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HYPERCYCLIC ABELIAN AFFINE GROUPS

ADLENE AYADI

ABSTRACT. In this paper, we give a characterization of hypercyclic abelian affine group \mathcal{G} . If \mathcal{G} is finitely generated, this characterization is explicit. We prove in particular that no abelian group generated by n affine maps on \mathbb{C}^n has a dense orbit.

1. INTRODUCTION

Let $M_n(\mathbb{C})$ be the set of all square matrices of order $n \geq 1$ with entries in \mathbb{C} and $GL(n, \mathbb{C})$ be the group of all invertible matrices of $M_n(\mathbb{C})$. A map $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is called an affine map if there exist $A \in M_n(\mathbb{C})$ and $a \in \mathbb{C}^n$ such that $f(x) = Ax + a$, $x \in \mathbb{C}^n$. We denote $f = (A, a)$, we call A the *linear part* of f . Denote by $MA(n, \mathbb{C})$ the set of all affine maps and $GA(n, \mathbb{C})$ the set of all invertible affine maps of $MA(n, \mathbb{C})$. $MA(n, \mathbb{C})$ is a vector space and for composition of maps, $GA(n, \mathbb{C})$ is a group.

Let \mathcal{G} be an abelian affine subgroup of $GA(n, \mathbb{C})$. For a vector $v \in \mathbb{C}^n$, we consider the orbit of \mathcal{G} through v : $\mathcal{G}(v) = \{f(v) : f \in \mathcal{G}\} \subset \mathbb{C}^n$. A subset $E \subset \mathbb{C}^n$ is called \mathcal{G} -invariant if $f(E) \subset E$ for any $f \in \mathcal{G}$; that is E is a union of orbits. Before stating our main results, we introduce the following notions:

A subset \mathcal{H} of \mathbb{C}^n is called an *affine subspace* of \mathbb{C}^n if there exist a vector subspace H of \mathbb{C}^n and $a \in \mathbb{C}^n$ such that $\mathcal{H} = H + a$. For $a \in \mathbb{C}^n$, denote by $T_a : \mathbb{C}^n \rightarrow \mathbb{C}^n$; $x \mapsto x + a$ the translation map by vector a , so $\mathcal{H} = T_a(H)$. We say that \mathcal{H} has dimension p ($0 \leq p \leq n$), denoted $\dim(\mathcal{H}) = p$, if H has dimension p .

Denote by \overline{A} the closure of a subset $A \subset \mathbb{C}^n$. A subset E of \mathbb{C}^n is called a *minimal set* of \mathcal{G} if E is closed in \mathbb{C}^n , non empty, \mathcal{G} -invariant and has no proper subset with these properties. It is equivalent to say that E is a \mathcal{G} -invariant set such that every orbit contained in E is dense in it. The group \mathcal{G} is called *hypercyclic* if there exists a vector $v \in \mathbb{C}^n$ such that $\mathcal{G}(v)$ is dense in \mathbb{C}^n . For an account of results and bibliography on hypercyclicity, we refer to the book [2] by Bayart and Matheron.

Define the map

$$\Phi : GA(n, \mathbb{C}) \rightarrow \Phi(GA(n, \mathbb{C})) \subset GL(n+1, \mathbb{C})$$

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$$f = (A, a) \mapsto \begin{bmatrix} 1 & 0 \\ a & A \end{bmatrix}$$

We have the following composition formula

$$\begin{bmatrix} 1 & 0 \\ a & A \end{bmatrix} \begin{bmatrix} 1 & 0 \\ b & B \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ Ab + a & AB \end{bmatrix}.$$

Then Φ is a homomorphism of groups.

Let \mathcal{G} be an abelian affine subgroup of $GA(n, \mathbb{C})$. Then $\Phi(\mathcal{G})$ is an abelian subgroup of $GL(n+1, \mathbb{C})$.

Denote by:

- $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $\mathbb{N}_0 = \mathbb{N} \setminus \{0\}$.

Let $n \in \mathbb{N}_0$ be fixed. For each $m = 1, 2, \dots, n+1$, denote by:

- $\mathcal{B}_0 = (e_1, \dots, e_{n+1})$ the canonical basis of \mathbb{C}^{n+1} and I_{n+1} the identity matrix of $GL(n+1, \mathbb{C})$.

- $\mathbb{T}_m(\mathbb{C})$ the set of matrices over \mathbb{C} of the form

$$\begin{bmatrix} \mu & & & 0 \\ a_{2,1} & \mu & & \\ \vdots & \ddots & \ddots & \\ a_{m,1} & \dots & a_{m,m-1} & \mu \end{bmatrix} \quad (1)$$

- $\mathbb{T}_m^*(\mathbb{C})$ the group of matrices of the form (1) with $\mu \neq 0$.

Let $r \in \mathbb{N}$ and $\eta = (n_1, \dots, n_r)$ be a sequence of positive integers such that $n_1 + \dots + n_r = n+1$. In particular, $r \leq n+1$.

Write

- $\mathcal{K}_{\eta,r}(\mathbb{C}) := \mathbb{T}_{n_1}(\mathbb{C}) \oplus \dots \oplus \mathbb{T}_{n_r}(\mathbb{C})$. In particular if $r = 1$, then $\mathcal{K}_{\eta,1}(\mathbb{C}) = \mathbb{T}_{n+1}(\mathbb{C})$ and $\eta = (n+1)$.
- $\mathcal{K}_{\eta,r}^*(\mathbb{C}) := \mathcal{K}_{\eta,r}(\mathbb{C}) \cap GL(n+1, \mathbb{C})$.

Define the map

$$\Psi : MA(n, \mathbb{C}) \longrightarrow \Psi(MA(n, \mathbb{C})) \subset M_{n+1}(\mathbb{C})$$

$$f = (A, a) \mapsto \begin{bmatrix} 0 & 0 \\ a & A \end{bmatrix}$$

We have Ψ is an isomorphism.

- $\mathcal{F}_{n+1} = \Psi(MA(n, \mathbb{C}))$.
- $\exp : M_{n+1}(\mathbb{C}) \longrightarrow GL(n+1, \mathbb{C})$ is the matrix exponential map; set $\exp(M) = e^M$.

There always exists a $P \in \Phi(GA(n, \mathbb{C}))$ and a partition η of $n+1$ such that $G' = P^{-1}GP \subset \mathcal{K}_{\eta,r}^*(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C}))$ (see Proposition 2.2). For such a choice of matrix P , we let

- $\mathfrak{g} = \exp^{-1}(G) \cap (P(\mathcal{K}_{\eta,r}(\mathbb{C}))P^{-1}) \cap \mathcal{F}_{n+1}$. If $G \subset \mathcal{K}_{\eta,r}^*(\mathbb{C})$, we have $\mathfrak{g} = \exp^{-1}(G) \cap \mathcal{K}_{\eta,r}(\mathbb{C}) \cap \mathcal{F}_{n+1}$.

- $\mathfrak{g}_u = \{Bu : B \in \mathfrak{g}\}$, $u \in \mathbb{C}^{n+1}$.
 - $\mathfrak{g} = \Psi^{-1}(\mathfrak{g})$.
 - $\mathfrak{g}_v = \{f(v), f \in \mathfrak{g}\}$, $v \in \mathbb{C}^n$.
 - $u_0 = [e_{1,1}, \dots, e_{r,1}]^T \in \mathbb{C}^{n+1}$ where $e_{k,1} = [1, 0, \dots, 0]^T \in \mathbb{C}^{n_k}$, for $k = 1, \dots, r$.
- One has $u_0 \in \{1\} \times \mathbb{C}^n$.
- $p_2 : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^n$ the projection defined by $p_2(x_1, \dots, x_{n+1}) = (x_2, \dots, x_{n+1})$.
 - $v_0 = Pu_0$. As $P \in \Phi(GA(n, \mathbb{C}))$, $v_0 \in \{1\} \times \mathbb{C}^n$.
 - $w_0 = p_2 v_0 \in \mathbb{C}^n$.
 - $e^{(k)} = [e_1^{(k)}, \dots, e_r^{(k)}]^T \in \mathbb{C}^{n+1}$ where

$$e_j^{(k)} = \begin{cases} 0 \in \mathbb{C}^{n_j} & \text{if } j \neq k \\ e_{k,1} & \text{if } j = k \end{cases} \quad \text{for every } 1 \leq j, k \leq r.$$

For groups of affine maps on \mathbb{K}^n ($\mathbb{K} = \mathbb{R}$ or \mathbb{C}), their dynamics were recently initiated for some classes in different point of view, (see for instance, [3], [4], [6],[5]). The purpose here is to give analogous results of that theorem for linear abelian subgroup of $GL(n, \mathbb{C})$ proved in [1] (see Proposition 3.1). Our main results are the following:

Theorem 1.1. *Let \mathcal{G} be an abelian subgroup of $GA(n, \mathbb{C})$. The following are equivalent:*

- (i) \mathcal{G} is hypercyclic.
- (ii) The orbit $\mathcal{G}(w_0)$ is dense in \mathbb{C}^n
- (iii) \mathfrak{g}_{w_0} is an additive subgroup dense in \mathbb{C}^n

For a *finitely generated* abelian subgroup $\mathcal{G} \subset GA(n, \mathbb{R})$, let introduce the following property. Consider the following rank condition on a collection of affine maps $f_1, \dots, f_p \in GA(n, \mathbb{C})$, where $f'_1, \dots, f'_p \in \mathfrak{g}$ such that $e^{\Psi(f'_k)} = \Phi(f_k)$, $k = 1, \dots, p$.

We say that f_1, \dots, f_p satisfy *property \mathcal{D}* if for every $(s_1, \dots, s_p; t_1, \dots, t_r) \in \mathbb{Z}^{p+r} \setminus \{0\}$:

$$\text{rank} \begin{bmatrix} \text{Re}(f'_1(w_0)) & \dots & \text{Re}(f'_p(w_0)) & 0 & \dots & 0 \\ \text{Im}(f'_1(w_0)) & \dots & \text{Im}(f'_p(w_0)) & 2\pi p_2 \circ Pe^{(1)} & \dots & 2\pi p_2 \circ Pe^{(r)} \\ s_1 & \dots & s_p & t_1 & \dots & t_r \end{bmatrix} = 2n+1.$$

For a vector $v \in \mathbb{C}^n$, we write $v = \text{Re}(v) + i\text{Im}(v)$ where $\text{Re}(v)$ and $\text{Im}(v) \in \mathbb{R}^n$. In this case, the Theorem can be stated as follows:

Theorem 1.2. *Let \mathcal{G} be an abelian subgroup of $GA(n, \mathbb{C})$ generated by f_1, \dots, f_p and let $f'_1, \dots, f'_p \in \mathfrak{g}$ such that $e^{\Psi(f'_1)} = \Phi(f_1), \dots, e^{\Psi(f'_p)} = \Phi(f_p)$. Then the following are equivalent:*

- (i) \mathcal{G} is hypercyclic.
- (ii) the maps f_1, \dots, f_p satisfy property \mathcal{D}
- (iii) $\mathfrak{g}_{w_0} = \sum_{k=1}^p \mathbb{Z}f'_k(w_0) + 2i\pi \sum_{k=1}^r \mathbb{Z}(p_2 \circ Pe^{(k)})$ is an additive group dense in \mathbb{C}^n .

Corollary 1.3. *If \mathcal{G} is of finite type p with $p \leq 2n - r + 1$, then it has no dense orbit.*

Corollary 1.4. *If \mathcal{G} is of finite type p with $p \leq n$, then it has no dense orbit.*

2. NOTATIONS AND LEMMAS

Denote by $\mathcal{L}_{\mathcal{G}}$ the set of the linear parts of all elements of \mathcal{G} and $\text{vect}(F)$ is the vector space generated by a subset $F \subset \mathbb{C}^{n+1}$. In the following, denote by I_m the identity matrix of $GL(m, \mathbb{C})$, for any $m \in \mathbb{N}_0$.

Proposition 2.1. ([1], Proposition 2.3) *Let L be an abelian subgroup of $GL(n, \mathbb{C})$. Then there exists $P \in GL(n, \mathbb{C})$ such that $P^{-1}LP$ is a subgroup of $\mathcal{K}_{\eta', r'}^*(\mathbb{C})$, for some $r' \leq n$ and $\eta' = (n'_1, \dots, n'_{r'}) \in \mathbb{N}_0^{r'}$.*

Proposition 2.2. *Let \mathcal{G} be an abelian subgroup of $GA(n, \mathbb{C})$ and $G = \Phi(\mathcal{G})$. Then there exists $P \in \Phi(GA(n, \mathbb{C}))$ such that $P^{-1}GP$ is a subgroup of $\mathcal{K}_{\eta, r}^*(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C}))$, for some $r \leq n + 1$ and $\eta = (n_1, \dots, n_r) \in \mathbb{N}_0^r$. In particular, $Pu_0 \in \{1\} \times \mathbb{C}^n$.*

Proof. We have $\mathcal{L}_{\mathcal{G}}$ is an abelian subgroup of $GL(n, \mathbb{C})$. By Proposition 2.1, there exists $Q \in GL(n, \mathbb{C})$ such that $Q^{-1}\mathcal{L}_{\mathcal{G}}Q$ is a subgroup of $\mathcal{K}_{\eta', r'}^*(\mathbb{C})$ for some $r' \leq n$ and $\eta' = (n'_1, \dots, n'_{r'}) \in \mathbb{N}_0^{r'}$ such that $n'_1 + \dots + n'_{r'} = n$. For every $A \in \mathcal{L}_{\mathcal{G}}$, $Q^{-1}AQ = \text{diag}(A_1, \dots, A_{r'})$ with $A_k \in \mathbb{T}_{n'_k}^*$ and μ_{A_k} is the only eigenvalue of A_k , $k = 1, \dots, r'$. Let $J = \{k \in \{1, \dots, r'\}, \mu_{A_k} = 1, \forall A \in \mathcal{L}_{\mathcal{G}}\}$. There are two cases:

- *Case1:* Suppose that $J = \{k_1, \dots, k_s\}$ for some $s \leq r'$. We can take $J = \{1, \dots, s\}$, otherwise, we replace P_1 by RP_1 for some permutation matrix R of $GL(n, \mathbb{C})$. Let $P_1 = \text{diag}(1, Q)$, so $P_1 \in \Phi(GA(n, \mathbb{C}))$ and

$$P_1^{-1}\Phi(f)P_1 = \begin{bmatrix} 1 & 0 \\ Q^{-1}a & Q^{-1}AQ \end{bmatrix}.$$

Write $E = \text{vect}(C_1, \dots, C_s)$ and $H = \text{vect}(C_{s+1}, \dots, C_{r'})$, so E and H are G -invariant vector spaces. Moreover, for every $f = (A, a) \in \mathcal{G}$ the restriction $A|_E$ has 1 as only eigenvalue and we have

$$P_1^{-1}\Phi(f)P_1 = \begin{bmatrix} 1 & 0 & 0 \\ a_1 & A_1 & 0 \\ a_2 & 0 & A_2 \end{bmatrix}, \quad (1)$$

where $A_1 = A|_E \in \mathbb{T}_p^*(\mathbb{C})$, $A_2 = A|_H \in \mathbb{T}_{n-p}^*(\mathbb{C})$, $a_1 \in \mathbb{C}^p$, $a_2 \in \mathbb{C}^{n-p}$, $p = n'_1 + \dots + n'_s$. On the other hand, there exists $f_0 = (B, b) \in \mathcal{G}$ such that $B_2 = B|_H$ has no eigenvalue equal to 1, so $B_2 - I_{n-p}$ is invertible. As in (1), write

$$P_1^{-1}\Phi(f_0)P_1 = \begin{bmatrix} 1 & 0 & 0 \\ b_1 & B_1 & 0 \\ b_2 & 0 & B_2 \end{bmatrix}.$$

Let $P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & I_p & 0 \\ b_2 & 0 & B_2 - I_{n-p} \end{bmatrix}$ and $P = P_1P_2^{-1}$, then $P = \begin{bmatrix} 1 & 0 \\ d & P_0 \end{bmatrix} \in$

$\Phi(GA(n, \mathbb{C}))$, where $P_0 = Q \cdot Q_1^{-1}$, $Q_1 = \begin{bmatrix} I_p & 0 \\ 0 & B_2 - I_{n-p} \end{bmatrix}$ and $d = -P_0[0, b_2]^T$.

For every $f = (A, a) \in \mathcal{G}$ we have by (1),

$$\begin{aligned} P^{-1}\Phi(f)P &= P_2P_1^{-1}\Phi(f)P_1P_2^{-1} \\ &= P_2 \begin{bmatrix} 1 & 0 & 0 \\ a_1 & A_1 & 0 \\ a_2 & 0 & A_2 \end{bmatrix} P_2^{-1} \\ &= \begin{bmatrix} & & 1 & & 0 & 0 \\ & & a_1 & & A_1 & 0 \\ -(A_2 - I_{n-p})b_2 + (B_2 - I_{n-p})a_2 & & 0 & & 0 & A_2 \end{bmatrix} \quad (2) \end{aligned}$$

Since G is abelian, so by the equality $P_1^{-1}\Phi(f)\Phi(f_0)P_1 = P_1^{-1}\Phi(f_0)\Phi(f)P_1$, we find $-(A_2 - I_{n-p})b_2 + (B_2 - I_{n-p})a_2 = 0$.

It follows by (2), that $P^{-1}GP$ is a subgroup of $\mathcal{K}_{\eta,r}^*(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C}))$, where $r = r' - s + 1$ and $\eta = (p + 1, n'_{s+1}, \dots, n'_{r'})$.

- *Case2:* Suppose that $J = \emptyset$, and denote by $Fix(G) = \{x \in \mathbb{C}^{n+1} : Bx = x, \forall B \in G\}$, then $Fix(G) = \mathbb{C}v$ for some $v = (1, v_1)$, $v_1 \in \mathbb{C}^n$. For every $f = (A, a) \in \mathcal{G}$, one has $\Phi(f)(1, v_1) = (1, f(v_1)) = (1, v_1)$ so $f(v_1) = Av_1 + a = v_1$.

Let $P = \begin{bmatrix} 1 & 0 \\ v_1 & P_1 \end{bmatrix} \in \Phi(GA(n, \mathbb{C}))$, then

$$\begin{aligned} P^{-1}\Phi(f)P &= \begin{bmatrix} 1 & 0 \\ -P_1^{-1}v_1 & P_1^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ a & A \end{bmatrix} \begin{bmatrix} 1 & 0 \\ v_1 & P_1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ P_1^{-1}(Av_1 + a - v_1) & P_1^{-1}AP_1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & P_1^{-1}AP_1 \end{bmatrix}. \end{aligned}$$

It follows that $P^{-1}GP$ is a subgroup of $\mathcal{K}_{\eta,r}^*(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C}))$, where $r = r' + 1$ and $\eta = (1, n'_1, \dots, n'_{r'})$.

Since $u_0 \in \{1\} \times \mathbb{C}^n$ and $P \in \Phi(GA(n, \mathbb{C}))$, so $Pu_0 \in \{1\} \times \mathbb{C}^n$. \square

Denote by:

- $G' = P^{-1}GP$.
- $g' = \exp^{-1}(G') \cap \mathcal{K}_{\eta,r}(\mathbb{C}) \cap \mathcal{F}_{n+1}$.
- $g_1 = \exp^{-1}(G) \cap (PK_{\eta,r}(\mathbb{C})P^{-1})$

Lemma 2.3. ([1], Proposition 3.2) $\exp(\mathcal{K}_{\eta,r}(\mathbb{C})) = \mathcal{K}_{\eta,r}^*(\mathbb{C})$.

Lemma 2.4. If $N \in PK_{\eta,r}(\mathbb{C})P^{-1}$ such that $e^N \in \Phi(GA(n, \mathbb{C}))$, so $N - 2ik\pi I_{n+1} \in \mathcal{F}_{n+1}$, for some $k \in \mathbb{Z}$.

Proof. Let $N' = P^{-1}NP \in \mathcal{K}_{\eta,r}(\mathbb{C})$, $M = e^N$ and $M' = P^{-1}MP$. We have $e^{N'} = M'$ and by Lemma 2.3, $M' \in \mathcal{K}_{\eta,r}^*(\mathbb{C})$. Write $M' = \text{diag}(M'_1, \dots, M'_r)$ and $N' = \text{diag}(N'_1, \dots, N'_r)$, $M'_k, N'_k \in \mathbb{T}_{n_k}(\mathbb{C})$, $k = 1, \dots, r$. Then $e^{N'} = \text{diag}(e^{N'_1}, \dots, e^{N'_r})$,

so $e^{N'_1} = M'_1$. As 1 is the only eigenvalue of M'_1 , N'_1 has an eigenvalue $\mu \in \mathbb{C}$ such that $e^\mu = 1$. Thus $\mu = 2ik\pi$ for some $k \in \mathbb{Z}$. Therefore, $N'' = N' - 2ik\pi I_{n+1} \in \mathcal{F}_{n+1}$ and satisfying $e^{N''} = e^{-2ik\pi} e^{N'} = M'$. It follows that $N - 2ik\pi I_{n+1} = PN''P^{-1} \in P\mathcal{F}_{n+1}P^{-1} = \mathcal{F}_{n+1}$, since $P \in \Phi(GA(n, \mathbb{C}))$. \square

Lemma 2.5. ([1], Lemma 4.2) *Under above notations, one has $\exp(\mathfrak{g}_1) = G$.*

As consequence, we obtain

Proposition 2.6. *We have:*

- (i) $\mathfrak{g}_1 = \mathfrak{g} + 2i\pi\mathbb{Z}I_{n+1}$.
- (ii) $\exp(\mathfrak{g}) = G$.

Proof. (i) By Lemma 2.5, $\exp(\mathfrak{g}_1) = G \subset \Phi(GA(n, \mathbb{C}))$, then by Lemma 2.4 we have $\mathfrak{g}_1 \subset \mathfrak{g} + 2i\pi\mathbb{Z}I_{n+1}$. Conversely is obvious since $\mathfrak{g} + 2i\pi\mathbb{Z}I_{n+1} \subset \mathcal{K}_{\eta,r}(\mathbb{C})$ and $\exp(\mathfrak{g} + 2i\pi\mathbb{Z}I_{n+1}) = \exp(\mathfrak{g}) \subset G$.

(ii) By Lemma 2.5 and (i) we have $G = \exp(\mathfrak{g}_1) = \exp(\mathfrak{g} + 2i\pi\mathbb{Z}I_{n+1}) = \exp(\mathfrak{g})$. \square

3. PROOF OF THEOREM 1.1

Let \tilde{G} be the group generated by G and $H = \{\lambda.I_{n+1} : \lambda \in \mathbb{C}^*\}$, and write $\tilde{\mathfrak{g}} = \exp^{-1}(\tilde{G} \cap \mathcal{K}_{\eta,r}(\mathbb{C}))$. Then \tilde{G} is an abelian subgroup of $GL(n+1, \mathbb{C})$. See that $Id_{\mathbb{C}^{n+1}} \in \exp^{-1}(H) \subset \tilde{\mathfrak{g}}$, so $\tilde{\mathfrak{g}} \setminus \Psi(MA(n, \mathbb{C})) \neq \emptyset$.

Denote by:

- $(\mathfrak{g}_1)_{v_0} = \{Bv_0 : B \in \mathfrak{g}_1\}$.
- $\tilde{\mathfrak{g}}_{v_0} = \{Bv_0 : B \in \tilde{\mathfrak{g}}\}$.

Proposition 3.1. ([1], Theorem 1.1) *Let G be an abelian subgroup of $GL(n+1, \mathbb{C})$.*

The following are equivalent:

- (i) G has a dense orbit in \mathbb{C}^{n+1}
- (ii) The orbit $G(v_0)$ is dense in \mathbb{C}^{n+1}
- (iii) $(\mathfrak{g}_1)_{v_0}$ is an additive subgroup dense in \mathbb{C}^{n+1}

Lemma 3.2. *The following assertions are equivalent:*

- (i) $\overline{\mathfrak{g}_{w_0}} = \mathbb{C}^n$.
- (ii) $\overline{\mathfrak{g}_{v_0}} = \{0\} \times \mathbb{C}^n$.
- (iii) $\overline{\tilde{\mathfrak{g}}_{v_0}} = \mathbb{C}^{n+1}$.

Proof. (i) \iff (ii) : For every $f' = (B, b) \in \mathfrak{g}$, one has

$$\begin{aligned} \Psi(f')v_0 &= \begin{bmatrix} 0 & 0 \\ b & B \end{bmatrix} \begin{bmatrix} 1 \\ w_0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ b + Bv_0 \end{bmatrix}. \end{aligned}$$

Then $\{0\} \times \mathfrak{g}_{w_0} = \mathfrak{g}_{v_0}$ and the equivalence is proved.

(ii) \iff (iii) : Firstly, remark that $\tilde{\mathfrak{g}} = \mathfrak{g}_1 + H$ and $v_0 \in \{1\} \times \mathbb{C}^n$, then

$\overline{\mathfrak{g}_{v_0}} = \overline{(\mathfrak{g}_1)_{v_0}} + \mathbb{C}v_0$. By Proposition 2.6.(i), $(\mathfrak{g}_1)_{v_0} = \mathfrak{g}_{v_0} + 2i\pi\mathbb{Z}v_0$, so $\overline{\mathfrak{g}_{v_0}} = \overline{\mathfrak{g}_{v_0}} + \mathbb{C}v_0$. Secondly, suppose that $\overline{\mathfrak{g}_{v_0}} = \{0\} \times \mathbb{C}^n$. Since $v_0 \notin \{0\} \times \mathbb{C}^n$ and $Id_{\mathbb{C}^{n+1}} \in \exp^{-1}(H \cap \mathcal{K}_{\eta,r}(\mathbb{C})) \subset \tilde{\mathfrak{g}}$, then $\mathbb{C}v_0 \subset \tilde{\mathfrak{g}_{v_0}}$. Therefore $\mathbb{C}^{n+1} = \overline{\mathfrak{g}_{v_0}} \oplus \mathbb{C}v_0 \subset \tilde{\mathfrak{g}_{v_0}}$. Conversely, suppose that $\overline{\mathfrak{g}_{v_0}} = \mathbb{C}^{n+1}$. Since $\overline{\mathfrak{g}_{v_0}} \subset \{0\} \times \mathbb{C}^n$ and $\mathbb{C}v_0 \cap (\{0\} \times \mathbb{C}^n) = \{0\}$, then $\overline{\mathfrak{g}_{v_0}} \oplus \mathbb{C}v_0 = \mathbb{C}^{n+1}$, thus $\overline{\mathfrak{g}_{v_0}} = \{0\} \times \mathbb{C}^n$. \square

Lemma 3.3. *Let $x \in \mathbb{C}^n$. Then the following assertions are equivalent:*

- (i) $\overline{\mathcal{G}(x)} = \mathbb{C}^n$.
- (ii) $\overline{G(1,x)} = \{1\} \times \mathbb{C}^n$.
- (iii) $\overline{\tilde{G}(1,x)} = \mathbb{C}^{n+1}$.

Proof. (i) \iff (ii) : The proof is obvious, since $\{1\} \times \mathcal{G}(x) = G(1,x)$ by construction.

(ii) \iff (iii) : Suppose that $\overline{\tilde{G}(1,x)} = \mathbb{C}^{n+1}$. If $\overline{G(1,x)} \neq \{1\} \times \mathbb{C}^n$, then there exists an open subset O of \mathbb{C}^n such that $(\{1\} \times O) \cap G(1,x) = \emptyset$. Let $y \in O$ and $(g_m)_m$ be a sequence in \tilde{G} such that $\lim_{m \rightarrow +\infty} g_m(1,x) = (1,y)$. Since \tilde{G} is abelian then $g_m = \lambda_m f_m$, with $f_m \in G$ and $\lambda_m \in \mathbb{C}^*$, thus $\lim_{m \rightarrow +\infty} \lambda_m = 1$. Therefore, $\lim_{m \rightarrow +\infty} f_m(1,x) = \lim_{m \rightarrow +\infty} \frac{1}{\lambda_m} g_m(1,x) = (1,y)$. Hence, $(1,y) \in \overline{G(1,x)} \cap (\{1\} \times O)$, a contradiction. Conversely, if $\overline{G(1,x)} = \{1\} \times \mathbb{C}^n$, then

$$\begin{aligned}
 \mathbb{C}^{n+1} &= \bigcup_{\lambda \in \mathbb{C}} \lambda(\{1\} \times \mathbb{C}^n) \\
 &= \bigcup_{\lambda \in \mathbb{C}} \lambda \overline{G(1,x)} \\
 &\subset \overline{\tilde{G}(1,x)}
 \end{aligned}$$

\square

3.1. Proof of Theorem 1.1. The proof of Theorem 1.1 results directly from Lemma 3.3, Proposition 3.1 and Lemma 3.2. \square

4. FINITELY GENERATED SUBGROUPS

4.1. Proof of Theorem 1.2.

Let $J_k = \text{diag}(J_{k,1}, \dots, J_{k,r})$ with $J_{k,i} = 0 \in \mathbb{T}_{n_i}(\mathbb{C})$ if $i \neq k$ and $J_{k,k} = I_{n_k}$.

Proposition 4.1. ([1], Proposition 8.1) *Let G be an abelian subgroup of $GL(n+1, \mathbb{C})$ generated by A_1, \dots, A_p . Let $B_1, \dots, B_p \in \mathfrak{g}$ such that $A_k = e^{B_k}$, $k = 1, \dots, p$ and $P \in GL(n+1, \mathbb{C})$ satisfying $P^{-1}GP \subset \mathcal{K}_{\eta,r}^*(\mathbb{C})$. Then:*

$$\mathfrak{g}_1 = \sum_{k=1}^p \mathbb{Z}B_k + 2i\pi \sum_{k=1}^r \mathbb{Z}PJ_kP^{-1} \quad \text{and} \quad (\mathfrak{g}_1)_{v_0} = \sum_{k=1}^p \mathbb{Z}B_kv_0 + \sum_{k=1}^r 2i\pi \mathbb{Z}Pe^{(k)}.$$

Proposition 4.2. (Under notations of Proposition 2.2) Let \mathcal{G} be an abelian subgroup of $GA(n, \mathbb{C})$ generated by f_1, \dots, f_p and let $f'_1, \dots, f'_p \in \mathfrak{g}$ such that $\Phi(f_k) = e^{\Psi(f'_k)}$, $k = 1, \dots, p$. Then:

$$\mathfrak{g}_{w_0} = \sum_{k=1}^p \mathbb{Z}f'_k(w_0) + \sum_{k=1}^r 2i\pi\mathbb{Z}(p_2 \circ Pe^{(k)}).$$

Proof. Let $G = \Phi(\mathcal{G})$. Then G is generated by $\Phi(f_1), \dots, \Phi(f_p)$. By proposition 4.1 we have

$$\mathfrak{g}_1 = \sum_{k=1}^p \mathbb{Z}\Psi(f'_k) + \sum_{k=1}^r 2i\pi\mathbb{Z}PJ^{(k)}P^{-1}.$$

Since $P \in \Phi(GA(n, \mathbb{C}))$ then $PJ^{(1)}P^{-1} \notin \mathcal{F}_{n+1}$ and $PJ^{(k)}P^{-1} \in \mathcal{F}_{n+1}$ for every $k = 2, \dots, r$. As $\mathfrak{g} = \mathfrak{g}_1 \cap \mathcal{F}_{n+1}$, then

$$\mathfrak{g} = \begin{cases} \sum_{k=1}^p \mathbb{Z}\Psi(f'_k) + \sum_{k=2}^r 2i\pi\mathbb{Z}PJ^{(k)}P^{-1}, & \text{if } r \geq 2 \\ \sum_{k=1}^p \mathbb{Z}\Psi(f'_k), & \text{if } r = 1 \end{cases}$$

By construction, one has $\mathfrak{g}_{w_0} = p_2(\mathfrak{g}_{v_0})$, $v_0 = Pu_0$, $J^{(k)}u_0 = e^{(k)}$ and $p_2(\Psi(f'_k)v_0) = f'_k(w_0)$. Then

$$\mathfrak{g}_{w_0} = \begin{cases} \sum_{k=1}^p \mathbb{Z}f'_k(w_0) + \sum_{k=2}^r 2i\pi\mathbb{Z}(p_2 \circ Pe^{(k)}), & \text{if } r \geq 2 \\ \sum_{k=1}^p \mathbb{Z}f'_k(w_0), & \text{if } r = 1 \end{cases} \quad (3)$$

The proof is completed. \square

Recall the following Proposition which was proven in [7]:

Proposition 4.3. (cf. [7], page 35). Let $F = \mathbb{Z}u_1 + \dots + \mathbb{Z}u_p$ with $u_k = (u_{k,1}, \dots, u_{k,n}) \in \mathbb{C}^n$ and $u_{k,i} = \text{Re}(u_{k,i}) + i\text{Im}(u_{k,i})$, $k = 1, \dots, p$, $i = 1, \dots, n$. Then F is dense in \mathbb{C}^n if and only if for every $(s_1, \dots, s_p) \in \mathbb{Z}^p \setminus \{0\}$:

$$\text{rank} \begin{bmatrix} \text{Re}(u_{1,1}) & \dots & \dots & \text{Re}(u_{p,1}) \\ \vdots & \vdots & \vdots & \vdots \\ \text{Re}(u_{1,n}) & \dots & \dots & \text{Re}(u_{p,n}) \\ \text{Im}(u_{1,1}) & \dots & \dots & \text{Im}(u_{p,1}) \\ \vdots & \vdots & \vdots & \vdots \\ \text{Im}(u_{1,n}) & \dots & \dots & \text{Im}(u_{p,n}) \\ s_1 & \dots & \dots & s_p \end{bmatrix} = 2n + 1.$$

Proof of Theorem 1.2: This follows directly from Theorem 1.1, Propositions 4.2 and 4.3.

4.2. Proof of Corollaries 1.3 and 1.4.

Proof of Corollary 1.3: We show first that if $F = \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_m$, $u_k \in \mathbb{C}^n$ with $m \leq 2n$, then F can not be dense: Write $u_k \in \mathbb{C}^n$, $u_k = \operatorname{Re}(u_k) + i\operatorname{Im}(u_k)$ and $v_k = [\operatorname{Re}(u_k); \operatorname{Im}(u_k); s_k]^T \in \mathbb{R}^{2n+1}$, $1 \leq k \leq m$. Since $m \leq 2n$, it follows that $\operatorname{rank}(v_1, \dots, v_m) \leq 2n$, and so F is not dense in \mathbb{C}^n by Proposition 4.3.

Now, by applying Theorem 1.2 and the fact that $m = p+r-1 \leq 2n$ (since $r \leq n+1$) and by (1), the Corollary 1.3 follows. \square

Proof of Corollary 1.4: Since $p \leq n$ and $r \leq n+1$ then $p+r-1 \leq 2n$. Corollary 1.4 follows from Corollary 1.3. \square

5. EXAMPLE

Example 5.1. Let \mathcal{G} the group generated by $f_1 = (A_1, a_1)$, $f_2 = (A_2, a_2)$, $f_3 = (A_3, a_3)$ and $f_4 = (A_4, a_4)$, where:

$$\begin{aligned} A_1 &= I_2, \quad a_1 = (1+i, 0), \quad A_2 = \operatorname{diag}(1, e^{-2+i}), \quad a_2 = (0, 0). \\ A_3 &= \operatorname{diag}\left(1, e^{\frac{-\sqrt{2}}{\pi} + i\left(\frac{\sqrt{2}}{2\pi} - \frac{\sqrt{7}}{2}\right)}\right), \quad a_3 = \left(\frac{-\sqrt{3}}{2\pi} + i\left(\frac{\sqrt{5}}{2} - \frac{\sqrt{3}}{2\pi}\right), 0\right), \\ A_4 &= I_2 \quad \text{and} \quad a_4 = (2i\pi, 0). \end{aligned}$$

Then every orbit in $V = \mathbb{C} \times \mathbb{C}^*$ is dense in \mathbb{C}^2 .

Proof. Denote by $G = \Phi(\mathcal{G})$. Then G generated by

$$\begin{aligned} \Phi(f_1) &= \begin{bmatrix} 1 & 0 & 0 \\ 1+i & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Phi(f_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-2+i} \end{bmatrix}, \\ \Phi(f_3) &= \begin{bmatrix} 1 & 0 & 0 \\ \frac{-\sqrt{3}}{2\pi} + i\left(\frac{\sqrt{5}}{2} - \frac{\sqrt{3}}{2\pi}\right) & 1 & 0 \\ 0 & 0 & e^{\frac{-\sqrt{2}}{\pi} + i\left(\frac{\sqrt{2}}{2\pi} - \frac{\sqrt{7}}{2}\right)} \end{bmatrix}, \end{aligned}$$

and

$$\Phi(f_4) = \begin{bmatrix} 1 & 0 & 0 \\ 2i\pi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $f'_1 = (B_1, b_1)$, $f'_2 = (B_2, b_2)$ and $f'_k = (B_k, b_k)$ such that $e^{\Psi(f'_k)} = A'_k$, $k = 1, 2, 3, 4$. We have

$$\begin{aligned} B_1 &= \operatorname{diag}(0, 0), \quad b_1 = (1+i, 0), \\ B_2 &= \operatorname{diag}(0, -2+i), \quad b_2 = (0, 0), \\ B_3 &= \operatorname{diag}\left(0, \frac{-\sqrt{2}}{\pi} + i\left(\frac{\sqrt{2}}{2\pi} - \frac{\sqrt{7}}{2}\right)\right), \quad b_3 = \left(\frac{-\sqrt{3}}{2\pi} + i\left(\frac{\sqrt{5}}{2} - \frac{\sqrt{3}}{2\pi}\right), 0\right), \\ B_4 &= \operatorname{diag}(0, 0) \quad \text{and} \quad b_4 = (2i\pi, 0). \end{aligned}$$

Here, we have:

- G is an abelian subgroup of $\mathcal{K}_{(2,1),2}^*(\mathbb{C})$.

- $P = I_3$, $r = 2$, $U = \mathbb{C}^* \times \mathbb{C} \times \mathbb{C}^*$, $u_0 = (1, 0, 1)$, $e^{(1)} = (1, 0, 0)$ and $e^{(2)} = (0, 0, 1)$
- $V = \mathbb{C} \times \mathbb{C}^*$, $w_0 = (0, 1)$.

In the other hand, for every $(s_1, s_2, s_3, s_4, t_2) \in \mathbb{Z}^5 \setminus \{0\}$, one has the determinant:

$$\begin{aligned} \Delta &= \begin{vmatrix} \operatorname{Re}(B_1 w_0 + b_1) & \operatorname{Re}(B_2 w_0 + b_2) & \operatorname{Re}(B_3 w_0 + b_3) & \operatorname{Re}(B_4 w_0 + b_4) & 0 \\ \operatorname{Im}(B_1 w_0 + b_1) & \operatorname{Im}(B_2 w_0 + b_2) & \operatorname{Im}(B_3 w_0 + b_3) & \operatorname{Im}(B_4 w_0 + b_4) & 2\pi e^{(2)} \\ & s_1 & s_2 & s_3 & s_4 & t_2 \end{vmatrix} \\ &= \begin{vmatrix} -\frac{\sqrt{3}}{2\pi} & 1 & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{\pi} & 0 & -2 & 0 & 0 \\ \frac{\sqrt{5}}{2} - \frac{\sqrt{3}}{2\pi} & 1 & 0 & 2\pi & 0 \\ \frac{\sqrt{2}}{2\pi} - \frac{\sqrt{7}}{2} & 0 & 1 & 0 & 2\pi \\ s_1 & s_2 & s_3 & s_4 & t_2 \end{vmatrix} \\ &= -2\pi \left((4s_1)\pi - (2s_3)\sqrt{2} + (2s_2)\sqrt{3} + s_4\sqrt{5} + t_2\sqrt{7} \right). \end{aligned}$$

Since π , $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ and $\sqrt{7}$ are rationally independent then $\Delta \neq 0$ for every $(s_1, s_2, s_3, t_1, t_2) \in \mathbb{Z}^5 \setminus \{0\}$. It follows that:

$$\operatorname{rank} \begin{bmatrix} -\frac{\sqrt{3}}{2\pi} & 1 & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{\pi} & 0 & -2 & 0 & 0 \\ \frac{\sqrt{5}}{2} - \frac{\sqrt{3}}{2\pi} & 1 & 0 & 2\pi & 0 \\ \frac{\sqrt{2}}{2\pi} - \frac{\sqrt{7}}{2} & 0 & 1 & 0 & 2\pi \\ s_1 & s_2 & s_3 & s_4 & t_2 \end{bmatrix} = 5$$

and by Theorem 1.2, \mathcal{G} has a dense orbit and every orbit of V is dense in \mathbb{C}^2 . \square

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