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Constraints for generalized mixtures of Weibull distributions with common shape parameter

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Abstract

Weibull mixtures have been widely used in many applications, and they can be generalized by allowing negative mixing weights. In this note, we investigate constraints on the mixing weights and parameters of components under which the generalized mixture of Weibull distributions to be a valid probability model, including the cases of exponential or Rayleigh components. In addition, characterizations are shown for generalized mixtures of three or fewer Weibull components. We use an example of two-fold Weibull mixture model where negative weights are estimated, and the constraints are illustrated with a graphical example of three components.

Key words: Weibull mixtures, generalized mixtures, characterization

AMS 2000 subject classification: Primary 60E05, secondary 62E10

1. Introduction

Weibull mixture models play a great role in reliability theory, since they exhibit a wide range of shapes for the density and failure rate functions, which makes them suitable for modelling complex survival or failure data sets. For instance, to the reliability modelling (e.g., see Patra and Dey (1999) and Bucar et al. (2004)), the Hang-Seng stock price index (Tsonas (2002)), survival data (Marín et al. (2005)), automobile warranty data (Attardi et al. (2005)), and probabilistic behavior of wind speeds (Carta and Ramírez (2007)). A basic feature of the finite mixture models is that the mixing weights are positive, but

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this is not a necessary condition to be a valid probability model (e.g., Titterington et al. (1985)). The generalized mixtures extend the standard mixed models and provide more flexibility in their estimations allowing negative mixing weights.

In this context, Jiang et al. (1999) derived generalized mixtures of Weibull distributions under the formation of series, parallel, k -out-of- n and consecutive k -out-of- n systems formed by Weibull components with the same shape parameter. In addition, some negative estimations for the weights of Weibull mixtures of three components with common shape parameter are obtained in three examples of Bucar et al. (2004), and they also pointed out it is crucial to be able to guarantee that these mixtures are valid models. Thus, it seems reasonable to use generalized mixtures of Weibull components as underlying distributions to model heterogeneous survival data, and reliability of a system when its structure is unknown or partially known, since one cannot use component lifetime distributions to derive the system's lifetime distribution. In both cases, sub-populations and components very often have the same distribution type, and it is important to know what parametric constraints must be required to have a legitimate probability model. So, we will concentrate only on Weibull components with common shape parameter, which is habitual in several applications, and includes as special cases the exponential or Rayleigh components.

In the literature, some authors have studied generalized mixtures of exponential distributions and properties related with these generalized mixtures, also called generalized hyperexponential models. Steutel (1967) gave necessary conditions to be a valid probability model, and Bartholomew (1969) obtained sufficient conditions, which characterize the two exponential components case. Likewise, Baggs and Nagaraja (1996) showed the characterization of generalized mixtures of three exponential distributions, and their reliability properties are discussed in Baggs and Nagaraja (1996) and Franco and Vivo (2002). Furthermore, Franco and Vivo (2006) analyzed the four exponential components case, and Franco and Vivo (2007) studied the generalized mixtures of a gamma and one or two exponential distributions.

Note that analytical and graphical methods have been used to approximate Weibull mixture models. For instance, moments, maximum likelihood, least-square and WPP

plot methods, and different numerical algorithms have been employed to approach the probabilistic behavior, among others quasi-Newton, Newton-Raphson, conjugate gradient and expectation-maximization algorithms. A brief review about these estimation methods and their applications to Weibull mixtures can be seen in Bucar et al. (2004), Murthy et al. (2004) and Carta and Ramírez (2007). Some of them may be applied to fit Weibull mixture models allowing negative mixing weights, which might yield a more suitable model by verifying its feasibility through determined restrictions. Therefore, we hope that the study of these constraints will attract its application.

The paper is organized as follows. In Section 2, we define a generalized mixture of Weibull distributions, and we discuss its necessary or sufficient conditions to be a valid probability model. In Section 3, the characterization of these mixtures of three or fewer Weibulls based on the mixing weights and the parameters of the components is deduced. In the last section, Weibull mixtures with some negative weights and the same shape parameter are arisen in the reliability field, and in a particular case simulated times to failure data are fitted by 2-fold Weibull mixtures. Moreover, the obtained restrictions are also illustrated with a specific example, which shows the non-equivalence between the necessary and sufficient conditions when a generalized mixture of more than two Weibull components is considered.

2. Constraints for generalized mixtures of Weibulls

First, we give the concept of generalized mixture of Weibull distributions with common shape parameter, which includes as special cases the generalized mixtures of exponential or Rayleigh components.

Let (X_1, X_2, \dots, X_n) be a random vector formed by Weibull components with common shape parameter $c > 0$ and density functions $f_i(x) = cb_i x^{c-1} e^{-b_i x^c}$ for all $x > 0$, and $b_i > 0$, $i = 1, \dots, n$. It is said that X is a generalized mixture of Weibull distributions with common shape parameter $c > 0$, if its density function (pdf) is given by

$$f(x) = \sum_{i=1}^n a_i f_i(x) \quad (1)$$

where $a_i \in \mathbb{R}$, $i = 1, \dots, n$, such that $\sum_{i=1}^n a_i = 1$.

Evidently, this generalized mixture may be defined by its distribution and survival functions, respectively,

$$F(x) = 1 - \sum_{i=1}^n a_i e^{-b_i x^c} \quad \text{and} \quad S(x) = \sum_{i=1}^n a_i e^{-b_i x^c}.$$

Moreover, taking into account that each component of (1) $f_i(x)$ is a pdf, it is clear that $\int_{\mathbb{R}} f(x) dx = \sum_{i=1}^n a_i = 1$, and consequently, $f(x)$ is a pdf if and only if $f(x) \geq 0$ for all x , i.e., a generalized mixture of Weibull distributions is a legitimate probability model if and only if $f(x)$ is everywhere positive. In addition, if some of the a_i 's are negative, $f(x)$ could be negative for some values of x and so may not be a pdf.

Likewise, we can suppose that b_i are different, $i = 1, \dots, n$, because in otherwise, the number of components of the generalized mixture of Weibull distributions is reduced to the different scale parameters b_i . For example, if we consider two identically distributed Weibull components, $b_1 = b_2 = b$, then their generalized mixture follows the same distribution than its components. So, without loss of generality, we may assume $0 < b_1 < b_2 < \dots < b_n$.

Let us see now constraints for a generalized mixture of Weibull distributions with common shape parameter, which are based on their weights and parameters.

Theorem 2.1. *Let $f(x)$ in (1) be a generalized mixture of Weibull density functions with common shape parameter $c > 0$, $0 < b_1 < b_2 < \dots < b_n$ and $a_i \in \mathbb{R}$, $i = 1, \dots, n$, such that $\sum_{i=1}^n a_i = 1$. If $f(x)$ is a pdf then $a_1 > 0$ and*

$$\sum_{i=1}^n a_i b_i \geq 0.$$

Proof. Firstly, in order to check $f(x) \geq 0$, the function $f(x)$ can be written as $f(x) = c x^{c-1} e^{-b_n x^c} g(x)$, where $g(x) = \sum_{i=1}^n a_i b_i e^{(b_n - b_i)x^c}$, and both $f(x)$ and $g(x)$ have the same sign.

If we suppose that $a_1 < 0$, then $\lim_{x \rightarrow \infty} g(x) = -\infty$, and consequently, there exists a x_0 such that $g(x) < 0$ for all $x > x_0$, which contradicts that $f(x)$ is a pdf. Therefore, $a_1 > 0$ is a necessary condition for $f(x) \geq 0$ everywhere. Note that $a_1 \neq 0$ because a_1 represents the coefficient of the Weibull component with less parameter b_i in the generalized mixture.

Moreover, it is required that $f(0) \geq 0$, and taking into account that $\lim_{x \rightarrow 0} g(x) = \sum_{i=1}^n a_i b_i$ we obtain that $\sum_{i=1}^n a_i b_i \geq 0$. \square

However, the conditions of Theorem 2.1 are not sufficient for a generalized mixture of Weibull distributions. So, the following result shows sufficient restrictions to ensure that $f(x)$ is a valid pdf.

Theorem 2.2. *Let $f(x)$ in (1) be a generalized mixture of Weibull density functions with common shape parameter $c > 0$, $0 < b_1 < b_2 < \dots < b_n$ and $a_i \in \mathbb{R}$, $i = 1, \dots, n$, such that $\sum_{i=1}^n a_i = 1$. If*

$$\sum_{k=1}^i a_k b_k \geq 0, \quad \text{for } i = 1, \dots, n \quad (2)$$

then $f(x)$ is a pdf.

Proof. Under the conditions of the statement, $f(x)$ can be rewritten as

$$f(x) = cx^{c-1} e^{-b_n x^c} \sum_{i=1}^n a_i b_i + cx^{c-1} \sum_{i=1}^{n-1} e^{-b_i x^c} (1 - e^{-(b_{i+1} - b_i)x^c}) \sum_{k=1}^i a_k b_k$$

where each term of $f(x)$ is positive if (2) is satisfied, and hence, $f(x)$ is a pdf. \square

Remark 2.3. *Theorems 2.1 and 2.2 show necessary conditions and sufficient conditions for a generalized mixture of exponential distributions when $c = 1$, which were given by Steutel (1967) and Bartholomew (1969), respectively. Moreover, when the shape parameter $c = 2$, the necessary or sufficient restrictions for a generalized mixture of Rayleigh components are also given by both theorems, respectively.*

3. Characterizations with three or fewer components

In this section, we characterize a generalized mixture of two or three Weibull distributions based on the mixing weights and parametric restrictions of their components.

In the two components cases, it is immediate to check that the necessary conditions of Theorem 2.1 and the sufficient conditions of Theorem 2.2 are equivalent.

Theorem 3.1. *Let $f(x) = cx^{c-1}(a_1 b_1 e^{-b_1 x^c} + a_2 b_2 e^{-b_2 x^c})$ be a generalized mixture of two Weibull density functions with common shape parameter $c > 0$, $0 < b_1 < b_2 < \infty$, $a_1 > 0$*

and $a_2 \in \mathbb{R}$ such that $a_1 + a_2 = 1$. Then, $f(x)$ is a pdf if and only if either $a_2 > 0$ or $a_1 b_1 + a_2 b_2 \geq 0$.

Nevertheless, the sufficient conditions (2) given by Theorem 2.2 are not necessary conditions when a generalized mixture of more than two Weibull components is considered. In this setting, the following result determines the constraints based on the mixing weights and parameters of a generalized mixture of three Weibull distributions to be a legitimate pdf.

Theorem 3.2. Let $f(x) = cx^{c-1}(a_1 b_1 e^{-b_1 x^c} + a_2 b_2 e^{-b_2 x^c} + a_3 b_3 e^{-b_3 x^c})$ be a generalized mixture of three Weibull density functions with common shape parameter $c > 0$, $0 < b_1 < b_2 < b_3 < \infty$, $a_1 > 0$ and $a_2, a_3 \in \mathbb{R}$ such that $a_1 + a_2 + a_3 = 1$.

1. If $a_2 > 0$ and $a_3 > 0$, then $f(x)$ is a pdf.
2. If $a_2 \in \mathbb{R}$ and $a_3 < 0$, then $f(x)$ is a pdf if and only if

$$a_1 b_1 + a_2 b_2 + a_3 b_3 \geq 0. \quad (3)$$

3. If $a_2 < 0$ and $a_3 > 0$. Then, $f(x)$ is a pdf if and only if one of the following cases holds:

(a) $m \leq 1$ and (3)

(b) $m > 1$ and $\log(m_1) \geq (b_3 - b_1)(b_2 - b_3)^{-1} \log(m)$

where

$$m = \frac{-a_3 b_3 (b_3 - b_1)}{a_2 b_2 (b_2 - b_1)} \quad \text{and} \quad m_1 = \frac{a_1 b_1 (b_2 - b_1)}{a_3 b_3 (b_3 - b_2)}.$$

Proof. Taking into account that $f(x)$ is a pdf if and only if $f(x)$ is nonnegative, we develop the proof depending on the signs of the a_i 's.

In the first place, if all weights are positive, $a_1 > 0$, $a_2 > 0$ and $a_3 > 0$, then $f(x)$ is a convex mixture of Weibull distributions; therefore, $f(x)$ is a true pdf.

In the case (2), $a_3 < 0$, we discuss the non-negativity of $f(x)$ depending on the sign of a_2 .

If $a_2 > 0$, the function $f(x)$ can be expressed as $f(x) = cx^{c-1} e^{-b_3 x^c} g_1(x)$, where

$$g_1(x) = a_1 b_1 e^{(b_3 - b_1)x^c} + a_2 b_2 e^{(b_3 - b_2)x^c} + a_3 b_3$$

thus, $f(x)$ and $g_1(x)$ have the same sign, and $g_1(0) = a_1b_1 + a_2b_2 + a_3b_3$. Moreover, $g_1(x)$ is increasing in $x > 0$. So, if $g_1(0) \geq 0$, which is restriction (3), then $g_1(x) \geq 0$, and consequently, $f(x)$ is a pdf.

Likewise, when $a_2 < 0$, the function $f(x)$ has the same sign that the function $g_2(x)$, where

$$g_2(x) = a_1b_1 + a_2b_2e^{-(b_2-b_1)x^c} + a_3b_3e^{-(b_3-b_1)x^c}$$

since $f(x) = cx^{c-1}e^{-b_1x^c}g_2(x)$. Therefore, taking into account that $g_2(x)$ is increasing in x and $g_2(0) = a_1b_1 + a_2b_2 + a_3b_3$, if $g_2(0) \geq 0$ then $g_2(x) \geq 0$, and consequently, $f(x)$ is a pdf.

Note that, in both cases $a_2 > 0$ and $a_2 < 0$, when (3) is not satisfied, then $g_i(x)$ changes of sign, $i = 1, 2$, and therefore, $f(x)$ cannot be a pdf, which proves that (3) is necessary.

Finally, to prove case (3), $a_1 > 0$, $a_2 < 0$ and $a_3 > 0$, we remark that $f(x)$ and $g_2(x)$ have the same sign, and the first derivative of $g_2(x)$ is given by

$$\begin{aligned} g_2'(x) &= -(b_2 - b_1)cx^{c-1}a_2b_2e^{-(b_2-b_1)x^c} - (b_3 - b_1)cx^{c-1}a_3b_3e^{-(b_3-b_1)x^c} \\ &= -cx^{c-1} \left[(b_2 - b_1)a_2b_2e^{-(b_2-b_1)x^c} + (b_3 - b_1)a_3b_3e^{-(b_3-b_1)x^c} \right]. \end{aligned}$$

Thus, taking into account the above term between bracket for all $x \in \mathbb{R}$, it is deduced that there exists x_0 such that $g_2'(x) \geq 0$ for $x \geq x_0$ and $g_2'(x) < 0$ for $x < x_0$, where

$$x_0 = \left(\frac{1}{b_3 - b_2} \log m \right)^{1/c} \quad \text{and} \quad m = -\frac{a_3b_3(b_3 - b_1)}{a_2b_2(b_2 - b_1)}.$$

So, if $m \leq 1$, then the function $g_2(x)$ is increasing for all $x > 0$, and hence $g_2(x) \geq g_2(0)$. Furthermore, under condition (3), the function $f(x)$ is a pdf; and if (3) is not verified, then $f(x)$ changes its sign.

Likewise, if $m > 1$, the function $g_2(x)$ changes its monotonicity from being decreasing to increasing for all $x > 0$, and consequently, $g_2(x)$ attains its minimum at x_0 , i.e., $g_2(x) \geq g_2(x_0)$. Therefore, $f(x)$ changes its sign when $g_2(x_0) < 0$, and it is a pdf when $g_2(x_0) \geq 0$.

Besides, $g_2(x_0)$ can be rewritten as

$$g_2(x_0) = a_1b_1 + e^{-(b_3-b_1)x_0^c} (a_2b_2e^{-(b_2-b_3)x_0^c} + a_3b_3)$$

thus, the condition $g_2(x_0) \geq 0$ can be expressed as

$$-a_1 b_1 \leq m^{-\frac{b_3-b_1}{b_3-b_2}} (a_2 b_2 m + a_3 b_3) = a_3 b_3 m^{-\frac{b_3-b_1}{b_3-b_2}} \left(\frac{b_3 - b_1}{-(b_2 - b_1)} + 1 \right)$$

or equivalently,

$$m_1 = \frac{a_1 b_1 (b_2 - b_1)}{a_3 b_3 (b_3 - b_2)} \geq m^{-\frac{b_3-b_1}{b_3-b_2}} \Leftrightarrow \log(m_1) \geq -\frac{b_3 - b_1}{b_3 - b_2} \log(m).$$

Note that, when $a_2 < 0$ and $a_3 > 0$, the restrictions of (3b) imply the necessary condition (3). \square

Remark 3.3. *When the shape parameter $c = 1$, Theorems 3.1 and 3.2 characterize generalized mixtures of two and three exponential components, which were obtained by Bartholomew (1969) and Baggs and Nagaraja (1996), respectively. In addition, when $c = 2$, the above theorems also establish the necessary and sufficient conditions for generalized mixtures of two and three Rayleigh components, respectively.*

4. Remarks

As mentioned in Section 1, it is habitual to use Weibull components in numerous applications. Here, we give a small sample of cases in the reliability field, which evidently satisfy the parametric restrictions given by Theorems 3.1 and 3.2.

We need to use the next notation. Let X_1 and X_2 be two neither necessarily independent nor identically distributed random variables with survival functions $S_i(x)$, $i = 1, 2$, and joint survival function $S(x_1, x_2) = \Pr(X_1 > x_1, X_2 > x_2)$. Let $T_1 = \min(X_1, X_2)$ and $T_2 = \max(X_1, X_2)$ denote their minimum and maximum order statistics, i.e., the lifetimes of the series and parallel systems, respectively; their survival functions are given by

$$S_{(1)}(x) = S(x, x) \quad \text{and} \quad S_{(2)}(x) = S_1(x) + S_2(x) - S_{(1)}(x). \quad (4)$$

In this setting, the generalized mixtures arise from some bivariate Weibull models, most of them have been constructed as extensions of bivariate exponential models, an excellent review on this topic can be found in Kotz et al. (2000) and Murthy et al. (2004). Thus, when marginals have a common shape parameter, their series and parallel systems are predominantly generalized mixtures of three or fewer Weibull components with the same shape parameter.

Example 4.1. Let $X_1 \sim W(\theta_1, c)$ and $X_2 \sim W(\theta_2, c)$ be two independent Weibull components with (θ_1, c) and (θ_2, c) parameters, respectively. From (4), $T_1 \sim W(\theta_1 + \theta_2, c)$, and its parallel system is a generalized mixture of three Weibull distributions with survival function $S_{(2)}(x) = e^{-\theta_1 x^c} + e^{-\theta_2 x^c} - e^{-(\theta_1 + \theta_2)x^c}$, i.e. $a_1 = a_2 = 1$ and negative weight $a_3 = -1$. Moreover, if $\theta_1 = \theta_2$ then T_2 is a generalized mixture of two Weibull components with weights $a_1 = 2$ and $a_2 = -1$.

Example 4.2. Let (X_1, X_2) be a bivariate Weibull distribution of Marshall-Olkin, which joint survival function is given by

$$S(x_1, x_2) = \exp(-\lambda_1 x_1^c - \lambda_2 x_2^c - \lambda_{12} \max(x_1^c, x_2^c))$$

for all $x_1, x_2 > 0$, where $\lambda_1 > 0, \lambda_2 > 0, \lambda_{12} > 0$ and $c > 0$. From (4), $T_1 \sim W(\lambda, c)$, and its parallel system has survival function $S_{(2)}(x) = e^{-(\lambda_1 + \lambda_{12})x^c} + e^{-(\lambda_2 + \lambda_{12})x^c} - e^{-\lambda x^c}$ where $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$, i.e., a generalized mixture of three Weibull distributions with weights $a_1 = a_2 = 1$ and $a_3 = -1$.

Other interesting application of generalized mixtures of Weibull distributions may be yielded in modelling occurrence of power system blackouts.

Example 4.3. Assuming independent Weibull distributions for the lifetimes X_1 of power system transmission lines and X_2 of power system protective relays, $X_1 \sim W(\theta_1, c)$ and $X_2 \sim W(\theta_2, c)$ respectively, then the conditional random variable $X = (X_1 | X_1 > X_2)$ is relevant to model blackouts when the protection system has failed, since power system blackouts due to the failure in transmission lines is related with hidden failure in protective relays. Thus, taking into account that $P(X_1 > X_2) = \theta_2(\theta_1 + \theta_2)^{-1}$ and $P(X_1 > x, X_1 > X_2) = e^{-\theta_1 x^c} - \theta_1(\theta_1 + \theta_2)^{-1}e^{-(\theta_1 + \theta_2)x^c}$, it is easy to prove that X follows a generalized mixture of two Weibull components with survival function

$$S(x) = (\theta_1 + \theta_2)\theta_2^{-1}e^{-\theta_1 x^c} - \theta_1\theta_2^{-1}e^{-(\theta_1 + \theta_2)x^c}.$$

where the mixing weights are $a_1 = (\theta_1 + \theta_2)\theta_2^{-1} > 1$ and $a_2 = -\theta_1\theta_2^{-1} < 0$.

In order to display a particular case where negative weights are estimated, we need to revisit the method of moments (MM), the least-square (LS) method and the maximum

likelihood (ML) method in the context of generalized mixtures of two Weibull distribution with common shape parameter.

Firstly, the MM determines the parameters by solving the system of equations formed by sample moments equal to population moments, which can be expressed as

$$k_r(c) = a_1\beta_1^r + (1 - a_1)\beta_2^r, \quad \text{for } r = 1, 2, 3, 4 \quad (5)$$

where $k_r(c) = (\Gamma(1 + r/c))^{-1} \sum_{i=1}^n x_i^r/n$ is the sample moment m_r of order r divided by the function Γ of Euler, and $\beta_j = b_j^{-1/c}$ with $j = 1, 2$. Thus, the solutions are restricted by the conditions of Theorem 3.1, or equivalently,

$$a_1 > 0, \beta_1 > \beta_2 > 0, a_1 \leq \beta_1^c(\beta_1^c - \beta_2^c)^{-1} \quad \text{and} \quad c > 0.$$

Remark that the above system of nonlinear equations can be simplified into one as follows. The first equation $r = 1$ allows to find the weight of the positive component $a_1 = (k_1(c) - \beta_2)/(\beta_1 - \beta_2)$. Substituting this expression for a_1 in the three remaining ones, it is easy to check that β_1 and β_2 are interchangeable in the second and third equations ($r = 2, 3$), and so they are reduce to a quadratic equation in β_j with $j = 1$ or 2 . Therefore, both parameters are determined by the roots of

$$(k_2(c) - k_1^2(c))\beta^2 + (k_1(c)k_2(c) - k_3(c))\beta + (k_1(c)k_3(c) - k_2^2(c)) = 0$$

and consequently, β_j will be calculated for each c in according to the restrictions, i.e. $\beta_1 > \beta_2$. Likewise, taking into account that both are also interchangeable in the fourth equation $r = 4$, the system (5) of nonlinear equations is reduced to

$$k_4(c)(k_2(c) - k_1^2(c)) - k_2^3(c) + 2k_1(c)k_2(c)k_3(c) - k_3^2(c) = 0$$

which has to be resolved by numerical algorithms of nonlinear programming such as Newton-Raphson (e.g., Bazaraa et al. (1993)) in order to determine the parameters using the MM. Unfortunately, this method could provide non-feasible solutions, and so the constraints must be checked to guarantee valid mixture models.

Second, the LS method estimates the parameters by looking for their numerical values, which minimize the sum of squares of the deviations between the experimental cumulative

relative frequencies and those expected ones under the assumed model, or equivalently, between the empirical survival function $S_E(x) = \sum_{i=1}^n I(x > x_i)/n$ and expected survival function of a generalized mixture of two Weibulls:

$$\begin{aligned} \min \quad & \sum_{i=1}^n (S_E(x_i) - a_1 e^{-b_1 x_i^c} - (1 - a_1) e^{-b_2 x_i^c})^2 \\ \text{subject to: } & a_1 > 0, 0 < b_1 < b_2, a_1 \leq b_2/(b_2 - b_1) \text{ and } c > 0 \end{aligned}$$

which can be numerically resolved by a quasi-Newton algorithm with exterior penalty functions (e.g., Bazaraa et al. (1993)).

The ML estimation method consists of finding the values of the parameters, which maximize the likelihood function from a given data set and the pdf of the assumed model, or applying its logarithmic transformation, i.e.

$$\begin{aligned} \max \quad & \sum_{i=1}^n (\log(cx_i^{c-1}) + \log(a_1 b_1 e^{-b_1 x_i^c} - (1 - a_1) b_2 e^{-b_2 x_i^c})) \\ \text{subject to: } & a_1 > 0, 0 < b_1 < b_2, a_1 \leq b_2/(b_2 - b_1) \text{ and } c > 0. \end{aligned}$$

The above optimization problem also can be approximated by a quasi-Newton algorithm with exterior penalty functions for nonlinear constraints.

Let us see now a particular case where a negative weight is obtained. Although, prior identification of the sample components of the mixture is not required for modelling data sets, in order to simulate times of failure data, we assume that a power system transmission lines and a power protective relays have Weibull distributions, with common shape parameter $c = 3$ and scale parameters $\theta_1 = 0.5$ and $\theta_2 = 5$, respectively.

Based on the chosen parameters and Example 4.3, simulated failure data sets are generated with $n = 50, 100, 200, 500$ data points. The first four sample moments for each n are given in Table 1. The three above estimation methods have been applied to model the failure times by using a two Weibull mixture allowing negative weights. The nonlinear programming techniques used to estimate the parameters require a starting point. As starting point in the LS method, we have used the results obtained from the application of the MM, and in the ML method the results obtained from the LS method, the estimated parameters obtained for the proposed mixture model are listed in Table 1.

No matter what kind of parameter estimation method is used, the selected mixture with the estimated parameters must be tested to determine whether it represents the existing data well. Therefore, the suitability of the fitted mixture models is judged from the Kolmogorov-Smirnov (KS) goodness-of-fit test, which are also listed in Table 1.

Table 1: Estimated parameters obtained for the two Weibull mixture allowing negative weights

n	m_1	m_2	m_3	m_4	Method	a_1	b_1	b_2	c	KS	P -value
50	1.1630	1.4864	2.0518	3.0214	MM	1.09074	0.51407	4.98759	3.06955	0.1076	0.6088
					LS	1.21629	0.45307	1.62170	3.20419	0.0600	0.9938
					ML	1.42286	0.42569	0.77992	3.38895	0.0702	0.9663
100	1.2093	1.5901	2.2336	3.3112	MM	1.00709	0.34681	46.4599	3.67983	0.0549	0.9238
					LS	0.99287	0.33012	7.11505	3.92043	0.0540	0.9322
					ML	1.05744	0.38210	2.43495	3.54033	0.0566	0.9059
200	1.1593	1.4640	1.9872	2.8691	MM	1.12350	0.53860	4.68518	3.10493	0.0343	0.9724
					LS	1.13215	0.56203	4.51713	3.01633	0.0277	0.9980
					ML	1.14477	0.54383	3.74945	3.11516	0.0394	0.9151
500	1.1921	1.5548	2.1850	3.2701	MM	1.07711	0.46268	6.11035	3.15325	0.0215	0.9751
					LS	1.14264	0.50950	3.48695	3.06687	0.0189	0.9942
					ML	1.27800	0.60092	2.69195	2.79064	0.0182	0.9965

Note that, in three fitted Weibull mixture models when $n = 100$, the estimated weight for the first component are closed to one. Nevertheless, from Table 1, the general conclusion is that Weibull mixture allowing negative weights should be considered, i.e. $a_2 = 1 - a_1 < 0$, because the results of the KS test indicated that the fitted generalized mixtures could not be rejected as an acceptable fit to each simulated failure data.

On the other hand, as it has been carried out, the necessary conditions of Theorem 2.1 are not equivalent to the sufficient restrictions of Theorem 2.2 when more than two Weibull components are considered. This can be easily illustrated through some particular generalized mixtures of three Weibull distributions. For instance, Figure 1 displays some pdf's of generalized mixtures of three Weibull components with shape parameter $c = 1.5$ and for varying values of the mixing weights and scale parameters, the solid line ($a_1 = 1$, $a_2 = -1.15 = -a_3$, $b_1 = 0.5$, $b_2 = 1.25$, $b_3 = 3.5$), the dashed line ($a_1 = 2.5 = -a_2$, $a_3 = 1$, $b_1 = 1.75$, $b_2 = 2.5$, $b_3 = 3$), the dotted line ($a_1 = 1.5$, $a_2 = -1.25$, $a_3 = 0.75$, $b_1 = 0.5$, $b_2 = 1.5$, $b_3 = 2.5$), and the dashed-dotted line ($a_1 = 2$, $a_2 = -1.5$, $a_3 = 0.5$, $b_1 = 0.75$, $b_2 = 2$, $b_3 = 3$), respectively. In the four cases, $a_1 b_1 + a_2 b_2 < 0$, and so (2) is not satisfied for $i = 2$. However, these four generalized mixtures of three Weibull distributions are

legitimate probability models, as they satisfy (3) of Theorem 3.2.

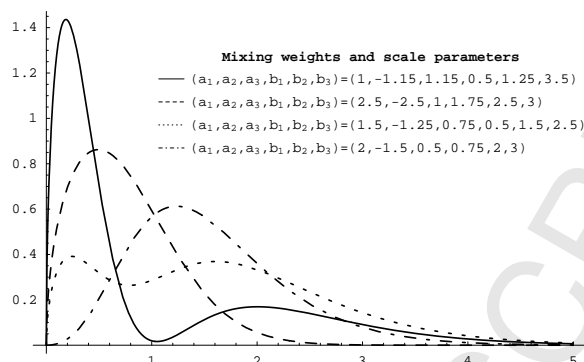


Figure 1: Pdf curves of generalized mixtures of three Weibulls with $c = 1.5$.

Finally, we point out that Bucar et al. (2004) obtained generalized mixtures of three Weibull components with common shape parameter by applying the multivariate regression method, but only two of those approximations verify Theorem 3.2. So, the other one must be discarded in their example because it is not a feasible model.

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