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Adlene Ayadi

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# DENSE ORBITS FOR ABELIAN AFFINE GROUPS

ADLENE AYADI

ABSTRACT. In this paper, we give a characterization of existence of a dense orbit for the action on  $\mathbb{C}^n$  of an abelian affine subgroup  $\mathcal{G}$  of  $GA(n, \mathbb{C})$ . We prove in particular that if  $\mathcal{G}$  has a dense orbit then the closure  $\overline{\mathcal{G}}$  is an affine space of dimension  $n$ . If  $\mathcal{G}$  is finitely generated, this characterization is explicit.

## 1. Introduction

Let  $M_n(\mathbb{C})$  be the set of all square matrices of order  $n \geq 1$  with entries in  $\mathbb{C}$  and  $GL(n, \mathbb{C})$  be the group of all invertible matrices of  $M_n(\mathbb{C})$ . A map  $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$  is called an affine map if there exist  $A \in M_n(\mathbb{C})$  and  $a \in \mathbb{C}^n$  such that  $f(x) = Ax + a$ ,  $x \in \mathbb{C}^n$ . We denote  $f = (A, a)$ , we call  $A$  the *linear part* of  $f$ . Denote by  $MA(n, \mathbb{C})$  the set of all affine maps and  $GA(n, \mathbb{C})$  the set of all invertible affine maps of  $MA(n, \mathbb{C})$ .  $MA(n, \mathbb{C})$  is a vector space and for composition of maps,  $GA(n, \mathbb{C})$  is a group.

Let  $\mathcal{G}$  be an abelian affine subgroup of  $GA(n, \mathbb{C})$ . For a vector  $v \in \mathbb{C}^n$ , we consider the orbit of  $\mathcal{G}$  through  $v$ :  $\mathcal{G}(v) = \{f(v) : f \in \mathcal{G}\} \subset \mathbb{C}^n$ . A subset  $E \subset \mathbb{C}^n$  is called  $\mathcal{G}$ -invariant if  $f(E) \subset E$  for any  $f \in \mathcal{G}$ ; that is  $E$  is a union of orbits. Before stating our main results, we introduce the following notions:

A subset  $\mathcal{H}$  of  $\mathbb{C}^n$  is called an *affine subspace* of  $\mathbb{C}^n$  if there exist a vector subspace  $H$  of  $\mathbb{C}^n$  and  $a \in \mathbb{C}^n$  such that  $\mathcal{H} = H + a$ . For  $a \in \mathbb{C}^n$ , denote by  $T_a : \mathbb{C}^n \rightarrow \mathbb{C}^n$ ;  $x \mapsto x + a$  the translation map by vector  $a$ , so  $\mathcal{H} = T_a(H)$ . We say that  $\mathcal{H}$  has dimension  $p$  ( $0 \leq p \leq n$ ), denoted  $\dim(\mathcal{H}) = p$ , if  $H$  has dimension  $p$ .

Denote by  $\overline{A}$  the closure of a subset  $A \subset \mathbb{C}^n$ . A subset  $E$  of  $\mathbb{C}^n$  is called a *minimal set* of  $\mathcal{G}$  if  $E$  is closed in  $\mathbb{C}^n$ , non empty,  $\mathcal{G}$ -invariant and has no proper subset with these properties. It is equivalent to say that  $E$  is a  $\mathcal{G}$ -invariant set such that every orbit contained in  $E$  is dense in it.

If  $V$  is a  $\mathcal{G}$ -invariant open set in  $\mathbb{C}^n$ , we say that  $E$  is a *minimal set in  $V$*  if it is a minimal set of the restriction  $\mathcal{G}|_V$  of  $\mathcal{G}$  to  $V$ .

Define the map

$$\Phi : GA(n, \mathbb{C}) \rightarrow \Phi(GA(n, \mathbb{C})) \subset GL(n+1, \mathbb{C})$$

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$$f = (A, a) \mapsto \begin{bmatrix} 1 & 0 \\ a & A \end{bmatrix}$$

We have the following composition formula

$$\begin{bmatrix} 1 & 0 \\ a & A \end{bmatrix} \begin{bmatrix} 1 & 0 \\ b & B \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ Ab + a & AB \end{bmatrix}.$$

Then  $\Phi$  is a homomorphism of groups.

Let  $\mathcal{G}$  be an abelian affine subgroup of  $GA(n, \mathbb{C})$ . Then  $\Phi(\mathcal{G})$  is an abelian subgroup of  $GL(n+1, \mathbb{C})$ .

Let  $\pi : \mathbb{C}^n \rightarrow \{1\} \times \mathbb{C}^n$  be the map defined by  $\pi(y) = (1, y)$ . Clearly  $\pi$  is an invertible affine maps and we have for every  $x \in \mathbb{C}^n$ ,

$$\pi(\mathcal{G}(x)) = \Phi(\mathcal{G})(1, x) = \{1\} \times \mathcal{G}(x) \subset \{1\} \times \mathbb{C}^n.$$

Let  $n \in \mathbb{N}_0$  be fixed. For each  $m = 1, 2, \dots, n+1$ , denote by:

- $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$  and  $\mathbb{N}_0 = \mathbb{N} \setminus \{0\}$ .
- $\mathcal{B}_0 = (e_1, \dots, e_{n+1})$  the canonical basis of  $\mathbb{C}^{n+1}$  and  $I_{n+1}$  the identity matrix of  $GL(n+1, \mathbb{C})$ .
- $\mathbb{T}_m(\mathbb{C})$  the set of matrices over  $\mathbb{C}$  of the form

$$\begin{bmatrix} \mu & & & 0 \\ a_{2,1} & \mu & & \\ \vdots & \ddots & \ddots & \\ a_{m,1} & \dots & a_{m,m-1} & \mu \end{bmatrix} \quad (1)$$

- $\mathbb{T}_m^*(\mathbb{C})$  the group of matrices of the form (1) with  $\mu \neq 0$ .

Let  $r \in \mathbb{N}$  and  $\eta = (n_1, \dots, n_r)$  be a sequence of positive integers such that  $n_1 + \dots + n_r = n+1$ . In particular,  $r \leq n+1$ .

Write

- $\mathcal{K}_{\eta,r}(\mathbb{C}) := \mathbb{T}_{n_1}(\mathbb{C}) \oplus \dots \oplus \mathbb{T}_{n_r}(\mathbb{C})$ . In particular if  $r = 1$ , then  $\mathcal{K}_{\eta,1}(\mathbb{C}) = \mathbb{T}_{n+1}(\mathbb{C})$  and  $\eta = (n+1)$ .
- $\mathcal{K}_{\eta,r}^*(\mathbb{C}) := \mathcal{K}_{\eta,r}(\mathbb{C}) \cap GL(n+1, \mathbb{C})$ .

Define the map

$$\Psi : MA(n, \mathbb{C}) \rightarrow \Psi(GA(n, \mathbb{C})) \subset M_{n+1}(\mathbb{C})$$

$$f = (A, a) \mapsto \begin{bmatrix} 0 & 0 \\ a & A \end{bmatrix}$$

We have  $\Psi$  is an isomorphism. Remark that  $\Phi(f) = \Psi(f) + D_1$ , where  $D_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in M_{n+1}(\mathbb{C})$ ,  $f \in GA(n, \mathbb{C})$ , so  $\Phi$  is an affine map.

Denote by:

- $\mathcal{F}_{n+1} = \left\{ \begin{bmatrix} 0 & 0 \\ b & B \end{bmatrix} \in M_{n+1}(\mathbb{C}), B \in M_n(\mathbb{C}), b \in \mathbb{C}^n \right\} = \Psi(MA(n, \mathbb{C}))$ .

- $\exp : \mathbb{M}_{n+1}(\mathbb{C}) \longrightarrow \mathrm{GL}(n+1, \mathbb{C})$  is the matrix exponential map; set  $\exp(M) = e^M$ .

There always exists a  $P \in \Phi(\mathrm{GA}(n, \mathbb{C}))$  and a partition  $\eta$  of  $n+1$  such that  $G' = P^{-1}GP \subset \mathcal{K}_{\eta,r}^*(\mathbb{C}) \cap \Phi(\mathrm{GA}(n, \mathbb{C}))$  (see Proposition 2.3). For such a choice of matrix  $P$ , we let

- $\mathfrak{g} = \exp^{-1}(G) \cap (P(\mathcal{K}_{\eta,r}(\mathbb{C}))P^{-1}) \cap \mathcal{F}_{n+1}$ . If  $G \subset \mathcal{K}_{\eta,r}^*(\mathbb{C})$ , we have  $\mathfrak{g} = \exp^{-1}(G) \cap \mathcal{K}_{\eta,r}(\mathbb{C}) \cap \mathcal{F}_{n+1}$ .
- $\mathfrak{g}_u = \{Bu : B \in \mathfrak{g}\}$ ,  $u \in \mathbb{C}^{n+1}$ .
- $\mathfrak{g} = \Psi^{-1}(\mathfrak{g})$ .
- $\mathfrak{g}_v = \{f(v), f \in \mathfrak{g}\}$ ,  $v \in \mathbb{C}^n$ .

Let  $\tilde{\pi} : \mathbb{C}^n \longrightarrow \{0\} \times \mathbb{C}^n$  be the map defined by  $\tilde{\pi}(y) = (0, y)$ . Clearly  $\tilde{\pi}$  is an isomorphism.

Denote by:

- $u_0 = [e_{1,1}, \dots, e_{r,1}]^T \in \mathbb{C}^{n+1}$  where  $e_{k,1} = [1, 0, \dots, 0]^T \in \mathbb{C}^{n_k}$ , for  $k = 1, \dots, r$ . One has  $u_0 \in \{1\} \times \mathbb{C}^n$ .
- $w_0 = \pi^{-1}(Pu_0) \in \mathbb{C}^n$ . As  $P \in \Phi(\mathrm{GA}(n, \mathbb{C}))$ ,  $Pu_0 \in \{1\} \times \mathbb{C}^n$ . So  $\pi(w_0) = (1, w_0) = Pu_0$ .
- $e^{(k)} = [e_1^{(k)}, \dots, e_r^{(k)}]^T \in \mathbb{C}^{n+1}$  where

$$e_j^{(k)} = \begin{cases} 0 \in \mathbb{C}^{n_j} & \text{if } j \neq k \\ e_{k,1} & \text{if } j = k \end{cases} \quad \text{for every } 1 \leq j, k \leq r.$$

- $p_2 : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^n$  the projection defined by  $p_2(x_1, \dots, x_{n+1}) = (x_2, \dots, x_{n+1})$ . Notice that  $\pi^{-1}$  is the restriction of  $p_2$  to  $\{1\} \times \mathbb{C}^n$ .
- $M^T$  is the transpose of any  $M \in M_m(\mathbb{C})$ ,  $m \leq n+1$ .
- $\overline{\mathcal{G}}$  is the closure of  $\mathcal{G}$  in  $MA(n, \mathbb{C})$ , which is considered as a vector space with the norm defined by  $\|f\| = \max(\|A\|, \|a\|)$  the maximum of  $\|A\|$  and  $\|a\|$ , for every  $f = (A, a) \in MA(n, \mathbb{C})$ .

For a *finitely generated* subgroup  $\mathcal{G} \subset \mathrm{GA}(n, \mathbb{R})$ , let introduce the following property. Consider the following rank condition on a collection of affine maps  $f_1, \dots, f_p \in \mathrm{GA}(n, \mathbb{C})$ , where  $f'_1, \dots, f'_p \in \mathfrak{g}$  such that  $e^{\Psi(f'_k)} = \Phi(f_k)$ ,  $k = 1, \dots, p$ .

We say that  $f_1, \dots, f_p$  satisfy *property D* if for every  $(s_1, \dots, s_p; t_1, \dots, t_r) \in \mathbb{Z}^{p+r} \setminus \{0\}$ :

$$\mathrm{rank} \begin{bmatrix} \mathrm{Re}(f'_1(w_0)) & \dots & \mathrm{Re}(f'_p(w_0)) & 0 & \dots & 0 \\ \mathrm{Im}(f'_1(w_0)) & \dots & \mathrm{Im}(f'_p(w_0)) & 2\pi e^{(1)} & \dots & 2\pi e^{(r)} \\ s_1 & \dots & s_p & t_1 & \dots & t_r \end{bmatrix} = n+1.$$

For a vector  $v \in \mathbb{C}^n$ , we write  $v = \mathrm{Re}(v) + i\mathrm{Im}(v)$  where  $\mathrm{Re}(v)$  and  $\mathrm{Im}(v) \in \mathbb{R}^n$ .

For groups of affine maps on  $\mathbb{K}^n$  ( $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ ), their dynamics were recently initiated for some classes in different point of view, (see for instance, [3], [4], [5],[6]). The purpose here is to give analogous results of that theorem for linear abelian subgroup of  $GL(n, \mathbb{C})$  proved in [1]. Our main results are the following:

**Theorem 1.1.** *Let  $\mathcal{G}$  be an abelian subgroup of  $GA(n, \mathbb{C})$ . The following are equivalent:*

- (i)  $\mathcal{G}$  has a dense orbit in  $\mathbb{C}^n$
- (ii) The orbit  $\mathcal{G}(w_0)$  is dense in  $\mathbb{C}^n$
- (iii)  $\mathfrak{g}_{w_0}$  is an additive subgroup dense in  $\mathbb{C}^n$

**Corollary 1.2.** *If  $\mathcal{G}$  has a dense orbit then  $\overline{\mathcal{G}}$  is an affine subspace of  $GA(n, \mathbb{C})$  of dimension  $n$ .*

For a finitely generated subgroup  $\mathcal{G}$ , the Theorem can be stated as follows:

**Theorem 1.3.** *Let  $\mathcal{G}$  be an abelian subgroup of  $GA(n, \mathbb{C})$  generated by  $f_1, \dots, f_p$  and let  $f'_1, \dots, f'_p \in \mathfrak{g}$  such that  $e^{\Psi(f'_1)} = \Phi(f_1), \dots, e^{\Psi(f'_p)} = \Phi(f_p)$ . Then the following are equivalent:*

- (i)  $\mathcal{G}$  has a dense orbit in  $\mathbb{C}^n$
- (ii) the maps  $f_1, \dots, f_p$  satisfy property  $\mathcal{D}$
- (iii)  $\mathfrak{g}_{w_0} = \sum_{k=1}^p \mathbb{Z}f'_k(w_0) + 2i\pi \sum_{k=1}^r \mathbb{Z}(p_2 \circ Pe^{(k)})$  is an additive group dense in  $\mathbb{C}^n$ .

**Corollary 1.4.** *If  $\mathcal{G}$  is of finite type  $p$  with  $p \leq 2n - r$ , then it has no dense orbit.*

**Corollary 1.5.** *If  $\mathcal{G}$  is of finite type  $p$  with  $p \leq n$ , then it has no dense orbit.*

## 2. Notations and Lemmas

Denote by  $\mathcal{L}_{\mathcal{G}}$  the set of the linear parts of all elements of  $\mathcal{G}$  and  $\text{vect}(F)$  is the vector space generated by a subset  $F \subset \mathbb{C}^{n+1}$ .

**Lemma 2.1.** *Let  $P \in \Phi(GA(n, \mathbb{C}))$  and  $h : M_{n+1}(\mathbb{C}) \rightarrow M_{n+1}(\mathbb{C})$  the isomorphism defined by  $h(A) = PAP^{-1}$ . Then  $h(\mathcal{F}_{n+1}) = \mathcal{F}_{n+1}$  and  $h(\Phi(GA(n, \mathbb{C}))) = \Phi(GA(n, \mathbb{C}))$ .*

*Proof.* Let  $f = (A, a) \in MA(n, \mathbb{C})$ . As  $P \in \Phi(GA(n, \mathbb{C}))$ , write

$$P = \begin{bmatrix} 1 & 0 \\ d & C \end{bmatrix}, \quad C \in GL(n, \mathbb{C}), \quad d \in \mathbb{C}^n.$$

Then

$$\begin{aligned} h(\Psi(f)) &= \begin{bmatrix} 1 & 0 \\ d & C \end{bmatrix} \begin{bmatrix} 0 & 0 \\ a & A \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -C^{-1}d & C^{-1} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ d + Ca - CAC^{-1}d & CAC^{-1} \end{bmatrix}, \end{aligned}$$

so  $h(\Psi(f)) \in \mathcal{F}_{n+1}$ . It follows that  $h(\mathcal{F}_{n+1}) \subset \mathcal{F}_{n+1}$ . Conversely, let  $h' = h^{-1}$ , so  $h' : M_{n+1}(\mathbb{C}) \rightarrow M_{n+1}(\mathbb{C})$  is the isomorphism defined by  $h(B) = P^{-1}BP$ . By replacing  $h$  by  $h'$  in above, we have  $h'(\mathcal{F}_{n+1}) \subset \mathcal{F}_{n+1}$ . It follows that

$$\mathcal{F}_{n+1} = h \circ h'(\mathcal{F}_{n+1}) \subset h(\mathcal{F}_{n+1}).$$

The same proof is used to show  $h(\Phi(GA(n, \mathbb{C}))) = \Phi(GA(n, \mathbb{C}))$ .  $\square$

**Proposition 2.2.** ([1], Proposition 2.3) *Let  $G$  be an abelian subgroup of  $GL(n, \mathbb{C})$ . Then there exists  $P \in GL(n, \mathbb{C})$  such that  $P^{-1}GP$  is a subgroup of  $\mathcal{K}_{\eta', r'}(\mathbb{C})$ , for some  $r' \leq n$  and  $\eta' = (n'_1, \dots, n'_{r'}) \in \mathbb{N}_0^{r'}$ .*

**Proposition 2.3.** *Let  $\mathcal{G}$  be an abelian subgroup of  $GA(n, \mathbb{C})$  and  $G = \Phi(\mathcal{G})$ . Then there exists  $P \in \Phi(GA(n, \mathbb{C}))$  such that  $P^{-1}GP$  is a subgroup of  $\mathcal{K}_{\eta, r}^*(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C}))$ , for some  $r \leq n + 1$  and  $\eta = (n_1, \dots, n_r) \in \mathbb{N}_0^r$ . In particular,  $Pu_0 \in \{1\} \times \mathbb{C}^n$ .*

*Proof.* We have  $\mathcal{L}_{\mathcal{G}}$  is an abelian subgroup of  $GL(n, \mathbb{C})$ . By Proposition 2.2, there exists  $P_1 \in GL(n, \mathbb{C})$  such that  $P_1^{-1}GP_1$  is a subgroup of  $\mathcal{K}_{\eta', r'}^*(\mathbb{C})$  for some  $r' \leq n$  and  $\eta' = (n'_1, \dots, n'_{r'}) \in \mathbb{N}_0^{r'}$  such that  $n'_1 + \dots + n'_{r'} = n$ . Write for every  $A \in \mathcal{L}_{\mathcal{G}}$ ,  $P_1^{-1}AP_1 = \text{diag}(A_1, \dots, A_{r'})$  with  $A_k \in \mathbb{T}_{n'_k}^*$  and  $\mu_{A_k}$  is the only eigenvalue of  $A_k$ ,  $k = 1, \dots, r'$ . Let  $J = \{k \in \{1, \dots, r'\}, \mu_{A_k} = 1, \forall A \in \mathcal{L}_{\mathcal{G}}\}$ . There are two cases:

- *Case1:* Suppose that  $J = \{k_1, \dots, k_s\}$  for some  $s \leq r'$ . We can take  $J = \{1, \dots, s\}$ , otherwise, we replace  $P_1$  by  $RP_1$  for some permutation matrix  $R$  of  $GL(n, \mathbb{C})$ . Let  $P = \text{diag}(1, P_1)$ , so  $P \in \Phi(GA(n, \mathbb{C}))$  and

$$P^{-1}\Phi(f)P = \begin{bmatrix} 1 & 0 \\ P_1^{-1}a & P_1^{-1}AP_1 \end{bmatrix}.$$

Write  $P(\mathcal{B}_0) = (e_1, \mathcal{C}_1, \dots, \mathcal{C}_{r'})$ , where  $\mathcal{C}_k = (e'_{k,1}, \dots, e'_{k,n'_k})$  is a basis of  $\mathbb{C}^{n'_k}$ ,  $k = 1, \dots, r'$ . Then  $H = \text{vect}(e_1, \mathcal{C}_1, \dots, \mathcal{C}_s)$  is  $G$ -invariant and the restriction  $\Phi(f)_{/H}$  has 1 as only eigenvalue for every  $f \in \mathcal{G}$ . Therefore for every  $f = (A, a) \in \mathcal{G}$  we have

$$P^{-1}\Phi(f)P = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix},$$

where  $A_1 \in \mathbb{T}_q^*(\mathbb{C})$ ,  $q = 1 + n'_1 + \dots + n'_s$  and  $A_2 \in \mathcal{K}_{\eta'', r'-s}(\mathbb{C})$  with  $\eta'' = (n'_{s+1}, \dots, n'_{r'})$ . As  $\mu_{A_1} = 1$  is the only eigenvalue of  $A_1$ , so  $P^{-1}\Phi(f)P \in \Phi(GA(n, \mathbb{C}))$ . It follows that  $P^{-1}GP$  is a subgroup of  $\mathcal{K}_{\eta, r}^*(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C}))$ , where  $r = r' - s + 1$  and  $\eta = (q, n'_{s+1}, \dots, n'_{r'})$ .

- *Case:2* Suppose that  $J = \emptyset$ , then 1 is the commune eigenvalue of all element of  $G$  and let  $v$  be the commune eigenvector of all element of  $G$  associated to 1. Since  $\{0\} \times \mathbb{C}^n$  is  $G$ -invariant, one has  $v \in \mathbb{C}^* \times \mathbb{C}^n$ , so we can take  $v = (1, v_1)$ ,  $v_1 \in \mathbb{C}^n$ , otherwise, if  $v = (\lambda, v'_1)$  we replace  $v$  by  $\frac{1}{\lambda}v$ . As  $\Phi(f)(1, v_1) = (1, f(v_1)) = (1, v_1)$

so  $f(v_1) = \pi^{-1}(\Phi(f)(1, v_1)) = \pi^{-1}(1, v_1) = v_1$ , hence  $f(v_1) = Av_1 + a = v_1$ . Let  $P = \begin{bmatrix} 1 & 0 \\ v_1 & P_1 \end{bmatrix} \in \Phi(GA(n, \mathbb{C}))$ , so for every  $f = (A, a) \in \mathcal{G}$

$$\begin{aligned} P^{-1}\Phi(f)P &= \begin{bmatrix} 1 & 0 \\ -P_1^{-1}v_1 & P_1^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ a & A \end{bmatrix} \begin{bmatrix} 1 & 0 \\ v_1 & P_1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ P_1^{-1}(Av_1 + a - v_1) & P_1^{-1}AP_1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & P_1^{-1}AP_1 \end{bmatrix}. \end{aligned}$$

It follows that  $P^{-1}GP$  is a subgroup of  $\mathcal{K}_{\eta, r}^*(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C}))$ , where  $r = r' + 1$  and  $\eta = (1, n'_1, \dots, n'_{r'})$ .

Since  $u_0 \in \{1\} \times \mathbb{C}^n$  and  $P \in \Phi(GA(n, \mathbb{C}))$ , so  $Pu_0 \in \{1\} \times \mathbb{C}^n$ .  $\square$

Denote by:

- $G' = P^{-1}GP$ .
- $\mathfrak{g}' = \exp^{-1}(G') \cap \mathcal{K}_{\eta, r}(\mathbb{C}) \cap \mathcal{F}_{n+1}$ .
- $\mathcal{C}(G') = \{B \in \mathcal{K}_{\eta, r}(\mathbb{C}), A'B = BA', A' \in G'\}$ .
- $\mathcal{C}(G) = P\mathcal{C}(G')P^{-1}$ .

**Proposition 2.4.** (Under above notation), we have  $\exp(\mathfrak{g}) = G$ . In particular,  $\exp(\mathfrak{g}') = G'$ .

The proof of Proposition 2.4 uses the following Lemmas:

**Lemma 2.5.** ([1], Lemma 4.2) Let  $G$  be an abelian subgroup of  $GL(n+1, \mathbb{C})$  and  $P \in GL(n+1, \mathbb{C})$  such that  $P^{-1}GP$  is a subgroup of  $\mathcal{K}_{\eta, r}(\mathbb{C})$ . If  $\mathfrak{g}_1 = \exp^{-1}(G) \cap (P\mathcal{K}_{\eta, r}(\mathbb{C})P^{-1})$  then:

- (i)  $\exp(\mathfrak{g}_1) = G$ .
- (ii)  $\mathfrak{g}_1 \subset \mathcal{C}(G)$ .

**Lemma 2.6.** ([1], Proposition 3.2)  $\exp(\mathcal{K}_{\eta, r}(\mathbb{C})) = \mathcal{K}_{\eta, r}^*(\mathbb{C})$ .

**Lemma 2.7.** (i) If  $N \in P\mathcal{K}_{\eta, r}(\mathbb{C})P^{-1}$  such that  $e^N \in \Phi(GA(n, \mathbb{C}))$ , so  $N - 2ik\pi I_{n+1} \in \mathcal{F}_{n+1}$ , for some  $k \in \mathbb{Z}$ .

(ii)  $\exp(\mathcal{F}_{n+1}) = \Phi(GA(n, \mathbb{C}))$ .

(iii)  $\exp(\mathcal{F}_{n+1} \cap \mathcal{K}_{\eta, r}(\mathbb{C})) = \Phi(GA(n, \mathbb{C})) \cap \mathcal{K}_{\eta, r}^*(\mathbb{C})$ .

*Proof.* (i) Let  $N' = P^{-1}NP \in \mathcal{K}_{\eta, r}(\mathbb{C})$ ,  $M = e^N$  and  $M' = P^{-1}MP$ . We have  $e^{N'} = M'$  and by Lemma 2.6,  $M' \in \mathcal{K}_{\eta, r}^*(\mathbb{C})$ . Write  $M' = \text{diag}(M'_1, \dots, M'_r)$  and  $N' = \text{diag}(N'_1, \dots, N'_r)$ ,  $M'_k, N'_k \in \mathbb{T}_{n_k}(\mathbb{C})$ ,  $k = 1, \dots, r$ . Then  $e^{N'} = \text{diag}(e^{N'_1}, \dots, e^{N'_r})$ , so  $e^{N'_1} = M'_1$ . As 1 is the only eigenvalue of  $M'_1$ ,  $N'_1$  has an eigenvalue  $\mu \in \mathbb{C}$  such that  $e^\mu = 1$ . Thus  $\mu = 2ik\pi$  for some  $k \in \mathbb{Z}$ . Therefore,  $N'' = N' - 2ik\pi I_{n+1} \in \mathcal{F}_{n+1}$  and satisfying  $e^{N''} = e^{-2ik\pi} e^{N'} = M'$ . It follows by Lemma 2.1, that  $N - 2ik\pi I_{n+1} = PN''P^{-1} \in P\mathcal{F}_{n+1}P^{-1} = \mathcal{F}_{n+1}$ , since  $P \in \Phi(GA(n, \mathbb{C}))$ .

(ii) let  $N = \begin{bmatrix} 0 & 0 \\ b & B \end{bmatrix} \in \mathcal{F}_{n+1}$ , we have  $N^k = \begin{bmatrix} 0 & 0 \\ B^{k-1}b & B^k \end{bmatrix} \in \mathcal{F}_{n+1}$  for every  $k \in \mathbb{N}_0$ , so

$$\begin{aligned} e^N &= I_{n+1} + \sum_{k \in \mathbb{N}_0} \frac{1}{k!} N^k \\ &= \begin{bmatrix} 1 & 0 \\ \sum_{k \in \mathbb{N}_0} \frac{1}{k!} B^{k-1}b & I_n + \sum_{k \in \mathbb{N}_0} \frac{1}{k!} B^k \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ \sum_{k \in \mathbb{N}_0} \frac{1}{k!} B^{k-1}b & e^B \end{bmatrix}. \end{aligned}$$

Then  $e^N \in \Phi(GA(n, \mathbb{C}))$ . It follows that  $\exp(\mathcal{F}_{n+1}) \subset \Phi(GA(n, \mathbb{C}))$ .

Conversely, let  $M = \begin{bmatrix} 1 & 0 \\ a & A \end{bmatrix} \in \Phi(GA(n, \mathbb{C}))$ . By applying Proposition 2.3 on the group generated by  $M$ , there exists  $P \in \Phi(GA(n, \mathbb{C}))$  such that  $M' = P^{-1}MP \in \mathcal{K}_{\eta, r}^*(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C}))$ . By Lemma 2.6, there exists  $N' \in \mathcal{K}_{\eta, r}(\mathbb{C})$  such that  $e^{N'} = M'$ . By (i),  $N'' = N' - 2ik\pi I_{n+1} \in \mathcal{F}_{n+1}$  for some  $k \in \mathbb{Z}$  and satisfying  $e^{N''} = M'$ . Hence,  $N = PN'P^{-1} \in P\mathcal{F}_{n+1}P^{-1}$  and satisfying  $e^N = Pe^{N''}P^{-1} = PM'P^{-1} = M$ . By Lemma 2.1,  $N \in \mathcal{F}_{n+1}$ . It follows that  $\Phi(GA(n, \mathbb{C})) \subset \exp(\mathcal{F}_{n+1})$ .

(iii) By Lemma 2.6 and (ii) we have  $\exp(\mathcal{F}_{n+1}) \cap \mathcal{K}_{\eta, r}(\mathbb{C}) \subset \Phi(GA(n, \mathbb{C})) \cap \mathcal{K}_{\eta, r}^*(\mathbb{C})$ .

Conversely, let  $M = \begin{bmatrix} 1 & 0 \\ a & A \end{bmatrix} \in \Phi(GA(n, \mathbb{C})) \cap \mathcal{K}_{\eta, r}^*(\mathbb{C})$ . By Lemma 2.6, there exists  $N' \in \mathcal{K}_{\eta, r}(\mathbb{C})$  such that  $e^{N'} = M$ . By (i),  $N = N' - 2ik\pi I_{n+1} \in \mathcal{F}_{n+1}$  for some  $k \in \mathbb{Z}$  and satisfying  $e^N = e^{-2ik\pi} e^{N'} = M$ . It follows that  $\Phi(GA(n, \mathbb{C})) \cap \mathcal{K}_{\eta, r}^*(\mathbb{C}) \subset \exp(\mathcal{F}_{n+1})$ . The proof is complete.  $\square$

*Proof of Proposition 2.4.* By Lemma 2.5.(i), we have  $\exp(\mathfrak{g}_1) = G$ . Since  $\mathfrak{g} = \mathfrak{g}_1 \cap \mathcal{F}_{n+1}$  then

$$\exp(\mathfrak{g}) \subset \exp(\mathfrak{g}_1) = G.$$

Conversely, let  $M \in G$ . By lemma 2.5.(i) we have  $\exp(\mathfrak{g}_1) = G$  then there exists  $N \in \mathfrak{g}_1 \subset \mathcal{K}_{\eta, r}(\mathbb{C})$  such that  $e^N = M$ . By Lemma 2.7.(i), there exists  $k \in \mathbb{Z}$  such that  $N' = N - 2ik\pi I_{n+1} \in \mathcal{F}_{n+1}$ . Since  $N$  and  $2ik\pi I_{n+1} \in P\mathcal{K}_{\eta, r}(\mathbb{C})P^{-1}$ ,  $N' \in \mathfrak{g}_1 \cap \mathcal{F}_{n+1} = \mathfrak{g}$ . It follows that  $M = e^{N'} \in \exp(\mathfrak{g})$ . The proof is complete. In particular if  $P = I_{n+1}$ , we have  $\exp(\mathfrak{g}') = G'$ .  $\square$



### 3. Parametrization

Let  $\mathcal{G}$  be an abelian subgroup of  $GA(n, \mathbb{C})$  and  $G = \Phi(\mathcal{G})$ . In all the following,  $P$  denotes the matrix defined in Proposition 2.3 (i.e.  $P \in \Phi(GA(n, \mathbb{C}))$ ) such that  $P^{-1}GP$  is a subgroup of  $\mathcal{K}_{\eta, r}^*(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C}))$ . Recall that  $G' = P^{-1}GP$  and  $\mathfrak{g}' = \exp^{-1}(G') \cap \mathcal{K}_{\eta, r}(\mathbb{C}) \cap \mathcal{F}_{n+1}$ , so every  $A' \in G'$  is in the form

$$A' = \text{diag}(A'_1, \dots, A'_r) \in \mathcal{K}_{\eta, r}^*(\mathbb{C}), \quad A'_k \in \mathbb{T}_{n_k}^*(\mathbb{C}), \quad k = 1, \dots, r$$

and every  $B' \in \mathfrak{g}'$  is in the form

$$B' = \text{diag}(B'_1, \dots, B'_r) \in \mathcal{K}_{\eta, r}(\mathbb{C}), \quad B'_k \in \mathbb{T}_{n_k}(\mathbb{C}), \quad k = 1, \dots, r$$

For every  $k = 1, \dots, r$ , denote by:

- $G'_k = \{A'_k, A' \in G'\}$ .
- $\mathfrak{g}'_k = \{B'_k, B' \in \mathfrak{g}'\}$ .
- $F_{G'_k} = \left\{ (A'_k - \mu_{A'_k} I_{n_k}) e_{k,i}, A' \in G', 1 \leq i \leq n_k - 1 \right\}$ .
- $F_{\mathfrak{g}'_k} = \left\{ (B'_k - \mu_{B'_k} I_{n_k}) e_{k,i}, B' \in \mathfrak{g}', 1 \leq i \leq n_k - 1 \right\}$ .

where  $\mu_{A'_k}$  (resp.  $\mu_{B'_k}$ ) is the only eigenvalue of  $A'_k$ . (resp.  $B'_k$ ) and  $(e_{k,1}, \dots, e_{k,n_k})$  is the canonical basis of  $\mathbb{C}^{n_k}$ .

- Let  $h : M_{n+1}(\mathbb{C}) \rightarrow M_{n+1}(\mathbb{C})$  be the isomorphism defined by  $h(B) = PBP^{-1}$ . one has  $h(G') = G$ .

See that  $\pi$  is an affine map, because  $\pi = (D_0, e_1)$ , where  $D_0 = \text{diag}(0, I_n)$ . Remark that  $\pi \notin GA(n, \mathbb{C})$ , however  $\pi : \mathbb{C}^n \rightarrow \{1\} \times \mathbb{C}^n$  is invertible.

Denote by :

- $U = \prod_{k=1}^r \mathbb{C}^* \times \mathbb{C}^{n_k-1}$ , we have  $U$  is a  $G'$ -invariant open set.
- $V = \pi^{-1} \circ P(U \cap (\{1\} \times \mathbb{C}^n))$ .

We have  $V$  is an open subset of  $\mathbb{C}^n$ , because  $V = \pi^{-1}(P(U) \cap P(\{1\} \times \mathbb{C}^n))$ ,  $\pi$  is continuous and  $P(U) \cap P(\{1\} \times \mathbb{C}^n) = P(U) \cap (\{1\} \times \mathbb{C}^n)$  is an open set of  $\{1\} \times \mathbb{C}^n$  (since  $P \in \Phi(GA(n, \mathbb{C}))$  and so  $P(\{1\} \times \mathbb{C}^n) = \{1\} \times \mathbb{C}^n$ ).

- $\mathcal{C}(\mathfrak{g}') = \{B \in \mathcal{K}_{\eta, r}(\mathbb{C}), A'B = BA', A' \in \mathfrak{g}'\}$ .
- $\mathcal{C}(\mathfrak{g}) = P\mathcal{C}(\mathfrak{g}')P^{-1}$  (i.e.  $\mathcal{C}(\mathfrak{g}) = h(\mathcal{C}(\mathfrak{g}'))$ ).
- $\mathcal{C}(\mathcal{G}) = \Phi^{-1}(\mathcal{C}(G) \cap \Phi(GA(n, \mathbb{C})))$ .
- $\mathcal{C}(\mathfrak{g}) = \Psi^{-1}(\mathcal{C}(\mathfrak{g}) \cap \mathcal{F}_{n+1})$ .

Notice that by construct,  $\mathcal{C}(\mathcal{G})$  is an affine subspace of  $MA(n, \mathbb{C})$ , since  $\Phi$  is affine map. However,  $\mathcal{C}(\mathfrak{g})$  is a vector subspace of  $MA(n, \mathbb{C})$ , since  $\Psi$  is a linear map.

Recall the following results proved in [1] for an abelian subgroup  $G'$  of  $\mathcal{K}_{\eta, r}^*(\mathbb{C})$ .

**Lemma 3.1.** ([1], Lemma 4.2) *We have  $\mathcal{C}(G') = \mathcal{C}(\mathfrak{g}')$ .*

By Lemma 3.1 we have  $\mathcal{C}(G') = \mathcal{C}(\mathfrak{g}')$  and so  $\mathcal{C}(G) = \mathcal{C}(\mathfrak{g})$ . By construct, this identification can not be true in the affine case (i.e.  $\mathcal{C}(\mathcal{G}) \neq \mathcal{C}(\mathfrak{g})$ ). Because  $f_0 = (0, 0) \notin \mathcal{C}(\mathcal{G})$  and  $\mathcal{C}(\mathfrak{g})$  is a vector space.

**Proposition 3.2.** ([1], Theorem 5.2 and Corollary 5.4) *Let  $G'$  be an abelian subgroup of  $\mathcal{K}_{\eta,r}^*(\mathbb{C})$ . If  $\text{rank}(F_{G'_k}) = n_k - 1$  (resp.  $\text{rank}(F_{\mathfrak{g}'_k}) = n_k - 1$ ),  $k = 1, \dots, r$ , then there exists a linear injective map  $\varphi : \mathbb{C}^{n+1} \rightarrow \mathcal{K}_{\eta,r}(\mathbb{C})$  (resp.  $\psi : \mathbb{C}^{n+1} \rightarrow \mathcal{K}_{\eta,r}(\mathbb{C})$ ) such that:*

- (i)  $\mathcal{C}(G') \subset \varphi(\mathbb{C}^{n+1})$ , (resp.  $\mathcal{C}(G') \subset \psi(\mathbb{C}^{n+1})$ ).
- (ii) For every  $v \in \mathbb{C}^{n+1}$ ,  $\varphi(v)u_0 = v$ , (resp.  $\psi(v)u_0 = v$ ). In particular  $\varphi(G'(u_0)) = G'$  and  $\varphi(\mathfrak{g}'_{u_0}) = \mathfrak{g}'$  (resp.  $\psi(G'(u_0)) = G'$  and  $\psi(\mathfrak{g}'_{u_0}) = \mathfrak{g}'$ ).

**Lemma 3.3.** ([1], Corollaries 2.7 and 3.2.(ii)) *The linear injective map  $\psi : \mathbb{C}^{n+1} \rightarrow \mathcal{K}_{\eta,r}(\mathbb{C})$  defined in Proposition 3.2 satisfies  $\exp(\mathcal{C}(\mathfrak{g}')) = \mathcal{C}(G') \cap \mathcal{K}_{\eta,r}^*(\mathbb{C})$ .*

**Corollary 3.4.** ([1], Proposition 6.2) *If  $\overline{G'(u_0)} = \mathbb{C}^{n+1}$  (resp.  $\overline{\mathfrak{g}'_{u_0}} = \mathbb{C}^{n+1}$ ) then  $\varphi$  (resp.  $\psi$ ) is an isomorphism of  $\mathbb{C}^{n+1}$  on  $\mathcal{C}(G')$ .*

**Corollary 3.5.** ([1], Corollary 3.7 and Propositions 3.6, 3.2) *The restriction  $\exp|_{\mathcal{K}_{\eta,r}(\mathbb{C})} : \mathcal{K}_{\eta,r}(\mathbb{C}) \rightarrow \mathcal{K}_{\eta,r}^*(\mathbb{C})$  is a local diffeomorphism.*

**Lemma 3.6.** *The restriction  $\exp|_{(\mathcal{K}_{\eta,r}(\mathbb{C}) \cap \mathcal{F}_{n+1})} : \mathcal{K}_{\eta,r}(\mathbb{C}) \cap \mathcal{F}_{n+1} \rightarrow \Phi(GA(n, \mathbb{C})) \cap \mathcal{K}_{\eta,r}^*(\mathbb{C})$  is a local diffeomorphism.*

*Proof.* The proof results from Corollary 3.5 and Lemma 2.7.(iii). □

Recall the following Proposition:

**Proposition 3.7.** ([1], Proposition 6.6)  *$U$  is  $G'$ -invariant dense open set in  $\mathbb{C}^{n+1}$  and all orbits of  $G'$  in  $U$  are minimal in  $U$ .*

**Lemma 3.8.** *Under notation of Proposition 3.2, we have:*

- (i)  $h(\mathfrak{g}') = \mathfrak{g}$  and  $P^{-1}(\mathfrak{g}_{(1,w_0)}) = \mathfrak{g}'_{u_0}$ .
- (ii)  $\varphi(\mathbb{C}^{n+1}) \cap \Phi(GA(n, \mathbb{C})) \subset \varphi(\{1\} \times \mathbb{C}^n)$  and  $\varphi(\mathbb{C}^{n+1}) \cap \mathcal{F}_{n+1} \subset \varphi(\{0\} \times \mathbb{C}^n)$ .
- (iii)  $h(\mathcal{C}(G')) = \mathcal{C}(G)$ .
- (iv)  $\tilde{\pi}(f'(u)) = \Psi(f')(1, u)$ , for every  $f' \in \mathfrak{g}$  and  $u \in \mathbb{C}^n$ . In particular,  $\tilde{\pi}(\mathfrak{g}_u) = \mathfrak{g}_{(1,u)}$ .

*Proof.* (i) Let  $B' \in \mathfrak{g}' = \exp^{-1}(G) \cap \mathcal{K}_{\eta,r}(\mathbb{C}) \cap \mathcal{F}_{n+1}$ , so  $e^{h(B')} = h(e^B) \in h(G') = G$  and since  $h(\mathcal{F}_{n+1}) = \mathcal{F}_{n+1}$  so  $h(B) \in \mathcal{F}_{n+1}$ , then

$$h(B) \in \exp^{-1}(G) \cap h(\mathcal{K}_{\eta,r}(\mathbb{C})) \cap \mathcal{F}_{n+1} = \mathfrak{g}.$$

Conversely, we use the same proof by replacing  $h$  by  $h^{-1}$ .  
 Since  $h(g') = \mathfrak{g}$ , so  $\mathfrak{g} = h^{-1}(g') = P^{-1}g'P$ . Then

$$\begin{aligned} P^{-1}(\mathfrak{g}_{(1,w_0)}) &= \{P^{-1}B(1, w_0) : B \in \mathfrak{g}\} \\ &= \{P^{-1}BPu_0 : B \in \mathfrak{g}\} \\ &= \{B'u_0 : B' \in \mathfrak{g}'\} \\ &= \mathfrak{g}'_{u_0} \end{aligned}$$

(ii) Let  $v = [x_1, \dots, x_{n+1}]^T \in \mathbb{C}^{n+1}$  such that  $\varphi(v) \in \Phi(GA(n, \mathbb{C}))$ , so by Proposition 3.2.(ii),  $\varphi(v)u_0 = v$ . It follows that  $x_1 = 1$  and so  $v \in \{1\} \times \mathbb{C}^n$ .

(iii) Since  $h(G') = G$  and  $AB = BA$  if and only if  $h(A)h(B) = h(B)h(A)$ , for every  $A, B \in M_{n+1}(\mathbb{C})$ , it follows that  $h(\mathcal{C}(G')) = \mathcal{C}(G)$ .

(iv) Let  $f' = (B, b) \in \mathfrak{g}$  and  $u \in \mathbb{C}^n$ , then

$$\Psi(f')(1, u) = \begin{bmatrix} 0 & 0 \\ b & B \end{bmatrix} \begin{bmatrix} 1 \\ u \end{bmatrix} = (0, f'(u)) = \tilde{\pi}(f'(u)),$$

so

$$\tilde{\pi}(\mathfrak{g}_u) = \Psi(\mathfrak{g})_{(1,u)} = \mathfrak{g}_{(1,u)} = \{0\} \times \mathfrak{g}_u \subset \{0\} \times \mathbb{C}^n.$$

□

**Proposition 3.9.** *Let  $\mathcal{G}$  be an abelian subgroup of  $GA(n, \mathbb{C})$ ,  $G = \Phi(\mathcal{G})$  and  $P \in \Phi(GA(n, \mathbb{C}))$  so that  $G' = P^{-1}GP$  is a subgroup of  $\mathcal{K}_{\eta,r}^*(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C}))$ . If  $\text{rank}(F_{G'_k}) = n_k - 1$  (resp.  $\text{rank}(F_{\mathfrak{g}'_k}) = n_k - 1$ ),  $k = 1, \dots, r - 1$ , then there exists two injective affine maps  $\gamma : \mathbb{C}^n \rightarrow GA(n, \mathbb{C})$  and  $\tilde{\gamma} : \mathbb{C}^n \rightarrow MA(n, \mathbb{C})$  (resp.  $\delta : \mathbb{C}^n \rightarrow GA(n, \mathbb{C})$  and  $\tilde{\delta} : \mathbb{C}^n \rightarrow MA(n, \mathbb{C})$ ) such that :*

(i)  $\mathcal{C}(\mathcal{G}) \subset \gamma(\mathbb{C}^n)$  and  $\mathcal{C}(\mathfrak{g}) \subset \tilde{\gamma}(\mathbb{C}^n)$  (resp.  $\mathcal{C}(\mathcal{G}) \subset \delta(\mathbb{C}^n)$  and  $\mathcal{C}(\mathfrak{g}) \subset \tilde{\delta}(\mathbb{C}^n)$ ).

(ii)  $\gamma(\mathcal{G}(w_0)) = \mathcal{G}$  and  $\tilde{\gamma}(\mathfrak{g}_{w_0}) = \mathfrak{g}$ . (resp.  $\delta(\mathcal{G}(w_0)) = \mathcal{G}$  and  $\tilde{\delta}(\mathfrak{g}_{w_0}) = \mathfrak{g}$ ).

*Proof.* Let's prove the Proposition for  $\text{rank}(F_{G_k}) = n_k - 1$ . The proof for  $\text{rank}(F_{\mathfrak{g}_k}) = n_k - 1$ ,  $k = 1, \dots, r$  is analogous.

(ii) As  $G(1, w_0) = \{1\} \times \mathcal{G}(w_0)$  we have  $\pi(\mathcal{G}(w_0)) = G(1, w_0)$ . Then

$$P^{-1} \circ \pi(\mathcal{G}(w_0)) = P^{-1}G(1, w_0) = P^{-1}GP(u_0) = G'(u_0) \quad (2)$$

By proposition 3.2, there exists a linear injective map  $\varphi : \mathbb{C}^{n+1} \rightarrow \mathcal{K}_{\eta,r}(\mathbb{C})$  such that  $\varphi(G'(u_0)) = G'$ . Then by (2) we have

$$h \circ \varphi(P^{-1} \circ \pi(\mathcal{G}(w_0))) = h \circ \varphi(G'(u_0)) = h(G') = G. \quad (3)$$

Since  $\Phi^{-1}(G) = \mathcal{G}$  then by (3), we have

$$\mathcal{G} = \Phi^{-1} \circ h \circ \varphi \circ P^{-1} \circ \pi(\mathcal{G}(w_0)).$$

It follows that  $\gamma = \Phi^{-1} \circ h \circ \varphi \circ P^{-1} \circ \pi$  is an injective affine map from  $\mathbb{C}^n$  to  $GA(n, \mathbb{C})$  and satisfying  $\gamma(\mathcal{G}(w_0)) = \mathcal{G}$ . (Because  $\pi$  and  $\Phi$  are invertible affine maps;  $h, P$  are isomorphisms and  $\varphi$  is an injective linear map)

Let  $\tilde{\gamma} = \Psi^{-1} \circ h \circ \varphi \circ P^{-1} \circ \tilde{\pi}$ , so  $\tilde{\gamma}$  is an injective affine map from  $\mathbb{C}^n$  to  $MA(n, \mathbb{C})$  and satisfying:

$$\begin{aligned}
\tilde{\gamma}(\mathfrak{g}_{w_0}) &= \Psi^{-1} \circ h \circ \varphi \circ P^{-1} \circ \tilde{\pi}(\mathfrak{g}_{w_0}) \\
&= \Psi^{-1} \circ h \circ \varphi \circ P^{-1}(\mathfrak{g}_{(1, w_0)}) && \text{(by Lemma 3.8.(iv))} \\
&= \Psi^{-1} \circ h \circ \varphi(\mathfrak{g}'_{u_0}) && \text{(by Lemma 3.8.(i))} \\
&= \Psi^{-1} \circ h(\mathfrak{g}') \\
&= \Psi^{-1}(\mathfrak{g}) && \text{(by Lemma 3.8.(i))} \\
&= \mathfrak{g}
\end{aligned}$$

By analogous proof, we replace  $\varphi$  by  $\psi$  in the construction of  $\gamma$  and  $\tilde{\gamma}$  and we find

$$\delta = \Phi^{-1} \circ h \circ \psi \circ P^{-1} \circ \pi \quad \text{and} \quad \tilde{\delta} = \Psi^{-1} \circ h \circ \psi \circ P^{-1} \circ \tilde{\pi} \quad (4)$$

(i) By Lemma 3.8.(iii),  $h(\mathcal{C}(G')) = \mathcal{C}(G)$  and by proposition 3.2,  $\mathcal{C}(G') \subset \varphi(\mathbb{C}^{n+1})$ . Then we have:

$$\begin{aligned}
\mathcal{C}(\mathcal{G}) &= \Phi^{-1}(\mathcal{C}(G) \cap \Phi(GA(n, \mathbb{C}))) \\
&= \Phi^{-1}(h(\mathcal{C}(G')) \cap \Phi(GA(n, \mathbb{C}))) \\
&\subset \Phi^{-1} \circ h(\varphi(\mathbb{C}^{n+1}) \cap h^{-1}(\Phi(GA(n, \mathbb{C})))) \\
&\subset \Phi^{-1} \circ h(\varphi(\mathbb{C}^{n+1}) \cap \Phi(GA(n, \mathbb{C}))) && \text{(by Lemma 2.1)} \\
&\subset \Phi^{-1} \circ h \circ \varphi(\{1\} \times \mathbb{C}^n) && \text{(by Lemma 3.8.(ii))} \quad (5)
\end{aligned}$$

On the other hand, we have  $P^{-1}(\{1\} \times \mathbb{C}^n) = \{1\} \times \mathbb{C}^n$ , because  $P \in \Phi(GA(n, \mathbb{C}))$ . As  $\{1\} \times \mathbb{C}^n = \pi(\mathbb{C}^n)$ , so  $\{1\} \times \mathbb{C}^n = P^{-1}(\{1\} \times \mathbb{C}^n) = P^{-1} \circ \pi(\mathbb{C}^n)$ . By (5), we have

$$\mathcal{C}(\mathcal{G}) \subset \Phi^{-1} \circ h \circ \varphi \circ P^{-1} \circ \pi(\mathbb{C}^n) = \gamma(\mathbb{C}^n).$$

By Lemma 3.1 and Proposition 3.2,  $\mathcal{C}(\mathfrak{g}') = \mathcal{C}(G') \subset \varphi(\mathbb{C}^{n+1})$ . By Lemma 3.8.(iii),  $h(\mathcal{C}(\mathfrak{g}')) = \mathcal{C}(\mathfrak{g})$ , then we have:

$$\begin{aligned}
\mathcal{C}(\mathfrak{g}) &= \Psi^{-1}(\mathcal{C}(\mathfrak{g}') \cap \mathcal{F}_{n+1}) \\
&= \Psi^{-1}(h(\mathcal{C}(\mathfrak{g}')) \cap h(\mathcal{F}_{n+1})) && \text{(by Lemma 2.1)} \\
&\subset \Psi^{-1} \circ h(\varphi(\mathbb{C}^{n+1}) \cap \mathcal{F}_{n+1}) \\
&\subset \Psi^{-1} \circ h \circ \varphi(\{0\} \times \mathbb{C}^n) && \text{(by Lemma 3.8.(i))} \quad (6)
\end{aligned}$$

On the other hand, we have  $P^{-1}(\{0\} \times \mathbb{C}^n) = \{0\} \times \mathbb{C}^n$ , because  $P \in \Phi(GA(n, \mathbb{C}))$ . As  $\{0\} \times \mathbb{C}^n = \tilde{\pi}(\mathbb{C}^n)$ , so  $\{0\} \times \mathbb{C}^n = P^{-1}(\{0\} \times \mathbb{C}^n) = P^{-1} \circ \tilde{\pi}(\mathbb{C}^n)$ . By (6), we have

$$\mathcal{C}(\mathfrak{g}) \subset \Psi^{-1} \circ h \circ \varphi \circ P^{-1} \circ \tilde{\pi}(\mathbb{C}^n) = \tilde{\gamma}(\mathbb{C}^n).$$

This completes the proof.  $\square$

**Corollary 3.10.** *If  $\overline{\mathcal{G}(w_0)} = \mathbb{C}^n$  (resp.  $\overline{\mathfrak{g}_{w_0}} = \mathbb{C}^n$ ) then  $\gamma : \mathbb{C}^n \rightarrow \mathcal{C}(\mathcal{G})$  and  $\tilde{\gamma} : \mathbb{C}^n \rightarrow \mathcal{C}(\mathfrak{g})$  (resp.  $\delta : \mathbb{C}^n \rightarrow \mathcal{C}(\mathcal{G})$  and  $\tilde{\delta} : \mathbb{C}^n \rightarrow \mathcal{C}(\mathfrak{g})$ ) are homeomorphisms.*

*Proof.* Suppose that  $\overline{\mathcal{G}(w_0)} = \mathbb{C}^n$ . Then by Proposition 3.9 there exists an injective affine map  $\gamma : \mathbb{C}^n \rightarrow GA(n, \mathbb{C})$  such that  $\mathcal{C}(\mathcal{G}) \subset \gamma(\mathbb{C}^n)$ . Let's prove that  $\gamma(\mathbb{C}^n) \subset \mathcal{C}(\mathcal{G})$ :

By proposition 3.9.(ii), we have  $\gamma(\mathcal{G}(w_0)) = \mathcal{G}$ . Then since  $\gamma$  is continuous, we have:

$$\gamma(\mathbb{C}^n) = \gamma(\overline{\mathcal{G}(w_0)}) \subset \overline{\gamma(\mathcal{G}(w_0))} = \overline{\mathcal{G}}.$$

By construction,  $\mathcal{C}(\mathcal{G})$  is an affine subspace of  $\mathbb{C}^n$  then it is closed and so  $\overline{\mathcal{G}} \subset \mathcal{C}(\mathcal{G})$ . Hence  $\gamma(\mathbb{C}^n) \subset \mathcal{C}(\mathcal{G})$ . We conclude that  $\gamma$  is an homeomorphism of  $\mathbb{C}^n$  on  $\mathcal{C}(\mathcal{G})$ .

The same proof is given for  $\tilde{\gamma}$ ,  $\delta$  and  $\tilde{\delta}$ . □

**Corollary 3.11.** *If  $\overline{\mathcal{G}(w_0)} = \mathbb{C}^n$  then the map  $\alpha : \mathbb{C}^n \rightarrow \mathbb{C}^n$  defined by  $\alpha = \gamma^{-1} \circ \Phi^{-1} \circ h \circ (\exp/\mathcal{K}_{\eta,r}(\mathbb{C}) \cap \mathcal{F}_{n+1}) \circ h^{-1} \circ \Psi \circ \tilde{\gamma}$  satisfies:*

- (i)  $\alpha$  is an open map
- (ii)  $\alpha^{-1}(\mathcal{G}(w_0)) = \mathfrak{g}_{w_0}$ .

*Proof.* If  $\overline{\mathcal{G}(w_0)} = \mathbb{C}^n$  then  $\varphi$  is invertible so are  $\gamma$  and  $\tilde{\gamma}$ . Hence  $\alpha$  is well defined.

- (i) Since  $\exp/\mathcal{K}_{\eta,r}(\mathbb{C})$  is a local diffeomorphism (Corollary 3.5) then  $\alpha$  is open.
- (ii) By Proposition 3.9.(ii), we have  $\tilde{\gamma}^{-1}(\mathfrak{g}) = \mathfrak{g}_{w_0}$  and  $\gamma(\mathcal{G}(w_0)) = \mathcal{G}$ . Then

$$\begin{aligned} \alpha^{-1}(\mathcal{G}(w_0)) &= \tilde{\gamma}^{-1} \circ \Psi^{-1} \circ h \circ (\exp/(\mathcal{K}_{\eta,r}(\mathbb{C}) \cap \mathcal{F}_{n+1}))^{-1} \circ h^{-1} \circ \Phi \circ \gamma(\mathcal{G}(w_0)) \\ &= \tilde{\gamma}^{-1} \circ \Psi^{-1} \circ h \circ (\exp/(\mathcal{K}_{\eta,r}(\mathbb{C}) \cap \mathcal{F}_{n+1}))^{-1} \circ h^{-1} \circ \Phi(\mathcal{G}) \\ &= \tilde{\gamma}^{-1} \circ \Psi^{-1} \circ h \circ (\exp/(\mathcal{K}_{\eta,r}(\mathbb{C}) \cap \mathcal{F}_{n+1}))^{-1} \circ h^{-1}(G) \\ &= \tilde{\gamma}^{-1} \circ \Psi^{-1} \circ h \circ (\exp/(\mathcal{K}_{\eta,r}(\mathbb{C}) \cap \mathcal{F}_{n+1}))^{-1}(G') \\ &= \tilde{\gamma}^{-1} \circ \Psi^{-1} \circ h(g') && \text{(by definition of } g') \\ &= \tilde{\gamma}^{-1} \circ \Psi^{-1}(g) && \text{(by Lemma 3.8.(i))} \\ &= \tilde{\gamma}^{-1}(\mathfrak{g}) \\ &= \mathfrak{g}_{w_0}. \end{aligned}$$

□

**Lemma 3.12.** *Let  $v \in \mathbb{C}^{n+1}$ . Then  $v \in U \cap (\{1\} \times \mathbb{C}^n)$  if and only if  $\psi(v) \in \mathcal{K}_{\eta,r}^*(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C}))$ . Moreover,  $\psi^{-1}(\mathcal{K}_{\eta,r}^*(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C}))) = U \cap (\{1\} \times \mathbb{C}^n)$ .*

*Proof.* Let  $x = [x_{1,1}, \dots, x_{1,n_1}; \dots; x_{r,1}, \dots, x_{r,n_r}]^T \in \mathbb{C}^{n+1}$ . By Proposition 3.2.(i),  $\psi(x) = \text{diag}(B_1, \dots, B_r) \in \mathcal{K}_{\eta,r}(\mathbb{C})$  with  $B_k \in \mathbb{T}_{n_k}(\mathbb{C})$ ,  $k = 1, \dots, r$ . By Proposition 3.2.(ii),  $x = \psi(x)u_0 = [B_1 e_{1,1}, \dots, B_r e_{r,1}]^T$ . Therefore  $B_1 e_{1,1} = [1, x_{1,2}, \dots, x_{1,n_1}]^T$  and  $B_k e_{k,1} = [x_{k,1}, \dots, x_{k,n_k}]^T$ ,  $k = 2, \dots, r$ . The proof of Lemma 3.12 follows, since  $B_k$  has  $x_{k,1}$  as only eigenvalue, for every  $k = 1, \dots, r$ . □

**Lemma 3.13.** ([1], Proposition 2.3) *If  $A, B \in \mathcal{K}_{\eta,r}(\mathbb{C})$  satisfy  $e^A e^B = e^B e^A$  then  $AB = BA$ .*

**Lemma 3.14.** *Under notation of Proposition 3.9. If  $\overline{\mathfrak{g}_{w_0}} = \mathbb{C}^n$  then:*

- (i)  $U \cap (\{1\} \times \mathbb{C}^n) = \psi^{-1}(\mathcal{C}(G') \cap \Phi(GA(n, \mathbb{C})) \cap \mathcal{K}_{\eta,r}^*(\mathbb{C}))$ .
- (ii)  $V = \delta^{-1}(\mathcal{C}(\mathcal{G}) \cap \Phi^{-1} \circ h(\mathcal{K}_{\eta,r}^*(\mathbb{C})))$ .
- (iii)  $\exp(\mathcal{C}(g') \cap \mathcal{F}_{n+1}) = \mathcal{C}(G') \cap \mathcal{K}_{\eta,r}^*(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C}))$ .

*Proof.* (i) By Lemma 3.12, we have

$$\psi^{-1}(\mathcal{C}(G') \cap \Phi(GA(n, \mathbb{C})) \cap \mathcal{K}_{\eta,r}^*(\mathbb{C})) \subset \psi^{-1}(\mathcal{K}_{\eta,r}^*(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C}))) = U \cap (\{1\} \times \mathbb{C}^n).$$

Conversely, let  $x \in U \cap (\{1\} \times \mathbb{C}^n)$ , so  $x_{k,1} \neq 0$ ,  $k = 2, \dots, r$ . Then  $\psi(x) \in \psi(\mathbb{C}^n) = \mathcal{C}(G')$  (Corollary 3.4). By Lemma 3.12,  $\psi(x) \in \mathcal{K}_{\eta,r}^*(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C}))$ . It follows that  $\psi(x) \in \mathcal{C}(G') \cap \Phi(GA(n, \mathbb{C})) \cap \mathcal{K}_{\eta,r}^*(\mathbb{C})$ .

(ii) Recall that  $\mathcal{C}(\mathcal{G}) = \Phi^{-1}(\mathcal{C}(G) \cap \Phi(GA(n, \mathbb{C})))$ , by Lemma 2.1,  $h(\Phi(GA(n, \mathbb{C}))) = \Phi(GA(n, \mathbb{C}))$  and by (4),  $\delta = \Phi^{-1} \circ h \circ \psi \circ P^{-1} \circ \pi$ . Then we have

$$\begin{aligned} V &= \pi^{-1} \circ P(U \cap (\{1\} \times \mathbb{C}^n)) \\ &= \pi^{-1} \circ P[\psi^{-1}(\mathcal{C}(G') \cap \Phi(GA(n, \mathbb{C})) \cap \mathcal{K}_{\eta,r}^*(\mathbb{C}))] && \text{(by (i))} \\ &= \pi^{-1} \circ P \circ \psi^{-1} \circ h^{-1}[\mathcal{C}(G) \cap \Phi(GA(n, \mathbb{C})) \cap h(\mathcal{K}_{\eta,r}^*(\mathbb{C}))] && \text{(by Lemma 3.8.(iii))} \\ &= (\pi^{-1} \circ P \circ \psi^{-1} \circ h^{-1} \circ \Phi) (\Phi^{-1}[\mathcal{C}(G) \cap \Phi(GA(n, \mathbb{C}))] \cap \Phi^{-1} \circ h(\mathcal{K}_{\eta,r}^*(\mathbb{C}))) \\ &= \delta^{-1}(\mathcal{C}(\mathcal{G}) \cap \Phi^{-1} \circ h(\mathcal{K}_{\eta,r}^*(\mathbb{C}))) \end{aligned}$$

(iii) By Lemma 3.3.(i),  $\exp(\mathcal{C}(g')) = \mathcal{C}(G') \cap \mathcal{K}_{\eta,r}^*(\mathbb{C})$  and by Lemma 2.7.(i),  $\exp(\mathcal{F}_{n+1}) = \Phi(GA(n, \mathbb{C}))$ . So  $\exp(\mathcal{C}(g') \cap \mathcal{F}_{n+1}) \subset \mathcal{C}(G') \cap \mathcal{K}_{\eta,r}^*(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C}))$ . Conversely, let  $M \in \mathcal{C}(G') \cap \mathcal{K}_{\eta,r}^*(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C}))$ . By Lemma 2.7.(iii), there exists  $N \in \mathcal{F}_{n+1} \cap \mathcal{K}_{\eta,r}(\mathbb{C})$  such that  $e^N = M$ . As  $M \in \mathcal{C}(G') = \mathcal{C}(g')$  (Lemma 3.1), so  $MA = AM$  for every  $A \in g'$  thus  $Me^A = e^A M$  then  $e^N e^A = e^A e^N$ . Since  $A, N \in \mathcal{K}_{\eta,r}(\mathbb{C})$ , then by Lemma 3.13,  $NA = AN$ , so  $N \in \mathcal{C}(g')$ . It follows that  $M = e^N \in \exp(\mathcal{C}(g') \cap \mathcal{F}_{n+1})$ .  $\square$

**Corollary 3.15.** *If  $\overline{\mathfrak{g}_{w_0}} = \mathbb{C}^n$  then the map  $\beta : \mathbb{C}^n \rightarrow \mathbb{C}^n$  defined by*

$$\beta = \delta^{-1} \circ \Phi^{-1} \circ h \circ (\exp|_{\mathcal{K}_{\eta,r}(\mathbb{C})}) \circ h^{-1} \circ \Psi \circ \tilde{\delta}$$

- (i)  $\beta(\mathfrak{g}_{w_0}) = \mathcal{G}(w_0)$ .
- (ii)  $\beta(\mathbb{C}^n) = V$ .

*Proof.* If  $\overline{\mathfrak{g}_{w_0}} = \mathbb{C}^n$  then by Corollary 3.10,  $\delta : \mathbb{C}^n \rightarrow \mathcal{C}(\mathcal{G})$  and  $\tilde{\delta} : \mathbb{C}^n \rightarrow \mathcal{C}(\mathfrak{g})$  are homeomorphisms, so  $\tilde{\delta}(\mathbb{C}^n) = \mathcal{C}(\mathfrak{g})$ .

(i) By proposition 2.4, we have  $\exp(\mathfrak{g}') = G'$ . By Proposition 3.9, we have  $\tilde{\delta}(\mathfrak{g}_{w_0}) = \mathfrak{g}$  and  $\delta(\mathcal{G}(w_0)) = \mathcal{G}$ . As  $\delta$  is invertible, then  $\delta^{-1}(\mathcal{G}) = \mathcal{G}(w_0)$ . Therefore :

$$\begin{aligned}
\beta(\mathfrak{g}_{w_0}) &= \delta^{-1} \circ \Phi^{-1} \circ h \circ (\exp/\mathcal{K}_{\eta,r}(\mathbb{C})) \circ h^{-1} \circ \Psi \circ \tilde{\delta}(\mathfrak{g}_{w_0}) \\
&= \delta^{-1} \circ \Phi^{-1} \circ h \circ (\exp/\mathcal{K}_{\eta,r}(\mathbb{C})) \circ h^{-1} \circ \Psi(\mathfrak{g}) \\
&= \delta^{-1} \circ \Phi^{-1} \circ h \circ (\exp/\mathcal{K}_{\eta,r}(\mathbb{C})) \circ h^{-1}(\mathfrak{g}) \\
&= \delta^{-1} \circ \Phi^{-1} \circ h \circ (\exp/\mathcal{K}_{\eta,r}(\mathbb{C})) (\mathfrak{g}') \\
&= \delta^{-1} \circ \Phi^{-1} \circ h(G') \\
&= \delta^{-1} \circ \Phi^{-1}(G) \\
&= \delta^{-1}(\mathcal{G}) \\
&= \mathcal{G}(w_0).
\end{aligned}$$

(ii) By Lemma 2.1,  $h(\mathcal{F}_{n+1}) = \mathcal{F}_{n+1}$  and  $h(\Phi(GA(n, \mathbb{C}))) = \Phi(GA(n, \mathbb{C}))$  and by Lemma 3.14, we have:

$$\begin{aligned}
\beta(\mathbb{C}^n) &= \delta^{-1} \circ \Phi^{-1} \circ h \circ (\exp/\mathcal{K}_{\eta,r}(\mathbb{C})) \circ h^{-1} \circ \Psi \circ \tilde{\delta}(\mathbb{C}^n) \\
&= \delta^{-1} \circ \Phi^{-1} \circ h \circ (\exp/\mathcal{K}_{\eta,r}(\mathbb{C})) \circ h^{-1} \circ \Psi(\mathcal{C}(\mathfrak{g})) \\
&= \delta^{-1} \circ \Phi^{-1} \circ h \circ (\exp/\mathcal{K}_{\eta,r}(\mathbb{C})) \circ h^{-1}(\mathcal{C}(\mathfrak{g}) \cap \mathcal{F}_{n+1}) \\
&= \delta^{-1} \circ \Phi^{-1} \circ h \circ (\exp/\mathcal{K}_{\eta,r}(\mathbb{C})) (\mathcal{C}(\mathfrak{g}') \cap h^{-1}(\mathcal{F}_{n+1})) \\
&= \delta^{-1} \circ \Phi^{-1} \circ h \circ (\exp/\mathcal{K}_{\eta,r}(\mathbb{C})) (\mathcal{C}(\mathfrak{g}') \cap \mathcal{F}_{n+1}) \\
&= \delta^{-1} \circ \Phi^{-1} \circ h(\mathcal{C}(G') \cap \mathcal{K}_{\eta,r}^*(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C}))) \\
&= \delta^{-1} \circ \Phi^{-1}(\mathcal{C}(G) \cap \Phi(GA(n, \mathbb{C})) \cap h(\mathcal{K}_{\eta,r}^*(\mathbb{C}))) \\
&= \delta^{-1}(\mathcal{C}(\mathcal{G}) \cap \Phi^{-1} \circ h(\mathcal{K}_{\eta,r}^*(\mathbb{C}))) \\
&= V.
\end{aligned}$$

□

**Proposition 3.16.**  *$V$  is  $\mathcal{G}$ -invariant dense open set of  $\mathbb{C}^n$  and every orbit of  $\mathcal{G}$ , contained in  $V$  is minimal in  $V$ .*

*Proof.* Let  $x \in V$  and  $f = (A, a) \in \mathcal{G}$ . Then  $\pi(x) = (1, x) \in P(U)$  and  $\Phi(f) \in G$ . By Proposition 3.7,  $U$  is  $G'$ -invariant so  $P(U)$  is  $G$ -invariant. Then  $\Phi(f)(1, x) = (1, f(x)) \in P(U)$ , so  $f(x) = \pi^{-1}(1, f(x)) \in \pi^{-1} \circ P(U) = V$ . It follows that  $V$  is  $\mathcal{G}$ -invariant.

By proposition 3.7, we have all orbits of  $G' = P^{-1}GP$  and contained in  $U$  are minimal in  $U$ . Let  $x \in V$  and  $y \in \overline{\mathcal{G}(x)} \cap V$ , so  $P^{-1} \circ \pi(x) = P^{-1}(1, x) \in U$  and also  $P^{-1}(1, y) \in U$ . We have

$$\{1\} \times \overline{\mathcal{G}(x)} = \overline{G(1, x)} = P \left( \overline{G'(P^{-1}(1, x))} \right). \quad (7)$$

As  $P^{-1}(1, x), P^{-1}(1, y) \in U$ , then

$$\overline{G'(P^{-1}(1, x))} \cap U = \overline{G'(P^{-1}(1, y))} \cap U.$$

Therefore

$$\begin{aligned} (\{1\} \times \overline{\mathcal{G}(x)}) \cap P(U) &= \overline{G(1, x)} \cap P(U) \\ &= P(\overline{G'(P^{-1}(1, x))} \cap U) \\ &= P(\overline{G'(P^{-1}(1, y))} \cap U) \\ &= \overline{G(1, y)} \cap P(U) \\ &= (\{1\} \times \overline{\mathcal{G}(y)}) \cap P(U) \quad (\text{by (7)}) \end{aligned}$$

Thus

$$(\{1\} \times \overline{\mathcal{G}(x)}) \cap P(U \cap (\{1\} \times \mathbb{C}^n)) = (\{1\} \times \overline{\mathcal{G}(y)}) \cap P(U \cap (\{1\} \times \mathbb{C}^n))$$

Then

$$\overline{\mathcal{G}(x)} \cap [\pi^{-1} \circ P(U \cap (\{1\} \times \mathbb{C}^n))] = \overline{\mathcal{G}(y)} \cap [\pi^{-1} \circ P(U \cap (\{1\} \times \mathbb{C}^n))].$$

So

$$\overline{\mathcal{G}(x)} \cap V = \overline{\mathcal{G}(y)} \cap V.$$

The proof is complete.  $\square$

**Lemma 3.17.**  $\mathbb{C}^n \setminus V$  is union of at most  $n$ ,  $\mathcal{G}$ -invariant affine spaces with dimension less to  $n - 1$ .

*Proof.* We have  $\mathbb{C}^{n+1} \setminus U = \bigcup_{k=1}^r H_k$ , where

$$H_k = \{u = (u_1, \dots, u_r) \in \mathbb{C}^{n+1} : u_k \in \{0\} \times \mathbb{C}^{n_k-1}, u_i \in \mathbb{C}^* \times \mathbb{C}^{n_i-1}, 1 \leq i \leq r, i \neq k\}$$

is a vector space  $G'$ -invariant with dimension  $n$ ,  $k = 1, \dots, r$ . Since  $r \leq n + 1$ ,  $\mathbb{C}^{n+1} \setminus U$  is a union of at most  $n + 1$ ,  $G'$ -invariant vector spaces with dimension  $n$ . Then  $\mathbb{C}^{n+1} \setminus P(U)$  is a union of at most  $n + 1$ ,  $G$ -invariant vector spaces with dimension  $n$ . As  $H_1 \cap (\{1\} \times \mathbb{C}^n) = \emptyset$ , so

$$\begin{aligned} (\{1\} \times \mathbb{C}^n) \setminus (U \cap (\{1\} \times \mathbb{C}^n)) &= (\mathbb{C}^{n+1} \setminus U) \cap (\{1\} \times \mathbb{C}^n) \\ &= \bigcup_{k=2}^r (H_k \cap (\{1\} \times \mathbb{C}^n)) \quad (8). \end{aligned}$$



Since  $P(\{1\} \times \mathbb{C}^n) = \{1\} \times \mathbb{C}^n$ , then by (8), we have

$$\begin{aligned} \mathbb{C}^n \setminus V &= \pi^{-1}(\{1\} \times \mathbb{C}^n) \setminus \pi^{-1} \circ P(U \cap (\{1\} \times \mathbb{C}^n)) \\ &= \pi^{-1} \circ P((\{1\} \times \mathbb{C}^n) \setminus (U \cap (\{1\} \times \mathbb{C}^n))) \\ &= \pi^{-1} \circ P\left(\bigcup_{k=2}^r (H_k \cap (\{1\} \times \mathbb{C}^n))\right) \\ &= \bigcup_{k=2}^r E_k, \end{aligned}$$

where  $E_k = \pi^{-1} \circ P(H_k \cap (\{1\} \times \mathbb{C}^n))$  is  $\mathcal{G}$ -invariant affine space for every  $2 \leq k \leq r$ . As  $H_k \neq \{1\} \times \mathbb{C}^n$ , then  $\dim(H_k \cap (\{1\} \times \mathbb{C}^n)) \leq n-1$ , so  $\dim(E_k) \leq n-1$ . It follows that  $\mathbb{C}^n \setminus V$  is union of at most  $n$  spaces  $E_k$ , since  $r \leq n+1$ .  $\square$

**Corollary 3.18.** *If  $\mathcal{G}$  has a dense orbit in  $\mathbb{C}^n$  then all orbits of  $V$  are dense in  $\mathbb{C}^n$ .*

*Proof.* Let  $u \in \mathbb{C}^n$  such that  $\overline{\mathcal{G}(u)} = \mathbb{C}^n$ . By Lemma 3.17,  $\mathbb{C}^n \setminus V$  is union of at most  $n$ ,  $\mathcal{G}$ -invariant affine spaces with dimension less to  $n-1$ , so  $u \in V$ . Let  $v \in V$ . Since  $V = \overline{\mathcal{G}(u)} \cap V$  then by Proposition 3.16,  $\overline{\mathcal{G}(v)} \cap V = \overline{\mathcal{G}(u)} \cap V = V$ . As  $\overline{V} = \mathbb{C}^n$ , one has  $\overline{\mathcal{G}(v)} = \mathbb{C}^n$ .  $\square$

#### 4. Proof of Theorem 1.1 and corollary 1.2

**4.1. Proof of Theorem 1.1.** Let  $\mathcal{G}$  be an abelian affine subgroup of  $GA(n, \mathbb{C})$  and let  $P \in \Phi(GA(n, \mathbb{C}))$  defined in Proposition 2.3. Recall that  $G = \Phi(\mathcal{G})$  and  $\mathfrak{g} = \exp^{-1}(G) \cap (PK_{\eta,r}(\mathbb{C})P^{-1}) \cap \mathcal{F}_{n+1}$ .

**ii)  $\implies$  i)** is clear.

**i)  $\implies$  ii)** This follows from Corollary 3.18, since  $u_0 \in U$  and so  $w_0 = \pi^{-1} \circ P(u_0) \in V$ .

**iii)  $\implies$  ii)** : Suppose that  $\overline{\mathfrak{g}_{w_0}} = \mathbb{C}^n$ . Then by Corollary 3.15 there exists a continuous map  $\beta : \mathbb{C}^n \rightarrow \mathbb{C}^n$  such that  $\beta(\mathfrak{g}_{w_0}) = \mathcal{G}(w_0)$  and  $\beta(\mathbb{C}^n) = V$ . By continuity, one has:

$$V = \beta(\mathbb{C}^n) = \beta(\overline{\mathfrak{g}_{w_0}}) \subset \overline{\beta(\mathfrak{g}_{w_0})} = \overline{\mathcal{G}(w_0)}$$

Since  $V$  is a dense open set in  $\mathbb{C}^n$  then  $\overline{\mathcal{G}(w_0)} = \mathbb{C}^n$ .

**ii)  $\implies$  iii)** : Suppose that  $\overline{\mathcal{G}(w_0)} = \mathbb{C}^n$ .

By Corollary 3.11, there exists an open map  $\alpha : \mathbb{C}^n \rightarrow \mathbb{C}^n$  such that  $\mathfrak{g}_{w_0} = \alpha^{-1}(\mathcal{G}(w_0))$ . Then:

$$\mathbb{C}^n = \alpha^{-1}(\mathbb{C}^n) = \alpha^{-1}(\overline{\mathcal{G}(w_0)}) \subset \overline{\alpha^{-1}(\mathcal{G}(w_0))} = \overline{\mathfrak{g}_{w_0}}$$

$\square$

**4.2. Proof of Corollary 1.2.** Let  $\mathcal{G}$  be an abelian subgroup of  $GA(n, \mathbb{C})$  having a dense orbit. By Theorem 1.1,  $\overline{\mathcal{G}(w_0)} = \mathbb{C}^n$ . Then by Proposition 3.2 and corollary 3.10, there exists an invertible affine map  $\gamma : \mathbb{C}^n \rightarrow \mathcal{C}(\mathcal{G})$  such that  $\gamma(\mathcal{G}(w_0)) = \mathcal{G}$ . Therefore:

$$\gamma(\mathbb{C}^n) = \gamma(\overline{\mathcal{G}(w_0)}) = \overline{\gamma(\mathcal{G}(w_0))} = \overline{\mathcal{G}}.$$

It follows that  $\overline{\mathcal{G}}$  is an affine space of dimension  $n$ .

## 5. Finitely generated subgroups

### 5.1. Proof of Theorem 1.3.

Denote by  $v_0 = Pu_0 = (1, w_0)$ . So  $w_0 = p_2(v_0)$ .

**Lemma 5.1.** *We have  $p_2(\Psi(f')v_0) = f'(w_0)$ , for every  $f' \in \mathfrak{g}$ . In particular,  $\mathfrak{g}_{w_0} = p_2(\mathfrak{g}_{v_0})$ .*

*Proof.* Let  $f' = (B, b) \in \mathfrak{g}$ , then

$$\begin{aligned} \Psi(f')v_0 &= \begin{bmatrix} 0 & 0 \\ b & B \end{bmatrix} \begin{bmatrix} 1 \\ w_0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ b + Bv_0 \end{bmatrix}. \end{aligned}$$

So  $p_2(\Psi(f')v_0) = b + Bv_0 = f'(w_0)$ .  $\square$

**Proposition 5.2.** ([1], Proposition 8.1) *Let  $G$  be a subgroup of  $GL(n+1, \mathbb{C})$  generated by  $A_1, \dots, A_p$ . Let  $B_1, \dots, B_p \in \mathfrak{g}$  such that  $A_k = e^{B_k}$ ,  $k = 1, \dots, p$ . Then:*

$$\mathfrak{g}_{v_0} = \sum_{k=1}^p \mathbb{Z}B_k v_0 + \sum_{k=1}^r 2i\pi \mathbb{Z}P e^{(k)}.$$

**Proposition 5.3.** (Under notations of Proposition 2.3) *Let  $\mathcal{G}$  be an abelian subgroup of  $GA(n, \mathbb{C})$  generated by  $f_1, \dots, f_p$  and let  $f'_1, \dots, f'_p \in \mathfrak{g}$  such that  $\Phi(f_k) = e^{\Psi(f'_k)}$ ,  $k = 1, \dots, p$ . Then:*

$$\mathfrak{g}_{w_0} = \sum_{k=1}^p \mathbb{Z}f'_k(w_0) + \sum_{k=1}^r 2i\pi \mathbb{Z}(p_2 \circ P e^{(k)}).$$

*Proof.* Let  $G = \Phi(\mathcal{G})$ . Then  $G$  is generated by  $\Phi(f_1), \dots, \Phi(f_p)$  and  $\mathfrak{g}$  is the additive group generated by  $\Psi(f'_1), \dots, \Psi(f'_p)$ . By proposition 5.2 we have

$$\mathfrak{g}_{v_0} = \sum_{k=1}^p \mathbb{Z}\Psi(f'_k)v_0 + \sum_{k=1}^r 2i\pi \mathbb{Z}P e^{(k)}.$$

By Lemma 5.1, we have  $\mathfrak{g}_{w_0} = p_2(\mathfrak{g}_{v_0})$  and  $p_2(\Psi(f'_k)v_0) = f'_k(w_0)$ . then

$$\begin{aligned} \mathfrak{g}_{w_0} &= p_2 \left( \sum_{k=1}^p \mathbb{Z}\Psi(f'_k)v_0 + \sum_{k=1}^r 2i\pi\mathbb{Z}Pe^{(k)} \right). \\ &= \sum_{k=1}^p \mathbb{Z}f'_k(w_0) + \sum_{k=1}^r 2i\pi\mathbb{Z}(p_2 \circ Pe^{(k)}). \end{aligned}$$

□

Recall the following Proposition which was proven in [2]:

**Proposition 5.4.** (cf. [2], page 35). *Let  $H = \mathbb{Z}u_1 + \dots + \mathbb{Z}u_p$  with  $u_k = (u_{k,1}, \dots, u_{k,n}) \in \mathbb{C}^n$  and  $u_{k,i} = \operatorname{Re}(u_{k,i}) + i\operatorname{Im}(u_{k,i})$ ,  $k = 1, \dots, p$ ,  $i = 1, \dots, n$ . Then  $H$  is dense in  $\mathbb{C}^n$  if and only if for every  $(s_1, \dots, s_p) \in \mathbb{Z}^p \setminus \{0\}$  :*

$$\operatorname{rank} \begin{bmatrix} \operatorname{Re}(u_{1,1}) & \dots & \dots & \operatorname{Re}(u_{p,1}) \\ \vdots & \vdots & \vdots & \vdots \\ \operatorname{Re}(u_{1,n}) & \dots & \dots & \operatorname{Re}(u_{p,n}) \\ \operatorname{Im}(u_{1,1}) & \dots & \dots & \operatorname{Im}(u_{p,1}) \\ \vdots & \vdots & \vdots & \vdots \\ \operatorname{Im}(u_{1,n}) & \dots & \dots & \operatorname{Im}(u_{p,n}) \\ s_1 & \dots & \dots & s_p \end{bmatrix} = 2n + 1.$$

*Proof of Theorem 1.3:* This follows directly from Theorem 1.1, Propositions 5.3 and 5.4.

## 5.2. Proof of Corollaries 1.4 and 1.5.

*Proof of Corollary 1.4:* We show first that if  $H = \mathbb{Z}u_1 + \dots + \mathbb{Z}u_m$ ,  $u_k \in \mathbb{C}^n$  with  $m \leq 2n$ , then  $H$  can not be dense: Write  $u_k \in \mathbb{C}^n$ ,  $u_k = \operatorname{Re}(u_k) + i\operatorname{Im}(u_k)$  and  $v_k = [\operatorname{Re}(u_k); \operatorname{Im}(u_k); s_k]^T \in \mathbb{R}^{2n+1}$ ,  $1 \leq k \leq m$ . Since  $m \leq 2n$ , it follows that  $\operatorname{rank}(v_1, \dots, v_m) \leq 2n$ , and so  $H$  is not dense in  $\mathbb{C}^n$  by Proposition 5.4.

Now, by applying Theorem 1.3 and the fact that  $m = p + r \leq 2n$ , the Corollary 1.4 follows. □

*Proof of Corollary 1.5:* Since  $p \leq n$  and  $r \leq n$  then  $p + r \leq 2n$ . Corollary 1.5 follows from Corollary 1.4. □

## 5.3. The case: $n = 1$ .

Let  $\mathcal{G}$  be a subgroup of  $GA(1, \mathbb{C})$ , generated by  $f_1 = (a_1, \lambda_1), \dots, f_p = (a_p, \lambda_p)$ , where  $a_k = \rho_k e^{i\theta_k}$ ,  $\lambda_k = r_k e^{i\alpha_k}$ ,  $\rho_k, r_k > 0$ ,  $\theta_k, \alpha_k \in \mathbb{R}$ ,  $1 \leq k \leq p$ . Therefore  $G = \Phi(\mathcal{G})$  is an abelian subgroup of  $GL(2, \mathbb{C})$ . Let  $f'_1, \dots, f'_p \in \mathfrak{g}$  such that  $e^{\Psi(f'_k)} = \Phi(f_k)$ ,  $1 \leq k \leq p$ . Since  $\mathcal{G}$  is abelian then there are two cases:

◇ *Case 1:  $\mathcal{G}$  is a group of translations.*

In this case  $\mathcal{G}(x) = T_x \left( \sum_{k=1}^p \lambda_k \mathbb{Z} \right)$ . Then  $\mathcal{G}$  has a dense orbit if and only if  $\mathcal{G}(0) =$

$\sum_{k=1}^p \lambda_k \mathbb{Z}$  has a dense orbit, which is equivalent, by Proposition 5.4, to:

For every  $(s_1, \dots, s_p, t) \in \mathbb{Z}^{p+1} \setminus \{0\}$ :

$$\text{rank} \begin{bmatrix} \log(r_1) & \dots & \log(r_p) & 0 \\ \alpha_1 & \dots & \alpha_p & 2\pi \\ s_1 & \dots & s_p & t \end{bmatrix} = 3.$$

◇ *Case 2 :  $\mathcal{G}$  contains no translation:*

In this case  $w_0 = (1, 1)$ ,  $a_k \neq 1$  for every  $k = 1, \dots, p$  and  $f'_k = (B_k, b_k)$  is defined by:

$$B_k = \begin{bmatrix} 0 & 0 \\ b_k & \mu_k \end{bmatrix}, \quad \text{where } b_k = \log(r_k) + i\theta_k \quad \text{and} \quad \mu_k = \frac{\lambda_k(\log(r_k) + i\theta_k)}{(a_k - 1)}.$$

Since  $w_0 = 1$  then

$$\begin{aligned} \mathfrak{g}_{w_0} &= f'_1(w_0)\mathbb{Z} + \dots + f'_p(w_0)\mathbb{Z} + 2i\pi\mathbb{Z} \\ &= (b_1 + \mu_1)\mathbb{Z} + \dots + (b_p + \mu_p)\mathbb{Z} + 2i\pi\mathbb{Z} \end{aligned}$$

Moreover, the property  $\mathcal{D}$  in this cases is as follow:

$f_1, \dots, f_p$  satisfy property  $\mathcal{D}$  if and only if for every  $(s_1, \dots, s_p, t) \in \mathbb{Z}^{p+1} - \{0\}$ :

$$\text{rank} \begin{bmatrix} \log|b_1 + \mu_1| & \dots & \log|b_p + \mu_p| & 0 \\ \arg(b_1 + \mu_1) & \dots & \arg(b_p + \mu_p) & 2\pi \\ s_1 & \dots & s_p & t \end{bmatrix} = 3.$$

where  $\arg(z) = \theta \in [0, 2\pi[$  for every  $z = re^{i\theta} \in \mathbb{C}$ ,  $r > 0$ . Therefore, from Theorem 1.3, we obtain:

**Proposition 5.5.** *If  $\mathcal{G}$  is a subgroup of  $GA(1, \mathbb{C})$  generated by  $f_1 = (a_1, \lambda_1), \dots, f_p = (a_p, \lambda_p)$ , where  $a_k = \rho_k e^{i\theta_k} \neq 1$ ,  $\lambda_k = r_k e^{i\alpha_k}$ ,  $\rho_k, r_k > 0$ ,  $\theta_k, \alpha_k \in \mathbb{R}$ ,  $1 \leq k \leq p$ . Then the following are equivalent:*

(i)  $G$  is minimal (i.e. every orbit of  $\mathcal{G}$  is dense in  $\mathbb{C}$ ).

(ii)  $G$  has a dense orbit

(iii)  $\sum_{k=1}^p (b_k + \mu_k)\mathbb{Z} + 2i\pi\mathbb{Z}$  is dense in  $\mathbb{C}$ , where  $b_k = \log(r_k) + i\theta_k$  and  $\mu_k = (\log \rho_k + i\theta_k) \left( 1 + \frac{\lambda_k}{a_k - 1} \right)$ .

(iv) for every  $(s_1, \dots, s_p, t) \in \mathbb{Z}^{p+1} \setminus \{0\}$ :

$$\text{rank} \begin{bmatrix} \log|b_1 + \mu_1| & \dots & \log|b_p + \mu_p| & 0 \\ \arg(b_1 + \mu_1) & \dots & \arg(b_p + \mu_p) & 2\pi \\ s_1 & \dots & s_p & t \end{bmatrix} = 3.$$

## 6. Examples

**Example 6.1.** Let  $a_k = e^{\pi(s_k + it_k)}$ ,  $s_k, t_k \in \mathbb{Q}$ ,  $1 \leq k \leq p$  and let  $\mathcal{G}$  be the subgroup of translations of  $\mathbb{C}$  generated by  $T_{a_1}, \dots, T_{a_p}$ . Then  $\mathcal{G}$  has no dense orbit.

*Proof.* We can write  $s_k = \frac{m_k}{q}$  and  $t_k = \frac{p_k}{q}$ ,  $q \in \mathbb{N}^*$  and  $m_k, p_k \in \mathbb{Z}$  for every  $1 \leq k \leq p$ . One has:  $\mathfrak{g}_x = T_x(\mathfrak{g}_0)$ , where

$$\mathfrak{g}_0 = \frac{\pi}{q} \left( \sum_{k=1}^p (m_k + ip_k)\mathbb{Z} + 2iq\mathbb{Z} \right) \subset \frac{\pi}{q}(\mathbb{Z} + i\mathbb{Z}).$$

So  $\overline{\mathfrak{g}_0} \neq \mathbb{C}$ . By Theorem 1.1,  $\overline{\mathcal{G}(0)} \neq \mathbb{C}$ . It follows by Proposition 5.5, that  $\overline{\mathcal{G}(x)} \neq \mathbb{C}$ .  $\square$

**Example 6.2.** Let  $\mathcal{G}$  the group generated by  $f_1 = (A_1, a_1)$ ,  $f_2 = (A_2, a_2)$  and  $f_3 = (A_3, a_3)$ , where:

$$A_1 = I_2, \quad a_1 = (1 + i, 0), \quad A_2 = \text{diag}(1, e^{-2+i}), \quad a_2 = (0, 0).$$

$$A_3 = \text{diag} \left( 1, e^{\frac{-\sqrt{2}}{\pi} + i \left( \frac{\sqrt{2}}{2\pi} - \frac{\sqrt{7}}{2} \right)} \right) \quad \text{and} \quad a_3 = \left( \frac{-\sqrt{3}}{2\pi} + i \left( \frac{\sqrt{5}}{2} - \frac{\sqrt{3}}{2\pi} \right), 0 \right).$$

Then every orbit in  $V = \mathbb{C} \times \mathbb{C}^*$  is dense in  $\mathbb{C}^2$ .

*Proof.* Denote by  $G = \Phi(\mathcal{G})$ . Then  $G$  generated by

$$A'_1 = \Phi(f_1) = \begin{bmatrix} 1 & 0 & 0 \\ 1+i & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A'_2 = \Phi(f_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-2+i} \end{bmatrix},$$

and

$$A'_3 = \Phi(f_3) = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-\sqrt{3}}{2\pi} + i \left( \frac{\sqrt{5}}{2} - \frac{\sqrt{3}}{2\pi} \right) & 1 & 0 \\ 0 & 0 & e^{\frac{-\sqrt{2}}{\pi} + i \left( \frac{\sqrt{2}}{2\pi} - \frac{\sqrt{7}}{2} \right)} \end{bmatrix}.$$

Let  $f'_1 = (B_1, b_1)$ ,  $f'_2 = (B_2, b_2)$  and  $f'_3 = (B_3, b_3)$  such that  $e^{\Psi(f'_k)} = A'_k$ ,  $k = 1, 2, 3$ . We have

$$B_1 = \text{diag}(0, 0), \quad B_2 = \text{diag}(0, -2 + i),$$

$$B_3 = \text{diag} \left( 0, \frac{-\sqrt{2}}{\pi} + i \left( \frac{\sqrt{2}}{2\pi} - \frac{\sqrt{7}}{2} \right) \right),$$

$$b_1 = (1 + i, 0), \quad b_2 = (0, 0), \quad \text{and} \quad b_3 = \left( \frac{-\sqrt{3}}{2\pi} + i \left( \frac{\sqrt{5}}{2} - \frac{\sqrt{3}}{2\pi} \right), 0 \right).$$

Here, we have:

-  $P = I_3$ ,  $U = \mathbb{C}^* \times \mathbb{C} \times \mathbb{C}^*$ ,  $u_0 = (1, 0, 1)$ .

-  $V = \mathbb{C} \times \mathbb{C}^*$ ,  $w_0 = (0, 1)$ .

The additive group  $g = \exp^{-1}(G) \cap (\mathcal{K}_{(2,1),2}(\mathbb{C})) \cap \mathcal{F}_3$  is generated by  $N_1, N_2$  and  $N_3$ , where

$$N_k = \Psi(f'_k) = \begin{bmatrix} 0 & 0 \\ b_k & B_k \end{bmatrix}, \quad 1 \leq k \leq 3$$

In the other hand, for every  $(s_1, s_2, s_3, t_1, t_2) \in \mathbb{Z}^5 \setminus \{0\}$ , one has the determinant:

$$\begin{aligned} \Delta &= \begin{vmatrix} \operatorname{Re}(B_1 w_0 + b_1) & \operatorname{Re}(B_2 w_0 + b_2) & \operatorname{Re}(B_3 w_0 + b_3) & 0 & 0 \\ \operatorname{Im}(B_1 w_0 + b_1) & \operatorname{Im}(B_2 w_0 + b_2) & \operatorname{Im}(B_3 w_0 + b_3) & 2\pi e_1 & 2\pi e_2 \\ & s_1 & s_2 & s_3 & t_1 & t_2 \\ & -\frac{\sqrt{3}}{2\pi} & 1 & 0 & 0 & 0 \\ & -\frac{\sqrt{2}}{\pi} & 0 & -2 & 0 & 0 \\ & \frac{\sqrt{5}}{2} - \frac{\sqrt{3}}{2\pi} & 1 & 0 & 2\pi & 0 \\ & \frac{\sqrt{2}}{2\pi} - \frac{\sqrt{7}}{2} & 0 & 1 & 0 & 2\pi \\ & s_1 & s_2 & s_3 & t_1 & t_2 \end{vmatrix} \\ &= -2\pi \left( (4s_1)\pi - (2s_3)\sqrt{2} + (2s_2)\sqrt{3} + t_1\sqrt{5} + t_2\sqrt{7} \right). \end{aligned}$$

Since  $\pi, \sqrt{2}, \sqrt{3}, \sqrt{5}$  and  $\sqrt{7}$  are rationally independent then  $\Delta \neq 0$  for every  $(s_1, s_2, s_3, t_1, t_2) \in \mathbb{Z}^5 \setminus \{0\}$ . It follows that:

$$\operatorname{rank} \begin{bmatrix} -\frac{\sqrt{3}}{2\pi} & 1 & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{\pi} & 0 & -2 & 0 & 0 \\ \frac{\sqrt{5}}{2} - \frac{\sqrt{3}}{2\pi} & 1 & 0 & 2\pi & 0 \\ \frac{\sqrt{2}}{2\pi} - \frac{\sqrt{7}}{2} & 0 & 1 & 0 & 2\pi \\ s_1 & s_2 & s_3 & t_1 & t_2 \end{bmatrix} = 5$$

and by Theorem 1.3,  $\mathcal{G}$  has a dense orbit and every orbit of  $V$  is dense in  $\mathbb{C}^2$ .  $\square$

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ADLENE AYADI, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES OF GAFSA, UNIVERSITY OF GAFSA, GAFSA, TUNISIA

*E-mail address:* adleneso@yahoo.fr