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Asymptotic models for the generation of internal waves by a moving ship, and the dead-water phenomenon

Vincent Duchêne*

December 21, 2010

Abstract

This paper deals with the dead-water phenomenon, which occurs when a ship sails in a stratified fluid, and experiences an important drag due to waves below the surface. More generally, we study the generation of internal waves by a disturbance moving at constant speed on top of two layers of fluids of different densities. Starting from the full Euler equations, we present several nonlinear asymptotic models, in the long wave regime. These models are rigorously justified by consistency or convergence results. A careful theoretical and numerical analysis is then provided, in order to predict the behavior of the flow and in which situations the dead-water effect appears.

1 Introduction

The so-called “dead-water” phenomenon has been first reported by Fridtjof Nansen [42], describing a severe (and then inexplicable) decrease of velocity encountered by a ship, sailing on calm seas. Bjerknes, and then Ekman [22] soon explained that this phenomenon occurs as the ship, on top of density-stratified fluids (due to variations in temperature or salinity), concedes a large amount of energy by generating internal waves. Our work is motivated by the recent work of Vasseur, Mercier and Dauxois [53], who performed experiments and produced videos that precisely demonstrate the phenomenon. Our aim is to produce simple models that are able to predict the apparition and the magnitude of the dead-water effect.

Relatively little consideration has been given to this problem, after the early works of Lamb [29] and Sretenskii [50]. Miloh, Tulin and Zilman in [40] produce a model and numerical simulations for the case of a semi-submersible moving steadily on the free surface. The authors assume that the density difference between the two layers is small, an assumption that is removed by Nguyen and Yeung [45]. Motygin and Kuznetsov [41] offer a rigorous mathematical treatment of the issue when the body is totally submerged in one of the two layers, and Ten and Kashiwagi [51] present a comparison between the theory and experiments, in the case of a body intersecting the interface as well as the surface. Finally, we would like to cite Lu and Chen [36] for a more general treatment of the problem. All of these works use linearized boundary conditions, that rely on the assumption that the generated waves amplitudes are small. The linear theory is convenient as it allows to obtain the flow field by a simple superposition of Green functions, replacing the moving body by a sum of singularities. The integral representation of the flow, as well as the wave resistance experienced by the body, are therefore found explicitly. However, the smallness assumption on the wave amplitudes, that is necessary to the linear theory, is quite restrictive.

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We show in this paper that nonlinear effects have an important role to play on the dead-water effect, and that simple nonlinear models duly predict the phenomenon.

In this paper, we introduce several different nonlinear asymptotic models, corresponding to several regimes (size of the parameters of the system), and several orders of approximation. Two different regimes are considered. Each time, we assume that the depth of the fluid layers are small when compared with the wave lengths (shallow water). In the one-layer case, the equations obtained with this only assumption are the shallow-water or Saint Venant equations (at first order), or the Green-Naghdi extension (one order further, therefore involving nonlinear dispersion terms). See Wu, Chen [54] and references therein for a numerical treatment of the waves generated by a moving ship in the one layer case, using the Green–Naghdi equations. The two regimes we study carry additional smallness assumptions, that allow to substantially simplify the models. The first regime considers the case of small surface deformations, and small differences between the densities of the two layers. Such additional assumptions are very natural in the oceanographic framework, and have been frequently used in the literature (see [26, 35, 39] for example). In the second regime, we assume that the magnitude of the produced internal waves, when compared with the depth of the two layers, are small and of the same order of magnitude as the shallowness parameter. This regime, known as the Boussinesq regime, is particularly interesting as it allows models with competing dispersion and nonlinearity. Along with the coupled Boussinesq-type model, we introduce the KdV approximation, which consists in a decomposition of the flow into two waves, each one being a solution of an independant forced Korteweg-de Vries equation (fKdV). The fKdV equation has been extensively studied in the framework of the one-layer water wave problem (where a moving topography, or pressure, is the forcing term that generates waves); see [55, 38, 34], for example.

The system we study consists in two layers of immiscible, homogeneous, ideal, incompressible fluids under the only influence of gravity. The bottom is assumed to be flat, and we use the rigid-lid approximation at the surface. However, the surface is *not flat*, but a given function, moving at constant speed, that reflects the presence of the ship. Moreover, as we are interested in unidirectional equations (the fKdV equations), we focus on the two-dimensional case, *i.e.* the horizontal dimension $d = 1$. However, the three-dimensional case could be easily treated with the method presented here. Starting from the governing equations of our problem, the so-called full Euler system, and armed with the analysis of [20, 19] in the free surface case, we are able to deduce asymptotic models for each of the regimes presented above. Each of the models presented here are justified by a consistency result, or a convergence theorem (in a sense precisely given in Section 1.3 page 9). We compute numerically the fully nonlinear system of the first regime, as well as the KdV approximation, which allows us to investigate the effect of different parameters of the system, such as the velocity of the boat, or the depth ratio of the two layers. The wave resistance encountered by the ship is also computed, so that we are able to predict in which situations the dead-water effect shows up. Consequences of our analysis include the following:

- i. a ship suffers from a positive drag when an internal elevation wave is generated at its stern;
- ii. this effect can be strong near critical Froude numbers, and is always small away from the critical value;
- iii. the maximum peak of the drag is reached at slightly subcritical values;
- iv. the depth ratio of the two layers plays a critical role in the phenomenon;

In particular, our analysis allows to study the behavior of the system depending on two parameters : the depth ratio of the two layers of fluid, and the velocity of the fluid. Our results are summarized in Table 1, below.

Velocity of the ship	subcritical case	critical case		supercritical case
Depth ratio between the layers		thicker lower layer	thicker upper layer	
Regime 1	The generated waves are small, and conclusions of Regime 2 apply.	Generation of dull elevation-depression wave below the body. Moderate wave resistance. See Figure 3, page 15	Generation of a sharp elevation wave below the body. Strong wave resistance. See Figure 4, page 16	The generated waves are small, and conclusions of Regime 2 apply.
Regime 2	Continuous generation of small up-streaming and down-streaming waves. Very weak wave resistance. See Figure 6, page 26	Periodic generation of up-streaming depression waves with very large time-period. Oscillating wave resistance, with positive mean. See Figure 9, page 28	Periodic generation of up-streaming elevation waves with very large time-period. Oscillating wave resistance, with mean zero. See Figure 9, page 28	Generation of small down-streaming waves. Convergence towards a steady state. No lasting wave resistance. See Figure 7, page 27

Table 1: Behavior of the flow in the two studied regimes, depending on the velocity of the body and the depth ratio of the fluids

One peculiar phenomenon that can be seen in the experiments described in [53] is not recovered by our models. Indeed, the dead-water effects exhibits a somewhat periodic behavior, where during each period, a wave is generated, slows down the ship, and then breaks. This discrepancy between our simulations and the experiments is due to the fact that our models are based on the assumption of a constant velocity for the traveling body, as their experiments are conducted with a constant force brought to the body [52]. We perform a numeric experiment to match this setting (the velocity is then adjusted at each time step as a function of the drag suffered by the ship), and recover this periodic behavior. This point is discussed in more details in page 16.

Outline of the paper : In Section 1.1, we introduce the governing equations of our problem: the so-called full Euler system. A specific care is given to an adapted rescaling, that leads to the dimensionless version of the full Euler system. In Section 1.3, we present precisely the shape of the results displayed in the paper, as well as the regimes at stake. Strongly nonlinear models are introduced in Section 2, and justified with consistency results. Numerical simulations are then displayed in Section 2.2. The weakly nonlinear models, *i.e.* the Boussinesq-type system and the KdV approximation are presented in Sections 3.1 and 3.2, respectively. The convergence of their solutions towards the solutions of the full Euler system are given in Propositions 3.1 and 3.2, respectively. An in-depth analysis of the forced Korteweg-de Vries equation, and its consequences to the dead-water effect, is displayed in Section 3.3. Calculations that lead to the intermediate fully nonlinear model, that we use as a support for the other simplified models, are postponed until Appendix A, and Appendix B contains the proof of Proposition 3.2. Finally, Appendix C is devoted to the analysis of the wave resistance encountered by the ship, from its definition to simple approximations in the different regimes used throughout the paper, and the numerical schemes used in the simulations are presented and justified in Appendix D.

Notations : We denote by $C(\lambda_1, \lambda_2, \dots)$ any positive constant, depending on the parameters $\lambda_1, \lambda_2, \dots$, and whose dependence on λ_j is always assumed to be nondecreasing.

Let $f(x_1, \dots, x_d)$ be a function defined on \mathbb{R}^d . We denote by $\partial_{x_i} f$ the derivative with respect to the variable x_i . If f depends only on x_i , then we use the notation

$$\frac{d}{dx_i} f \equiv \partial_{x_i} f.$$

We denote by $L^2 = L^2(\mathbb{R})$ the Hilbert space of Lebesgue measurable functions $F : \mathbb{R} \rightarrow \mathbb{R}^n$ with the standard norm $\|F\|_2 = \left(\int_{\mathbb{R}} |F(x)|^2 dx \right)^{1/2} < \infty$. Its inner product is denoted by $(F_1, F_2) = \int_{\mathbb{R}} F_1 \cdot F_2$.

Then, we denote by $H^s = H^s(\mathbb{R})$ the L^2 -based Sobolev spaces. Their norm is written $\|\cdot\|_{H^s} = \|\Lambda^s \cdot\|_2$, with the pseudo-differential operator $\Lambda \equiv (1 - \partial_x^2)^{1/2}$. It is also convenient to introduce the following norm on the Sobolev space H^{s+1} (for $\varepsilon > 0$ a small parameter):

$$\|U\|_{H_\varepsilon^{s+1}}^2 = \|U\|_{H^s}^2 + \varepsilon \|U\|_{H^{s+1}}^2.$$

Let $0 < T \leq \infty$ and $f(t, x)$ be a function defined on $[0, T] \times \mathbb{R}$. Then one has $f \in L^\infty([0, T]; H^s)$ if f is bounded in H^s , uniformly with respect to $t \in [0, T]$. Moreover, $f \in W^{1,\infty}([0, T]; H^s)$ if $f, \partial_t f \in L^\infty([0, T]; H^s)$. Their respective norm is written $\|\cdot\|_{L^\infty H^s}$ and $\|\cdot\|_{W^{1,\infty} H^s}$.

1.1 The basic equations

We present here as briefly as possible the governing equations of our problem, and refer to [20] for more details. The system we study consists in two layers of immiscible, homogeneous, ideal,

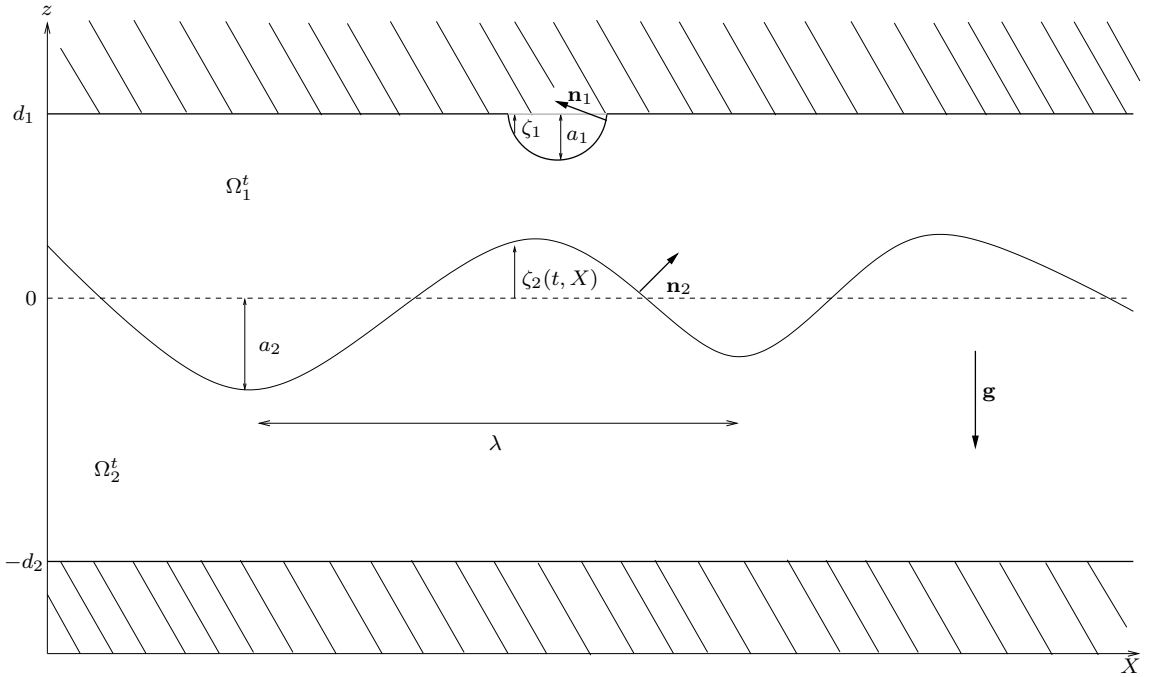


Figure 1: Sketch of the domain

incompressible fluids under the only influence of gravity (see Figure 1). Since we are interested in unidirectional models (the KdV approximation), we focus on the two-dimensional case, *i.e.* the horizontal dimension $d = 1$. However, most of this study could easily be extended to the case $d = 2$, using the techniques presented here.

We denote by ρ_1 and ρ_2 the density of, respectively, the upper and the lower fluid. Since we assume that each fluid is incompressible, ρ_1 and ρ_2 are constant and the velocity potentials ϕ_i ($i = 1, 2$), respectively associated to the upper and lower fluid layers, satisfy the Laplace equation

$$\partial_x^2 \phi_i + \partial_z^2 \phi_i = 0.$$

Moreover, it is presumed that the surface and the interface are given as the graph of functions (respectively, $\zeta_1(t, x)$ and $\zeta_2(t, x)$) which express the deviation from their rest position (respectively, (x, d_1) and $(x, 0)$) at the spatial coordinate x and at time t . The bottom is assumed to be flat, and the surface is flat away from the location of the ship moving at constant speed c_s , so that ζ_1 match the submerged part of the ship and is given by

$$\zeta_1(t, x) = \zeta_1(x - c_s t).$$

Therefore, at each time $t \geq 0$, the domains of the upper and lower fluid (denoted, respectively, Ω_1^t and Ω_2^t), are given by

$$\begin{aligned} \Omega_1^t &= \{ (x, z) \in \mathbb{R}^d \times \mathbb{R}, \quad \zeta_2(t, x) \leq z \leq d_1 + \zeta_1(x - c_s t) \}, \\ \Omega_2^t &= \{ (x, z) \in \mathbb{R}^d \times \mathbb{R}, \quad -d_2 \leq z \leq \zeta_2(t, x) \}. \end{aligned}$$

The fluids being ideal, they satisfy the Euler equation (or Bernoulli equation when written in terms of the velocity potentials), and kinematic boundary conditions are given through the assumption that no fluid particle crosses the surface, the bottom or the interface. Finally, the set of

equations is closed by the continuity of the stress tensor at the interface, which takes into account the effects of surface tension.

Altogether, the governing equations of our problem are given by the following

$$(1.1) \quad \begin{cases} \partial_x^2 \phi_i + \partial_z^2 \phi_i = 0 & \text{in } \Omega_i^t, \quad i = 1, 2, \\ \partial_t \phi_i + \frac{1}{2} |\nabla_{x,z} \phi_i|^2 = -\frac{P}{\rho_i} - gz & \text{in } \Omega_i^t, \quad i = 1, 2, \\ \partial_t \zeta_1 = \sqrt{1 + |\partial_x \zeta_1|^2} \partial_{n_1} \phi_1 & \text{on } \Gamma_1 \equiv \{(x, z), z = d_1 + \zeta_1(t, x)\}, \\ \partial_t \zeta_2 = \sqrt{1 + |\partial_x \zeta_2|^2} \partial_{n_2} \phi_1 = \sqrt{1 + |\partial_x \zeta_2|^2} \partial_{n_2} \phi_2 & \text{on } \Gamma_2 \equiv \{(x, z), z = \zeta_2(t, x)\}, \\ \partial_z \phi_2 = 0 & \text{on } \Gamma_b \equiv \{(x, z), z = -d_2\}, \\ \llbracket P \rrbracket = \sigma \partial_x \left(\frac{\partial_x \zeta_2}{\sqrt{1 + |\partial_x \zeta_2|^2}} \right) & \text{on } \Gamma_2, \end{cases}$$

where ∂_{n_i} is the upward normal derivative at Γ_i :

$$\partial_{n_i} \equiv n_i \cdot \nabla_{x,z}, \quad \text{with } n_i \equiv \frac{1}{\sqrt{1 + |\partial_x \zeta_i|^2}} (-\partial_x \zeta_i, 1)^T;$$

we denote $\sigma > 0$ the surface tension coefficient and $\llbracket P \rrbracket$ the jump of the pressure at the interface:

$$\llbracket P(t, x) \rrbracket \equiv \lim_{\varepsilon \rightarrow 0} \left(P(t, x, \zeta_2(t, x) + \varepsilon) - P(t, x, \zeta_2(t, x) - \varepsilon) \right).$$

This system can be reduced into evolution equations located at the surface and at the interface. Indeed, when we define the trace of the potentials at the surface and at the interface

$$\psi_1(t, x) \equiv \phi_1(t, x, d_1 + \zeta_1(t, x)), \quad \text{and} \quad \psi_2(t, x) \equiv \phi_2(t, x, \zeta_2(t, x)),$$

then ϕ_1 and ϕ_2 are uniquely given by the Laplace equation, and the (Dirichlet or Neumann) boundary conditions (see Figure 2):

$$(1.2) \quad \begin{cases} (\partial_x^2 + \partial_z^2) \phi_2 = 0 & \text{in } \Omega_2, \\ \phi_2 = \psi_2 & \text{on } \Gamma_2, \\ \partial_z \phi_2 = 0 & \text{on } \Gamma_b, \end{cases} \quad \text{and} \quad \begin{cases} (\partial_x^2 + \partial_z^2) \phi_1 = 0 & \text{in } \Omega_1, \\ \phi_1 = \psi_1 & \text{on } \Gamma_1, \\ \partial_{n_2} \phi_1 = \partial_{n_2} \phi_2 & \text{on } \Gamma_2. \end{cases}$$

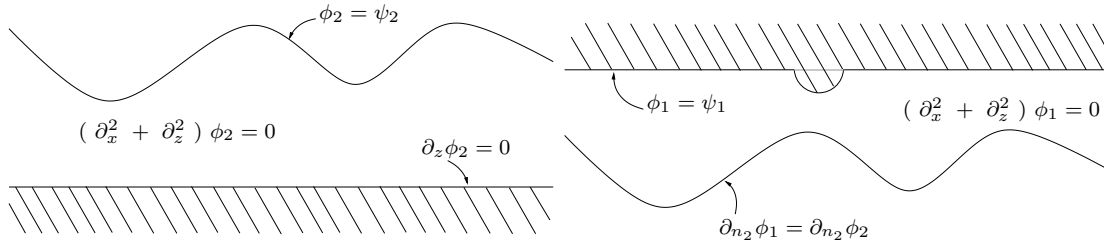


Figure 2: Laplace problems in the two domains.

Therefore, the following operators are well-defined:

$$\begin{aligned} G_1[\zeta_1, \zeta_2](\psi_1, \psi_2) &\equiv \sqrt{1 + |\partial_x \zeta_1|^2} (\partial_{n_1} \phi_1) \Big|_{z=d_1+\zeta_1}, \\ G_2[\zeta_2](\psi_2) &\equiv \sqrt{1 + |\partial_x \zeta_2|^2} (\partial_{n_2} \phi_2) \Big|_{z=\zeta_2}, \\ H[\zeta_1, \zeta_2](\psi_1, \psi_2) &\equiv \partial_x (\phi_1 \Big|_{z=\zeta_2}). \end{aligned}$$

Then, the chain rule and straightforward combinations of the equations of (1.1) lead to the following equivalent system:

$$(1.3) \quad \begin{cases} \partial_t \zeta_1 - G_1[\zeta_1, \zeta_2](\psi_1, \psi_2) = 0, \\ \partial_t \zeta_2 - G_2[\zeta_2]\psi_2 = 0, \\ \partial_t \left(\rho_2 \partial_x \psi_2 - \rho_1 H[\zeta_1, \zeta_2](\psi_1, \psi_2) \right) + g(\rho_2 - \rho_1) \partial_x \zeta_2 \\ \quad + \frac{1}{2} \partial_x \left(\rho_2 |\partial_x \psi_2|^2 - \rho_1 |H[\zeta_1, \zeta_2](\psi_1, \psi_2)|^2 \right) = \partial_x \mathcal{N} + \sigma \partial_x^2 \left(\frac{\partial_x \zeta_2}{\sqrt{1 + |\partial_x \zeta_2|^2}} \right), \end{cases}$$

with

$$\mathcal{N} = \frac{\rho_2 (G_2[\zeta_2]\psi_2 + (\partial_x \zeta_2)(\partial_x \psi_2))^2 - \rho_1 (G_2[\zeta_2]\psi_2 + (\partial_x \zeta_2)H[\zeta_1, \zeta_2](\psi_1, \psi_2))^2}{2(1 + |\partial_x \zeta_2|^2)}.$$

1.2 Nondimensionalization of the system

The next step consists in nondimensionalizing the system. The study of the linearized system (see [31], for example), which can be solved explicitly, leads to a well-adapted rescaling. Moreover, it is convenient to write the equations in the frame of reference of the ship.

Let a_1 and a_2 be the maximum amplitude of the deformation of, respectively, the surface and the interface. We denote by λ a characteristic horizontal length, say the wavelength of the interface. Then the typical velocity of small propagating internal waves (or wave celerity) is given by

$$c_0 = \sqrt{g \frac{(\rho_2 - \rho_1)d_1 d_2}{\rho_2 d_1 + \rho_1 d_2}}.$$

Consequently, we introduce the dimensionless variables

$$\tilde{z} \equiv \frac{z}{d_1}, \quad \tilde{x} \equiv \frac{x + c_s t}{\lambda}, \quad \tilde{t} \equiv \frac{c_0}{\lambda} t,$$

the dimensionless unknowns

$$\tilde{\zeta}_i(\tilde{x}) \equiv \frac{\zeta_i(x)}{a_i}, \quad \tilde{\psi}_i(\tilde{x}) \equiv \frac{d_1}{a_2 \lambda c_0} \psi_i(x),$$

and the seven independent dimensionless parameters

$$\gamma = \frac{\rho_1}{\rho_2}, \quad \epsilon_1 \equiv \frac{a_1}{d_1}, \quad \epsilon_2 \equiv \frac{a_2}{d_1}, \quad \mu \equiv \frac{d_1^2}{\lambda^2}, \quad \delta \equiv \frac{d_1}{d_2}, \quad \text{Fr} \equiv \frac{c_s}{c_0}, \quad \text{Bo} \equiv \frac{c_0^2 \lambda^2 \rho_2}{d_1 \sigma}.$$

With this rescaling, the full Euler system (1.1) becomes (we have withdrawn the tildes for the sake of readability)

$$(1.4) \quad \begin{cases} -\epsilon_1 \text{Fr} \partial_x \zeta_1 - \frac{\epsilon_2}{\mu} G_1(\psi_1, \psi_2) = 0, \\ (\partial_t - \text{Fr} \partial_x) \zeta_2 - \frac{1}{\mu} G_2 \psi_2 = 0, \\ (\partial_t - \text{Fr} \partial_x) \left(\partial_x \psi_2 - \gamma H(\psi_1, \psi_2) \right) + (\gamma + \delta) \partial_x \zeta_2 \\ \quad + \frac{\epsilon_2}{2} \partial_x \left(|\partial_x \psi_2|^2 - \gamma |H(\psi_1, \psi_2)|^2 \right) = \mu \epsilon_2 \partial_x \mathcal{N} + \frac{1}{\text{Bo}} \partial_x^2 \left(\frac{\partial_x \zeta_2}{\sqrt{1 + \mu \epsilon_2^2 |\partial_x \zeta_2|^2}} \right), \end{cases}$$

with

$$\mathcal{N} \equiv \frac{\left(\frac{1}{\mu}G_2\psi_2 + \epsilon_2(\partial_x\zeta_2)(\partial_x\psi_2)\right)^2 - \gamma\left(\frac{1}{\mu}G_2\psi_2 + \epsilon_2(\partial_x\zeta_2)H(\psi_1, \psi_2)\right)^2}{2(1 + \mu|\epsilon_2\partial_x\zeta_2|^2)},$$

and the dimensionless Dirichlet-to-Neumann operators defined by

$$\begin{aligned} G_1(\psi_1, \psi_2) &\equiv G_1^{\mu, \delta}[\epsilon_1\zeta_1, \epsilon_2\zeta_2](\psi_1, \psi_2) \equiv -\mu\epsilon_1\left(\frac{d}{dx}\zeta_1\right)(\partial_x\phi_1)\Big|_{z=1+\epsilon_1\zeta_1} + (\partial_z\phi_1)\Big|_{z=1+\epsilon_1\zeta_1}, \\ G_2\psi_2 &\equiv G_2^{\mu, \delta}[\epsilon_2\zeta_2]\psi_2 \equiv -\mu\epsilon_2\partial_x\zeta_2(\partial_x\phi_2)\Big|_{z=\epsilon_2\zeta_2} + (\partial_z\phi_2)\Big|_{z=\epsilon_2\zeta_2}, \\ H(\psi_1, \psi_2) &\equiv H^{\mu, \delta}[\epsilon_1\zeta_1, \epsilon_2\zeta_2](\psi_1, \psi_2) \equiv (\partial_x\phi_1)\Big|_{z=\epsilon_2\zeta_2} + \epsilon_2\partial_x\zeta_2(\partial_z\phi_1)\Big|_{z=\epsilon_2\zeta_2}, \end{aligned}$$

where ϕ_1 and ϕ_2 are the solutions of the Laplace problems

$$\begin{cases} \left(\mu\partial_x^2 + \partial_z^2\right)\phi_2 = 0 & \text{in } \Omega_2 \equiv \{(x, z) \in \mathbb{R}^2, -\frac{1}{\delta} < z < \epsilon_2\zeta_2(x)\}, \\ \phi_2 = \psi_2 & \text{on } \Gamma_2 \equiv \{z = \epsilon_2\zeta_2\}, \\ \partial_z\phi_2 = 0 & \text{on } \Gamma_b \equiv \{z = -\frac{1}{\delta}\}, \end{cases} \quad \begin{cases} \left(\mu\partial_x^2 + \partial_z^2\right)\phi_1 = 0 & \text{in } \Omega_1 \equiv \{(x, z) \in \mathbb{R}^2, \epsilon_2\zeta_2(x) < z < 1 + \epsilon_1\zeta_1(x)\}, \\ \phi_1 = \psi_1 & \text{on } \Gamma_1 \equiv \{z = 1 + \epsilon_1\zeta_1\}, \\ \partial_{n_2}\phi_1 = G_2\psi_2 & \text{on } \Gamma_2. \end{cases}$$

Remark 1.1. *Let us keep in mind that in our case, the function ζ_1 is not an unknown, but a fixed data of the problem:*

$$\zeta_1(t, x) = \zeta_1(x),$$

where ζ_1 is the submerged part of the ship (independent of time thanks to the change in the frame of reference). In that way, the first line of (1.4) is a relation connecting ψ_1 with ψ_2 , and (1.4) can be reduced into a system of two equations with two unknowns. This reduction is computed explicitly in the following asymptotic models.

Remark 1.2. *The Cauchy problem associated to the full Euler system at the interface of two fluids is known to be ill-posed in Sobolev spaces in the absence of surface tension, as Kelvin-Helmholtz instabilities appear. However, in [31], Lannes proved that adding a small amount of surface tension guarantees the well-posedness of the Cauchy problem (with a flat rigid lid), with a time of existence that may remain quite large even with a small surface tension coefficient, and thus is consistent with the observations. The idea behind this result is that the Kelvin-Helmholtz instabilities appear for high frequencies, where the regularization effect of the surface tension is relevant, even if Bo the Bond number measuring the ratio of gravity forces over capillary forces is very large.¹ On the contrary, the main profile of the wave that we want to capture is located at lower frequencies, and will therefore be unaffected by surface tension. Driven by this analysis, we decide to omit the surface tension term in the models of the regime 2 presented below, as they do not give rise to Kelvin-Helmholtz instabilities. On the contrary, we keep the surface tension term in the fully nonlinear models of regime 1. As a matter of fact the regularization effect of the surface tension in this case plays a critical role in our numerical simulations, as it stabilizes the scheme, even if Bo^{-1} is one order smaller than the coefficients in front of all other terms.*

The following terminology is used throughout the paper.

¹As an example, let us consider the values of the experiments of Vasseur, Mercier and Dauxois [53]. One has $d_1 = 0.05\text{m}$, $d_2 = 0.12\text{m}$, $\rho_1 = 1,000.5\text{kg}\cdot\text{m}^{-3}$, $\rho_2 = 1,022.7\text{kg}\cdot\text{m}^{-3}$. Taking advantage of the analysis of Lannes [31], we use $\sigma = 0.095\text{N}\cdot\text{m}^{-1}$ as a typical value for the interfacial tension coefficient. As a conclusion, one has $\text{Bo} = \frac{\rho_2 d_2^3}{\sigma} \frac{1}{\mu} \approx \frac{40}{\mu}$. The coefficient in front of the surface tension term is therefore much smaller than the coefficients in front of any other term.

Definition 1.3 (Adapted solutions). *We call adapted solution of (1.4) any strong solution $(\zeta_1, \zeta_2, \psi_1, \psi_2)$, bounded in $W^{1,\infty}([0, T]; H^{s+t_0})$ with $T > 0$, $s > 1$ and t_0 big enough, and such that $\zeta_1(t, x) = \zeta_1(x)$ (in the frame reference of the ship) and the domains of the fluids remain strictly connected, i.e. there exists $h_{\min} > 0$ such that*

$$(1.5) \quad 1 + \epsilon_1 \zeta_1(x) - \epsilon_2 \zeta_2(t, x) \geq h_{\min} > 0 \quad \text{and} \quad \frac{1}{\delta} + \epsilon_2 \zeta_2(t, X) \geq h_{\min} > 0.$$

From the discussion above, it is legitimate to assume that such a smooth, uniformly bounded family of solutions of (1.4) indeed exists.

1.3 Description of the results, and the regimes under study

The models displayed in the following sections are justified with a consistency, or a convergence result, in the following sense (the convergence theorems are precisely disclosed in Proposition 3.1 and 3.2).

Definition 1.4 (Consistency). *The full Euler system (1.4) is consistent with a system of two equations (Σ) on $[0, T]$ if any adapted solution U defines, via the changes of variables explained throughout the paper, a pair of functions satisfying (Σ) up to a small residual, called the precision of the model. The order of the precision will be $\mathcal{O}(\varepsilon)$, if there exists $s_0, t_0 \geq 0$ such that if $U \in W^{1,\infty}([0, T]; H^{s+t_0})$ with $s > s_0$, then the $L^\infty([0, T]; H^s)$ norm of the residual is bounded by $C_0 \varepsilon$, with C_0 a constant independent of ε .*

Definition 1.5 (Convergence). *The full Euler system (1.4) and a well-posed system (Σ) of two equations are convergent at order $\mathcal{O}(\varepsilon)$ on $[0, T]$ if any adapted solution with small initial data U defines, via the changes of variables explained throughout the paper, a pair of functions V such that \tilde{V} the solution of (Σ) with same initial data satisfies*

$$\|V - \tilde{V}\|_{L^\infty([0, T]; H^s)} \leq C_0 \varepsilon,$$

with C_0 independent of ε .

The small parameter ε in these definitions is a function of some of the dimensionless parameters of the system that are assumed to be small. The regimes that we study throughout the paper have been briefly presented in the introduction; let us describe them here in more details. When nothing is specifically said in the regimes below, we assume the parameters to be fixed as

$$\gamma \in (0, 1), \quad \epsilon_1, \epsilon_2 \in (0, 1), \quad 0 < \delta_{\min} \leq \delta \leq \delta_{\max}, \quad \text{Fr} \in [0, \text{Fr}_{\max}], \quad 0 < \text{Bo}_{\min} \leq \text{Bo}.$$

In particular, the two layers are assumed to be of finite, comparable depth.

The mutual and crucial assumption between the two regimes is that the depth of the two layers of fluids is small when compared with the internal wavelength. This is commonly called *shallow water regime* and is simply given by (with our notation)

$$\mu \ll 1.$$

The shallow water regime has been widely used in the framework of gravity waves. In the one layer, free surface case, it leads at order $\mathcal{O}(\mu)$ to the shallow water equations [18], and at order $\mathcal{O}(\mu^2)$ to the Green-Naghdi equations [23]. The analysis has been extended to the case of two layers with a free surface in [20], and the models presented here can be recovered from models in this situation, when fixing the surface as a data of the problem.

Regime 1 (Small sized boats).

$$\mu \ll 1 ; \quad \alpha \equiv \frac{\epsilon_1}{\epsilon_2} = \mathcal{O}(\mu) , \quad 1 - \gamma = \mathcal{O}(\mu).$$

The two additional smallness assumptions of this regime are very natural. First, we assume that the depth of the submerged part of the ship is small when compared with the depth of the fluid, and the attainable size of the deformation. The numerical simulations of our model show that even with this assumption, the waves generated by the small disturbance can be very large (of the order of the depth of the two layers, and therefore very large when compared with the variation of the surface induced by the ship). This explains why the dead-water phenomenon can be so powerful. What is more, we assume that the densities of the two fluids are almost equal. This is known as the Boussinesq approximation, and is valid most of the time (the value of $1 - \gamma$ reported in Celtic Sea in [46] is $\approx 10^{-3}$, and the value in the experiments of [53] is $\approx 10^{-2}$).

Regime 2 (Small wave amplitudes).

$$\mu \ll 1 ; \quad \epsilon_2 = \mathcal{O}(\mu) , \quad \alpha \equiv \frac{\epsilon_1}{\epsilon_2} = \mathcal{O}(\mu).$$

In this regime, we assume that the internal wave generated by the disturbance will remain small when compared with the depth of the two fluids. As previously, we assume that the waves generated by the ship are large when compared with the depth of the disturbance, so that the ship suffers from a relatively large wave resistance. In that way, the models we obtain involve only weak nonlinearities, and will be much easier to study. In particular, we are able to obtain well-posedness and convergence results for both of the models presented here. The counterpart is that these models remain valid only for relatively small waves. Again, this regime has been widely used in the literature. The one layer, free surface equivalent systems of the models presented here are the classical Boussinesq system [7, 8] and the KdV approximation [27]. These models have been extended to the case of two layers, with a free surface in [19].

2 Strongly nonlinear models

In this section, we introduce different strongly nonlinear asymptotic models for the full Euler system (1.4). The weakly nonlinear models (Regime 2) are presented in Section 3. The first assumption we use is the so-called shallow water assumption

$$\mu \ll 1,$$

which states that the depth of the two layers of fluids are small when compared with the characteristic wavelength of the system. The key ingredient in the construction of the asymptotic models in the shallow water regime, lies in the construction and the justification of the following expansion of the operators G_1 , G_2 and H :

Lemma 2.1. *Let $\zeta_1, \zeta_2, \psi_1, \psi_2$, such that $(\zeta_1, \zeta_2, \partial_x \psi_1, \partial_x \psi_2) \in W^{1,\infty}([0, T]; H^{s+t_0}(\mathbb{R}))$, with*

$s > 1$ and $t_0 \geq 9/2$, and such that (1.5) is satisfied. Then one has

$$\begin{aligned} & \left| G_1(\psi_1, \psi_2) + \mu \partial_x (h_1 \partial_x \psi_1 + h_2 \partial_x \psi_2) - \mu^2 \partial_x \left(\mathcal{T}[h_1, h_2] \partial_x \psi_1 + \mathcal{T}[h_2, 0] \partial_x \psi_2 \right. \right. \\ & \quad \left. \left. - \frac{1}{2} (h_1^2 \partial_x^2 (h_2 \partial_x \psi_2)) - (h_1 \epsilon_1 \frac{d}{dx} \zeta_1 \partial_x (h_2 \partial_x \psi_2)) \right) \right|_{L^\infty H^s} \leq \mu^3 C_0, \\ & \left| G_2 \psi_2 + \mu \partial_x (h_2 \partial_x \psi_2) - \mu^2 \partial_x \mathcal{T}[h_2, 0] \partial_x \psi_2 \right|_{L^\infty H^s} \leq \mu^3 C_0, \\ & \left| H(\psi_1, \psi_2) - \partial_x \psi_1 - \mu \partial_x \left(h_1 (\partial_x (h_1 \partial_x \psi_1) + \partial_x (h_2 \partial_x \psi_2)) \right. \right. \\ & \quad \left. \left. - \frac{1}{2} h_1^2 \partial_x^2 \psi_1 - h_1 \epsilon_1 (\partial_x \psi_1) \frac{d}{dx} \zeta_1(x) \right) \right|_{W^{1, \infty} H^s} \leq \mu^2 C_0, \end{aligned}$$

with $C_0 = C_0 \left(\frac{1}{h_{\min}}, |U|_{W^{1, \infty} H^{s+t_0}} \right)$, and the operator \mathcal{T} defined by

$$\mathcal{T}[h, b]V \equiv -\frac{1}{3} \partial_x (h^3 \partial_x V) + \frac{1}{2} (\partial_x (h^2 (\partial_x b) V)) - h^2 (\partial_x b) (\partial_x V) + h (\partial_x b)^2 V.$$

These estimates have been proved for $L^\infty([0, T]; H^s)$ norms in Propositions 2.2, 2.5 and 2.7 of [20]. The proof can easily be adapted to work with the time derivative of the functions, following in the proof of [20, Proposition 2.12].

The idea is then simply to plug these expansions into the full Euler system (1.4), and withdraw all $\mathcal{O}(\mu^2)$ terms. Then, using the fact that ζ_1 is a forced parameter of our problem, the system reduces to two evolution equations for (ζ_2, v) , with v the *shear velocity* defined by

$$(2.1) \quad v \equiv \partial_x \left((\phi_2 - \gamma \phi_1) \Big|_{z=\epsilon_2 \zeta_2} \right) = \partial_x \psi_2 - \gamma H(\psi_1, \psi_2).$$

These calculations are postponed to Appendix A for the sake of readability, and we directly present here the system thus obtained:

$$(2.2) \quad \begin{cases} (\partial_t - \text{Fr} \partial_x) \zeta_2 + \partial_x \left(\frac{h_2}{h_1 + \gamma h_2} (h_1 v + \gamma \alpha \text{Fr} \zeta) \right) + \mu \partial_x (\mathcal{L}(v_1, v_2)) = 0, \\ (\partial_t - \text{Fr} \partial_x) v + (\gamma + \delta) \partial_x \zeta_2 + \frac{\epsilon_2}{2} \partial_x \left(\frac{|h_1 v + \gamma \alpha \text{Fr} \zeta|^2 - \gamma |h_2 v - \alpha \text{Fr} \zeta|^2}{(h_1 + \gamma h_2)^2} \right) \\ \quad + \mu \epsilon_2 \partial_x (\mathcal{Q}[v_1, v_2]) = \frac{1}{\text{Bo}} \partial_x^2 \left(\frac{\partial_x \zeta_2}{\sqrt{1 + \mu \epsilon_2^2 |\partial_x \zeta_2|^2}} \right), \end{cases}$$

where \mathcal{L} and \mathcal{Q} are respectively linear and quadratic in (v_1, v_2) , the latter being the approximation at order $\mathcal{O}(\mu)$ of $(\partial_x \psi_1, \partial_x \psi_2)$ given by

$$(2.3) \quad v_1 \equiv -\frac{h_2 v - \alpha \text{Fr} \zeta_1}{h_1 + \gamma h_2}, \quad \text{and} \quad v_2 \equiv \frac{h_1 v + \gamma \alpha \text{Fr} \zeta_1}{h_1 + \gamma h_2}.$$

Precisely, the operators \mathcal{L} and \mathcal{Q} are defined by

$$\begin{aligned} (h_1 + \gamma h_2) \mathcal{L}(v_1, v_2) &= \gamma h_2 \mathcal{T}[h_1, h_2] v_1 - h_1 \mathcal{T}[h_2, 0] v_2 \\ &+ \gamma h_1 h_2 \left(\partial_x \left(\frac{h_1^2}{2} \partial_x v_1 - h_1 (\partial_x h_2) v_1 \right) + \frac{h_1}{2} \partial_x^2 (h_2 v_2) - (\partial_x h_2) \partial_x (h_2 v_2) \right), \\ (h_1 + \gamma h_2) \mathcal{Q}[v_1, v_2] &= \gamma (v_2 - v_1) (\mathcal{T}[h_1, h_2] v_1 + \mathcal{T}[h_2, 0] v_2) \\ &+ \gamma (v_2 - v_1) h_1 \left(\partial_x \left(\frac{h_1^2}{2} \partial_x v_1 - h_1 (\partial_x h_2) v_1 \right) + \frac{h_1^2}{2} \partial_x^2 (h_2 v_2) - h_1 (\partial_x h_2) \partial_x (h_2 v_2) \right) \\ &- \frac{1}{2} (h_1 + \gamma h_2) ((h_2 \partial_x v_2)^2 - \gamma ((\partial_x h_2) v_1 - \partial_x (h_2 v_2))^2). \end{aligned}$$

The full Euler system (1.4) is consistent with this system at order $\mathcal{O}(\mu^2)$ on $[0, T]$, $T > 0$, as stated in Proposition A.1, page 31.

The next step consists in deducing from system (2.2) simpler models, using the additional assumptions of Regime 1 and Regime 2 (see page 10). The full Euler system (1.4) is still consistent at order $\mathcal{O}(\mu^2)$ with the models thus obtained, that is (see below) the strongly nonlinear systems (2.4) and (2.6), the Boussinesq-type system (3.2) and its symmetrized version (3.3). The KdV approximation (3.12) is then deduced from the symmetric Boussinesq-type system.

2.1 The fully nonlinear model in Regime 1

The system (2.2) only presumes some smallness on the shallowness parameter μ , that is to say the depth of the fluids are small when compared with the characteristic wavelength of the system. In this section, we use the additional smallness assumptions of Regime 1 in order to introduce a simplified system, which remains consistent at order $\mathcal{O}(\mu^2)$. Therefore, we assume that

$$\mu \ll 1; \quad \alpha \equiv \frac{\epsilon_1}{\epsilon_2} = \mathcal{O}(\mu), \quad 1 - \gamma = \mathcal{O}(\mu),$$

where we recall that γ is the density ratio and α the ratio between the amplitudes of the deformations at the surface and at the interface. As it has been said in the introduction, these assumptions are very natural in the framework of our study. The first one suppose that the deformation induced by the presence of the ship at the surface is small when compared with the depth of the two layers of fluid, and small when compared with the attainable size of the wave generated at the interface. The second one is the classical Boussinesq approximation.

The first obvious observation is that the total depth of the fluid is then approximatively constant, equal to $h \equiv 1 + \frac{1}{\delta}$ at order $\mathcal{O}(\mu)$. What is more, one has

$$h_1 + \gamma h_2 = h + \mathcal{O}(\mu) \equiv 1 + \frac{1}{\delta} + \mathcal{O}(\mu).$$

Therefore, the approximations of $(\partial_x \psi_1, \partial_x \psi_2)$ at order $\mathcal{O}(\mu)$ given in (2.3) are now simply

$$h_2 v = -h v_1 + \mathcal{O}(\mu), \quad h_1 v = h v_2 + \mathcal{O}(\mu).$$

It follows then some substantial simplifications, and one obtains in the end the system

$$(2.4) \quad \begin{cases} (\partial_t - \text{Fr} \partial_x) \zeta_2 + \partial_x \left(\frac{h_1 h_2}{h_1 + \gamma h_2} v + h_2 \frac{\alpha \text{Fr}}{h} \zeta_1 \right) + \mu \partial_x (\mathcal{P}_1 v) = 0, \\ (\partial_t - \text{Fr} \partial_x) v + (\gamma + \delta) \partial_x \zeta_2 + \epsilon_2 \partial_x \left(\frac{1}{2} \frac{|h_1 v|^2 - \gamma |h_2 v|^2}{(h_1 + \gamma h_2)^2} + \frac{\alpha \text{Fr}}{h} \zeta_1 v \right) \\ + \mu \epsilon_2 \partial_x (\mathcal{P}_2[v]) = \frac{1}{\text{Bo}} \partial_x^2 \left(\frac{\partial_x \zeta_2}{\sqrt{1 + \mu \epsilon_2^2 |\partial_x \zeta_2|^2}} \right), \end{cases}$$

where $h \equiv 1 + \frac{1}{\delta}$ and the operators \mathcal{P}_1 and \mathcal{P}_2 are defined by

$$\begin{aligned} h^2 \mathcal{P}_1 v &= \frac{1}{3} (h_2 \partial_x (h_1^3 \partial_x (h_2 v)) + h_1 \partial_x (h_2^3 \partial_x (h_1 v))), \\ h^2 \mathcal{P}_2[v] &= \frac{v}{3} \partial_x [h_1^3 \partial_x (h_2 v) - h_2^3 \partial_x (h_1 v)] + \frac{1}{2} ((h_1 \partial_x (h_2 v))^2 - (h_2 \partial_x (h_1 v))^2). \end{aligned}$$

Linearizing the system (2.4) around the rest state, leads to

$$\begin{cases} (\partial_t - \text{Fr} \partial_x) \zeta_2 + \frac{1}{\gamma + \delta} \partial_x v + \mu \frac{1}{3\delta(1 + \delta)} \partial_x^3 v = \frac{\alpha \text{Fr}}{1 + \delta} \partial_x \zeta_1, \\ (\partial_t - \text{Fr} \partial_x) v + (\gamma + \delta) \partial_x \zeta_2 = \frac{1}{\text{Bo}} \partial_x^3 \zeta_2, \end{cases}$$

from which we can easily deduce dispersion relation of the system, without forcing. Indeed, setting $\alpha \equiv 0$, the wave frequency $\omega(k)$, corresponding to plane-wave solutions $e^{ik \cdot X - i\omega t}$, is solution of the quadratic equation

$$(2.5) \quad (\text{Fr} k + \omega)^2 = k^2 \left(1 + \frac{k^2}{\text{Bo}(\gamma + \delta)} \right) \left(1 - \mu k^2 \frac{\delta + \gamma}{3\delta(1 + \delta)} \right).$$

Therefore, for high frequencies, there is no real-valued solution of (2.5), which means that the system (2.4) is linearly ill-posed, and leads to instabilities. In order to deal with this issue, we use a nonlinear change of variables on the shear velocity v , that leads to a formally equivalent system. This idea is not new (see [5, 44, 15, 20] for example), and usually relies on a “shear velocity” constructed from horizontal velocities *at any depth* in the upper and lower layers. The choice of the depth allows then to control properties of the linear relation dispersion. Here, we define w with

$$\frac{h_1 h_2}{h_1 + \gamma h_2} w \equiv \frac{h_1 h_2}{h_1 + \gamma h_2} v + \mu \mathcal{P}_1 v \implies v = w - \mu \frac{h \mathcal{P}_1 w}{(h - h_2) h_2} + \mathcal{O}(\mu^2).$$

This leads immediately to the following system, equivalent at order $\mathcal{O}(\mu^2)$:

$$(2.6) \quad \begin{cases} (\partial_t - \text{Fr} \partial_x) \zeta_2 + \partial_x \left(\frac{h_1 h_2}{h_1 + \gamma h_2} w + h_2 \frac{\alpha \text{Fr}}{h} \zeta_1 \right) = 0, \\ (\partial_t - \text{Fr} \partial_x) (w - \mu \mathcal{S}_1 w) + (\gamma + \delta) \partial_x \zeta_2 + \epsilon_2 \partial_x \left(\frac{1}{2} \frac{h_1^2 - \gamma h_2^2}{(h_1 + \gamma h_2)^2} w^2 + w \frac{\alpha \text{Fr}}{h} \zeta_1 \right) \\ \quad + \mu \epsilon_2 \partial_x (w \mathcal{S}_2 w) = \frac{1}{\text{Bo}} \partial_x^2 \left(\frac{\partial_x \zeta_2}{\sqrt{1 + \mu \epsilon_2^2 |\partial_x \zeta_2|^2}} \right), \end{cases}$$

with $h \equiv 1 + \frac{1}{\delta}$, $h_1 \equiv 1 + \epsilon_1 \zeta_1 - \epsilon_2 \zeta_2$, $h_2 \equiv \frac{1}{\delta} + \epsilon_2 \zeta_2$, and the operators

$$\begin{aligned} \mathcal{S}_1 w &\equiv \frac{1}{3} (\partial_x^2 (h_1 h_2 w) - (\partial_x h_2)^2 w), \\ \mathcal{S}_2 w &\equiv \frac{h_1 h_2}{3h} ((\partial_x^2 h_2) w + 2(\partial_x w)(\partial_x h_2)) + \frac{h_1 - h_2}{2h} (\partial_x h_2)^2 w. \end{aligned}$$

Under this form, our system corresponds to the Green-Naghdi model presented in the one-layer case in [1, 2], and proved to be well-posed and convergent, in the sense that their solutions provide an approximation at order $\mathcal{O}(\mu^2)$ in $L^\infty([0, T]; H^s)$ to the solutions of the one layer water wave equations. When withdrawing all $\mathcal{O}(\mu)$ terms, one obtains a shallow water model, similar to the ones derived in [16] and [17] in the flat rigid lid case (the forcing terms induced by the presence of the body do not appear). Such a system has been studied in details and analyzed as a hyperbolic system (leading to well-posedness results under reasonable assumptions on the initial data) in [39, 24, 9].

One can check that the issue of linear ill-posedness is now solved. Indeed, linearizing the system (2.6) around the rest state, leads to

$$\begin{cases} (\partial_t - \text{Fr} \partial_x) \zeta_2 + \frac{1}{\gamma + \delta} \partial_x w = \frac{\alpha \text{Fr}}{1 + \delta} \partial_x \zeta_1, \\ (\partial_t - \text{Fr} \partial_x) \left(1 - \frac{\mu}{3\delta} \partial_x^2\right) w + (\gamma + \delta) \partial_x \zeta_2 = \frac{1}{\text{Bo}} \partial_x^3 \zeta_2. \end{cases}$$

The dispersion relation of this linear system, when setting $\alpha \equiv 0$, is

$$(2.7) \quad (\text{Fr} k + \omega)^2 \left(1 + \frac{\mu}{3\delta} k^2\right) = k^2 \left(1 + \frac{1}{(\gamma + \delta) \text{Bo}} k^2\right),$$

which leads to real-valued waves frequencies ω , for any values of $k \in \mathbb{R}$.

Let us remark that under the assumptions of regime 1, the relations (2.5) and (2.7) are asymptotically equivalent at order $\mathcal{O}(\mu^2 k^4)$, so that the effect of the nonlinear change of variable only affects high frequencies.

We now state that the two systems (2.4) and (2.6) are equivalently justified as models for the full Euler system, with the following consistency result.

Proposition 2.2. *Assuming that $\alpha = \mathcal{O}(\mu)$ and $1 - \gamma = \mathcal{O}(\mu)$, the full Euler system (1.4) is consistent with the models (2.4) and (2.6), both at precision $\mathcal{O}(\mu^2)$ on $[0, T]$, with $T > 0$.*

Proof. Let $U \equiv (\zeta_1, \zeta_2, \psi_1, \psi_2)$ be a strong solution of (1.4), bounded in $W^{1,\infty}([0, T]; H^{s+t_0})$ with $s > 1$ and $t_0 \geq 9/2$, and such that (1.5) is satisfied with $\zeta_1(t, x) \equiv \zeta_1(x)$. The consistency result of Proposition A.1 states that (ζ_2, v) , with $v \equiv \partial_x \psi_2 - \gamma H(\psi_1, \psi_2)$, satisfies (2.2), up to $R_1 = (r_1, r_2)^T \in L^\infty([0, T]; H^s)^2$, with (for $i = 1, 2$)

$$|r_i|_{L^\infty H^s} \leq \mu^2 C_0 \left(\frac{1}{h_{\min}}, |U|_{W^{1,\infty} H^{s+t_0}} \right).$$

Since \mathcal{R}_1 , \mathcal{R}_2 , \mathcal{T} and $\partial_x \mathcal{H}$ (defined in Appendix A) involve two spatial derivatives, and thanks to straightforward calculations using the smallness assumptions of regime 1, one has for $i = 1, 2$

$$|\mathcal{L}_i - \mathcal{P}_i|_{L^\infty([0,t]; H^s)} \leq \mu^2 C_0 \left(\frac{1}{h_{\min}}, |(h_1, h_2, v)|_{L^\infty H^{s+2}} \right) \leq \mu^2 C_0 \left(\frac{1}{h_{\min}}, |U|_{W^{1,\infty} H^{s+t_0}} \right).$$

The consistency of (1.4) with (2.4) is therefore proved.

Now, we set $w \equiv v + \mu \frac{h_1 + \gamma h_2}{h_1 h_2} \mathcal{P}_1[h_2]v$, and one has immediately, using the fact that $H^s(\mathbb{R})$ is an algebra for $s > 1/2$,

$$\begin{aligned} \left| v - w + \mu \frac{h \mathcal{P}_1[h_2]w}{(h - h_2)h_2} \right|_{W^{1,\infty} H^s} &\leq \mu^2 C_0 \left(\frac{1}{h_{\min}}, |(h_1, h_2, v)|_{W^{1,\infty} H^{s+2}} \right) \\ &\leq \mu^2 C_0 \left(\frac{1}{h_{\min}}, |U|_{W^{1,\infty} H^{s+t_0}} \right). \end{aligned}$$

It is then straightforward to deduce from the previous consistency result, the consistency of (1.4) with (2.6). \square

2.2 Numerical simulations

In figures 3 and 4, we plot the behavior of the flow predicted by the model (2.6), for different values of the parameters. In these simulations, and throughout the paper, we use zero initial data conditions, and a function $\zeta_1(x)$ defined by

$$\zeta_1(x) \equiv \begin{cases} -\exp\left(-\frac{x^2}{(1-x)(x+1)}\right) & \text{if } x \in (-1, 1), \\ 0 & \text{otherwise.} \end{cases}$$

The parameters of the system ϵ_2 , μ , $\alpha \equiv \epsilon_1/\epsilon_2$, δ , γ , Fr and Bo, are specified below each of the figures. The schemes used are described and justified in Appendix D.

Here, we decide to study the effect of the depth ratio coefficient δ . We choose two different values for the depth ratio: $\delta = 5/12$ (corresponding to the experiments of [53]), or $\delta = 12/5$ for a thicker upper layer. Each time, we set Fr = 1, since the amplitude of the generated waves, and therefore the magnitude of the wave resistance coefficient, is significantly smaller away from this critical value. In that way, for subcritical and supercritical values, the weakly nonlinear models are justified, and the analysis of these cases is postponed until Section 3.3.

Each figure contains three panels. The left panel represents the interface deformation, depending on space ($x \in [-20, 20]$) and time ($t \in [0, 15]$) variables. The right panel contains the time evolution of the wave resistance coefficient, computed thanks to formula (C.4). Finally, we plot in the bottom panel the situation of the system (*i.e.* the surface and interface deformations) at final time $t = 15$.

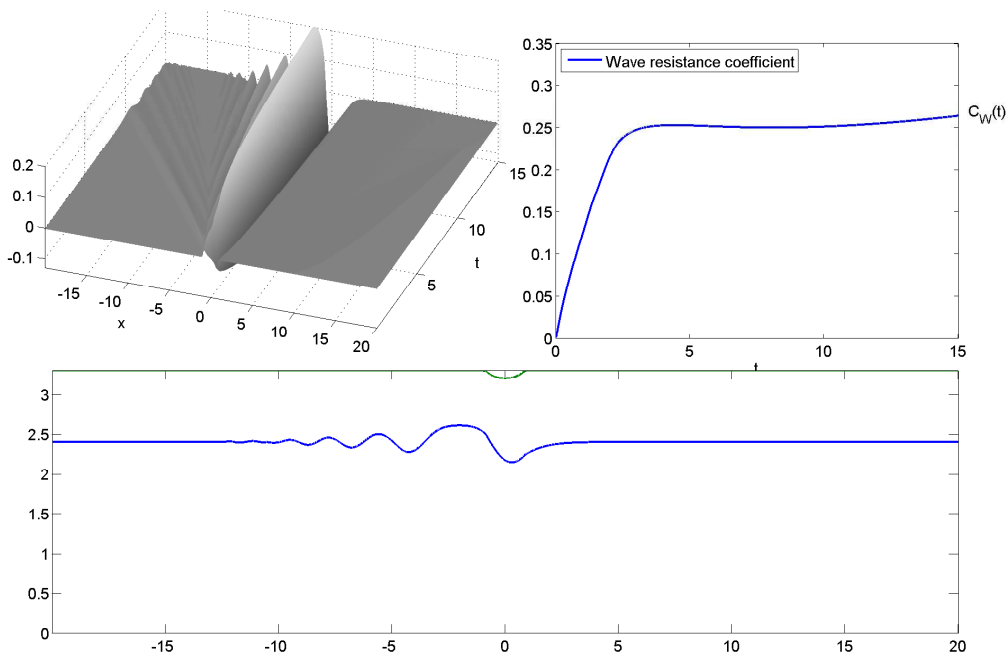


Figure 3: Flow predicted by (2.6), with steady initial data and a thicker lower layer.
 $\epsilon_2 = 1$, $\alpha = \mu = 0.1$, $\gamma = 0.99$, Fr = 1, Bo = 100, $\delta = 5/12$.

Let us express a few remarks on these results. There are some similarities for each of the situations. First, one sees that there is no deformation on the right hand side of the plots,

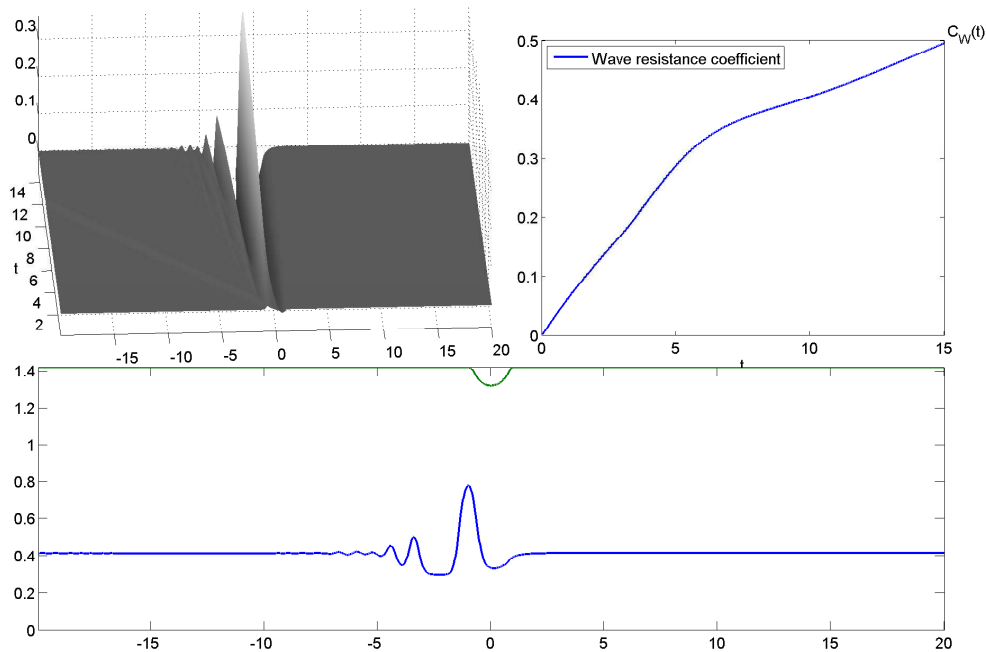


Figure 4: Flow predicted by (2.6), with steady initial data and a thicker upper layer.
 $\epsilon_2 = 1$, $\alpha = \mu = 0.1$, $\gamma = 0.99$, $\text{Fr} = 1$, $\text{Bo} = 100$, $\delta = 12/5$.

which corresponds to the up-streaming part. This is due to the fact that we have set $\text{Fr} = 1$, which corresponds to the maximum gravitational wave velocity in the flat rigid lid case. On the contrary, on the left-hand side (down-streaming part), one remarks a small elevation wave, followed by even smaller perturbations, progressing with velocity $c_- = -2$ (in the frame of the ship). This corresponds to the η_- part of the KdV approximation decomposition, and is studied with more details in Section 3.3. Finally, the most important part takes place just behind the location of the body. Each time, an important wave of elevation is generated just behind the body, producing a severe wave resistance (see Remark C.1 page 38). This wave comes with a tail of smaller waves, that are located away from the boat, and therefore do not produce any drag.

However, one can see that the shape and time-behavior of this elevation wave is quite different, depending on the value of the depth ratio δ . When the upper fluid domain is thinner than the lower fluid's, the generated wave is flattening as it is growing up. Its height is relatively low, but the deformation carries on with a depression wave, located just below the body. On the contrary, when the upper fluid domain is the thicker one, then the generated wave remains sharp and is continuously growing. It is separated with its tail, while all of the energy produced by the ship contributes to the elevation of the wave. The wave produced wave resistance is greater in this case, and the dead-water effect appears therefore much stronger.

The experiments conducted in [53] (this has also been reported in [37] for example) exhibit an interesting behavior, that is not predicted by our models. Indeed, with their setting, the dead-water phenomenon appears to be periodic in some sense, as the generated wave increases its amplitude, slows down the boat, breaks, and the process repeats. This cannot be seen in any of our simulations in figures 3 and 4. As a matter of fact, even when running simulations for much longer (around ten times) time intervals, the solution of (2.6) does not display such a phenomenon. It is interesting to see that this is a major difference with the solutions of the KdV

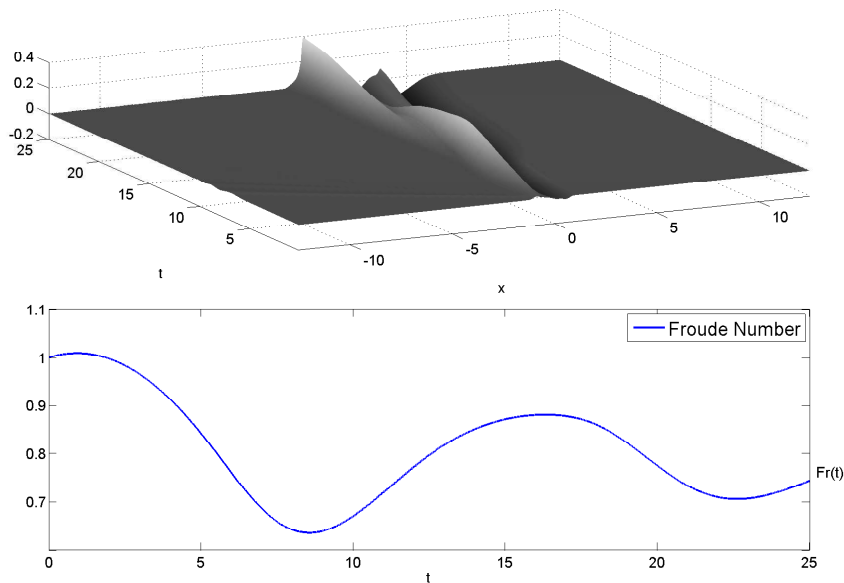


Figure 5: Flow predicted by (2.6), when the velocity of the body Fr is not constant.
 $\epsilon_2 = 1$, $\delta = 1$, $\alpha = \mu = 0.1$, $\gamma = 0.99$, $Bo = 100$.

approximation, which generate periodically up-streaming waves, inducing an oscillation in the related wave resistance (see Figure 9 page 28). However, even with this model, the time-period of generation of these solitons ($\Delta T \approx 100$ in our situation) is too large to explain the phenomenon. This discrepancy is due to our assumption of a constant velocity for the body, as the setting in [53] (presented in more details in [52]) constrains a constant power. As a numeric experiment, we performed simulations with a Froude number Fr adjusted at each step:

$$Fr((n+1)\Delta t) \equiv Fr(n\Delta t) - Cstt_1(C_W(n\Delta t) - Cstt_2).$$

This roughly corresponds to a case where the acceleration of the ship is given by a constant force brought to the ship, minus the resistance suffered by the body. We present in Figure 5 the results of this computation.

A periodicity can clearly be seen in the wave resistance (with a time period of order $\Delta T \approx 10$). One can explain the phenomenon as follows. The velocity of the ship, suffering from the drag generated by the wave resistance, decreases down to a value where the generated wave resistance is very small. Released from its drag, the ship speeds up, and the phenomenon repeats periodically.

3 Weakly nonlinear models

The aim of this section is to introduce simple weakly nonlinear models for our problem, using the smallness assumptions of Regime 2:

$$\mu \ll 1; \quad \epsilon_2 = \mathcal{O}(\mu), \quad \alpha = \mathcal{O}(\mu).$$

We recall that μ stands for the shallowness parameter, that is the ratio between the depth of the upper fluid layer and the internal wavelength. The parameter ϵ_2 measures the magnitude of the deformation at the interface when compared with the upper fluid layer, and α is the ratio

between the amplitudes of the deformations at the surface and at the interface. As discussed in Remark 1.2, the effect of the surface tension term will be relevant only if Kelvin-Helmholtz instabilities, located at high frequencies, appear. The models we obtain in this section do not give rise to these instabilities, and we therefore neglect the surface tension term, and set $\text{Bo}^{-1} \equiv 0$ ($\text{Bo}^{-1} = \mathcal{O}(\mu^2)$ would in fact suffice).

Starting with the strongly nonlinear model (2.2), we deduce easily a Boussinesq-type model and its symmetrized version. Then, using a classical BKW expansion, we obtain a rougher approximation, that consists in two *uncoupled Korteweg-de Vries equation, with a forcing term*. These two equations are studied in details in Section 3.3. The symmetric Boussinesq-type model, as well as the KdV approximation, are justified thanks to convergence results.

3.1 The Boussinesq-type models

Let us withdraw $\mathcal{O}(\mu^2)$ terms in system (2.2), using the assumptions of Regime 2. One obtains straightforwardly the following system

$$(3.1) \quad \begin{cases} (\partial_t - \text{Fr} \partial_x) \zeta_2 + \frac{1}{\delta + \gamma} \partial_x v + \epsilon_2 \frac{\delta^2 - \gamma}{(\gamma + \delta)^2} \partial_x (\zeta_2 v) + \mu \frac{1 + \gamma \delta}{3\delta(\delta + \gamma)^2} \partial_x^3 v = -\alpha \frac{\text{Fr} \gamma}{\delta + \gamma} \frac{d}{dx} \zeta_1, \\ (\partial_t - \text{Fr} \partial_x) v + (\gamma + \delta) \partial_x \zeta_2 + \frac{\epsilon_2}{2} \frac{\delta^2 - \gamma}{(\gamma + \delta)^2} \partial_x (|v|^2) = 0. \end{cases}$$

This Boussinesq-type system can be written in a compact form as

$$(3.2) \quad \partial_t U + A_0 \partial_x U + \epsilon_2 A_1(U) \partial_x U - \mu A_2 \partial_x^3 U = \alpha b_0(x),$$

with $U = (\zeta_2, v)^T$, $b_0 = -\text{Fr} \frac{\gamma}{\delta + \gamma} (\frac{d}{dx} \zeta_1, 0)^T$, and

$$A_0 = \begin{pmatrix} -\text{Fr} & \frac{1}{\delta + \gamma} \\ \delta + \gamma & -\text{Fr} \end{pmatrix}, \quad A_1(U) = \frac{\delta^2 - \gamma}{(\gamma + \delta)^2} \begin{pmatrix} v & \zeta_2 \\ 0 & v \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & -\frac{1 + \gamma \delta}{3\delta(\delta + \gamma)^2} \\ 0 & 0 \end{pmatrix}.$$

Following the classical theory of hyperbolic quasilinear equations, our aim is now to obtain an appropriate symmetrizer of this system. Let us define

$$\begin{aligned} S(U) &\equiv S_0 + \epsilon_2 S_1(U) - \mu S_2 \partial_x^2 \\ &= \begin{pmatrix} \delta + \gamma & 0 \\ 0 & \frac{1}{\delta + \gamma} \end{pmatrix} + \epsilon_2 \frac{\delta^2 - \gamma}{(\gamma + \delta)^2} \begin{pmatrix} 0 & -v \\ -v & \zeta_2 \end{pmatrix} - \mu \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1 + \gamma \delta}{3\delta(\delta + \gamma)^2} \end{pmatrix} \partial_x^2. \end{aligned}$$

Multiplying (3.2) on the left by $S(U) - \mu K S_0 \partial_x^2$, and withdrawing the $\mathcal{O}(\mu^2)$ terms, leads to the following equivalent system

$$(3.3) \quad (S_0 + \epsilon_2 S_1(U) - \mu(S_2 + K S_0) \partial_x^2) \partial_t U + (\Sigma_0 + \epsilon_2 \Sigma_1(U) - \mu \Sigma_2) \partial_x U = \alpha b(x),$$

with $b = -\text{Fr} \gamma (\frac{d}{dx} \zeta_1, 0)^T$, and

$$\begin{aligned} \Sigma_0 &= \begin{pmatrix} -\text{Fr}(\gamma + \delta) & 1 \\ 1 & \frac{-\text{Fr}}{\gamma + \delta} \end{pmatrix}, \quad \Sigma_1(U) = \frac{\delta^2 - \gamma}{\gamma + \delta} \begin{pmatrix} 0 & \zeta_2 + \frac{\text{Fr} v}{\gamma + \delta} \\ \zeta_2 + \frac{\text{Fr} v}{\gamma + \delta} & \frac{-\text{Fr} \zeta_2}{\gamma + \delta} \end{pmatrix}, \\ \Sigma_2 &= -\frac{1 + \gamma \delta}{3\delta(\delta + \gamma)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + K \Sigma_0. \end{aligned}$$

As we can see, the system (3.3) is perfectly symmetric, and K can be chosen so that S_0 and $S_2 + KS_0$ are definite positive. With these properties, we are able to use energy methods in order to prove that (3.3) is well-posed, and convergent with the full Euler system (1.4), at order $\mathcal{O}(\mu)$ up to times of order $\mathcal{O}(1/\mu)$. More precisely, one has

Proposition 3.1. *Let $s > 3/2$, $t_0 \geq 9/2$ and $U = (\zeta_1, \zeta_2, \psi_1, \psi_2)$ be an adapted solution of the full Euler system (1.4), bounded in $W^{1,\infty}([0, T/\mu]; H^{s+t_0})$. We define $V \equiv (\zeta_2, v)$ by*

$$v \equiv \partial_x (\phi_2|_{z=\epsilon_2\zeta_2} - \gamma\phi_1|_{z=\epsilon_2\zeta_2}) \equiv \partial_x \psi_2 - \gamma H(\psi_1, \psi_2).$$

Moreover, let us assume that there exists a constant C_0 such that $\epsilon_2 \leq C_0\mu$ and $\alpha \leq C_0\mu$. Then there exists a constant $C_1 = C(\frac{1}{\gamma+\delta}, \gamma + \delta, C_0) > 0$ such that if $\epsilon_2|V|_{t=0}|_{H_\mu^{s+1}} \leq \frac{1}{C_1}$, there exists $T > 0$, independent of μ , and a unique solution $V_B \in C^0([0, T/\mu]; H_\mu^{s+1}) \cap C^1([0, T/\mu]; H_\mu^s)$, bounded in $W^{1,\infty}([0, T/\mu]; H_\mu^s)$, of the Cauchy problem (3.3) with $V_B|_{t=0} = V|_{t=0}$. Moreover, one has for all $t \in [0, T/\mu]$,

$$|V - V_B|_{L^\infty([0,t]; H^s)} \leq \mu^2 t C,$$

with $C = C(\frac{1}{h_{\min}}, \frac{1}{\gamma+\delta}, \gamma + \delta, |V|_{W^{1,\infty} H^{s+t_0}}, T)$.

Before starting with the proof, let us remark that the proposition is not empty only if there actually exists a family of solutions of (1.4), smooth and bounded in $W^{1,\infty}([0, T/\mu]; H^{s+t_0})$. As discussed in Remark 1.2 page 8, this requires adding a surface tension term. However, this surface tension term is very small in practical cases, so that we can assume $\frac{1}{\text{Bo}} = \mathcal{O}(\mu^2)$, and the result is obtained as in the proof presented below.

Proof. Step 1: Well-posedness. In order to prove the well-posedness of the symmetric system (3.3), we use techniques of [19]. It is proved in Proposition 2.4 that systems of the form (3.3) (with four equations instead of two and without the right hand size), satisfying

- i. The matrices $S_0, \Sigma_0, S_2, \Sigma_2$ are symmetric,
- ii. $S_1(\cdot)$ and $\Sigma_1(\cdot)$ are linear mappings, and for all $U \in \mathbb{R}^4$, $S_1(U)$ and $\Sigma_1(U)$ are symmetric,
- iii. S_0 and S_2 are definite positive,

are well-posed and satisfy an energy estimate. The proof is easily adapted for the case of a non-zero right hand side, and we briefly give the arguments here. The key point of the proof relies on a differential inequality, satisfied by the following energy

$$E_s(U) \equiv 1/2(S_0\Lambda^s U, \Lambda^s U) + \epsilon_2/2(S_1(U)\Lambda^s U, \Lambda^s U) + \mu/2(S_2\Lambda^s \partial_x U, \Lambda^s \partial_x U).$$

This energy is easily proved, thanks to the positiveness of S_0 and S_2 and using the smallness assumptions of Regime 2, to be equivalent to the $|\cdot|_{H_\mu^{s+1}}$ norm, that is to say there exists $C_0 > 0$ such that

$$\frac{1}{C_0} \left(|U|_{H^s}^2 + \mu |U|_{H^{s+1}}^2 \right) \leq E_s(U) \leq C_0 \left(|U|_{H^s}^2 + \mu |U|_{H^{s+1}}^2 \right).$$

Then, we prove that there exists $C_1 = C_1(\frac{1}{\gamma+\delta}, \gamma + \delta) > 0$ such that if $\epsilon_2|U|_{H_\mu^{s+1}} \leq \frac{1}{C_1}$, then the operator defined by $P(U, \partial_x) = S_0 + \epsilon_2 S_1(U) - \mu S_2 \partial_x^2 : H^{s+1} \rightarrow H^{s-1}$ is one-to-one and onto, and that $P(U, \partial_x)^{-1}(\Sigma_0 + \epsilon_2 \Sigma_1(U) - \Sigma_2 \partial_x^2)$ is uniformly bounded $H_\epsilon^s \rightarrow H_\epsilon^s$, so that any solution V of (3.3) will satisfy the a priori estimate

$$|\partial_t V|_{H_\epsilon^s} = \left| P(V, \partial_x)^{-1} \left((\Sigma_0 + \epsilon_2 \Sigma_1(V) - \Sigma_2 \partial_x^2) \partial_x V - \alpha b \right) \right|_{H_\epsilon^s} \leq C_2 \left(|V|_{H_\epsilon^{s+1}} + \alpha |b|_{H^s} \right),$$

with C_2 independent of ϵ_2 and μ , as long as $\epsilon_2|V|_{H_\mu^{s+1}} \leq \frac{1}{C_1}$. It follows that $E_s(V)$ satisfies

$$\begin{aligned} \frac{d}{dt}E_s(V) &= \epsilon_2/2(S_1(\partial_t V)\Lambda^s V, \Lambda^s V) - \epsilon_2([\Lambda^s, S_1(V)]\partial_t V, \Lambda^s V) + \epsilon_2/2((\Sigma_1(\partial_x V)\Lambda^s V), \Lambda^s V) \\ &\quad - \epsilon_2([\Lambda^s, \Sigma_1(V)]\partial_x V, \Lambda^s V) + \alpha(\Lambda^s b, \Lambda^s V) \\ &\leq C_3 \left(\mu|V|_{H^s}^2 (|V|_{H_\mu^{s+1}} + \alpha|b|_{H^s}) + \alpha|b|_{H^s}|V|_{H^s} \right) \\ &\leq C_4 \left(\epsilon_2 E_s(V)^{3/2} + \alpha|b|_{H^s} E_s(V)^{1/2} \right). \end{aligned}$$

The last inequalities come from Cauchy-Schwarz inequality, Sobolev embeddings, Kato-Ponce commutator estimates and the above a priori estimate (see Appendix A of [19]). The Gronwall-Bihari's Lemma allows to conclude that as long as $\epsilon_2|V|_{H_\mu^{s+1}} \leq \frac{1}{C_0}$, one has

$$(3.4) \quad |V|_{H_\mu^{s+1}} \leq C_0 E(R_s)^{1/2} \leq C_5 \alpha |f|_{H^s} t.$$

Using the assumptions of Regime 2 ($\alpha = \mathcal{O}(\mu)$, $\epsilon_2 = \mathcal{O}(\mu)$), one can then follow the classical Friedrichs proof, and obtain the existence of $T(\gamma + \delta, \frac{1}{\gamma + \delta}, (\epsilon_2|V^0|_{H_\mu^{s+1}})^{-1}) > 0$ and a solution V of (3.3), defined over times $[0, T/\mu)$, such that $V \in C^0([0, T/\mu); H^{s+1}) \cap C^1([0, T/\mu); H^s)$.

Finally, the uniqueness of the solution is obtained in the same way, applying the energy estimate to the difference of two solutions. Indeed, let $V_1, V_2 \in C^0([0, T/\epsilon); H^{s+1}) \cap C^1([0, T/\epsilon); H^s)$ be two solutions of the Cauchy problem (3.3) with same initial value $V_1|_{t=0} = V_2|_{t=0} = V^0$. The functions V_1 and V_2 are uniformly bounded thanks to (3.4). One can immediately check that $R \equiv V_1 - V_2$ satisfies

$$(3.5) \quad \begin{aligned} &\left(S_0 + \epsilon_2 S_1(V_1) - \mu S_2 \partial_x^2 \right) \partial_t \Lambda^s R + \left(\Sigma_0 + \epsilon_2 \Sigma_1(V_1) - \mu \Sigma_2 \partial_x^2 \right) \partial_x \Lambda^s R \\ &\quad + \epsilon_2 [\Lambda^s, S_1(V_1)] \partial_t R + \epsilon_2 [\Lambda^s, \Sigma_1(V_1)] \partial_x R = \epsilon_2 F, \end{aligned}$$

with $F = -\Lambda^s (S_1(R)\partial_t V_2 + \Sigma_1(R)\partial_x V_2)$. Then, we can carry out the above method on R with a modified right hand side, and obtain the equivalent energy estimate

$$\frac{d}{dt}E_s(R) \leq \epsilon_2 C_6 (|V_1|_{H^s} + |V_2|_{H^s}) E_s,$$

with $C_6 = C(\frac{1}{\gamma + \delta}, \delta + \gamma, |U^0|_{H_\epsilon^{s+1}})$. From Gronwall-Bihari's inequality, using the uniform boundedness of V_1 and V_2 on $[0, T/\epsilon)$, and since $E_s(R)|_{t=0} = 0$, one has immediately $E_s(R) = 0$ on $[0, T/\epsilon)$, and finally $V_1 = V_2$.

Step 2: Consistency. We prove that, assuming that $\alpha = \mathcal{O}(\mu)$ and $\epsilon_2 = \mathcal{O}(\mu)$, the full Euler system (1.4) is consistent with the models (3.2) and (3.3), both at precision $\mathcal{O}(\mu^2)$ on $[0, T]$, $T > 0$.

Let $U \equiv (\zeta_1, \zeta_2, \psi_1, \psi_2)$ be a strong solution of (1.4), bounded in $W^{1,\infty}([0, T]; H^{s+t_0})$ with $s > 1$ and $t_0 \geq 9/2$, and such that (1.5) is satisfied with $\zeta_1(t, x) \equiv \zeta_1(x)$. The consistency result of Proposition A.1 states that (ζ_2, v) , with $v \equiv \partial_x \psi_2 - \gamma H(\psi_1, \psi_2)$, satisfies (2.2), up to $R_1 = (r_1, r_2)^T \in L^\infty([0, T]; H^s)^2$, satisfying (for $i = 1, 2$)

$$|r_i|_{L^\infty H^s} \leq \mu^2 C_0 \left(\frac{1}{h_{\min}}, |U|_{W^{1,\infty} H^{s+t_0}} \right).$$

It is the straightforward, as in the proof of Proposition 2.2, to show that the terms dropped in (3.1), and later in (3.3), are all bounded by $\mu^2 C_0 \left(\frac{1}{h_{\min}}, |U|_{W^{1,\infty} H^{s+4}} \right)$ in $L^\infty([0, T]; H^s)$

norm. Finally, (ζ_2, v) satisfies (3.2) and (3.3) with modified right hand sides, *i.e.*, respectively,

$$\alpha \tilde{b}_0 = \alpha b_0 + \mu^2 f_0, \quad \text{and} \quad \alpha \tilde{b} = \alpha b + \mu^2 f,$$

with f_0, f uniformly bounded in $L^\infty([0, T]; H^s)$.

Step 3: Convergence. The convergence estimates of the proposition follow easily from the calculations of *Step 1*, together with *Step 2*. Indeed, thanks to the consistency result, $V - V_B$ satisfy (3.5), with modified right hand side

$$\tilde{F} \equiv F + \mu^2 f, \quad f \in L^\infty([0, T]; H^s).$$

It follows that $E_s(V - V_B)$ satisfies

$$\frac{d}{dt} E_s(V - V_B) \leq \epsilon_2 C_6 E_s(V - V_B) + \mu^2 C_7 |f|_{H^s} E_s(V - V_B)^{1/2},$$

and the Gronwall-Bihari's Lemma leads to

$$|V - V_B|_{L^\infty([0, T/\mu]; H_\mu^{s+1})} \leq C_0 E_s(V - V_B)^{1/2} \leq \frac{C_7 \mu^2}{C_6 \epsilon_2} |f|_{H^s} (e^{C_6 \epsilon_2 t} - 1).$$

The convergence estimate of the proposition follow from $\epsilon_2 = \mathcal{O}(\mu)$. \square

3.2 The Korteweg-de Vries approximation

In this section, we use a BKW expansion, in order to deduce from the symmetric Boussinesq-type system (3.3), an approximated model that consists in two uncoupled forced Korteweg-de Vries (fKdV) equations. This method has been used, for example, in [19, 32], and is briefly presented in the following.

The idea is to look for an approximate solution of the Cauchy problem (3.3) with initial data U^0 , under the form

$$U_{\text{app}}(t, x) = U_0(\mu t, t, x) + \mu U_1(\mu t, t, x),$$

with the profiles $U_0(\tau, t, x)$ and $\mu U_1(\tau, t, x)$ satisfying $U_0|_{t=\tau=0} = U^0$ and $U_1|_{t=\tau=0} = 0$.

Plugging the *Ansatz* into (3.3) leads to the following equation

$$(3.6) \quad (S_0 \partial_t + \Sigma_0 \partial_x) U_0 + \mu S_0 \partial_\tau U_0 + \epsilon_2 (S_1(U_0) \partial_t U_0 + \Sigma_1(U_0) \partial_x U_0) - \mu (S_2 \partial_x^2 \partial_t U_0 + \Sigma_2 \partial_x^3 U_0) + \mu (S_0 \partial_t + \Sigma_0 \partial_x) U_1 + \mu^2 R = 0.$$

We now deduce $U_0(\tau, t, x)$ and $U_1(\tau, t, x)$, by solving (3.6) at each order.

At order $\mathcal{O}(1)$: We solve

$$(3.7) \quad (S_0 \partial_t + \Sigma_0 \partial_x) U_0 = 0.$$

Let us define $\mathbf{e}_\pm \equiv \frac{1}{\sqrt{2}} (\pm \frac{1}{\sqrt{\gamma+\delta}}, \sqrt{\gamma+\delta})^T$. One can check that the basis satisfies

$$\mathbf{e}_i \cdot \Sigma_0 \mathbf{e}_j = c_i \delta_{i,j}, \quad \text{and} \quad \mathbf{e}_i \cdot S_0 \mathbf{e}_j = \delta_{i,j},$$

with $\delta_{i,j}$ the classical Kronecker delta symbol, and $c_\pm = \pm 1 - \text{Fr}$.

Therefore, when we define $u_\pm \equiv \mathbf{e}_\pm \cdot S_0 U_0$ (and hence $U_0 = u_+ \mathbf{e}_+ + u_- \mathbf{e}_-$), the scalar product of (3.7) with \mathbf{e}_\pm leads to

$$(\partial_t + c_\pm \partial_x) u_\pm = 0.$$

Finally, since u_{\pm} satisfies a scalar transport equation, we use the notation

$$(3.8) \quad u_{\pm}(\tau, t, x) = u_{\pm}(\tau, x - c_{\pm}t) = u_{\pm}(\tau, x_{\pm}),$$

with initial data $u_{\pm}(0, x_{\pm}) = \mathbf{e}_{\pm} \cdot S_0 U^0(x_{\pm})$.

At order $\mathcal{O}(\mu)$: We solve

$$(3.9) \quad S_0 \partial_{\tau} U_0 + \frac{\epsilon_2}{\mu} (\Sigma_1(U_0) \partial_x U_0 + S_1(U_0) \partial_t U_0) - \Sigma_2 \partial_x^3 U_0 - S_2 \partial_x^2 \partial_t U_0 + (S_0 \partial_t + \Sigma_0 \partial_x) U_1 = \frac{\alpha}{\mu} b(x),$$

that we can split in

$$(3.10) \quad \partial_{\tau} u_{\pm} + \lambda_{\pm} u_{\pm} \partial_{x_{\pm}} u_{\pm} + \nu_{\pm} \partial_{x_{\pm}}^3 u_{\pm} = \beta_{\pm}(x),$$

with $\lambda_{\pm} \equiv \frac{\epsilon_2}{\mu} \mathbf{e}_{\pm} \cdot (\Sigma_1 - c_{\pm} S_1)(\mathbf{e}_{\pm}) \mathbf{e}_{\pm}$, $\nu_{\pm} \equiv \mathbf{e}_{\pm} \cdot (-\Sigma_2 + c_{\pm} S_2) \mathbf{e}_{\pm}$ and $\beta_{\pm} \equiv \frac{\alpha}{\mu} \mathbf{e}_{\pm} \cdot b$; and in the other hand,

$$(3.11) \quad (\partial_t + c_i \partial_x) \mathbf{e}_i \cdot S_0 U_1 + \sum_{(j,k) \neq (i,i)} \alpha_{ijk} u_k(\tau, x - c_k t) \partial_x u_j(\tau, x - c_j t) = \sum_{j \neq i} \beta_{ij} \partial_x^3 u_j(\tau, x - c_j t),$$

with $\alpha_{ijk} \equiv \mathbf{e}_i \cdot (\Sigma_1(\mathbf{e}_k) - c_j S_1(\mathbf{e}_k)) \mathbf{e}_j$ and $\beta_{ij} \equiv \mathbf{e}_i \cdot (\Sigma_2 - c_j S_2) \mathbf{e}_j$.

It is clear that u_i satisfies (3.8) and (3.10), if and only if $u_i(\epsilon t, t, x)$ satisfies the Korteweg-de Vries equation:

$$\partial_t u_{\pm} + c_{\pm} \partial_x u_{\pm} + \mu (\lambda_{\pm} u_{\pm} \partial_x u_{\pm} + \nu_{\pm} \partial_x^3 u_{\pm}) = \mu \beta_{\pm}(x).$$

Finally, simple calculations show that in our case, we can decompose

$$\zeta_2 \equiv \eta_+ + \eta_-, \quad \text{with } \eta_{\pm} \equiv \pm \frac{1}{\sqrt{2(\gamma + \delta)}} u_{\pm}$$

satisfying precisely the following KdV equation

$$\partial_t \eta_{\pm} + (-\text{Fr} \pm 1) \partial_x \eta_{\pm} \pm \epsilon_2 \frac{3\delta^2 - \gamma}{2(\gamma + \delta)} \eta_{\pm} \partial_x \eta_{\pm} \pm \mu \frac{1}{6} \frac{1 + \gamma\delta}{\delta(\gamma + \delta)} \partial_x^3 \eta_{\pm} = -\alpha \text{Fr} \gamma \frac{d}{dx} \zeta_1.$$

Unsurprisingly, we recover the KdV approximation with a flat rigid lid, when $\alpha = 0$ (see [19] and references therein).

Using these forced Korteweg-de Vries equations as an approximation of the full problem is justified up to times of order $\mathcal{O}(1/\mu)$ by the following proposition.

Proposition 3.2. *Let $s > 1/2$, $t_0 \geq 5 + 5/2$ and $U = (\zeta_1, \zeta_2, \psi_1, \psi_2)$ be an adapted solution of the full Euler system (1.4), bounded in $W^{1,\infty}([0, T/\mu]; H^{s+t_0})$. We define $V \equiv (\zeta_2, v)$ by*

$$v \equiv \partial_x (\phi_2|_{z=\epsilon_2 \zeta_2} - \gamma \phi_1|_{z=\epsilon_2 \zeta_2}) \equiv \partial_x \psi_2 - \gamma H(\psi_1, \psi_2).$$

Then there exists η_+ and η_- , the two solutions of the following forced Korteweg-de Vries equation

$$(3.12) \quad \partial_t \eta_{\pm} + (-\text{Fr} \pm 1) \partial_x \eta_{\pm} \pm \epsilon_2 \frac{3\delta^2 - \gamma}{2(\gamma + \delta)} \eta_{\pm} \partial_x \eta_{\pm} \pm \mu \frac{1}{6} \frac{1 + \gamma\delta}{\delta(\gamma + \delta)} \partial_x^3 \eta_{\pm} = -\alpha \text{Fr} \gamma \frac{d}{dx} \zeta_1(x),$$

with $\eta_{\pm}|_{t=0} = \frac{1}{2}(\zeta_2 \pm \frac{1}{\gamma + \delta} v)|_{t=0}$. Moreover, if there exists a constant C_0 such that $\epsilon_2 \leq C_0 \mu$ and $\alpha \leq C_0 \mu$, then one has for all $t \in [0, T/\mu)$,

$$|\zeta_2 - (\eta_+ + \eta_-)|_{L^\infty([0,t]; H^s)} + |v - (\gamma + \delta)(\eta_+ - \eta_-)|_{L^\infty([0,t]; H^s)} \leq \mu \sqrt{t} C,$$

with $C = C(\frac{1}{h_{\min}}, \frac{1}{\gamma+\delta}, \gamma + \delta, |U|_{W^{1,\infty}H^{s+t_0}}, C_0)$.

Additionally, if there exists a $a > 1/2$ such that $(1+x^2)^a V|_{t=0} \in H^{s+5}$, then one has the improved estimate

$$|\zeta_2 - (\eta_+ + \eta_-)|_{L^\infty([0,t];H^s)} + |v - (\gamma + \delta)(\eta_+ - \eta_-)|_{L^\infty([0,t];H^s)} \leq \epsilon C',$$

with $C' = C(\frac{1}{h_{\min}}, \frac{1}{\gamma+\delta}, \gamma + \delta, |U|_{W^{1,\infty}H^{s+t_0}}, |(1+x^2)^a V|_{t=0}|_{H^{s+5}})$.

The proposition is obtained by a simple adaptation of the techniques presented in [19], with additional forcing terms in the Korteweg-de Vries equations. The proof is given in Appendix B, for the sake of completeness.

3.3 Analysis of the forced Korteweg-de Vries equation

The forced Korteweg-de Vries equation

$$(3.13) \quad \partial_t u + c\partial_x u + \varepsilon\lambda u\partial_x u + \varepsilon\nu\partial_x^3 u = \varepsilon\frac{d}{dx}f(x)$$

has attracted a lot of interests, especially in the framework of the one layer water wave problem (where a moving topography, or pressure, is the forcing term that generates waves). Of particular interest is the problem of the generation of solitons, that have first been numerically discovered by Wu and Wu [43], and validated with experiments by Lee [33]. Using the Boussinesq-type system or the KdV approximation, they found that starting with a zero initial data, the solution can generate periodically an infinite number of solitary waves. Numerous work have then tackled this problem, including [55, 34, 49, 47]. It appeared that the Froude number (which is given by $1 - c$ in (3.13)) is playing a predominant role in this phenomenon, as the generation of solitons only occurs for a narrow band of its values. One could roughly summarize the observations by the existence of $F_c > 1$ such that

- i. if $\text{Fr} > F_c$, then the flow approaches a steady state, symmetric and localized at the site of forcing;
- ii. for $\text{Fr} < F_c$, solitons are periodically generated at the site of forcing and radiated up-stream;
- iii. the amplitude of the generated solitons goes to zero as $\text{Fr} \rightarrow -\infty$.

The existence of steady solitary waves of (3.13), and their stability is therefore essential. This issue has been studied for specific forcing terms by Camassa and Wu in [11, 10], and for general forcing terms by Choi, Lin, Sun and Whang in [14, 13]. They prove that for Fr sufficiently large, there exists a unique small steady solution. Moreover, this solution is proved to be symmetric and localized at the site of forcing, and stable in $H^1(\mathbb{R})$ (in the Lyapunov sense). The behavior when the Froude number Fr approaches unity is more peculiar and depends on the sign of the forcing term. Subcritical Froude numbers are not studied.

In the framework of our analysis, the values of the coefficients λ and ν depend on the parameters γ and δ , and their order of magnitude depends on ε_2 and μ (the magnitude of the forcing term depends mostly on α). We use the factor ε to keep in mind that all of these coefficients are assumed to be small. More precisely, we recall

$$c_\pm = \pm 1 - \text{Fr}, \quad \varepsilon\lambda_\pm = \pm\varepsilon_2\frac{3\delta^2 - \gamma}{2\gamma + \delta}, \quad \varepsilon\nu_\pm = \pm\mu\frac{1}{6}\frac{1 + \gamma\delta}{\delta(\gamma + \delta)}, \quad \varepsilon f(x) = -\text{Fr}\alpha\gamma\zeta_1(x).$$

When $\delta^2 - \gamma \neq 0$, a simple change of parameters allows to recover values of the one layer problem, and one obtains straightforwardly similar results. Our approach is quite different. In the following

section, we prove that away from the critical speed ($c = 0$), the solution of (3.13) with small initial data will remain small (in $H^s(\mathbb{R})$ norm) for long times, using smallness assumptions on the forcing terms and the coefficients $\varepsilon\lambda$ and $\varepsilon\nu$. Numerically, it appears that the solution converges locally towards a negative steady state in the supercritical case, and that small solitons are continuously generated otherwise. Then, in Section 3.3.2, we study numerically the generation of up-streaming solitons for Froude numbers around the critical value.

The consequences of this study to the dead-water effect are the following:

- i. away from the critical Froude number, the drag suffered by the ship is always small;
- ii. the peak of wave resistance occurs for Froude numbers just below the critical value;
- iii. the time-period of the generation of solitons predicted by the KdV approximation is bigger than the time-scale of relevance of the model, and another explanation has to be provided to explain the periodic aspect described among others in [53].

3.3.1 Non-critical Froude numbers

In the following proposition, we obtain an improved growth rate for the solution of the forced Korteweg-de Vries equation (3.13), if the velocity coefficient c is away from zero, when assuming smallness on the nonlinearity and dispersion coefficients λ and ν , and on the initial data and the forcing term. This phenomenon is easily explained, when looking at the linear transport equation related to (3.13):

$$(3.14) \quad \partial_t v + c\partial_x v = \varepsilon \frac{d}{dx} f(x).$$

The Cauchy problem, with initial data $v|_{t=0} = \varepsilon u_0$, is solved by

$$v \equiv \varepsilon u_0(x - ct) + \frac{\varepsilon}{c} \left(f(x) - f(x - ct) \right).$$

The solution is therefore bounded for all times as soon as $c \neq 0$, and small if the initial data and the forcing terms are small. We will obtain improved bound estimates on u the solution of the forced KdV equation, by estimating the difference $|u - v|_{H^s}$.

Proposition 3.3. *Let $s > 3/2$ and u be the solution of*

$$\partial_t u + c\partial_x u + \varepsilon\lambda u\partial_x u + \varepsilon\nu\partial_x^3 u = \varepsilon \frac{d}{dx} f(x),$$

such that $u|_{t=0} = \varepsilon u_0$. Let us assume that there exists $M > 0$ such that

$$\left| \frac{1}{c} \right|, |\lambda|, |\nu|, |u_0|_{H^{s+3}}, |f|_{H^{s+3}} \leq M.$$

Then there exists $T(M, s) > 0$ and $C = C(M, s) > 0$, such that the function u is bounded on $[0, T/\varepsilon]$ by

$$|u(t)|_{L^\infty([0, T/\varepsilon]; H^s)} \leq C\varepsilon.$$

Proof. Thanks to [6], we know that the function u exists and is unique. We define as above

$$v \equiv \varepsilon u_0(x - ct) + \frac{\varepsilon}{c} \left(f(x) - f(x - ct) \right),$$

the solution of the transport equation

$$\partial_t v + c \partial_x v = \varepsilon \frac{d}{dx} f(x),$$

with same initial data $v|_{t=0} = \varepsilon u_0$. Since the function v is uniformly bounded by

$$|v(t)|_{L^\infty(\mathbb{R}; H^s)} \leq C\varepsilon,$$

the estimate of the proposition follows from the same estimate on the difference $r \equiv u - v$. It is easy to check that r satisfies

$$(3.15) \quad \partial_t r + c \partial_x r + \varepsilon \lambda(r+v) \partial_x(r+v) + \varepsilon \nu \partial_x^3(r+v) = 0,$$

and $r|_{t=0} = 0$. When multiplying (3.15) by $\Lambda^{2s} r$, and integrating, one obtains

$$\frac{1}{2} \frac{d}{dt} (\Lambda^s r, \Lambda^s r) + c (\Lambda^s \partial_x r, \Lambda^s r) + \varepsilon \lambda (\Lambda^s ((r+v) \partial_x(r+v)), \Lambda^s r) - \varepsilon \nu (\Lambda^s \partial_x^3(r+v), \Lambda^s \partial_x r) = 0,$$

denoting $(f, g) \equiv \int_{\mathbb{R}} fg$ the L^2 -based inner product. It follows then

$$\frac{1}{2} \frac{d}{dt} (|r|_{H^s}^2) = -\varepsilon \lambda (\Lambda^s ((r+v) \partial_x(r+v)), \Lambda^s r) - \varepsilon \nu (\Lambda^s \partial_x^3 v, \Lambda^s r).$$

Some of the terms of the right-hand side are straightforwardly estimated using Cauchy-Schwarz inequality, and the algebraic properties of $H^s(\mathbb{R})$, $s > 1/2$:

$$\begin{aligned} |(\Lambda^s \partial_x^3 v, \Lambda^s r)| &\leq |v|_{H^{s+3}} |r|_{H^s}, \\ |(\Lambda^s (v \partial_x v), \Lambda^s r)| &\leq C_0(s) |v|_{H^{s+1}} |v|_{H^s} |r|_{H^s}, \\ |(\Lambda^s (r \partial_x v), \Lambda^s r)| &\leq C_0(s) |v|_{H^{s+1}} |r|_{H^s}^2. \end{aligned}$$

As for the remaining terms, we integrate by part and obtain

$$\begin{aligned} (\Lambda^s (r \partial_x r), \Lambda^s r) &= -\frac{1}{2} (\partial_x r \Lambda^s r, \Lambda^s r) + ([\Lambda^s, r] \partial_x r, \Lambda^s r), \\ (\Lambda^s (v \partial_x r), \Lambda^s r) &= -\frac{1}{2} (\partial_x v \Lambda^s r, \Lambda^s r) + ([\Lambda^s, v] \partial_x r, \Lambda^s r), \end{aligned}$$

with $[T, f]$ the commutator defined by $[T, f]g \equiv T(fg) - fT(g)$. Since for $f \in H^s$, $s > 3/2$, one has $|\partial_x f|_{L^\infty} \leq |f|_{H^s}$, one can use the classical Kato-Ponce Lemma [25] to deduce the following estimates

$$\begin{aligned} |(\Lambda^s (v \partial_x r), \Lambda^s r)| &\leq C_0(s) |v|_{H^s} |r|_{H^s}^2, \\ |(\Lambda^s (r \partial_x r), \Lambda^s r)| &\leq C_0(s) |r|_{H^s}^3. \end{aligned}$$

Altogether, it follows that $|r|_{H^s}$ satisfies the differential inequality

$$\frac{1}{2} \frac{d}{dt} (|r|_{H^s}^2) \leq C_0(s) \varepsilon \lambda (|v|_{H^{s+1}}^2 |r|_{H^s} + |v|_{H^{s+1}} |r|_{H^s}^2 + |r|_{H^s}^3) + \varepsilon \nu |v|_{H^{s+3}} |r|_{H^s},$$

with $C_0(s)$ a constant depending only on the parameter $s > 3/2$. Now, using the assumptions of the proposition, it is straightforward to see that one can rewrite

$$\frac{d}{dt} |r|_{H^s} \leq C_0 \varepsilon^2 + C_0(s) \varepsilon (\varepsilon^2 + \varepsilon |r|_{H^s} + |r|_{H^s}^2).$$

Now, let us define T^* the maximum time such that $|r|_{H^s} \leq \varepsilon$ on $[0, T^*]$ (this is true at $t = 0$, so that $T^* > 0$ by a continuity argument). Then we deduce from the above inequality, and Gronwall's Lemma, that for all $t \in [0, T^*]$,

$$|r|_{H^s} \leq C(s)\varepsilon^2 t,$$

with a constant $C(s)$ depending only on the parameter $s > 3/2$. From this very a priori estimate, we deduce that $T^* \geq (C(s)\varepsilon)^{-1}$, and the proposition is proved. \square

Remark 3.4. *One of the immediate consequences of Proposition 3.3 to our problem is that, in the decomposition of the KdV approximation (Proposition 3.2), the function η_- remains small on the time interval where the KdV approximation is a relevant model. Indeed, as $c_- \equiv -1 - \text{Fr} < -1$, then if $\eta_-|_{t=0} = 0$ (small in H^{s+3} would suffice), then for $t \in [0, T/\mu)$, one has*

$$|\eta_-(t)|_{H^s} \leq C\left(\frac{1}{\gamma + \delta}, \gamma + \delta, |\zeta_1|_{H^{s+3}}\right)\mu.$$

Therefore, we focus in the following on η_+ , the solution of

$$(3.16) \quad \partial_t \eta_+ + (1 - \text{Fr})\partial_x \eta_+ + \varepsilon_2 \frac{3\delta^2 - \gamma}{2(\gamma + \delta)} \eta_+ \partial_x \eta_+ + \mu \frac{1}{6} \frac{1 + \gamma\delta}{\delta(\gamma + \delta)} \partial_x^3 \eta_+ = -\alpha \text{Fr} \gamma \frac{d}{dx} \zeta_1.$$

In Figures 6 and 7, we compute the solution of (3.16), with zero initial data, in the supercritical ($\text{Fr} = 1.5$) and subcritical ($\text{Fr} = 0.5$) cases. For each of the figures, we plot the flow, depending on space and time variables, as well as the evolution of the related wave resistance coefficient C_W , calculated with formula (C.5) page 37. We use two values for the depth ratio: $\delta \in \{5/12, 12/5\}$.

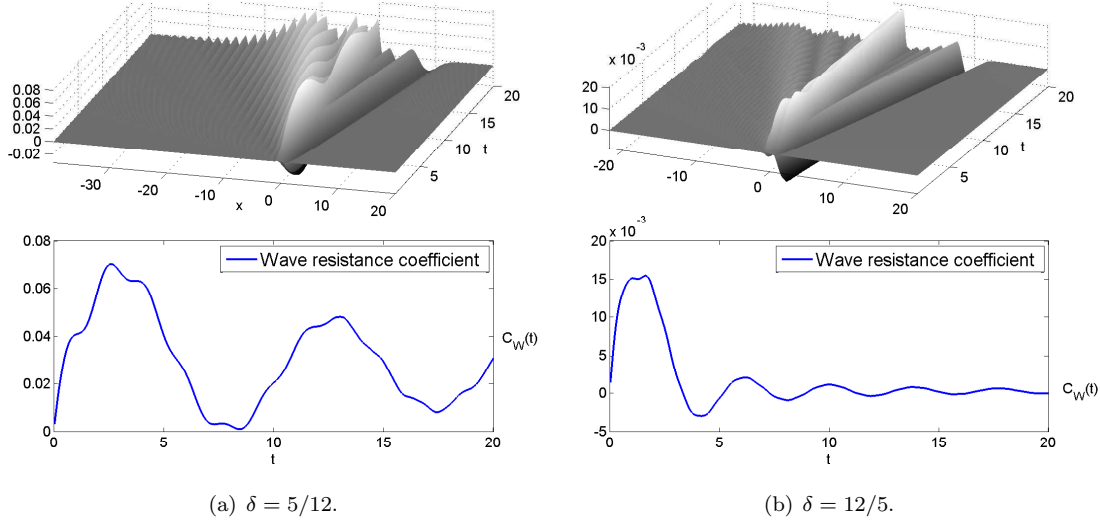


Figure 6: Subcritical flow: $\text{Fr} = 0.5$. $\alpha = \varepsilon_2 = \mu = 0.1$, $\gamma = 0.99$.

As we are away from the critical Froude number, the amplitudes of the generated waves, as well as the magnitude of the wave resistance coefficient, are small (at most of the order $\mathcal{O}(\mu)$). One remarks that the behavior of the flow is roughly independent of the value of the depth ratio δ . This is easily explained, as the quadratic nonlinearities do not play an important role when the deformation is small (as they are of order $\mathcal{O}(\mu^3)$). Therefore, the variations of δ are mostly seen

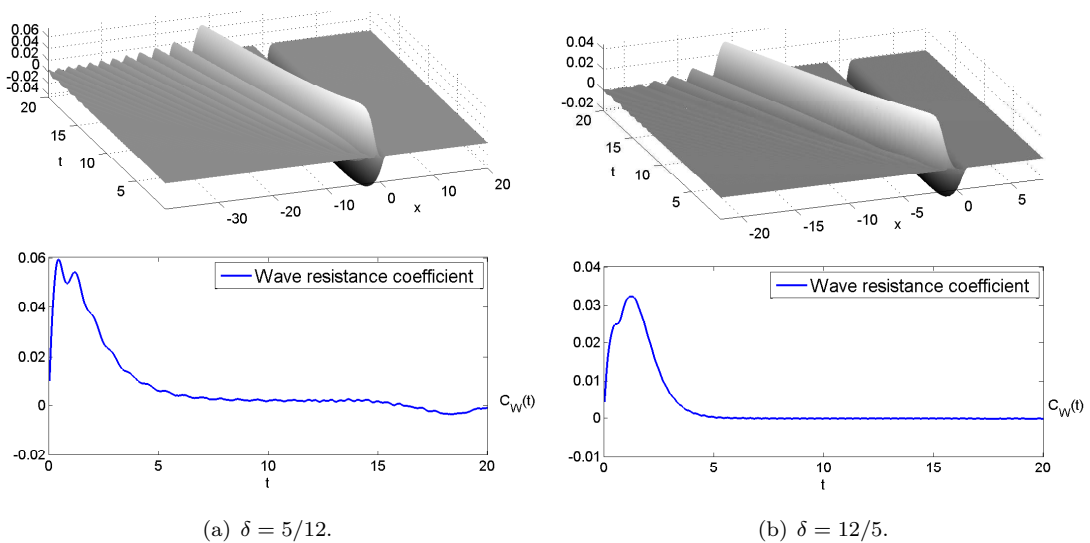


Figure 7: Supercritical flow: $Fr = 1.5$. $\alpha = \epsilon_2 = \mu = 0.1$, $\gamma = 0.99$.

through a variation of the size of the dispersion coefficient ν , which do not change the behavior pattern of the solution.

On the contrary, the Froude number parameter has an essential role on the behavior of the solution. One sees that for subcritical flows (Figure 6), up-streaming solitary waves are continuously generated, as a widening oscillatory tail is generated at the down-streaming area. The former will generate most of the wave resistance, as they overtake the body. However, the drag remains small, and possibly tends to zero in the long time limit. The behavior of supercritical flows (Figure 7) are quite different. A down-streaming solitary wave, with a small oscillatory tail, is generated and travels at constant velocity $c_0 \approx -1/2$. It remains a symmetric, depression wave at the location of the body, that do not generates any wave resistance. Unsurprisingly, this corresponds to the observations of the one-layer theory in the supercritical case.

3.3.2 Critical Froude numbers

As we have seen in the previous section, the solution of the fKdV equation, and therefore the wave resistance coefficient (C.5), are small when the Froude number is away from its critical number $Fr = 1$. We present in Figure 8 the behavior of the wave resistance at time $t = 10$, depending on the Froude number, for different values of the depth ratio δ , and different values of the shallowness parameter μ .

It appears that the maximum of the critical wave is obtained below the critical speed, at approximate value $Fr \approx 1 - Cstt \lambda_+ = 1 - Cstt \mu \frac{1+\gamma\delta}{\gamma+\delta}$. The fact that the maximum peak is slightly subcritical has been obtained using the linear theory in [40], and explained as a result of dispersion effects (without the nonlinear and dispersion terms, the peak is infinite and obtained at $Fr = 1$). This effect is therefore preserved in the nonlinear theory.

In Figure 9, we compute the solution of (3.16), with zero initial data, in the critical case ($Fr = 1$). Our aim is to exhibit the generation of up-streaming solitary waves, as in [55]. This requires a rather long period of time computation, as discussed in the following. Again, we plot the flow, depending on space and time variables, as well as the wave resistance coefficient, and we set $\delta \in \{5/12, 12/5\}$.

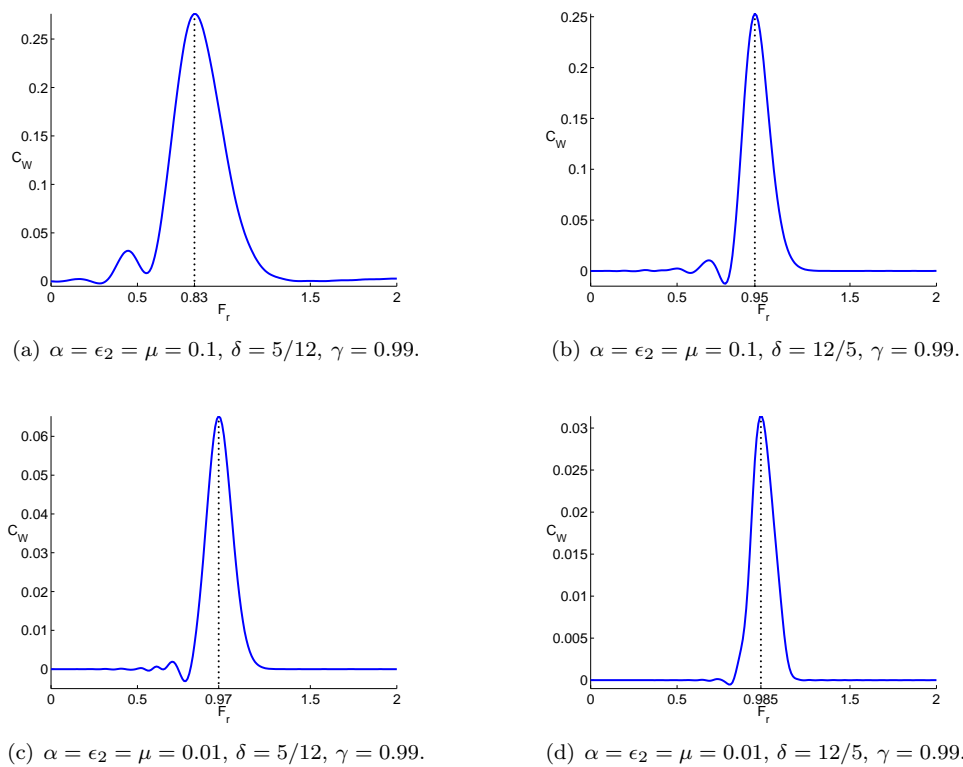


Figure 8: Dependence of the wave resistance coefficient on the Froude number, at time $T = 10$.

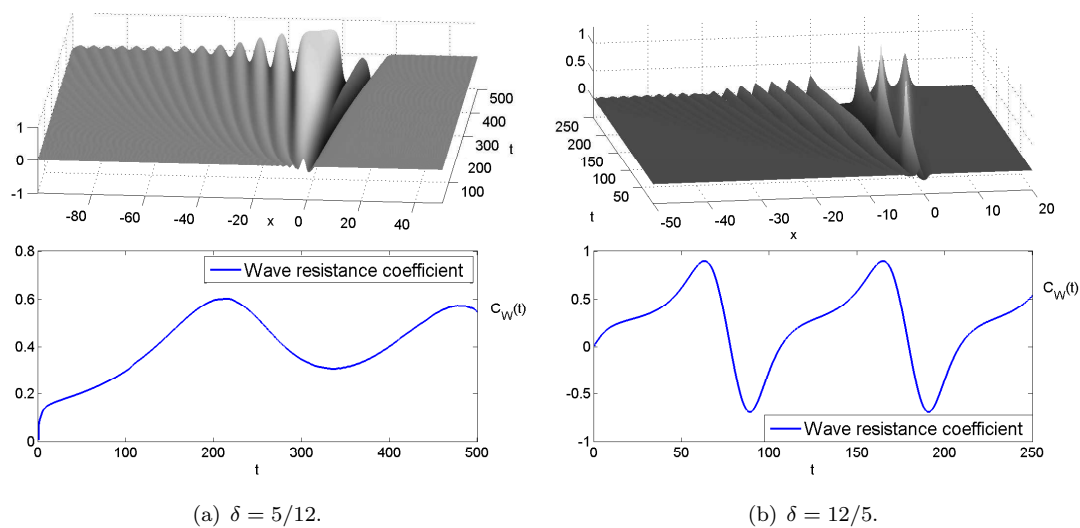


Figure 9: Critical flow: $Fr = 1. \alpha = \epsilon_2 = \mu = 0.1, \gamma = 0.99.$

As we can see, the depth ratio δ now plays an essential role. Upstreaming solitary waves are generated for both of the values of δ , but these waves are of elevation if the upper fluid is thicker, and of depression in the opposite case. These waves induce a periodic aspect to the wave resistance. However, we do not believe that these waves are the cause of the periodic behavior recorded in [53], based on the two following facts.

- i. The time period of the generation of waves ΔT is very large. A simple scaling argument shows that $\Delta T \propto 1/\mu$ if $c = 0$ and $\lambda, \alpha, \nu \propto \mu$, and the proportionality constant appears to be big (see also scaling arguments in [55]). By Proposition 3.2, the KdV approximation is a valid model only up to times of order $\mathcal{O}(1/\mu)$, so that nothing shows that the solutions of the full Euler equations (1.4) follow this behavior.
- ii. As a matter of fact, the solutions of the strongly nonlinear model (2.6) do not exhibit such a phenomenon, as the generated internal wave stays in the trail of the body, and never overtake the ship, even for long times.

In Figure 5 page 17, we present the results of a simulation where the velocity of the ship is affected by the generated wave resistance. A similar periodic behavior is exhibited, with a more reasonable time period ($\Delta T \approx 10$). What is more, in this case, the up-streaming solitons vanish as soon as they overtake and slow down the body. This phenomenon corresponds to the observations of [53], but cannot be seen in Figure 9.

A Derivation of the Green-Naghdi type model

In this section, we derive the strongly nonlinear model (2.2). This model only requires the shallow water assumption

$$\mu \ll 1,$$

and is a convenient intermediate step to construct the models of Regimes 1 and 2 used throughout the paper. The full Euler system is proved to be consistent with our model at order $\mathcal{O}(\mu^2)$, in Proposition A.1 below.

Let us first plug the expansions of Lemma 2.1 into (1.4), and withdraw $\mathcal{O}(\mu^2)$ terms. One can easily check that we obtain the approximate system

$$(A.1) \quad \left\{ \begin{array}{l} -\alpha \text{Fr} \frac{d}{dx} \zeta_1 + \partial_x (h_1 \partial_x \psi_1) + \partial_x (h_2 \partial_x \psi_2) = \mu \partial_x (\mathcal{R}_1(\partial_x \psi_1, \partial_x \psi_2)), \\ (\partial_t - \text{Fr} \partial_x) \zeta_2 + \partial_x (h_2 \partial_x \psi_2) = \mu \partial_x \mathcal{T}[h_2, 0] \partial_x \psi_2, \\ (\partial_t - \text{Fr} \partial_x) (\partial_x \psi_2 - \gamma \partial_x \psi_1) + (\gamma + \delta) \partial_x \zeta_2 + \frac{\epsilon_2}{2} \partial_x (|\partial_x \psi_2|^2 - \gamma |\partial_x \psi_1|^2) \\ \quad = \mu \left(\gamma (\partial_t - \text{Fr} \partial_x) \partial_x \mathcal{H}(\partial_x \psi_1, \partial_x \psi_2) + \epsilon_2 \partial_x \mathcal{R}_2[\partial_x \psi_1, \partial_x \psi_2] \right) \\ \quad \quad \quad + \frac{1}{\text{Bo}} \partial_x^2 \left(\frac{\partial_x \zeta_2}{\sqrt{1 + \mu \epsilon_2^2 |\partial_x \zeta_2|^2}} \right), \end{array} \right.$$

where we have used the following notations:

$$\begin{aligned}
 \mathcal{T}[h, b]V &\equiv -\frac{1}{3}\partial_x(h^3\partial_x V) + \frac{1}{2}(\partial_x(h^2(\partial_x b)V)) - h^2(\partial_x b)(\partial_x V) + h(\partial_x b)^2V, \\
 \mathcal{N}[V_1, V_2] &\equiv \frac{((\partial_x h)V_2 - \partial_x(hV_2))^2 - \gamma((\partial_x h)V_1 - \partial_x(hV_2))^2}{2}, \\
 \mathcal{H}(V_1, V_2) &\equiv h_1(\partial_x(h_1V_1) + \partial_x(h_2V_2)) - \frac{1}{2}h_1\partial_x V_1 - \partial_x(h_1 + h_2)V_1, \\
 \mathcal{R}_1(V_1, V_2) &\equiv \mathcal{T}[h_1, h_2]V_1 + \mathcal{T}[h_2, 0]V_2 - \frac{1}{2}\partial_x(h_1^2\partial_x(h_2V_2)) - h_1\partial_x h_2\partial_x(h_2V_2), \\
 \mathcal{R}_2[V_1, V_2] &\equiv \gamma V_1\partial_x(\mathcal{H}(V_1, V_2)) + \mathcal{N}[V_1, V_2].
 \end{aligned}$$

Now, thanks to the fact that ζ_1 is a forced parameter of our problem, the system reduces to two evolution equations for (ζ_2, v) , with v the *shear velocity* defined by

$$v \equiv \partial_x(\phi_2|_{z=\epsilon_2\zeta_2} - \gamma\phi_1|_{z=\epsilon_2\zeta_2}) = \partial_x\psi_2 - \gamma\mathcal{H}(\psi_1, \psi_2).$$

From the last estimate of Lemma 2.1, one has immediately

$$(A.2) \quad v = \partial_x\psi_2 - \gamma\partial_x\psi_1 - \mu\gamma\partial_x\mathcal{H}(\partial_x\psi_1, \partial_x\psi_2) + \mathcal{O}(\mu^2).$$

Now, one deduces from the first equation of (A.1), and (A.2), the following relations

$$(A.3) \quad \begin{aligned} h_1\partial_x\psi_1 + h_2\partial_x\psi_2 &= \alpha \text{Fr} \zeta_1 + \mu \mathcal{R}_1(\partial_x\psi_1, \partial_x\psi_2) + \mathcal{O}(\mu^2), \\ v &= \partial_x\psi_2 - \gamma\partial_x\psi_1 - \mu \gamma \partial_x\mathcal{H}(\partial_x\psi_1, \partial_x\psi_2) + \mathcal{O}(\mu^2). \end{aligned}$$

It follows that any linear operator defined above can be approximated as in the following example

$$\mathcal{H}(\partial_x\psi_1, \partial_x\psi_2) = \mathcal{H}\left(-\frac{h_2v - \alpha \text{Fr} \zeta_1}{h_1 + \gamma h_2}, \frac{h_1v + \gamma\alpha \text{Fr} \zeta_1}{h_1 + \gamma h_2}\right) + \mathcal{O}(\mu).$$

For the sake of readability, we do not precise the arguments in the following, and simply write

$$\mathcal{H} \equiv \mathcal{H}\left(-\frac{h_2v - \alpha \text{Fr} \zeta_1}{h_1 + \gamma h_2}, \frac{h_1v + \gamma\alpha \text{Fr} \zeta_1}{h_1 + \gamma h_2}\right), \quad \mathcal{R}_1 \equiv \mathcal{R}_1\left(-\frac{h_2v - \alpha \text{Fr} \zeta_1}{h_1 + \gamma h_2}, \frac{h_1v + \gamma\alpha \text{Fr} \zeta_1}{h_1 + \gamma h_2}\right).$$

Using (A.3), one can approximate $\partial_x\psi_1$ and $\partial_x\psi_2$ at order $\mathcal{O}(\mu^2)$ with

$$(A.4) \quad \begin{aligned} (h_1 + \gamma h_2)\partial_x\psi_1 &= -h_2v + \alpha \text{Fr} \zeta + \mu(\mathcal{R}_1 - \gamma h_2\partial_x\mathcal{H}) + \mathcal{O}(\mu^2), \\ (h_1 + \gamma h_2)\partial_x\psi_2 &= h_1v + \gamma\alpha \text{Fr} \zeta + \mu\gamma(\mathcal{R}_1 + h_1\partial_x\mathcal{H}) + \mathcal{O}(\mu^2). \end{aligned}$$

Using these formulae, the system (A.1) becomes (withdrawing $\mathcal{O}(\mu^2)$ terms)

$$(A.5) \quad \begin{cases} (\partial_t - \text{Fr} \partial_x)\zeta_2 + \partial_x\left(\frac{h_2}{h_1 + \gamma h_2}(h_1v + \gamma\alpha \text{Fr} \zeta)\right) + \mu\partial_x\mathcal{L} = 0, \\ (\partial_t - \text{Fr} \partial_x)v + (\gamma + \delta)\partial_x\zeta_2 + \frac{\epsilon_2}{2}\partial_x\left(\frac{|h_1v + \gamma\alpha \text{Fr} \zeta|^2 - \gamma|h_2v - \alpha \text{Fr} \zeta|^2}{(h_1 + \gamma h_2)^2}\right) + \mu\epsilon_2\partial_x\mathcal{Q} \\ \qquad \qquad \qquad = \frac{1}{\text{Bo}}\partial_x^2\left(\frac{\partial_x\zeta_2}{\sqrt{1 + \mu\epsilon_2^2|\partial_x\zeta_2|^2}}\right), \end{cases}$$

where \mathcal{L} and \mathcal{Q} are defined by

$$\begin{aligned}\mathcal{L} &= \gamma \frac{h_2}{h_1 + \gamma h_2} (\mathcal{R}_1 + h_1 \partial_x \mathcal{H}) - \mathcal{T}[h_2, 0] \left(\frac{h_1 v + \gamma \alpha \text{Fr } \zeta_1}{h_1 + \gamma h_2} \right), \\ \mathcal{Q} &= \gamma \frac{(h_1 + h_2)v - (1 - \gamma)\alpha \text{Fr } \zeta_1}{(h_1 + \gamma h_2)^2} \mathcal{R}_1 - \mathcal{R}_2 \left[-\frac{h_2 v - \alpha \text{Fr } \zeta_1}{h_1 + \gamma h_2}, \frac{h_1 v + \gamma \alpha \text{Fr } \zeta_1}{h_1 + \gamma h_2} \right] \\ &\quad + \gamma \frac{((h_1^2 - \gamma h_2^2)v + \gamma(h_1 + h_2)\alpha \text{Fr } \zeta_1)}{(h_1 + \gamma h_2)^2} \partial_x \mathcal{H}.\end{aligned}$$

It is now simple to check that the system (A.5) is exactly the system (2.2).

This model is justified by the following proposition:

Proposition A.1. *The full Euler system (1.4) is consistent with (A.5), at precision $\mathcal{O}(\mu^2)$ on $[0, T]$, with $T > 0$.*

Proof. Let $U \equiv (\zeta_1, \zeta_2, \psi_1, \psi_2)$ be a strong solution of (1.4), bounded in $W^{1,\infty}([0, T]; H^{s+t_0})$ with $s > 1$ and $t_0 \geq 9/2 + 5$, and such that (1.5) is satisfied with $\zeta_1(t, x) \equiv \zeta_1(x)$. Using Lemma 2.1, it is clear that $(\zeta_1, \zeta_2, \partial_x \psi_1, \partial_x \psi_2)$ satisfies (A.1), up to $R_1 = (r_1, r_2, r_3, r_4)^T \in L^\infty([0, T]; H^{s+5})^4$, satisfying (for $i = 1 \dots 4$)

$$|r_i|_{W^{1,\infty} H^{s+5}} \leq \mu^2 C_0 \left(\frac{1}{h_{\min}}, |U|_{W^{1,\infty} H^{s+t_0}} \right).$$

Now, we set $v \equiv \partial_x \psi_2 - \gamma H(\psi_1, \psi_2)$, and v satisfies (A.2), up to $R_2 \in W^{1,\infty}([0, T]; H^{s+5})$, with (again, thanks to Lemma 2.1)

$$|R_2|_{W^{1,\infty} H^{s+5}} \leq \mu^2 C_0 \left(\frac{1}{h_{\min}}, |U|_{W^{1,\infty} H^{s+t_0}} \right).$$

Therefore, since \mathcal{R}_1 and $\partial_x \mathcal{H}$ involve two spatial derivatives, one has

$$\begin{cases} (h_1 + \gamma h_2) \partial_x \psi_1 &= -h_2 v + \alpha \text{Fr } \zeta + \mu R_3, \\ (h_1 + \gamma h_2) \partial_x \psi_2 &= h_1 v + \gamma \alpha \text{Fr } \zeta + \mu R_4, \end{cases}$$

with $R_3, R_4 \in L^\infty([0, T]; H^{s+3})$. Consequently, using the fact that $H^s(\mathbb{R})$ is an algebra for $s > 1/2$ and that (1.5) is satisfied, we deduce that (A.4) is satisfied, up to $R_5 \in L^\infty([0, T]; H^{s+1})$, with

$$|R_5|_{L^\infty H^{s+1}} \leq \mu^2 (|R_3|_{L^\infty H^{s+3}} + |R_4|_{L^\infty H^{s+3}}) C_0 \left(\frac{1}{h_{\min}}, |U|_{W^{1,\infty} H^{s+t_0}} \right).$$

Finally, when plugging (v, ζ_2) into (A.1), the residuals are clearly bounded by $\mu^2 C_0$, uniformly in $L^\infty([0, T]; H^s)$, and the proposition is proved. \square

B Proof of Proposition 3.2

The sketch of the proof is the following. We will first prove the convergence between

$$U_{\text{app}}(t, x) = U_0(\mu t, t, x) + \mu U_1(\mu t, t, x),$$

defined by (3.8)–(3.11), and the solutions of the symmetric Boussinesq-type system (3.3). This is obtained thanks to a consistency result, together with energy estimates. The convergence towards solutions of the full Euler system (1.4) follows then immediately from Proposition 3.1. Finally, the proposition is completed when remarking that the corrector term U_1 obeys to a sublinear growth. This result is a consequence of the following Lemma, that proceeds from Propositions 3.2 and 3.5 of [30]:

Lemma B.1. *Let u be the solution of*

$$(B.1) \quad \begin{cases} (\partial_t + c\partial_x)u = g(v_1, v_2), & \text{with } \forall i \in \{1, 2\}, \\ u|_{t=0} = 0, \end{cases} \quad \begin{cases} (\partial_t + c_i\partial_x)v_i = 0, \\ v_i|_{t=0} = v_i^0, \end{cases}$$

with $c_1 \neq c_2$, $v_1^0, v_2^0 \in H^s(\mathbb{R})$, $s > 1/2$, and g is a bilinear mapping defined on \mathbb{R}^2 and with values in \mathbb{R} . Then one has the following estimates:

i. If $c = c_1$, then $\lim_{t \rightarrow \infty} \frac{1}{\sqrt{t}} |u(t, \cdot)|_{H^s(\mathbb{R})} = 0$.

ii. If $c \neq c_1 \neq c_2$, then $\frac{1}{\sqrt{t}} |u(t, \cdot)|_{H^s(\mathbb{R})} = \mathcal{O}(1)$.

Moreover, if there exists $a > 1/2$ such that $v_1^0(1+x^2)^a$, and $v_2^0(1+x^2)^a \in H^s(\mathbb{R})$, then one has the better estimate

$$|u|_{L^\infty H^s(\mathbb{R})} \leq C_0 |v_1^0(1+x^2)^a|_{H^s(\mathbb{R})} |v_2^0(1+x^2)^a|_{H^s(\mathbb{R})},$$

with $C_0 = C(c, c_1, c_2)$.

We can now proceed with the proof.

Step 1: Well-posedness of $U_{app}(t, x)$. The global well-posedness of the forced Korteweg-de Vries equation is given by Bona and Zhang in [6].

The proof relies on an *a priori* estimate, that we recall here, as it will be useful for the following arguments. Let u be a solution of

$$\partial_t u + c\partial_x u + \lambda u\partial_x u + \nu\partial_x^3 u = b(x).$$

As we multiply the equation by $\Lambda^{2s'} u$ (with $s' > 3/2$), and integrate with respect to the space variable, one obtains

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (\Lambda^{s'} u)^2 dx = \left| \lambda \int_{\mathbb{R}} \Lambda^{s'} (u\partial_x u) \Lambda^{s'} u dx \right| + \left| \int_{\mathbb{R}} (\Lambda^{s'} b) (\Lambda^{s'} u) dx \right|.$$

Thanks to the Kato-Ponce Lemma, we estimate the right-hand side as follows:

$$\left| \int_{\mathbb{R}} \Lambda^{s'} (u\partial_x u) \Lambda^{s'} u dx \right| \leq \left| \frac{1}{2} \int_{\mathbb{R}} \partial_x u (\Lambda^{s'} u)^2 dx \right| + \left| \int_{\mathbb{R}} [\Lambda^{s'}, u] \partial_x u (\Lambda^{s'} u) dx \right| \leq C_s |u|_{H^{s'}}^2 |\partial_x u|_{L^\infty},$$

and the Cauchy-Schwarz inequality leads to

$$\frac{1}{2} \frac{d}{dt} |u|_{H^{s'}}^2 \leq C_0 \lambda |u|_{H^{s'}}^2 |\partial_x u|_{L^\infty} + |u|_{H^{s'}} |b|_{H^{s'}}.$$

It follows

$$\sup_{t \in [0, T]} |u|_{H^{s'}} \leq \exp \left(\lambda \int_0^T |\partial_x u(s, \cdot)|_{L^\infty} ds \right) (|u|_{t=0}|_{H^{s'}} + T|b|_{H^{s'}}).$$

In the particular case of (3.12), since $\alpha = \mathcal{O}(\mu)$, $\epsilon_2 = \mathcal{O}(\mu)$ and $\eta_{\pm}|_{t=0} = \frac{1}{2}(\zeta_2 \pm \frac{1}{\gamma+\delta}v)|_{t=0}$, it is straightforward that there exists $T = T(\gamma + \delta, \frac{1}{\gamma+\delta}) > 0$ such that for all $s' > 3/2$,

$$(B.2) \quad \sup_{t \in [0, T/\mu]} |u|_{H^{s'}} \leq C_0 \left(\frac{d}{dx} \zeta_1|_{H^{s'}}, |V|_{t=0}|_{H^{s'}}, T \right).$$

We can then exhibit U_1 . Indeed, (3.11) can be written in a simplified form as

$$(B.3) \quad (\partial_t + c_i \partial_x) \mathbf{e}_i \cdot S_0 U_1 = \sum_{(j,k) \neq (i,i)} f_{ijk}(\tau, t, x) + \sum_{j \neq i} \partial_x g_{ij}(\tau, t, x),$$

with

$$f_{ijk}(\tau, t, x) \equiv \alpha_{ijk} u_j(\tau, x - c_j t) \partial_x u_k(\tau, x - c_k t) \quad \text{and} \quad g_{ij}(\tau, t, x) \equiv \beta_{ij} \partial_x^2 u_j(\tau, x - c_j t).$$

Thanks to estimate (B.2), one has for $s' > 1/2$,

$$\|f_{ijk}\|_{L^\infty([0,T] \times [0,T/\epsilon]; H^{s'+1})} + \|g_{ij}\|_{L^\infty([0,T]; H^{s'})} \leq C_0 \left(\left| \frac{d}{dx} \zeta_1 \right|_{H^{s'+2}}, |V|_{t=0} \Big|_{H^{s'+2}}, T \right).$$

with $C_0 = C_0(\frac{1}{\gamma+\delta}, \gamma + \delta)$. Hence, for $s' > 1/2$, one can set

$$(B.4) \quad \begin{aligned} \mathbf{e}_\pm \cdot S_0 U_1(\tau, t, x) &= \sum_{(j,k) \neq (i,i)} \int_0^t f_{ijk}(\tau, s, x + c_i(s-t)) ds \\ &\quad + \sum_{j \neq i} \frac{1}{c_i - c_j} (g_{ij}(\tau, x - c_j t) - g_{ij}(\tau, x - c_i t)) \\ &\equiv \sum_{(j,k) \neq (i,i)} U^{ijk} + \sum_{j \neq i} V^{ij}. \end{aligned}$$

One checks immediately that $U_1 \in L^\infty([0, T] \times [0, T/\mu]; H^{s'})$ and $U_1|_{\tau=t=0} = 0$.

Step 2: Estimate on U_1 . Let us estimate each term of the decomposition (B.4). Thanks to the uniform estimates of Step 1. above, one has $g_{ij} \in L^\infty([0, T]; H^{s'})$, so that it follows immediately

$$\forall j \neq i, \quad \|V^{ij}\|_{L^\infty(\mathbb{R}^+ \times [0, T]; H^{s'})} \leq C_0 \left(\left| \frac{d}{dx} \zeta_1 \right|_{H^{s'+2}}, |V|_{t=0} \Big|_{H^{s'+2}}, T \right).$$

Moreover, for $j \neq i$, we remark that f_{ijj} can be written as

$$f_{ijj}(\tau, t, x) \equiv \alpha_{ijj} u_j(\tau, x - c_j t) \partial_x u_j(\tau, x - c_j t) \equiv \partial_x h_{ij}(\tau, x - c_j t),$$

so that U^{ijj} has actually the same form as V^{ij} , and can be treated in the same way. Since $f_{ijj} \in L^\infty([0, T] \times [0, T/\epsilon]; H^{s'+1})$, U^{ijj} is bounded in $H^{s'}$ by $C_0 \left(\left| \frac{d}{dx} \zeta_1 \right|_{H^{s'}}, |V|_{t=0} \Big|_{H^{s'}}, T \right)$, for all $j \neq i$.

Finally, for all $(j, k) \neq (i, i)$ with $j \neq k$, U^{ijk} satisfies the hypothesis of Lemma B.1, with $f_{ijk} = g(u_j, \partial_x u_k)$. Therefore, we deduce

$$(B.5) \quad \|U_1\|_{L^\infty(\mathbb{R}^+ \times [0, T]; H^{s'})} \leq \sqrt{t} C_0 \left(\left| \frac{d}{dx} \zeta_1 \right|_{H^{s'+2}}, |V|_{t=0} \Big|_{H^{s'+2}}, T \right).$$

As for the second estimate of the proposition, let us first remark that the estimates of V^{ij} and U^{ijj} are time-independent, and in agreement with the improved estimate. Therefore, the only remaining terms we have to control are U^{ijk} with $j \neq k$. Of course, we will use the second case of Lemma B.1, but we have to check first that for every $\tau \in \mathbb{R}^+$, the initial data $u_j(\tau, 0, x)$ and $\partial_x u_k(\tau, 0, x)$ are localised in space, that is

$$\forall \tau \in \mathbb{R}^+, \quad \left| (1+x^2)^a u_\pm(\tau, 0, x) \right|_{H^{s'}} + \left| (1+x^2)^a \partial_x u_\pm(\tau, 0, x) \right|_{H^{s'}} < \infty.$$

This property is true at $\tau = 0$ (by hypothesis of the proposition), and is propagated to $\tau > 0$, using the fact that $u_{\pm}(\tau, x_{\pm})$ satisfies the KdV equation (3.10). This propagation of the localisation in space has been proved for $a = 2$ by Schneider and Wayne in [48, Lemma 6.4]. Their proof, relying on classical energy estimates and Gronwall's inequality in weighted Sobolev space, actually holds for any $a > 1/2$. We do not recall it here, and make direct use of the statement:

Lemma B.2. *If $(1 + x^2)^a V|_{t=0} \in H^{s'+2}$, for any $a > 0$, then there exists $C_1, \tilde{C}_1 > 0$ such that*

$$|(1 + x^2)^a u_{\pm}(\tau, 0, x)|_{H^{s'+1}} \leq C_1 |(1 + x^2)^a u_{\pm}|_{\tau=t=0} |_{H^{s'+2}} \leq \tilde{C}_1 |(1 + x^2)^a V|_{t=0} |_{H^{s'+2}}.$$

This Lemma, together with the second estimate of Lemma B.1, allows to control uniformly $U^{ijk} \in L^{\infty}(\mathbb{R}^+ \times [0, T]; H^s)$. Every term of the decomposition (B.4) has been controled, and one has the following estimate:

$$(B.6) \quad |U_1|_{L^{\infty}(\mathbb{R}^+ \times [0, T]; H^{s'})} \leq C_0 \left(\left| \frac{d}{dx} \zeta_1 \right|_{H^{s'+2}}, |V|_{t=0} |_{H^{s'+2}}, |(1 + x^2)^a V|_{t=0} |_{H^{s'+2}}, T \right).$$

Step 3: Consistency results. We first prove the consistency of the Boussinesq-type system (3.3) with our approximation. The precision is $\mathcal{O}(\mu^{3/2})$ in the general case, and $\mathcal{O}(\mu^2)$ in the second case. Here and in the following, we set $\alpha = \epsilon_2 = \mu$ in order to simplify the notations. The general case $\alpha = \mathcal{O}(\mu)$, $\epsilon_2 = \mathcal{O}(\mu)$ is obtained with slight obvious modifications.

Plugging $U_{\text{app}}(t, x)$ into (3.6), we see from (3.7)-(3.11) that the only remaining term we have to control is $\mu^2 R(\mu t, t, x)$, with

$$R \equiv \partial_{\tau} U_1 + \Sigma_1(U_0) \partial_x U_1 + \Sigma_1(U_1) \partial_x U_0 + S_1(U_0) \partial_t U_1 + S_1(U_1) \partial_t U_0 \\ - \Sigma_2 \partial_x^3 U_1 - S_2 \partial_x^2 \partial_t U_1 + \mu \Sigma_1(U_1) \partial_x U_1 + \mu S_1(U_1) \partial_x U_1,$$

where $U_0(\mu t, t, x) = u_+(\mu t, x - c_+ t) \mathbf{e}_i + u_-(\mu t, x - c_- t) \mathbf{e}_-$. We bound each term of the right hand side in the Sobolev H^s -norm, with $s > 1/2$ as in the proposition.

The estimate (B.5), with $s' = s + 3$, leads to

$$|\Sigma_2 \partial_x^3 U_1|_{H^s} \leq \sqrt{t} C_0 \left(\left| \frac{d}{dx} \zeta_1 \right|_{H^{s+5}}, |V|_{t=0} |_{H^{s+5}}, T \right).$$

Then, from (3.11), one has

$$\mathbf{e}_i \cdot \partial_t S_0 U_1 = -c_i \mathbf{e}_i \cdot \partial_x S_0 U_1 + f_i$$

with $f_i \in L^{\infty}([0, T] \times [0, T/\mu]; H^{s+2})$, and $|f_i|_{H^{s+2}} \leq C_0 \left(\left| \frac{d}{dx} \zeta_1 \right|_{H^{s+5}}, |V|_{t=0} |_{H^{s+5}}, T \right)$, so that

$$|S_2 \partial_x^2 \partial_t U_1|_{H^s} \leq \sqrt{t} C_0 |\partial_t U_1|_{H^{s+2}} \leq \sqrt{T/\mu} C_0 \left(\left| \frac{d}{dx} \zeta_1 \right|_{H^{s+5}}, |V|_{t=0} |_{H^{s+5}}, T \right).$$

One obtains in the same way the desired estimates for $\Sigma_1(U_0) \partial_x U_1$, $\Sigma_1(U_1) \partial_x U_0$, $S_1(U_0) \partial_t U_1$, $S_1(U_1) \partial_t U_0$, $\Sigma_1(U_1) \partial_x U_1$ and $S_1(U_1) \partial_x U_1$.

Finally, in order to estimate $\partial_{\tau} U_1$, we differentiate (3.11) with respect to τ . Since u_i satisfies (3.10), one has $\partial_{\tau} u_i \in L^{\infty}([0, T]; H^{s+2})$. We are on the framework of Lemma B.1, so that we can obtain, as for (B.5), that $\partial_{\tau} U_1 \in L^{\infty}([0, T]; H^s)$, with

$$|\partial_{\tau} U_1|_{H^s} \leq \sqrt{t} C_0 \left(\left| \frac{d}{dx} \zeta_1 \right|_{H^{s+5}}, |V|_{t=0} |_{H^{s+5}}, T \right).$$

Hence, the residual $\mu^2 R$ is uniformly bounded in $L^{\infty}([0, T/\mu]; H^s)$, and

$$|R|_{H^s} \leq C_0 \mu^{-1/2} C_0 \left(\left| \frac{d}{dx} \zeta_1 \right|_{H^{s+5}}, |V|_{t=0} |_{H^{s+5}}, T \right),$$

which gives the consistency at order $\mathcal{O}(\mu^{3/2})$.

The consistency at order $\mathcal{O}(\mu^2)$, for initial data sufficiently decreasing in space, is obtained in the same way, using the second estimate (B.6).

Step 4: Convergence results. The convergence is deduced from the consistency, as in the proof of Proposition 3.1. Indeed, when setting $R^\mu \equiv U_{\text{app}} - V \equiv U_0(\mu t, t, x) + \mu U_1(\mu t, t, x) - V(t, x)$, one can check that R^μ satisfies the following equation:

$$(B.7) \quad (S_0 - \mu S_2 \partial_x^2 + \mu S_1(U_{\text{app}})) \partial_t R^\mu + (\Sigma_0 - \mu \Sigma_2 \partial_x^2 + \mu \Sigma_1(U_{\text{app}})) \partial_x R^\mu = \mu^{3/2} f + \mu \mathcal{A} + \mu \mathcal{B},$$

with $\mathcal{A} = \partial_t S_1(U_{\text{app}}) R^\mu - S_1(R^\mu) \partial_t V$, $\mathcal{B} = \partial_x \Sigma_1(U_{\text{app}}) R^\mu - \Sigma_1(R^\mu) \partial_x V$ and the function f proved to be uniformly bounded in H^s , by the consistency result. We define the energy as

$$E_s(R^\mu) \equiv \frac{1}{2} (S_0 \Lambda^s R^\mu, \Lambda^s R^\mu) + \frac{\mu}{2} (S_2 \partial_x \Lambda^s R^\mu, \partial_x \Lambda^s R^\mu) + \frac{\mu}{2} (S_1(U_{\text{app}}) \Lambda^s R^\mu, \Lambda^s R^\mu),$$

as in the proof of Proposition 3.1. The same calculations lead to the following differential inequality:

$$\frac{d}{dt} E_s(R^\mu) \leq C_0 \mu E_s(R^\mu) + C_0 \mu^{3/2} (E_s(R^\mu))^2,$$

with $C_0 = C_0(|\frac{d}{dx} \zeta_1|_{H^{s+5}}, |V|_{t=0}|_{H^{s+5}}, T)$. Then, Gronwall-Bihari's theorem allows to obtain $(E_s(R^\mu))^{1/2} \leq C_0 \mu^{1/2} (e^{C_0 \mu t} - 1)$, and finally for $\mu t \leq T(\frac{1}{\gamma + \delta}, \gamma + \delta)$,

$$|U_{\text{app}} - V|_{H^s} \leq C_0 (E_s(R^\mu))^{1/2} \leq \mu^{3/2} t C_0 (|\frac{d}{dx} \zeta_1|_{H^{s+5}}, |V|_{t=0}|_{H^{s+5}}, T).$$

The first estimate of the proposition is now a direct consequence of (B.5), and

$$|\frac{d}{dx} \zeta_1(x)|_{H^{s+5}} + |V(t, x)|_{H^{s+5}} \leq |U(t, x)|_{H^{s+t_0}},$$

thanks to an appropriate estimate of the operator $H(\psi_1, \psi_2)$; see [20, Proposition 2.7].

The second estimate of the proposition follows in the same way, using (B.6) and the consistency at order $\mathcal{O}(\mu^2)$.

C Wave resistance and the dead-water phenomenon

Following Lamb [29] and Kostyukov [28], we assume that the drag experienced by ships is mostly due to the wave (making) resistance. This section is devoted to the analysis of this wave resistance. We first work with the variables *with dimension*, and deduce an explicit formula for the wave resistance R_W , depending on the flow, given as a solution of the full Euler equation (1.1). Accordingly with the nondimensionalization performed in Section 1.2, we introduce the dimensionless version of the wave resistance, that we call *wave resistance coefficient* and denote C_W . Finally, we derive simple approximations, for the two regimes considered throughout the paper.

The wave resistance acknowledges the energy required from the body to push the water away, and is defined by (see [41, 40, 54] and references therein)

$$R_W \equiv \int_{\Gamma_{\text{ship}}} P(-\mathbf{e}_x \cdot \mathbf{n}) \, dS,$$

where Γ_{ship} is the exterior domain of the ship, P is the pressure, \mathbf{e}_x is the horizontal unit vector and \mathbf{n} the normal unit vector exterior to the ship. The pressure being constant ($\equiv P_\infty$) on the non-submerged part of the ship, one has

$$\begin{aligned} R_W &= \int_{\Gamma_{\text{ship}}} (P - P_\infty)(-\mathbf{e}_x \cdot \mathbf{n}) \, dS = \int_{\mathbb{R}} (P|_{d_1+\zeta_1} - P_\infty)(-\partial_x \zeta_1) \, dx \\ &= - \int_{\mathbb{R}} P|_{d_1+\zeta_1} \partial_x \zeta_1 \, dx. \end{aligned}$$

As a solution of the Bernoulli equation in (1.1), the pressure P is given in the upper domain by

$$\frac{P(x, z)}{\rho_1} = -\partial_t \phi_1(x, z) - \frac{1}{2} |\nabla_{x,z} \phi_1(x, z)|^2 - gz,$$

so that the wave resistance satisfies

$$\begin{aligned} R_W &= \rho_1 \int_{\mathbb{R}} g(1 + \zeta_1) \partial_x \zeta_1 - \left(\partial_x \partial_t \phi_1 + \frac{1}{2} \partial_x (\partial_x \phi_1)^2 + \frac{1}{2} \partial_x (\partial_z \phi_1)^2 \right) \Big|_{z=d_1+\zeta_1} \zeta_1 \, dx. \\ &\equiv \int_{\mathbb{R}} F_W[\zeta_2, \phi_1] (d_1 + \zeta_1) \zeta_1 \, dx. \end{aligned}$$

Let us now construct the dimensionless version of this formula. Using the same change of variables as in Section 1.2, it is straightforward to obtain the dimensionless wave resistance (that we call *wave resistance coefficient*, and denote C_W):

$$\begin{aligned} \text{(C.1)} \quad C_W &\equiv \frac{\rho_1 \lambda}{a_2 c_0^2} R_W = \int_{\mathbb{R}} \tilde{F}_W[\tilde{\phi}_1](\tilde{x}) \tilde{\zeta}_1(\tilde{x}) \, d\tilde{x}, \quad \text{with} \\ &- \tilde{F}_W[\tilde{\phi}_1] \equiv (\partial_{\tilde{t}} - \text{Fr} \partial_{\tilde{x}}) \partial_{\tilde{x}} \tilde{\phi}_1 + \frac{\epsilon_2}{2} \partial_{\tilde{x}} \left((\partial_{\tilde{x}} \tilde{\phi}_1)^2 + \frac{\epsilon_2}{2\mu} (\partial_{\tilde{z}} \tilde{\phi}_1)^2 \right). \end{aligned}$$

Again, we omit the tildes in the following, for readability. Let us remark that by definition of the Dirichlet-Neumann operator G_1 , and using (1.4), one has

$$\text{(C.2)} \quad -F_W[\phi_1] = (\partial_t - \text{Fr} \partial_x) \partial_x \phi_1 + \frac{\epsilon_2}{2} \partial_x \left((\partial_x \phi_1)^2 + \left(-\alpha \text{Fr} \partial_x \zeta_1 + \epsilon_1 \frac{d}{dx} \zeta_1 \partial_x \phi_1 \right)^2 \right).$$

For practical purposes, we use approximations to compute the wave resistance coefficient, corresponding to the leading order term in the asymptotic expansion of C_W , for the different regimes at stake. These formulae use the variables of Sections 2 and 3, *i.e.* the surface and interface deviations ζ_1 and ζ_2 and the *shear velocity*, defined by

$$v \equiv \partial_x (\phi_2|_{z=\epsilon_2 \zeta_2} - \gamma \phi_1|_{z=\epsilon_2 \zeta_2}).$$

First of all, we use the following estimate, justified in [20]:

$$\phi_1(x, z) = \psi_1(x) + \mathcal{O}(\mu).$$

When combined with the first equation of (A.4):

$$\partial_x \psi_1 = \frac{-h_2 v + \alpha \text{Fr} \zeta_1}{h_1 + \gamma h_2} + \mathcal{O}(\mu), \quad \text{with} \quad h_1 \equiv 1 + \epsilon_1 \zeta_1, \quad \text{and} \quad h_2 \equiv \frac{1}{\delta} + \epsilon_2 \zeta_2,$$

one has immediately that (C.2) simplifies into

$$(C.3) \quad -F_W[\phi_1] = (\partial_t - \text{Fr} \partial_x) \left(\frac{-h_2 v + \alpha \text{Fr} \zeta_1}{h_1 + \gamma h_2} \right) + \frac{\epsilon_2}{2} \partial_x \left(\frac{(-h_2 v + \alpha \text{Fr} \zeta_1)^2 + (\alpha \frac{d}{dx} \zeta_1)^2 ((h_1 + \gamma h_2) \text{Fr} + \epsilon_2 (h_2 v - \alpha \text{Fr} \zeta_1))^2}{(h_1 + \gamma h_2)^2} \right) + \mathcal{O}(\mu).$$

Now, $\partial_t h_2$, and $\partial_t v$ can be written using only ζ_1 , ζ_2 , v , and their spatial derivatives, since they satisfy system (A.5) up to $\mathcal{O}(\mu^2)$. Therefore, the leading order term of the wave resistance coefficient C_W can be deduced from the knowledge of the solution (ζ_2, v) . We do not write explicitly this expression, as the models used in our simulations benefit from extra smallness assumptions, and simpler formulae are deduced in these cases.

Case of Regime 1 :

$$\mu \ll 1 ; \quad \alpha \equiv \frac{\epsilon_1}{\epsilon_2} = \mathcal{O}(\mu) , \quad 1 - \gamma = \mathcal{O}(\mu).$$

The first immediate simplification in this regime is

$$h_1 + \gamma h_2 = h_1 + h_2 + \mathcal{O}(\mu) = 1 + \frac{1}{\delta} + \mathcal{O}(\mu).$$

Then, we use that (ζ_2, v) satisfies system (2.4) up to $\mathcal{O}(\mu^2)$ to deduce² (with $\bar{h}_1 \equiv 1 + \frac{1}{\delta} - h_2$)

$$\begin{aligned} \partial_t(h_2 v) &= \text{Fr} \partial_x(h_2 v) + (1 + \delta) h_2 \partial_x \zeta_2 + \epsilon_2 \frac{\delta}{1 + \delta} \left(v \partial_x (\bar{h}_1 h_2 v) + \frac{h_2}{2} \partial_x (\bar{h}_1 - h_2) v^2 \right) + \mathcal{O}(\mu) \\ &= \text{Fr} \partial_x(h_2 v) + (1 + \delta) h_2 \partial_x h_2 + \epsilon_2 \partial_x(h_2 v) + \frac{3\epsilon_2}{2} \frac{\delta}{1 + \delta} \partial_x(h_2^2 v^2). \end{aligned}$$

Finally, using these approximations into (C.1) and (C.3), one has

$$(C.4) \quad \begin{aligned} C_W &= \frac{\delta}{1 + \delta} \int_{\mathbb{R}} \left((1 + \delta) h_2 \partial_x \zeta_2 + \epsilon_2 \partial_x(h_2 v) + \epsilon_2 \frac{\delta}{1 + \delta} \partial_x(h_2^2 v^2) \right) \zeta_1(x) \, dx + \mathcal{O}(\mu^2) \\ &= - \int_{\mathbb{R}} \left(\left(\zeta_2 + \frac{\epsilon_2}{2} \zeta_2^2 \right) + \epsilon_2 \frac{\delta}{1 + \delta} h_2 v + \epsilon_2 \left(\frac{\delta}{1 + \delta} h_2 v \right)^2 \right) \frac{d}{dx} \zeta_1(x) \, dx + \mathcal{O}(\mu). \end{aligned}$$

This is the formula used in Figures 3–5, in Section 2.2.

Case of Regime 2 :

$$\mu \ll 1 ; \quad \epsilon_2 = \mathcal{O}(\mu) , \quad \alpha = \mathcal{O}(\mu).$$

Most of the terms of (C.3) are now of size $\mathcal{O}(\mu)$. The first order of system (2.6) leads immediately to

$$\partial_t(h_2 v) - \text{Fr} \partial_x(h_2 v) = -\frac{\gamma + \delta}{\delta} \partial_x \zeta_2 + \mathcal{O}(\mu),$$

so that we obtain simply

$$(C.5) \quad C_W = - \int_{\mathbb{R}} \zeta_2 \frac{d}{dx} \zeta_1 \, dx + \mathcal{O}(\mu).$$

This formula is used in Figures 6–9, in Section 3.3.

²as we use $\text{Bo}^{-1} = \mu^2$ in our simulations, we do not take here into account the surface tension term.

Remark C.1. As we can see in (C.4) and (C.5), the wave resistance coefficient will be small in the two following cases

- i. The internal wave is small, or is not located below the ship,
- ii. The internal wave is symmetric and centered at the location of the ship.

The first case is obvious, and the last case reflects the fact that the integrands of (C.4) and (C.5) are odd if ζ_2 , v and ζ_1 are even.

On the contrary, the ship will encounter a strong positive wave resistance if the functions ζ_2 and h_2v are decreasing at the location of the body. This is the case when an internal wave of elevation is located just behind the ship. The dead-water effect is then explained in that way: the ship, traveling in a density-stratified water, generates internal waves of elevation in its trail, and therefore suffers from a severe wave resistance.

D The Numerical schemes

We present in this section the numerical method used to obtain the figures of this study. For all of the simulations, we use a scheme based on the Crank-Nicholson method, and take care of the nonlinearities using a predictive step. This method has been introduced in [4, 3], and used in the water wave framework in [12, 21, 19]. We give the general directions of such a method in the following, and then present the exact schemes we used for each of the models.

Time discretization. Denoting Δt the time step of the scheme, we approximate a function $u(x, t)$ at time $t = n\Delta t$ by $u(x, n\Delta t) \equiv u^n(x)$. Then, we approximate the time derivative by

$$\partial_t u \approx \frac{u^{n+1} - u^n}{\Delta t}$$

and any linear function of u by

$$F(u) \approx \frac{F(u^{n+1}) + F(u^n)}{2}.$$

Now we deal with the nonlinearities using a predictive step, defined by

$$\frac{u^{n+\frac{1}{2}} + u^{n-\frac{1}{2}}}{2} = u^n.$$

The discretization of the quadratic nonlinearity $F(u, v)$ is then a linear combination of the two possible discretizations, namely

$$F(u, v) \approx \alpha F\left(u^{n+\frac{1}{2}}, \frac{v^{n+1} + v^n}{2}\right) + (1 - \alpha) F\left(\frac{u^{n+1} + u^n}{2}, v^{n+\frac{1}{2}}\right).$$

The parameter α can be chosen so that natural quantities are conserved.

Space discretization. With Δx the space step of the scheme, the functions are discretized spatially with a central difference. In particular,

$$u(x, t) \approx M(\beta)u \text{ with } (M(\beta)u^n)_i = (1 - \beta)u_i^n + \frac{\beta}{2}(u_{i+1}^n + u_{i-1}^n),$$

and again, the parameter β is chosen so that natural quantities are conserved. The spatial derivatives are given by

$$\partial_x u \approx \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} \equiv (D_1 u^n)_i, \quad \partial_x^2 u \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} \equiv (D_2 u^n)_i, \quad \dots$$

We use periodic boundary conditions.

D.1 The forced Korteweg-de-Vries equation

We want to solve numerically the following generic forced KdV equation

$$(D.1) \quad \partial_t u + c \partial_x u + \lambda u \partial_x u + \nu \partial_x^3 u = f(x).$$

When there is no forcing term ($f \equiv 0$), it is known that the KdV equation preserves the energy $E \equiv |u|_{L^2}^2$. The following scheme has been presented and studied (without the forcing term) in [21], and has the property to preserve the discrete energy when $f \equiv 0$.

$$(D.2) \quad \frac{u_i^{n+1} - u_i^n}{\Delta t} + c \left(D_1 \frac{u^{n+1} + u^n}{2} \right)_i + \nu \left(D_3 \frac{u^{n+1} + u^n}{2} \right)_i + \frac{\lambda}{3} \left(\left(M(1) \frac{u_i^{n+1} + u_i^n}{2} \right)_i \left(D_1 u^{n+\frac{1}{2}} \right)_i \right. \\ \left. + 2 \left(M(1/2) u^{n+\frac{1}{2}} \right)_i \left(D_1 \frac{u^{n+1} + u^n}{2} \right)_i \right) = f(i\Delta x).$$

Proposition D.1. *If $f \equiv 0$, then the scheme (D.2) preserves the discrete l^2 -norm:*

$$\forall n \in \mathbb{N}, \quad |u^n|_{l^2}^2 \equiv \sum_i |u_i^n|^2 = \sum_i |u_i^0|^2.$$

The proof is given in [21, Theorem 2].

D.2 The fully nonlinear model of Regime 1

The system we deal with now is (2.6), that we can write under the compact form

$$(D.3) \quad (\partial_t - \text{Fr} \partial_x) (U - \mu \mathcal{R}_1[U]) + \partial_x ((\mathcal{A}[U] + \epsilon_1 \mathcal{B}(x))U) + \mu \epsilon_2 \partial_x (\mathcal{R}_2[U] \cdot U) = b(x) + \frac{1}{\text{Bo}} \partial_x^2 (\mathcal{T}[U]),$$

with $U \equiv (\zeta_2, w)^T$, $b(x) \equiv -\alpha \text{Fr} / (1 + \delta) (\zeta_1(x), 0)^T$, and

$$\mathcal{A}[U] \equiv \begin{pmatrix} 0 & \frac{h_1 h_2}{h_1 + \gamma h_2} \\ \gamma + \delta & \epsilon_2 \frac{h_1^2 - \gamma h_2^2}{(h_1 + \gamma h_2)^2} w \end{pmatrix}, \quad \mathcal{B}(x) \equiv \frac{\text{Fr}}{h} \begin{pmatrix} \zeta_1(x) & 0 \\ 0 & \zeta_1(x) \end{pmatrix}, \\ \mathcal{R}_i[U] \equiv \begin{pmatrix} 0 \\ \mathcal{S}_i[h_2]w \end{pmatrix}, \quad \mathcal{T}[U] \equiv \begin{pmatrix} 0 \\ \frac{\partial_x \zeta_2}{\sqrt{1 + \mu \epsilon_2^2 |\partial_x \zeta_2|^2}} \end{pmatrix}.$$

It is convenient to denote, with $h_1 \equiv 1 + \epsilon_1 \zeta_1 - \epsilon_2 \zeta_2$ and $h_2 \equiv \frac{1}{\delta} + \epsilon_2 \zeta_2$,

$$f[\zeta_2] \equiv \frac{h_1 h_2}{h_1 + \gamma h_2}, \quad g[\zeta_2] \equiv \epsilon_2 \frac{h_1^2 - \gamma h_2^2}{(h_1 + \gamma h_2)^2}, \quad \text{and} \quad t[\zeta_2] \equiv \frac{\partial_x \zeta_2}{\sqrt{1 + \mu \epsilon_2^2 |\partial_x \zeta_2|^2}}.$$

Advised by the above work on the KdV equation, we use the following time discretization:

$$f[\zeta]w \approx f[\zeta^{n+\frac{1}{2}}] \frac{w^n + w^{n+1}}{2}, \quad g[\zeta]w^2 \approx g[\zeta^{n+\frac{1}{2}}] w^{n+\frac{1}{2}} \frac{w^n + w^{n+1}}{2}, \\ t[\zeta] \approx \frac{1}{\sqrt{1 + \mu \epsilon_2^2 |\partial_x \zeta^{n+\frac{1}{2}}|^2}} \partial_x \left(\frac{\zeta^{n+1} + \zeta^n}{2} \right), \\ \partial_t (\mathcal{S}_1[h]w) \approx \mathcal{S}_1[h^{n+\frac{1}{2}}] \frac{w^{n+1} - w^n}{\Delta t} + \tilde{\mathcal{S}}_1[h^{n+\frac{1}{2}}, w^{n+\frac{1}{2}}] \frac{\zeta^{n+1} - \zeta^n}{\Delta t}, \\ \text{with} \quad \tilde{\mathcal{S}}_1[h, w]\zeta \equiv \frac{\epsilon_2}{3} \partial_x^2 (\zeta(1 + 1/\delta - 2h)w) - 2\epsilon_2 (\partial_x \zeta)(\partial_x h)w, \\ w \mathcal{S}_2[H]w \approx \frac{1}{3} w^{n+\frac{1}{2}} \mathcal{S}_2[H^{n+\frac{1}{2}}] \left(\frac{w^n + w^{n+1}}{2} \right) + \frac{2}{3} \frac{w^n + w^{n+1}}{2} \mathcal{S}_2[H^{n+\frac{1}{2}}] w^{n+\frac{1}{2}}.$$

Finally, after the space discretization, this leads to the following scheme (with $h^n \equiv \frac{1}{\delta} + \epsilon_2 \zeta^n$)

(D.4)

$$\begin{aligned} \frac{\zeta_i^{n+1} - \zeta_i^n}{\Delta t} - \text{Fr} \left(D_1 \frac{\zeta^{n+1} + \zeta^n}{2} \right)_i + \alpha \frac{\text{Fr} \delta}{1 + \delta} \left(D_1 \left(\left(\frac{1}{\delta} + \epsilon_2 \frac{\zeta^{n+1} + \zeta^n}{2} \right) \zeta_1(i\Delta x) \right) \right)_i \\ + \frac{1}{3} D_1 \left(M(1) f[h^{n+\frac{1}{2}}] \frac{w^n + w^{n+1}}{2} + f[h^{n+\frac{1}{2}}] M(1/2) (w^n + w^{n+1}) \right)_i = 0, \end{aligned}$$

(D.5)

$$\begin{aligned} \frac{(I - \mu \mathcal{S}_1[h^{n+\frac{1}{2}}])(w^{n+1} - w^n) - \mu \tilde{\mathcal{S}}_1[h^{n+\frac{1}{2}}, w^{n+\frac{1}{2}}](\zeta^{n+1} - \zeta^n)}{\Delta t} + (\gamma + \delta) \left(D_1 \frac{\zeta^n + \zeta^{n+1}}{2} \right)_i \\ - \text{Fr} \left(D_1 (I - \mu \mathcal{S}_1[h^{n+\frac{1}{2}}]) \frac{w^{n+1} + w^n}{2} \right)_i + \alpha \frac{\text{Fr} \delta}{1 + \delta} \left(D_1 \left(\frac{w^{n+1} + w^n}{2} \zeta_1(i\Delta x) \right) \right)_i \\ + \frac{\epsilon_2}{3} D_1 \left(M(1) g[h^{n+\frac{1}{2}}] w^{n+\frac{1}{2}} \frac{w^n + w^{n+1}}{2} + g[h^{n+\frac{1}{2}}] w^{n+\frac{1}{2}} M(1/2) (w^n + w^{n+1}) \right)_i \\ + \frac{\mu \epsilon_2}{3} D_1 \left(w^{n+\frac{1}{2}} \mathcal{S}_2[H^{n+\frac{1}{2}}] \left(\frac{w^n + w^{n+1}}{2} \right) + (w^n + w^{n+1}) \mathcal{S}_2[H^{n+\frac{1}{2}}] w^{n+\frac{1}{2}} \right)_i \\ = \frac{1}{\text{Bo}} \left(D_2 \frac{1}{\sqrt{1 + \mu \epsilon_2^2 |\partial_x \zeta^{n+\frac{1}{2}}|^2}} D_1 \left(\frac{\zeta^{n+1} + \zeta^n}{2} \right) \right)_i. \end{aligned}$$

D.3 Validation of the method

In order to validate the proposed schemes, we use known explicit solutions of the forced Korteweg-de Vries equation (3.13). The first test function we use is the travelling wave

$$(D.6) \quad u_0 \equiv -\text{sech}^2(k(x - c_{sw}t)),$$

with $c_{sw} = c + 4\nu k^2$, $k = \sqrt{-\lambda/(12\nu)}$, corresponding to the classical case where there is no forcing.

The second test function is the steady solution

$$(D.7) \quad u_1 \equiv \pm \text{sech}^2(kx),$$

with $k^2 = \frac{\lambda}{12\nu}$ and the corresponding forcing

$$f(x) = \left(c + \frac{\lambda}{3} \right) \text{sech}^2(kx).$$

These two functions are exact solutions of the forced KdV equations

$$\partial_t u + c \partial_x u + \lambda u \partial_x u + \nu \partial_x^3 u = \frac{d}{dx} f(x).$$

In the absence of non-trivial solutions system (2.6), we use the same functions as reference. After an appropriate change of variables, u_0 and u_1 will satisfy system (2.6) up to small terms. Therefore, adding the corresponding forcing term to the equations, u_0 and u_1 are exact solutions of the modified system, and we are able to compare the results of our numerical schemes with the theoretical solution.

These comparisons are presented in Tables 2 and 3. The error are givens in term of the normalized l^2 norm. The results show a convergence behavior of order $\mathcal{O}((\Delta x)^2 + (\Delta t)^2)$, and allow to validate our schemes.

$\Delta x = \Delta t$	L	T	KdV scheme	fully nonlinear scheme
0.1	20	10	$8.1530 \cdot 10^{-4}$	$4.9498 \cdot 10^{-4}$
0.05	20	10	$2.0393 \cdot 10^{-4}$	$1.5154 \cdot 10^{-4}$
0.01	20	10	$8.1604 \cdot 10^{-6}$	$1.8363 \cdot 10^{-5}$

Table 2: Numerical errors of the KdV and Green-Naghdi schemes for the initial data (D.6). We choose the parameters $\mu = \epsilon_2 = \alpha = 0.1$, $\gamma = 0.9$, $\delta = 5/12$, $\text{Fr} = 1.1$, $\text{Bo} = 100$.

$\Delta x = \Delta t$	L	T	KdV scheme	fully nonlinear scheme
0.1	20	10	$4.7397 \cdot 10^{-4}$	$1.7844 \cdot 10^{-4}$
0.05	20	10	$1.1850 \cdot 10^{-4}$	$4.4557 \cdot 10^{-5}$
0.01	20	10	$4.7409 \cdot 10^{-6}$	$8.1604 \cdot 10^{-6}$

Table 3: Numerical errors of the KdV and Green-Naghdi schemes for the initial data (D.7). We choose the parameters $\mu = \epsilon_2 = \alpha = 0.1$, $\gamma = 0.9$, $\delta = 5/12$, $\text{Fr} = 1.1$, $\text{Bo} = 100$.

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