T'} \int_{\Omega} M_2^{(0)} \psi_2 \cdot \psi_2 dx dt,$$

with  $\psi_2 = e^{-s\eta} Z$ .

Using the fact that  $Z(\cdot, T') = \gamma_2 \tilde{U}(\cdot, T')$ , we obtain

$$\begin{aligned} \int_{\Omega} e^{-2s\eta(\cdot, T')} |\gamma_2 \tilde{U}(\cdot, T')|^2 dx &\leq C s^{5/2} \lambda^{2+\epsilon} \iint_{Q_\omega} e^{-2s\eta} \varphi^4 |Z|^2 dx dt \\ &\quad + C s^{-9/2} \lambda^{-6+\epsilon} \iint_Q e^{-2s\eta} \varphi^{-3} |\gamma_1 \partial_t f(\tilde{U}, \tilde{V})|^2 dx dt \\ &\quad + C s^{-3/2} \lambda^{-2+2\epsilon} \iint_Q e^{-2s\eta} |\gamma_2 \partial_t \tilde{U}|^2 dx dt. \end{aligned} \quad (15)$$

By adding (14) and (15), we have

$$\begin{aligned} \lambda^{-4+\epsilon} \int_{\Omega} e^{-2s\eta(\cdot, T')} \varphi^{-3}(\cdot, T') |\gamma_1 f(\tilde{U}, \tilde{V})(\cdot, T')|^2 dx &+ \int_{\Omega} e^{-2s\eta(\cdot, T')} |\gamma_2 \tilde{U}(\cdot, T')|^2 dx \\ &\leq C s^5 \lambda^{2+\epsilon} \iint_{Q_\omega} e^{-2s\eta} \varphi^4 |Z|^2 dx dt + C s \lambda^{-2+2\epsilon} \iint_Q e^{-2s\eta} |\gamma_2 \partial_t \tilde{U}|^2 dx dt \\ &\quad + C s^{-2} \lambda^{-6+\epsilon} \iint_Q e^{-2s\eta} \varphi^{-3} |\gamma_1 \partial_t f(\tilde{U}, \tilde{V})|^2 dx dt. \end{aligned} \quad (16)$$

By [14], we can have  $\tilde{U} \in H^1((t_0, T); H^2(\Omega))$  for suitable boundary and initial conditions, so  $\partial_t \tilde{U} \in L^2((t_0, T); H^2(\Omega))$ . Moreover, by Assumption 1.2-(3)  $\partial_t f(\tilde{U}, \tilde{V}) \in L^2((t_0, T); H^2(\Omega))$ . Then for  $n \leq 3$ ,  $\partial_t \tilde{U}$  and  $\partial_t f(\tilde{U}, \tilde{V})$  are in  $L^2((t_0, T); L^\infty(\Omega))$  by classical Sobolev imbedding. Thus, using Assumptions 1.1-(2) and 1.2-(2), the inequality (16), for  $0 < \epsilon < 1$  and  $\lambda$  sufficiently large, yields

$$\lambda^{-4+\epsilon} \int_{\Omega} e^{-2s\eta} \varphi^{-3} |\gamma_1|^2 dx + \int_{\Omega} e^{-2s\eta} |\gamma_2|^2 dx \leq C s^5 \lambda^{2+\epsilon} \iint_{Q_\omega} e^{-2s\eta} \varphi^4 |Z|^2 dx dt.$$

Then, the proof of Theorem 1.3 is complete.

*Remark 3.* Denote that in this result we need the use of the two large parameters  $s$  and  $\lambda$ . This technical point allows to obtain the observation in  $L^2$ -norm.

We have thus the following uniqueness result:

**Corollary 3.2.** *Under the same assumptions as in Theorem 1.3 and if*

$$(\partial_t V - \partial_t \tilde{V})(x, t) = 0 \quad \text{in } Q_\omega,$$

*then  $a_{21} = \tilde{a}_{21}$  and  $a_{13} = \tilde{a}_{13}$  in  $\Omega$ .*

## References

- [1] A. Benabdallah, M. Cristofol, P. Gaitan and M. Yamamoto, *Inverse problem for a parabolic system with two components by measurements of one component*, Applicable Analysis, 88, no. 5 (2009), 683–710.
- [2] A. Benabdallah, M. Cristofol, P. Gaitan and L. De Teresa, *A new Carleman inequality for parabolic systems with a single observation and applications*, C. R. Acad. Sci. Paris, Ser. I 348, 2529, (2010).
- [3] M. Choulli, E. M Ouhabaz, and M. Yamamoto, *Stable determination of a semilinear term in a parabolic equation*, Communications in Pure and Applied Analysis **5** (2006), 447–462.
- [4] M. Cristofol, P. Gaitan and H. Ramoul, *Inverse problems for a  $2 \times 2$  reaction-diffusion system using a Carleman estimate with one observation*, Inverse Problems, 22, 1561-1573, (2006).
- [5] M. Cristofol, J. Garnier, F. Hamel and L. Roques, *Uniqueness from pointwise observations in a multi-parameter inverse problem*, accepted in Communication in Pure and Applied Analysis, (2011).
- [6] M. Cristofol and L. Roques, *An inverse problem involving two coefficients in a nonlinear reaction-diffusion equation*, submitted in Comptes rendus de l'Academie des Sciences .

- [7] P. DuChateau and W. Rundell, *Unicity in an inverse problem for an unknown reaction term in a reaction-diffusion equation*, Journal of Differential Equations **59** (1985), 155–164.
- [8] H. Egger, H. W. Engl, and M. Klibanov, *Global uniqueness and Hölder stability for recovering a nonlinear source term in a parabolic equation*, Inverse Problems **21** (2005), 271–290.
- [9] A. Fursikov, *Optimal control of distributed systems*, Translations of Mathematical Monographs, 187. American Mathematical Society, Providence, RI, 2000.
- [10] E. Fernández-Cara, S. Guerrero, *Global Carleman inequalities for parabolic systems and application to controllability*, SIAM J. Control Optim. **45**, 1395–1446, (2006).
- [11] O. Yu. Immanuvilov and M. Yamamoto, *Lipschitz stability in inverse parabolic problems by the Carleman estimate*, Inverse Problems, **14**, 1229–1245, (1998).
- [12] O. Yu. Immanuvilov and M. Yamamoto, *Carleman for parabolic equations in Sobolev spaces of negative order and exact controllability for semilinear parabolic equations*, Publ. RIMS, Kyoto Univ, **39**, 227–274, (2003).
- [13] M. Klibanov, *Global uniqueness of a multidimensional inverse problem for a nonlinear parabolic equation by a Carleman estimate*, Inverse Problems, **20**, 1003–1032, (2004).
- [14] O. A. Ladyzenskaya, V. A. Solonnikov, N. N. Ural'ceva, *Linear and quasilinear equations of parabolic type*, AMS (1968).
- [15] A. Lorenzi, *An inverse problem for a semilinear parabolic equation*, Annali di Matematica Pura ed Applicata **131** (1982), 145–166.
- [16] A. Okuro and S. Levin, *Diffusion and ecological problems : modern perspectives*, Springer-Verlag, 2001.
- [17] L. Roques and M. Cristofol, *On the determination of the nonlinearity from localized measurements in a reaction-diffusion equation*, Nonlinearity **23**, 675–686, (2010).
- [18] N. Shigesada and K. Kawasaki, *Biological Invasions: Theory and Practice*, Oxford University Press, 1997.
- [19] J. Smoller, *Shock waves and reaction-diffusion equations*, Springer-Verlag, 1983.