# HYPERCYCLIC ABELIAN AFFINE GROUPS

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Abstract. In this paper, we give a characterization of hypercyclic abelian affine group $\mathcal{G}$. If $\mathcal{G}$ is finitely generated, this characterization is explicit. We prove in particular that no abelian group generated by $n$ affine maps on $\mathbb{C}^n$ has a dense orbit.

## 1. Introduction

Let $M_n(\mathbb{C})$ be the set of all square matrices of order $n \geq 1$ with entries in $\mathbb{C}$ and $GL(n,\ \mathbb{C})$ be the group of all invertible matrices of $M_n(\mathbb{C})$. A map $f:\ \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is called an affine map if there exist $A \in M_n(\mathbb{C})$ and $a \in \mathbb{C}^n$ such that $f(x) = Ax + a$, $x \in \mathbb{C}^n$. We denote $f = (A, a)$, we call $A$ the *linear part* of $f$. Denote by $MA(n,\ \mathbb{C})$ the set of all affine maps and $GA(n,\ \mathbb{C})$ the set of all invertible affine maps of $MA(n, \mathbb{C})$. $MA(n, \mathbb{C})$ is a vector space and for composition of maps, $GA(n,\ \mathbb{C})$ is a group.

Let $\mathcal{G}$ be an abelian affine subgroup of $GA(n,\ \mathbb{C})$. For a vector $v \in \mathbb{C}^n$, we consider the orbit of $\mathcal{G}$ through $v$: $\mathcal{G}(v) = \{f(v):\ f \in \mathcal{G}\} \subset \mathbb{C}^n$. A subset $E \subset \mathbb{C}^n$ is called *$\mathcal{G}$-invariant* if $f(E) \subset E$ for any $f \in \mathcal{G}$; that is $E$ is a union of orbits. Before stating our main results, we introduce the following notions:

A subset $\mathcal{H}$ of $\mathbb{C}^n$ is called *an affine subspace* of $\mathbb{C}^n$ if there exist a vector subspace $H$ of $\mathbb{C}^n$ and $a \in \mathbb{C}^n$ such that $\mathcal{H} = H + a$. For $a \in \mathbb{C}^n$, denote by $T_a:\ \mathbb{C}^n \longrightarrow \mathbb{C}^n$; $x \longmapsto x + a$ the translation map by vector $a$, so $\mathcal{H} = T_a(H)$. We say that $\mathcal{H}$ has dimension $p$ $(0 \leq p \leq n)$, denoted $\dim(\mathcal{H}) = p$, if $H$ has dimension $p$.

Denote by $\overline{A}$ the closure of a subset $A \subset \mathbb{C}^n$. A subset $E$ of $\mathbb{C}^n$ is called *a minimal set* of $\mathcal{G}$ if $E$ is closed in $\mathbb{C}^n$, non empty, $\mathcal{G}$-invariant and has no proper subset with these properties. It is equivalent to say that $E$ is a $\mathcal{G}$-invariant set such that every orbit contained in $E$ is dense in it. The group $\mathcal{G}$ is called *hypercyclic* if there exists a vector $v \in \mathbb{C}^n$ such that $\mathcal{G}(v)$ is dense in $\mathbb{C}^n$. For an account of results and bibliography on hypercyclicity, we refer to the book [2] by Bayart and Matheron.

Define the map

$$\Phi\ :\ GA(n,\ \mathbb{C})\ \longrightarrow\ \Phi(GA(n, \mathbb{C})) \subset GL(n+1,\ \mathbb{C})$$

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$$f = (A, a) \longmapsto \begin{bmatrix} 1 & 0 \\ a & A \end{bmatrix}$$

We have the following composition formula

$$\begin{bmatrix} 1 & 0 \\ a & A \end{bmatrix} \begin{bmatrix} 1 & 0 \\ b & B \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ Ab + a & AB \end{bmatrix}.$$

Then $\Phi$ is a homomorphism of groups.

Let $\mathcal{G}$ be an abelian affine subgroup of $GA(n, \mathbb{C})$. Then $\Phi(\mathcal{G})$ is an abelian subgroup of $GL(n+1, \mathbb{C})$.

Denote by:
• $\mathbb{C}^* = \mathbb{C} \backslash \{0\}$ and $\mathbb{N}_0 = \mathbb{N} \backslash \{0\}$.
Let $n \in \mathbb{N}_0$ be fixed. For each $m = 1, 2, \dots, n+1$, denote by:
• $\mathcal{B}_0 = (e_1, \dots, e_{n+1})$ the canonical basis of $\mathbb{C}^{n+1}$ and $I_{n+1}$ the identity matrix of $GL(n+1, \mathbb{C})$.

• $\mathbb{T}_m(\mathbb{C})$ the set of matrices over $\mathbb{C}$ of the form

$$\begin{bmatrix} \mu & & & 0 \\ a_{2,1} & \mu & & \\ \vdots & \ddots & \ddots & \\ a_{m,1} & \dots & a_{m,m-1} & \mu \end{bmatrix} \qquad (1)$$

• $\mathbb{T}_m^*(\mathbb{C})$ the group of matrices of the form (1) with $\mu \neq 0$.

Let $r \in \mathbb{N}$ and $\eta = (n_1, \dots, n_r)$ be a sequence of positive integers such that $n_1 + \dots + n_r = n + 1$. In particular, $r \leq n + 1$.
Write
• $\mathcal{K}_{\eta,r}(\mathbb{C}) := \mathbb{T}_{n_1}(\mathbb{C}) \oplus \dots \oplus \mathbb{T}_{n_r}(\mathbb{C})$. In particular if $r = 1$, then $\mathcal{K}_{\eta,1}(\mathbb{C}) = \mathbb{T}_{n+1}(\mathbb{C})$ and $\eta = (n+1)$.
• $\mathcal{K}_{\eta,r}^*(\mathbb{C}) := \mathcal{K}_{\eta,r}(\mathbb{C}) \cap \mathrm{GL}(n+1, \mathbb{C})$.

Define the map

$$\begin{aligned} \Psi \ : \ MA(n, \mathbb{C}) \ &\longrightarrow \ \Psi(MA(n, \mathbb{C})) \subset M_{n+1}(\mathbb{C}) \\ f = (A, a) \ &\longmapsto \ \begin{bmatrix} 0 & 0 \\ a & A \end{bmatrix} \end{aligned}$$

We have $\Psi$ is an isomorphism.
• $\mathcal{F}_{n+1} = \Psi(MA(n, \mathbb{C}))$.
• $\exp : \mathbb{M}_{n+1}(\mathbb{C}) \longrightarrow \mathrm{GL}(n+1, \mathbb{C})$ is the matrix exponential map; set $\exp(M) = e^M$.

There always exists a $P \in \Phi(\mathrm{GA}(n, \mathbb{C}))$ and a partition $\eta$ of $n+1$ such that $G' = P^{-1}GP \subset \mathcal{K}_{\eta,r}^*(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C}))$ (see Proposition 2.2). For such a choice of matrix $P$, we let

• $\mathrm{g} = \exp^{-1}(G) \cap \left(P(\mathcal{K}_{\eta,r}(\mathbb{C}))P^{-1}\right) \cap \mathcal{F}_{n+1}$. If $G \subset \mathcal{K}_{\eta,r}^*(\mathbb{C})$, we have $\mathrm{g} = \exp^{-1}(G) \cap \mathcal{K}_{\eta,r}(\mathbb{C}) \cap \mathcal{F}_{n+1}$.

• $g_u = \{Bu : \ B \in g\}, \ u \in \mathbb{C}^{n+1}$.
• $\mathfrak{g} = \Psi^{-1}(g)$.
• $\mathfrak{g}_v = \{f(v), \ \ f \in \mathfrak{g}\}, \ v \in \mathbb{C}^n$.
• $u_0 = [e_{1,1}, \dots, e_{r,1}]^T \in \mathbb{C}^{n+1}$ where $e_{k,1} = [1, 0, \dots, 0]^T \in \mathbb{C}^{n_k}$, for $k = 1, \dots, r$. One has $u_0 \in \{1\} \times \mathbb{C}^n$.
• $p_2 : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^n$ the projection defined by $p_2(x_1, \dots, x_{n+1}) = (x_2, \dots, x_{n+1})$.
• $v_0 = Pu_0$. As $P \in \Phi(GA(n, \mathbb{C}))$, $v_0 \in \{1\} \times \mathbb{C}^n$.
• $w_0 = p_2 v_0 \in \mathbb{C}^n$.
• $e^{(k)} = [e_1^{(k)}, \dots, e_r^{(k)}]^T \in \mathbb{C}^{n+1}$ where

$$e_j^{(k)} = \begin{cases} 0 \in \mathbb{C}^{n_j} & \text{if } \ j \neq k \\ e_{k,1} & \text{if } \ j = k \end{cases} \qquad for \ \ every \ \ 1 \leq j, k \leq r.$$

For groups of affine maps on $\mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), their dynamics were recently initiated for some classes in different point of view, (see for instance, [3], [4], [6],[5]). The purpose here is to give analogous results of that theorem for linear abelian subgroup of $GL(n, \mathbb{C})$ proved in [1] (see Proposition 3.1). Our main results are the following:

**Theorem 1.1.** *Let $\mathcal{G}$ be an abelian subgroup of $GA(n, \mathbb{C})$. The following are equivalent:*
*(i) $\mathcal{G}$ is hypercyclic.*
*(ii) The orbit $\mathcal{G}(w_0)$ is dense in $\mathbb{C}^n$*
*(iii) $\mathfrak{g}_{w_0}$ is an additive subgroup dense in $\mathbb{C}^n$*

For a *finitely generated* abelian subgroup $\mathcal{G} \subset \mathrm{GA}(n, \mathbb{R})$, let introduce the following property. Consider the following rank condition on a collection of affine maps $f_1, \dots, f_p \in GA(n, \mathbb{C})$, where $f'_1, \dots, f'_p \in \mathfrak{g}$ such that $e^{\Psi(f'_k)} = \Phi(f_k)$, $k = 1, \dots, p$.

We say that $f_1, \dots, f_p$ satisfy *property $\mathcal{D}$* if for every $(s_1, \dots, s_p; \ t_1, \dots, t_r) \in \mathbb{Z}^{p+r} \backslash \{0\}$:

$$rank \begin{bmatrix} Re(f'_1(w_0)) & \dots & Re(f'_p(w_0)) & 0 & \dots & 0 \\ Im(f'_1(w_0)) & \dots & Im(f'_p(w_0)) & 2\pi p_2 \circ Pe^{(1)} & \dots & 2\pi p_2 \circ Pe^{(r)} \\ s_1 & \dots & s_p & t_1 & \dots & t_r \end{bmatrix} = 2n+1.$$

For a vector $v \in \mathbb{C}^n$, we write $v = Re(v) + iIm(v)$ where $Re(v)$ and $Im(v) \in \mathbb{R}^n$. In this case, the Theorem can be stated as follows:

**Theorem 1.2.** *Let $\mathcal{G}$ be an abelian subgroup of $GA(n, \mathbb{C})$ generated by $f_1, \dots, f_p$ and let $f'_1, \dots, f'_p \in \mathfrak{g}$ such that $e^{\Psi(f'_1)} = \Phi(f_1), \dots, e^{\Psi(f'_p)} = \Phi(f_p)$. Then the following are equivalent:*
*(i) $\mathcal{G}$ is hypercyclic.*
*(ii) the maps $f_1, \dots, f_p$ satisfy property $\mathcal{D}$*
*(iii) $\mathfrak{g}_{w_0} = \sum_{k=1}^{p} \mathbb{Z} f'_k(w_0) + 2i\pi \sum_{k=1}^{r} \mathbb{Z}(p_2 \circ Pe^{(k)})$ is an additive group dense in $\mathbb{C}^n$.*

**Corollary 1.3.** *If $\mathcal{G}$ is of finite type $p$ with $p \leq 2n - r + 1$, then it has no dense orbit.*

**Corollary 1.4.** *If $\mathcal{G}$ is of finite type $p$ with $p \leq n$, then it has no dense orbit.*

## 2. Notations and Lemmas

Denote by $\mathcal{L}_{\mathcal{G}}$ the set of the linear parts of all elements of $\mathcal{G}$ and $vect(F)$ is the vector space generated by a subset $F \subset \mathbb{C}^{n+1}$. In the following, denote by $I_m$ the identity matrix of $GL(m, \mathbb{C})$, for any $m \in \mathbb{N}_0$.

**Proposition 2.1.** ([1], *Proposition 2.3*) *Let $L$ be an abelian subgroup of $GL(n, \mathbb{C})$. Then there exists $P \in GL(n, \mathbb{C})$ such that $P^{-1}LP$ is a subgroup of $\mathcal{K}_{\eta',r'}(\mathbb{C})$, for some $r' \leq n$ and $\eta' = (n'_1, \dots, n'_{r'}) \in \mathbb{N}_0^{r'}$.*

**Proposition 2.2.** *Let $\mathcal{G}$ be an abelian subgroup of $GA(n, \mathbb{C})$ and $G = \Phi(\mathcal{G})$. Then there exists $P \in \Phi(GA(n, \mathbb{C}))$ such that $P^{-1}GP$ is a subgroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C}))$, for some $r \leq n+1$ and $\eta = (n_1, \dots, n_r) \in \mathbb{N}_0^r$. In particular, $Pu_0 \in \{1\} \times \mathbb{C}^n$.*

*Proof.* We have $\mathcal{L}_{\mathcal{G}}$ is an abelian subgroup of $GL(n, \mathbb{C})$. By Proposition 2.1, there exists $Q \in GL(n, \mathbb{C})$ such that $Q^{-1}\mathcal{L}_G Q$ is a subgroup of $\mathcal{K}^*_{\eta',r'}(\mathbb{C})$ for some $r' \leq n$ and $\eta' = (n'_1, \dots, n'_{r'}) \in \mathbb{N}_0^{r'}$ such that $n'_1 + \dots, n'_{r'} = n$. For every $A \in \mathcal{L}_{\mathcal{G}}$, $Q^{-1}AQ = \mathrm{diag}(A_1, \dots, A_{r'})$ with $A_k \in \mathbb{T}^*_{n'_k}$ and $\mu_{A_k}$ is the only eigenvalue of $A_k$, $k = 1, \dots, r'$. Let $J = \{k \in \{1, \dots, r'\},\ \ \mu_{A_k} = 1,\ \forall\ A \in \mathcal{L}_{\mathcal{G}}\}$. There are two cases:

- *Case1:* Suppose that $J = \{k_1, \dots, k_s\}$ for some $s \leq r'$. We can take $J = \{1, \dots, s\}$, otherwise, we replace $P_1$ by $RP_1$ for some permutation matrix $R$ of $GL(n, \mathbb{C})$. Let $P_1 = \mathrm{diag}(1, Q)$, so $P_1 \in \Phi(GA(n, \mathbb{C}))$ and

$$P_1^{-1}\Phi(f)P_1 = \begin{bmatrix} 1 & 0 \\ Q^{-1}a & Q^{-1}AQ \end{bmatrix}.$$

Write $E = \mathrm{vect}(\mathcal{C}_1, \dots, \mathcal{C}_s)$ and $H = \mathrm{vect}(\mathcal{C}_{s+1}, \dots, \mathcal{C}_{r'})$, so $E$ and $H$ are $G$-invariant vector spaces. Moreover, for every $f = (A, a) \in \mathcal{G}$ the restriction $A_{/E}$ has 1 as only eigenvalue and we have

$$P_1^{-1}\Phi(f)P_1 = \begin{bmatrix} 1 & 0 & 0 \\ a_1 & A_1 & 0 \\ a_2 & 0 & A_2 \end{bmatrix}, \quad (1)$$

where $A_1 = A_{/E} \in \mathbb{T}^*_p(\mathbb{C})$, $A_2 = A_{/H} \in \mathbb{T}^*_{n-p}(\mathbb{C})$, $a_1 \in \mathbb{C}^p$, $a_2 \in \mathbb{C}^{n-p}$, $p = n'_1 + \dots + n'_s$. On the other hand, there exists $f_0 = (B, b) \in \mathcal{G}$ such that $B_2 = B_{/H}$ has no eigenvalue equal to 1, so $B_2 - I_{n-p}$ is invertible. As in (1), write

$$P_1^{-1}\Phi(f_0)P_1 = \begin{bmatrix} 1 & 0 & 0 \\ b_1 & B_1 & 0 \\ b_2 & 0 & B_2 \end{bmatrix}.$$

Let $P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & I_p & 0 \\ b_2 & 0 & B_2 - I_{n-p} \end{bmatrix}$ and $P = P_1 P_2^{-1}$, then $P = \begin{bmatrix} 1 & 0 \\ d & P_0 \end{bmatrix} \in \Phi(GA(n, \mathbb{C}))$, where $P_0 = Q.Q_1^{-1}$, $Q_1 = \begin{bmatrix} I_p & 0 \\ 0 & B_2 - I_{n-p} \end{bmatrix}$ and $d = -P_0[0, b_2]^T$.

For every $f = (A, a) \in \mathcal{G}$ we have by (1),

$$\begin{aligned}P^{-1}\Phi(f)P &= P_2P_1^{-1}\Phi(f)P_1P_2^{-1}\\ &= P_2\begin{bmatrix} 1 & 0 & 0\\ a_1 & A_1 & 0\\ a_2 & 0 & A_2\end{bmatrix}P_2^{-1}\\ &= \begin{bmatrix} 1 & 0 & 0\\ a_1 & A_1 & 0\\ -(A_2-I_{n-p})b_2+(B_2-I_{n-p})a_2 & 0 & A_2\end{bmatrix} \qquad (2)\end{aligned}$$

Since $G$ is abelian, so by the equality $P_1^{-1}\Phi(f)\Phi(f_0)P_1 = P_1^{-1}\Phi(f_0)\Phi(f)P_1$, we find $-(A_2 - I_{n-p})b_2 + (B_2 - I_{n-p})a_2 = 0$.

It follows by (2), that $P^{-1}GP$ is a subgroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C}))$, where $r = r' - s + 1$ and $\eta = (p + 1, n'_{s+1}, \dots, n'_{r'})$.

- *Case2:* Suppose that $J = \emptyset$, and denote by $Fix(G) = \{x \in \mathbb{C}^{n+1} : Bx = x, \forall B \in G\}$, then $Fix(G) = \mathbb{C}v$ for some $v = (1, v_1)$, $v_1 \in \mathbb{C}^n$. For every $f = (A, a) \in \mathcal{G}$, one has $\Phi(f)(1, v_1) = (1, f(v_1)) = (1, v_1)$ so $f(v_1) = Av_1 + a = v_1$. Let $P = \begin{bmatrix} 1 & 0\\ v_1 & P_1\end{bmatrix} \in \Phi(GA(n, \mathbb{C}))$, then

$$\begin{aligned}P^{-1}\Phi(f)P &= \begin{bmatrix} 1 & 0\\ -P_1^{-1}v_1 & P_1^{-1}\end{bmatrix}\begin{bmatrix} 1 & 0\\ a & A\end{bmatrix}\begin{bmatrix} 1 & 0\\ v_1 & P_1\end{bmatrix}\\ &= \begin{bmatrix} 1 & 0\\ P_1^{-1}(Av_1+a-v_1) & P_1^{-1}AP_1\end{bmatrix}\\ &= \begin{bmatrix} 1 & 0\\ 0 & P_1^{-1}AP_1\end{bmatrix}.\end{aligned}$$

It follows that $P^{-1}GP$ is a subgroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C}))$, where $r = r' + 1$ and $\eta = (1, n'_1, \dots, n'_{r'})$.
Since $u_0 \in \{1\} \times \mathbb{C}^n$ and $P \in \Phi(GA(n, \mathbb{C}))$, so $Pu_0 \in \{1\} \times \mathbb{C}^n$. $\square$

Denote by:
- $G' = P^{-1}GP$.
- $\mathrm{g}' = exp^{-1}(G') \cap \mathcal{K}_{\eta,r}(\mathbb{C}) \cap \mathcal{F}_{n+1}$.
- $\mathrm{g}_1 = exp^{-1}(G) \cap \left(P\mathcal{K}_{\eta,r}(\mathbb{C})P^{-1}\right)$

**Lemma 2.3.** ([1], *Proposition 3.2*) *$exp(\mathcal{K}_{\eta,r}(\mathbb{C})) = \mathcal{K}^*_{\eta,r}(\mathbb{C})$.*

**Lemma 2.4.** *If $N \in P\mathcal{K}_{\eta,r}(\mathbb{C})P^{-1}$ such that $e^N \in \Phi(GA(n, \mathbb{C}))$, so $N - 2ik\pi I_{n+1} \in \mathcal{F}_{n+1}$, for some $k \in \mathbb{Z}$.*

*Proof.* Let $N' = P^{-1}NP \in \mathcal{K}_{\eta,r}(\mathbb{C})$, $M = e^N$ and $M' = P^{-1}MP$. We have $e^{N'} = M'$ and by Lemma 2.3, $M' \in \mathcal{K}^*_{\eta,r}(\mathbb{C})$. Write $M' = \mathrm{diag}(M'_1, \dots, M'_r)$ and $N' = \mathrm{diag}(N'_1, \dots, N'_r)$, $M'_k, N'_k \in \mathbb{T}_{n_k}(\mathbb{C})$, $k = 1, \dots, r$. Then $e^{N'} = \mathrm{diag}(e^{N'_1}, \dots, e^{N'_r})$, so $e^{N'_1} = M'_1$. As 1 is the only eigenvalue of $M'_1$, $N'_1$ has an eigenvalue $\mu \in \mathbb{C}$ such that $e^{\mu} = 1$. Thus $\mu = 2ik\pi$ for some $k \in \mathbb{Z}$. Therefore, $N'' = N' - 2ik\pi I_{n+1} \in \mathcal{F}_{n+1}$ and satisfying $e^{N''} = e^{-2ik\pi} e^{N'} = M'$. It follows that $N - 2ik\pi I_{n+1} = PN''P^{-1} \in P\mathcal{F}_{n+1}P^{-1} = \mathcal{F}_{n+1}$, since $P \in \Phi(GA(n,\mathbb{C}))$. □

**Lemma 2.5.** ([1], *Lemma 4.2*) *Under above notations, one has* $exp(\mathrm{g}_1) = G$.

As consequence, we obtain

**Proposition 2.6.** *We have:*
(*i*) $\mathrm{g}_1 = \mathrm{g} + 2i\pi\mathbb{Z}I_{n+1}$.
(*ii*) $exp(\mathrm{g}) = G$.

*Proof.* (i) By Lemma 2.5, $exp(\mathrm{g}_1) = G \subset \Phi(GA(n,\mathbb{C}))$, then by Lemma 2.4 we have $\mathrm{g}_1 \subset \mathrm{g} + 2i\pi\mathbb{Z}I_{n+1}$. Conversely is obvious since $\mathrm{g} + 2i\pi\mathbb{Z}I_{n+1} \subset \mathcal{K}_{\eta,r}(\mathbb{C})$ and $exp(\mathrm{g} + 2i\pi\mathbb{Z}I_{n+1}) = exp(\mathrm{g}) \subset G$.
(ii) By Lemma 2.5 and (i) we have $G = exp(g_1) = exp(\mathrm{g} + 2i\pi\mathbb{Z}I_{n+1}) = exp(\mathrm{g})$. □

## 3. Proof of Theorem 1.1

Let $\widetilde{G}$ be the group generated by $G$ and $H = \{\lambda.I_{n+1} : \quad \lambda \in \mathbb{C}^*\}$, and write $\widetilde{\mathrm{g}} = exp^{-1}(\widetilde{G} \cap \mathcal{K}_{\eta,r}(\mathbb{C}))$. Then $\widetilde{G}$ is an abelian subgroup of $GL(n+1,\mathbb{C})$. See that $Id_{\mathbb{C}^{n+1}} \in exp^{-1}(H) \subset \widetilde{\mathrm{g}}$, so $\widetilde{\mathrm{g}} \backslash \Psi(MA(n,\mathbb{C})) \neq \emptyset$.
Denote by:
- $(\mathrm{g}_1)_{v_0} = \{Bv_0 \ : \ B \in \mathrm{g}_1\}$.
- $\widetilde{\mathrm{g}}_{v_0} = \{Bv_0 \ : \ B \in \widetilde{\mathrm{g}}\}$.

**Proposition 3.1.** ([1], *Theorem1.1*) *Let $G$ be an abelian subgroup of $GL(n+1,\mathbb{C})$. The following are equivalent:*
*(i) $G$ has a dense orbit in $\mathbb{C}^{n+1}$*
*(ii) The orbit $G(v_0)$ is dense in $\mathbb{C}^{n+1}$*
*(iii) $(\mathrm{g}_1)_{v_0}$ is an additive subgroup dense in $\mathbb{C}^{n+1}$*

**Lemma 3.2.** *The following assertions are equivalent:*
(*i*) $\overline{\mathfrak{g}_{w_0}} = \mathbb{C}^n$.
(*ii*) $\overline{\mathrm{g}_{v_0}} = \{0\} \times \mathbb{C}^n$.
(*iii*) $\overline{\widetilde{\mathrm{g}}_{v_0}} = \mathbb{C}^{n+1}$.

*Proof.* (*i*) $\Longleftrightarrow$ (*ii*) : For every $f' = (B,b) \in \mathfrak{g}$, one has

$$\begin{aligned}\Psi(f')v_0 &= \begin{bmatrix} 0 & 0 \\ b & B \end{bmatrix} \begin{bmatrix} 1 \\ w_0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ b + Bv_0 \end{bmatrix}.\end{aligned}$$

Then $\{0\} \times \mathfrak{g}_{w_0} = \mathrm{g}_{v_0}$ and the equivalence is proved.

(*ii*) $\Longleftrightarrow$ (*iii*) : Firstly, remark that $\widetilde{\mathrm{g}} = \mathrm{g}_1 + H$ and $v_0 \in \{1\} \times \mathbb{C}^n$, then

$\overline{\widetilde{\mathrm{g}}_{v_0}} = \overline{(\mathrm{g}_1)_{v_0}} + \mathbb{C}v_0$. By Proposition 2.6.(i), $(\mathrm{g}_1)_{v_0} = \mathrm{g}_{v_0} + 2i\pi\mathbb{Z}v_0$, so $\overline{\widetilde{\mathrm{g}}_{v_0}} = \overline{\mathrm{g}_{v_0}} + \mathbb{C}v_0$. Secondly, suppose that $\overline{\mathrm{g}_{v_0}} = \{0\} \times \mathbb{C}^n$. Since $v_0 \notin \{0\} \times \mathbb{C}^n$ and $Id_{\mathbb{C}^{n+1}} \in exp^{-1}(H \cap \mathcal{K}_{\eta,r}(\mathbb{C})) \subset \widetilde{\mathrm{g}}$, then $\mathbb{C}_{v_0} \subset \widetilde{\mathrm{g}}_{v_0}$. Therefore $\mathbb{C}^{n+1} = \overline{\mathrm{g}_{v_0}} \oplus \mathbb{C}v_0 \subset \overline{\widetilde{\mathrm{g}}_{v_0}}$. Conversely, suppose that $\overline{\widetilde{\mathrm{g}}_{v_0}} = \mathbb{C}^{n+1}$. Since $\overline{\mathrm{g}_{v_0}} \subset \{0\} \times \mathbb{C}^n$ and $\mathbb{C}v_0 \cap (\{0\} \times \mathbb{C}^n) = \{0\}$, then $\overline{\mathrm{g}_{v_0}} \oplus \mathbb{C}v_0 = \mathbb{C}^{n+1}$, thus $\overline{\mathrm{g}_{v_0}} = \{0\} \times \mathbb{C}^n$. □

**Lemma 3.3.** *Let $x \in \mathbb{C}^n$. Then the following assertions are equivalent:*
*(i)* $\overline{\mathcal{G}(x)} = \mathbb{C}^n$.
*(ii)* $\overline{G(1,x)} = \{1\} \times \mathbb{C}^n$.
*(iii)* $\overline{\widetilde{G}(1,x)} = \mathbb{C}^{n+1}$.

*Proof.* $(i) \iff (ii)$ : The proof is obvious, since $\{1\} \times \mathcal{G}(x) = G(1,x)$ by construction.
$(ii) \iff (iii)$ : Suppose that $\overline{\widetilde{G}(1,x)} = \mathbb{C}^{n+1}$. If $\overline{G(1,x)} \neq \{1\} \times \mathbb{C}^n$, then there exists an open subset $O$ of $\mathbb{C}^n$ such that $(\{1\} \times O) \cap G(1,x) = \emptyset$. Let $y \in O$ and $(g_m)_m$ be a sequence in $\widetilde{G}$ such that $\underset{m\to+\infty}{lim} g_m(1,x) = (1,y)$. Since $\widetilde{G}$ is abelian then $g_m = \lambda_m f_m$, with $f_m \in G$ and $\lambda_m \in \mathbb{C}^*$, thus $\underset{m\to+\infty}{lim} \lambda_m = 1$. Therefore, $\underset{m\to+\infty}{lim} f_m(1,x) = \underset{m\to+\infty}{lim} \frac{1}{\lambda_m} g_m(1,x) = (1,y)$. Hence, $(1,y) \in \overline{G(1,x)} \cap (\{1\} \times O)$, a contradiction. Conversely, if $\overline{G(1,x)} = \{1\} \times \mathbb{C}^n$, then

$$\begin{aligned}\mathbb{C}^{n+1} &= \bigcup_{\lambda\in\mathbb{C}} \lambda\left(\{1\} \times \mathbb{C}^n\right)\\ &= \bigcup_{\lambda\in\mathbb{C}} \lambda\overline{G(1,x)}\\ &\subset \overline{\widetilde{G}(1,x)}\end{aligned}$$

□

3.1. **Proof of Theorem 1.1.** The proof of Theorem 1.1 results directly from Lemma 3.3, Proposition 3.1 and Lemma 3.2. □

## 4. Finitely generated subgroups

4.1. **Proof of Theorem 1.2.**
Let $J_k = \mathrm{diag}(J_{k,1}, \dots, J_{k,r})$ with $J_{k,i} = 0 \in \mathbb{T}_{n_i}(\mathbb{C})$ if $i \neq k$ and $J_{k,k} = I_{n_k}$.

**Proposition 4.1.** ([1], *Proposition 8.1) Let $G$ be an abelian subgroup of $GL(n+1,\mathbb{C})$ generated by $A_1, \dots, A_p$. Let $B_1, \dots, B_p \in \mathrm{g}$ such that $A_k = e^{B_k}$, $k = 1, \dots, p$ and $P \in GL(n+1,\mathbb{C})$ satisfying $P^{-1}GP \subset \mathcal{K}^*_{\eta,r}(\mathbb{C})$. Then:*

$$\mathrm{g}_1 = \sum_{k=1}^{p} \mathbb{Z}B_k + 2i\pi\sum_{k=1}^{r} \mathbb{Z}PJ_kP^{-1} \quad \text{and} \quad (\mathrm{g}_1)_{v_0} = \sum_{k=1}^{p} \mathbb{Z}B_kv_0 + \sum_{k=1}^{r} 2i\pi\mathbb{Z}Pe^{(k)}.$$

**Proposition 4.2.** (*Under notations of Proposition 2.2*) *Let $\mathcal{G}$ be an abelian subgroup of $GA(n,\mathbb{C})$ generated by $f_1,\dots,f_p$ and let $f'_1,\dots,f'_p\in\mathfrak{g}$ such that $\Phi(f_k)=e^{\Psi(f'_k)}$, $k=1,..,p$. Then:*

$$\mathfrak{g}_{w_0}=\sum_{k=1}^{p}\mathbb{Z}f'_k(w_0)+\sum_{k=1}^{r}2i\pi\mathbb{Z}(p_2\circ Pe^{(k)}).$$

*Proof.* Let $G=\Phi(\mathcal{G})$. Then $G$ is generated by $\Phi(f_1),\dots,\Phi(f_p)$. By proposition 4.1 we have

$$\mathrm{g}_1=\sum_{k=1}^{p}\mathbb{Z}\Psi(f'_k)+\sum_{k=1}^{r}2i\pi\mathbb{Z}PJ^{(k)}P^{-1}.$$

Since $P\in\Phi(GA(n,\mathbb{C}))$ then $PJ^{(1)}P^{-1}\notin\mathcal{F}_{n+1}$ and $PJ^{(k)}P^{-1}\in\mathcal{F}_{n+1}$ for every $k=2,\dots,r$. As $\mathrm{g}=\mathrm{g}_1\cap\mathcal{F}_{n+1}$, then

$$\mathrm{g}=\begin{cases}\sum\limits_{k=1}^{p}\mathbb{Z}\Psi(f'_k)+\sum\limits_{k=2}^{r}2i\pi\mathbb{Z}PJ^{(k)}P^{-1}, & if\ r\geq 2\\ \sum\limits_{k=1}^{p}\mathbb{Z}\Psi(f'_k), & if\ r=1\end{cases}$$

By construction, one has $\mathfrak{g}_{w_0}=p_2\left(\mathrm{g}_{v_0}\right)$, $v_0=Pu_0$, $J^{(k)}u_0=e^{(k)}$ and $p_2(\Psi(f'_k)v_0)=f'_k(w_0)$. Then

$$\mathfrak{g}_{w_0}=\begin{cases}\sum\limits_{k=1}^{p}\mathbb{Z}f'_k(w_0)+\sum\limits_{k=2}^{r}2i\pi\mathbb{Z}(p_2\circ Pe^{(k)}), & if\ r\geq 2\\ \sum\limits_{k=1}^{p}\mathbb{Z}f'_k(w_0), & if\ r=1\end{cases}\tag{3}$$

The proof is completed. □

Recall the following Proposition which was proven in [7]:

**Proposition 4.3.** (*cf.* [7], *page 35*). *Let $F=\mathbb{Z}u_1+\dots+\mathbb{Z}u_p$ with $u_k=(u_{k,1},\dots,u_{k,n})\in\mathbb{C}^n$ and $u_{k,i}=Re(u_{k,i})+iIm(u_{k,i})$, $k=1,\dots,p$, $i=1,\dots,n$. Then $F$ is dense in $\mathbb{C}^n$ if and only if for every $(s_1,\dots,s_p)\in\mathbb{Z}^p\backslash\{0\}$ :*

$$rank\begin{bmatrix} Re(u_{1,1}) & \dots & \dots & Re(u_{p,1})\\ \vdots & \vdots & \vdots & \vdots\\ Re(u_{1,n}) & \dots & \dots & Re(u_{p,n})\\ Im(u_{1,1}) & \dots & \dots & Im(u_{p,1})\\ \vdots & \vdots & \vdots & \vdots\\ Im(u_{1,n}) & \dots & \dots & Im(u_{p,n})\\ s_1 & \dots & \dots & s_p\end{bmatrix} = 2n+1.$$

*Proof of Theorem 1.2:* This follows directly from Theorem 1.1, Propositions 4.2 and 4.3.

4.2. **Proof of Corollaries 1.3 and 1.4.**
*Proof of Corollary 1.3:* We show first that if $F = \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_m$, $u_k \in \mathbb{C}^n$ with $m \leq 2n$, then $F$ can not be dense: Write $u_k \in \mathbb{C}^n$, $u_k = Re(u_k) + iIm(u_k)$ and $v_k = [Re(u_k); Im(u_k); s_k]^T \in \mathbb{R}^{2n+1}$, $1 \leq k \leq m$. Since $m \leq 2n$, it follows that $rank(v_1, \dots, v_m) \leq 2n$, and so $F$ is not dense in $\mathbb{C}^n$ by Proposition 4.3.
Now, by applying Theorem 1.2 and the fact that $m = p+r-1 \leq 2n$ (since $r \leq n+1$) and by (1), the Corollary 1.3 follows. □

*Proof of Corollary 1.4:* Since $p \leq n$ and $r \leq n+1$ then $p+r-1 \leq 2n$. Corollary 1.4 follows from Corollary 1.3. □

## 5. Example

**Example 5.1.** Let $\mathcal{G}$ the group generated by $f_1 = (A_1, a_1)$, $f_2 = (A_2, a_2)$, $f_3 = (A_3, a_3)$ and $f_4 = (A_4, a_4)$, where:

$$A_1 = I_2, \quad a_1 = (1+i, 0), \qquad A_2 = \mathrm{diag}(1, e^{-2+i}), \quad a_2 = (0,\ 0).$$

$$A_3 = \mathrm{diag}\left(1,\ e^{\frac{-\sqrt{2}}{\pi} + i\left(\frac{\sqrt{2}}{2\pi} - \frac{\sqrt{7}}{2}\right)}\right), \qquad a_3 = \left(\frac{-\sqrt{3}}{2\pi} + i\left(\frac{\sqrt{5}}{2} - \frac{\sqrt{3}}{2\pi}\right), 0\right),$$

$$A_4 = I_2 \quad \text{and} \quad a_4 = (2i\pi,\ 0).$$

Then every orbit in $V = \mathbb{C} \times \mathbb{C}^*$ is dense in $\mathbb{C}^2$.

*Proof.* Denote by $G = \Phi(\mathcal{G})$. Then $G$ generated by

$$\Phi(f_1) = \begin{bmatrix} 1 & 0 & 0 \\ 1+i & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \Phi(f_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-2+i} \end{bmatrix},$$

$$\Phi(f_3) = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-\sqrt{3}}{2\pi} + i\left(\frac{\sqrt{5}}{2} - \frac{\sqrt{3}}{2\pi}\right) & 1 & 0 \\ 0 & 0 & e^{\frac{-\sqrt{2}}{\pi} + i\left(\frac{\sqrt{2}}{2\pi} - \frac{\sqrt{7}}{2}\right)} \end{bmatrix},$$

and

$$\Phi(f_4) = \begin{bmatrix} 1 & 0 & 0 \\ 2i\pi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $f'_1 = (B_1, b_1)$, $f'_2 = (B_2, b_2)$ and $f'_2 = (B_2, b_2)$ such that $e^{\Psi(f'_k)} = A'_k$, $k = 1, 2, 3, 4$. We have

$$B_1 = \mathrm{diag}(0,\ 0), b_1 = (1+i,\ 0)\ ,$$

$$B_2 = \mathrm{diag}(0,\ -2+i), \qquad b_2 = (0,\ 0),$$

$$B_3 = \mathrm{diag}\left(0,\quad \frac{-\sqrt{2}}{\pi} + i\left(\frac{\sqrt{2}}{2\pi} - \frac{\sqrt{7}}{2}\right)\right), \qquad b_3 = \left(\frac{-\sqrt{3}}{2\pi} + i\left(\frac{\sqrt{5}}{2} - \frac{\sqrt{3}}{2\pi}\right),\ 0\right),$$

$$B_4 = \mathrm{diag}(0, 0) \quad \text{and} \quad b_4 = (2i\pi,\ 0).$$

Here, we have:
- $G$ is an abelian subgroup of $\mathcal{K}^*_{(2,1),2}(\mathbb{C})$.

- $P = I_3$, $r = 2$, $U = \mathbb{C}^* \times \mathbb{C} \times \mathbb{C}^*$, $u_0 = (1,0,1)$, $e^{(1)} = (1,0,0)$ and $e^{(2)} = (0,0,1)$
- $V = \mathbb{C} \times \mathbb{C}^*$, $w_0 = (0,1)$.

In the other hand, for every $(s_1, s_2, s_3, s_4, t_2) \in \mathbb{Z}^5 \backslash \{0\}$, one has the determinant:

$$\Delta = \begin{vmatrix} Re(B_1w_0+b_1) & Re(B_2w_0+b_2) & Re(B_3w_0+b_3) & Re(B_4w_0+b_4) & 0 \\ Im(B_1w_0+b_1) & Im(B_2w_0+b_2) & Im(B_3w_0+b_3) & Im(B_4w_0+b_4) & 2\pi e^{(2)} \\ s_1 & s_2 & s_3 & s_4 & t_2 \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{\sqrt{3}}{2\pi} & 1 & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{\pi} & 0 & -2 & 0 & 0 \\ \frac{\sqrt{5}}{2} - \frac{\sqrt{3}}{2\pi} & 1 & 0 & 2\pi & 0 \\ \frac{\sqrt{2}}{2\pi} - \frac{\sqrt{7}}{2} & 0 & 1 & 0 & 2\pi \\ s_1 & s_2 & s_3 & s_4 & t_2 \end{vmatrix}$$

$$= -2\pi \left( (4s_1)\pi - (2s_3)\sqrt{2} + (2s_2)\sqrt{3} + s_4\sqrt{5} + t_2\sqrt{7} \right).$$

Since $\pi$, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ and $\sqrt{7}$ are rationally independent then $\Delta \neq 0$ for every $(s_1, s_2, s_3, t_1, t_2) \in \mathbb{Z}^5 \backslash \{0\}$. It follows that:

$$rank \begin{bmatrix} -\frac{\sqrt{3}}{2\pi} & 1 & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{\pi} & 0 & -2 & 0 & 0 \\ \frac{\sqrt{5}}{2} - \frac{\sqrt{3}}{2\pi} & 1 & 0 & 2\pi & 0 \\ \frac{\sqrt{2}}{2\pi} - \frac{\sqrt{7}}{2} & 0 & 1 & 0 & 2\pi \\ s_1 & s_2 & s_3 & s_4 & t_2 \end{bmatrix} = 5$$

and by Theorem 1.2, $\mathcal{G}$ has a dense orbit and every orbit of $V$ is dense in $\mathbb{C}^2$. $\square$

## References


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