

# 2D electrostatic problems with rounded corners

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**Abstract**—The second order terms of a multiscale expansion for dealing with rounded corners in 2D electrostatic problems are studied. The main ideas and the sequence of problems to consider are recalled and finite element simulations are presented to assess the accuracy of the method.

## I. INTRODUCTION

High-voltage applications require a precise knowledge of the electric field in the area where the geometry of the structure is sharp. In a real device, the geometry is not “truly sharp” but present rounded edges or corners. The accurate description of this rounded shape can be cumbersome in a numerical model and the description of the rounded shape can be only statistical due to the tolerance in the manufacturing. A theoretical work propose methods that could be answers to the “rounded shape” issue [2]. We put this strategy numerically in practice on a 2D electrostatic problem in [3]. More precisely, we considered  $v_\epsilon$  the solution on  $\Omega_\epsilon$  (see Fig. 1(a)) of the following problem:

$$\begin{cases} \Delta v_\epsilon = 0, & \text{on } \Omega_\epsilon, \\ v_\epsilon = 0, & \text{on } \Gamma_\epsilon^0, \text{ and } v_\epsilon = 1, \text{ on } \Gamma^1, \\ \partial_n v_\epsilon = 0, & \text{on } \Gamma^N, \end{cases} \quad (1)$$

where  $\epsilon$  characterizes the “size” of the rounded corner and  $\partial_n$  denotes  $n \cdot \nabla$ ,  $n$  being the unitary outward normal on the boundary of the domain. The expansion of  $v_\epsilon$  studied in [3] came from the following remarks:

- the exact solutions *close to the corner*, computed for several values of the curvature radius  $\epsilon$ , are quasi-similar, up to a “scaling factor” (related to  $\epsilon$ ). It is also noticed that the “shape” of the solutions *close to the corner* (their “shape” but not their amplitude) weakly depend on other elements of the studied structure, such as the distance to the boundaries: if the corner geometry is *self-similar*<sup>1</sup>, it is also said that the dominant term of the solutions close to the corner are *self-similar*.
- the exact solutions *far from the corner* are weakly influenced by the change of the curvature radius  $\epsilon$ , and they converge to the solution on the domain with the sharp corner when  $\epsilon$  goes to zero.

<sup>1</sup>Roughly, it means that a single parameter, here  $\epsilon$ , and a basic geometry are sufficient to describe the corners for any value of  $\epsilon$ . For a precise definition, refer to [2].

Nonetheless, in [3], only the first order of the approximation is considered but it has been shown that higher orders are required when one considers a non-symmetric structure. The second order terms of the same expansion are proposed here and numerical examples are given.

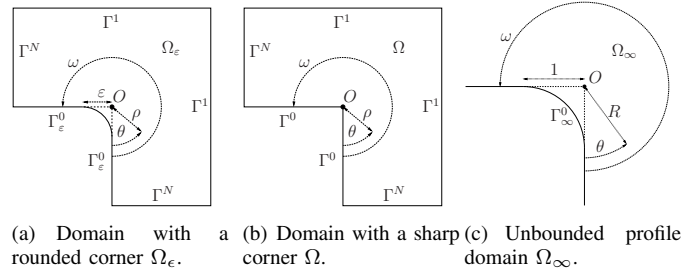


Fig. 1. Considered domains  $\Omega_\epsilon$ ,  $\Omega$  and  $\Omega_\infty$ .

## II. PROBLEMS AND NOTATIONS

The heuristics for the construction of the two first orders are detailed in [2, subsection 4.1]. The two terms for the approximation *far from the corner* are computed on the singular domain  $\Omega$  (Fig. 1(b)) and the two terms for the approximation *close to the corner* are computed on the so-called *profile domain*  $\Omega_\infty$  (Fig. 1(c)), which is an unbounded domain with a unitary corner.

The roughest approximation  $v^0$  of  $v_\epsilon$ , far from the corner, is the solution on  $\Omega$  of the following problem:

$$\begin{cases} \Delta v^0 = 0, & \text{on } \Omega, \\ v^0 = 0, & \text{on } \Gamma^0, \text{ and } v^0 = 1, \text{ on } \Gamma^1, \\ \partial_n v^0 = 0, & \text{on } \Gamma^N. \end{cases} \quad (2)$$

Note that in the neighborhood of the origin:

$$v^0(x) \underset{\rho \rightarrow 0}{\simeq} \sum_{p=1}^{\infty} a_p \rho^{p\alpha} \sin(\alpha\theta) = \sum_{p=1}^{\infty} a_p s^{p\alpha}(\rho, \theta), \quad (3)$$

where  $\alpha = \pi/\omega$ , and  $(\rho, \theta)$  are the polar coordinates. As the behaviors of  $v^0$  and  $v_\epsilon$  are different in the corner,  $v^0$  should be truncated in the corner. A radial cut-off function  $\varphi$  is introduced for this purpose:

$$\varphi(\rho) = \begin{cases} 1, & \text{if } \rho \geq d_1, \\ 0, & \text{if } \rho \leq d_0, \end{cases} \quad (4)$$

where  $d_0, d_1$  are fixed distances from the corner with  $d_0 < d_1$ . It can be shown that [2]:

$$v_\epsilon = \varphi \left( \frac{\cdot}{\epsilon} \right) v^0 + \mathcal{O}_{H^1}(\epsilon^\alpha), \quad (5)$$

which means that:

$$\exists \tilde{\epsilon} > 0, \forall \epsilon < \tilde{\epsilon}, \int_{\Omega_\epsilon} \|\nabla(v_\epsilon - v^0)\|^2 dx < C\epsilon^\alpha, \quad (6)$$

i.e. the energy norm of the error converges as  $\epsilon^\alpha$  to 0.

Expansion (3) provides the coefficient  $a_1$  for a first correction in the corner. The first order expansion is then:

$$v_\epsilon = \varphi \left( \frac{\cdot}{\epsilon} \right) v^0 + \epsilon^\alpha (1 - \varphi) a_1 V^\alpha \left( \frac{\cdot}{\epsilon} \right) + \mathcal{O}_{H^1}(\epsilon^{2\alpha}). \quad (7)$$

In (7)  $V^\alpha$  is a profile term that satisfies, for  $p = 1$ ,

$$\begin{cases} -\Delta_X V^{p\alpha} = [\Delta_X; \varphi] \mathfrak{s}^{p\alpha}, & \text{in } \Omega_\infty, \\ V^{p\alpha}|_{\Gamma_\infty} = 0, \\ \lim_{R \rightarrow +\infty} V^{p\alpha} = 0, \end{cases} \quad (8)$$

where for any couple  $(\nu, u)$ ,  $[\Delta; \nu]u = \Delta(\nu u) - \nu \Delta u$ . Note that in the neighborhood of  $+\infty$ :

$$V^\alpha(X) = \sum_{p=1}^{+\infty} A_p \mathfrak{s}^{-p\alpha}(X). \quad (9)$$

For the second order, the expansion becomes:

$$\begin{aligned} v_\epsilon &= \varphi \left( \frac{\cdot}{\epsilon} \right) [v^0 + \epsilon^{2\alpha} a_1 A_1 v^{2\alpha}] \\ &+ \epsilon^\alpha (1 - \varphi) \left[ a_1 V^\alpha \left( \frac{\cdot}{\epsilon} \right) + \epsilon^\alpha a_2 V^{2\alpha} \left( \frac{\cdot}{\epsilon} \right) \right] + \mathcal{O}_{H^1}(\epsilon^{3\alpha}), \end{aligned} \quad (10)$$

where  $V^{2\alpha}$  is a profile term that satisfies (8) for  $p = 2$  and  $v^{2\alpha}$  is a correction for the behavior far from the corner that satisfies:

$$\begin{cases} -\Delta v^{2\alpha} = [\Delta; (1 - \varphi)] \mathfrak{s}^{-\alpha}, & \text{on } \Omega, \\ v^{2\alpha} = 0, & \text{on } \Gamma^0, \text{ and } v = 0, \text{ on } \Gamma^1, \\ \frac{\partial v^{2\alpha}}{\partial n} = 0, & \text{on } \Gamma^N. \end{cases} \quad (11)$$

In [3], we have considered a problem of the form:

$$\begin{cases} -\Delta_X U^{p\alpha} = 0, & \text{in } \Omega_\infty, \\ U^{p\alpha}|_{\Gamma_\infty} = 0, \\ \lim_{R \rightarrow +\infty} (U^{p\alpha} - \mathfrak{s}^{p\alpha}) = 0, \end{cases} \quad (12)$$

instead of (8) because for  $p = 1$  or 2:

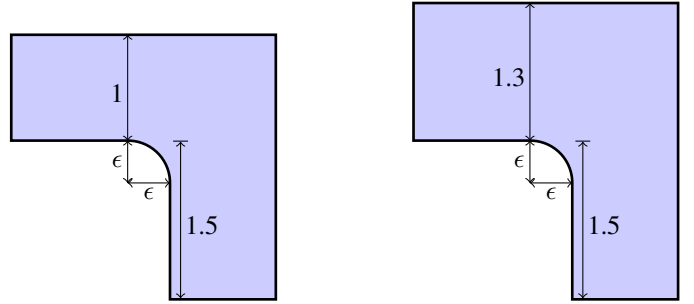
$$V^{p\alpha} = U^{p\alpha} - \varphi \mathfrak{s}^{p\alpha}. \quad (13)$$

Nonetheless, the relations are more complicated for  $p > 2$  [2].

### III. NUMERICAL RESULTS

Two L-shape geometries with a rounded corner are considered; their dimensions are recalled on Fig. 2. The finite element method has been used for solving (1), (2), (8), and (11) as it was done in [3].

It is shown on Fig. 3 that the behaviors of the error in the energy norm for approximate expansions (7) and (10) behave respectively as  $\epsilon^{2\alpha}$  and  $\epsilon^{3\alpha}$ , independently of the geometry. It

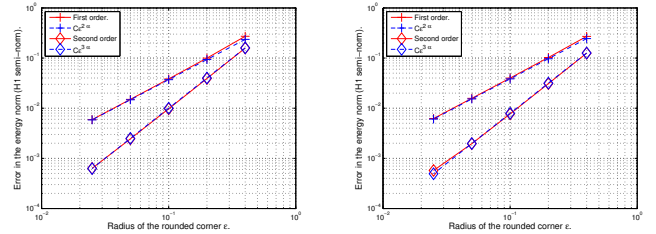


(a) Symmetric geometry.

(b) Non symmetric geometry.

Fig. 2. Two considered problems and their dimensions.  $\epsilon = 0.4$ .

is remarkable because  $a_1 = 0$  in the symmetric configuration (see Fig. 2(a)) and we could think that (7) would have lead to a better approximation than in the non symmetric configuration (see Fig. 2(b)). Nonetheless, this intuition is obviously not correct and the correction  $v^{2\alpha}$  far from the corner plays an equivalent role than the correction close to the corner regarding the energy norm in the whole domain.

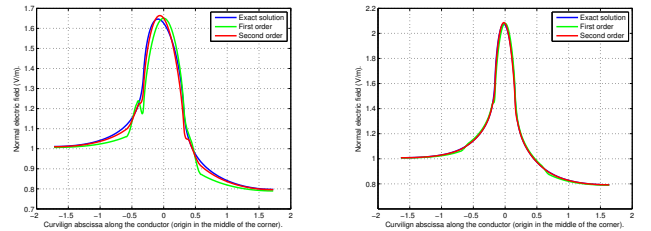


(a) Symmetric geometry.

(b) Non symmetric geometry.

Fig. 3. Convergence of the error in the energy norm.

It is shown on Fig. 4 the normal electric field along the electrodes in the non symmetric configuration (from the bottom right to the top left of the electrode, see Fig. 2(b)) for two values of  $\epsilon$ . This normal field has been computed respectively from  $v_\epsilon$ , from first order expansion (7) and from second order expansion (10). The behavior of the normal field is “more closely respected” for the order 2, in particular the location of the maximum is roughly equivalent to the correct solution. Nonetheless, on this example, the maximum electric field is slightly overestimated by both approximations.



(a) Rounded corner with  $\epsilon = 0.4$ .

(b) Rounded corner with  $\epsilon = 0.2$ .

Fig. 4. Normal electric field along the conductors for the exact solution and the two first order approximations. The non symmetric geometry is considered.

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