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DENSE ORBITS FOR ABELIAN AFFINE GROUPS

ADLENE AYADI

Abstract. In this paper, we give a characterization of existence of a dense orbit for the action on $\mathbb{C}^n$ of an abelian affine subgroup $\mathcal{G}$ of $GA(n,\mathbb{C})$. We prove in particular that if $\mathcal{G}$ has a dense orbit then the closure $\overline{\mathcal{G}}$ is an affine space of dimension n. If $\mathcal{G}$ is finitely generated, this characterization is explicit.

1. Introduction

Let $M_n(\mathbb{C})$ be the set of all square matrices of order $n \geq 1$ with entries in $\mathbb{C}$ and $GL(n,\ \mathbb{C})$ be the group of all invertible matrices of $M_n(\mathbb{C})$. A map $f:\ \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is called an affine map if there exist $A \in M_n(\mathbb{C})$ and $a \in \mathbb{C}^n$ such that $f(x) = Ax + a$, $x \in \mathbb{C}^n$. We denote $f = (A, a)$, we call A the *linear part* of f. Denote by $MA(n,\ \mathbb{C})$ the set of all affine maps and $GA(n,\ \mathbb{C})$ the set of all invertible affine maps of $MA(n,\mathbb{C})$. $MA(n,\mathbb{C})$ is a vector space and for composition of maps, $GA(n,\ \mathbb{C})$ is a group.

Let $\mathcal{G}$ be an abelian affine subgroup of $GA(n,\ \mathbb{C})$. For a vector $v \in \mathbb{C}^n$, we consider the orbit of $\mathcal{G}$ through v: $\mathcal{G}(v) = \{f(v):\ f \in \mathcal{G}\} \subset \mathbb{C}^n$. A subset $E \subset \mathbb{C}^n$ is called *$\mathcal{G}$-invariant* if $f(E) \subset E$ for any $f \in \mathcal{G}$; that is E is a union of orbits. Before stating our main results, we introduce the following notions:

A subset $\mathcal{H}$ of $\mathbb{C}^n$ is called *an affine subspace* of $\mathbb{C}^n$ if there exist a vector subspace H of $\mathbb{C}^n$ and $a \in \mathbb{C}^n$ such that $\mathcal{H} = H + a$. For $a \in \mathbb{C}^n$, denote by $T_a:\ \mathbb{C}^n \longrightarrow \mathbb{C}^n$; $x \longmapsto x + a$ the translation map by vector a, so $\mathcal{H} = T_a(H)$. We say that $\mathcal{H}$ has dimension p $(0 \leq p \leq n)$, denoted $\dim(\mathcal{H}) = p$, if H has dimension p.

Denote by $\overline{A}$ the closure of a subset $A \subset \mathbb{C}^n$. A subset E of $\mathbb{C}^n$ is called *a minimal set* of $\mathcal{G}$ if E is closed in $\mathbb{C}^n$, non empty, $\mathcal{G}$-invariant and has no proper subset with these properties. It is equivalent to say that E is a $\mathcal{G}$-invariant set such that every orbit contained in E is dense in it.

If V is a $\mathcal{G}$-invariant open set in $\mathbb{C}^n$, we say that E is a *minimal set in V* if it is a minimal set of the restriction $\mathcal{G}_{/V}$ of $\mathcal{G}$ to V.

Define the map

$$\Phi\ :\ GA(n,\ \mathbb{C})\ \longrightarrow\ \Phi(GA(n,\mathbb{C})) \subset GL(n+1,\ \mathbb{C})$$

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$$f=(A,a)\ \longmapsto\ \begin{bmatrix}1 & 0\\ a & A\end{bmatrix}$$

We have the following composition formula

$$\begin{bmatrix}1 & 0\\ a & A\end{bmatrix}\begin{bmatrix}1 & 0\\ b & B\end{bmatrix}=\begin{bmatrix}1 & 0\\ Ab+a & AB\end{bmatrix}.$$

Then Φ is a homomorphism of groups.

Let $\mathcal{G}$ be an abelian affine subgroup of $GA(n,\ \mathbb{C})$. Then $\Phi(\mathcal{G})$ is an abelian subgroup of $GL(n+1,\mathbb{C})$.

Let $\pi\ :\ \mathbb{C}^n\ \longrightarrow\ \{1\}\times\mathbb{C}^n$ be the map defined by $\pi(y)=(1,y)$. Clearly π is an invertible affine maps and we have for every $x\in\mathbb{C}^n$,

$$\pi(\mathcal{G}(x))=\Phi(\mathcal{G})(1,x)=\{1\}\times\mathcal{G}(x)\subset\{1\}\times\mathbb{C}^n.$$

Let $n\in\mathbb{N}_0$ be fixed. For each $m=1,2,\ldots,n+1$, denote by:
• $\mathbb{C}^*=\mathbb{C}\backslash\{0\}$ and $\mathbb{N}_0=\mathbb{N}\backslash\{0\}$.
• $\mathcal{B}_0=(e_1,\ldots,e_{n+1})$ the canonical basis of $\mathbb{C}^{n+1}$ and I_{n+1} the identity matrix of $GL(n+1,\mathbb{C})$.

• $\mathbb{T}_m(\mathbb{C})$ the set of matrices over $\mathbb{C}$ of the form

$$\begin{bmatrix}\mu & & & 0\\ a_{2,1} & \mu & & \\ \vdots & \ddots & \ddots & \\ a_{m,1} & \ldots & a_{m,m-1} & \mu\end{bmatrix}\qquad(1)$$

• $\mathbb{T}_m^*(\mathbb{C})$ the group of matrices of the form (1) with $\mu\neq 0$.

Let $r\in\mathbb{N}$ and $\eta=(n_1,\ldots,n_r)$ be a sequence of positive integers such that $n_1+\cdots+n_r=n+1$. In particular, $r\leq n+1$.
Write
• $\mathcal{K}_{\eta,r}(\mathbb{C}):=\mathbb{T}_{n_1}(\mathbb{C})\oplus\cdots\oplus\mathbb{T}_{n_r}(\mathbb{C})$. In particular if $r=1$, then $\mathcal{K}_{\eta,1}(\mathbb{C})=\mathbb{T}_{n+1}(\mathbb{C})$ and $\eta=(n+1)$.
• $\mathcal{K}^*_{\eta,r}(\mathbb{C}):=\mathcal{K}_{\eta,r}(\mathbb{C})\cap\mathrm{GL}(n+1,\ \mathbb{C})$.

Define the map

$$\Psi\ :\ MA(n,\ \mathbb{C})\ \longrightarrow\ \Psi(GA(n,\mathbb{C}))\subset M_{n+1}(\mathbb{C})$$

$$f=(A,a)\ \longmapsto\ \begin{bmatrix}0 & 0\\ a & A\end{bmatrix}$$

We have Ψ is an isomorphism. Remark that $\Phi(f)=\Psi(f)+D_1$, where $D_1=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\in M_{n+1}(\mathbb{C})$, $f\in GA(n,\mathbb{C})$, so Φ is an affine map.

Denote by:
• $\mathcal{F}_{n+1}=\left\{\begin{bmatrix}0 & 0\\ b & B\end{bmatrix}\in M_{n+1}(\mathbb{C}),\quad B\in M_n(\mathbb{C}),b\in\mathbb{C}^n\right\}=\Psi(MA(n,\mathbb{C}))$.

• $\exp : \mathbb{M}_{n+1}(\mathbb{C}) \longrightarrow \mathrm{GL}(n+1,\mathbb{C})$ is the matrix exponential map; set $\exp(M) = e^M$.

There always exists a $P \in \Phi(\mathrm{GA}(n,\mathbb{C}))$ and a partition η of $n+1$ such that $G' = P^{-1}GP \subset \mathcal{K}^*_{\eta,r}(\mathbb{C}) \cap \Phi(GA(n,\mathbb{C}))$ (see Proposition 2.3). For such a choice of matrix P, we let

• $\mathrm{g} = \exp^{-1}(G) \cap \left(P(\mathcal{K}_{\eta,r}(\mathbb{C}))P^{-1}\right) \cap \mathcal{F}_{n+1}$. If $G \subset \mathcal{K}^*_{\eta,r}(\mathbb{C})$, we have $\mathrm{g} = \exp^{-1}(G) \cap \mathcal{K}_{\eta,r}(\mathbb{C}) \cap \mathcal{F}_{n+1}$.
• $\mathrm{g}_u = \{Bu : \ B \in \mathrm{g}\}, \ u \in \mathbb{C}^{n+1}$.
• $\mathfrak{g} = \Psi^{-1}(\mathrm{g})$.
• $\mathfrak{g}_v = \{f(v), \quad f \in \mathfrak{g}\}, \ v \in \mathbb{C}^n$.

Let $\widetilde{\pi} : \ \mathbb{C}^n \longrightarrow \{0\} \times \mathbb{C}^n$ be the map defined by $\widetilde{\pi}(y) = (0,y)$. Clearly $\widetilde{\pi}$ is an isomorphism.

Denote by:
• $u_0 = [e_{1,1}, \ldots, e_{r,1}]^T \in \mathbb{C}^{n+1}$ where $e_{k,1} = [1,0,\ldots,0]^T \in \mathbb{C}^{n_k}$, for $k = 1,\ldots,r$. One has $u_0 \in \{1\} \times \mathbb{C}^n$.
• $w_0 = \pi^{-1}(Pu_0) \in \mathbb{C}^n$. As $P \in \Phi(GA(n,\mathbb{C}))$, $Pu_0 \in \{1\} \times \mathbb{C}^n$. So $\pi(w_0) = (1,w_0) = Pu_0$.
• $e^{(k)} = [e_1^{(k)}, \ldots, e_r^{(k)}]^T \in \mathbb{C}^{n+1}$ where

$$e_j^{(k)} = \begin{cases} 0 \in \mathbb{C}^{n_j} & \text{if} \quad j \neq k \\ e_{k,1} & \text{if} \quad j = k \end{cases} \qquad for \ \ every \ \ 1 \leq j,k \leq r.$$

• $p_2 : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^n$ the projection defined by $p_2(x_1,\ldots,x_{n+1}) = (x_2,\ldots,x_{n+1})$. Notice that π^{-1} is the restriction of p_2 to $\{1\} \times \mathbb{C}^n$.
• M^T is the transpose of any $M \in M_m(\mathbb{C})$, $m \leq n+1$.
• $\overline{\mathcal{G}}$ is the closure of $\mathcal{G}$ in $MA(n,\mathbb{C})$, which is considered as a vector space with the norm defined by $\|f\| = \max(\|A\|, \ \|a\|)$ the maximum of $\|A\|$ and $\|a\|$, for every $f = (A,a) \in MA(n,\mathbb{C})$.

For a *finitely generated* subgroup $\mathcal{G} \subset \mathrm{GA}(n,\mathbb{R})$, let introduce the following property. Consider the following rank condition on a collection of affine maps $f_1,\ldots,f_p \in GA(n,\mathbb{C})$, where $f_1',\ldots,f_p' \in \mathfrak{g}$ such that $e^{\Psi(f_k')} = \Phi(f_k)$, $k = 1,\ldots,p$.

We say that $f_1,\ldots,f_p$ satisfy *property* $\mathcal{D}$ if for every $(s_1,\ldots,s_p; \ t_1,\ldots,t_r) \in \mathbb{Z}^{p+r}\backslash\{0\}$:

$$rank \begin{bmatrix} Re(f_1'(w_0)) & \ldots & Re(f_p'(w_0)) & 0 & \ldots & 0 \\ Im(f_1'(w_0)) & \ldots & Im(f_p'(w_0)) & 2\pi e^{(1)} & \ldots & 2\pi e^{(r)} \\ s_1 & \ldots & s_p & t_1 & \ldots & t_r \end{bmatrix} = n+1.$$

For a vector $v \in \mathbb{C}^n$, we write $v = Re(v) + iIm(v)$ where $Re(v)$ and $Im(v) \in \mathbb{R}^n$.

For groups of affine maps on $\mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), their dynamics were recently initiated for some classes in different point of view, (see for instance, [3], [4], [5],[6]). The purpose here is to give analogous results of that theorem for linear abelian subgroup of $GL(n,\mathbb{C})$ proved in [1]. Our main results are the following:

Theorem 1.1. *Let $\mathcal{G}$ be an abelian subgroup of $GA(n,\mathbb{C})$. The following are equivalent:*
(i) $\mathcal{G}$ has a dense orbit in $\mathbb{C}^n$
(ii) The orbit $\mathcal{G}(w_0)$ is dense in $\mathbb{C}^n$
(iii) $\mathfrak{g}_{w_0}$ is an additive subgroup dense in $\mathbb{C}^n$

Corollary 1.2. *If $\mathcal{G}$ has a dense orbit then $\overline{\mathcal{G}}$ is an affine subspace of $GA(n,\mathbb{C})$ of dimension n.*

For a finitely generated subgroup $\mathcal{G}$, the Theorem can be stated as follows:

Theorem 1.3. *Let $\mathcal{G}$ be an abelian subgroup of $GA(n,\mathbb{C})$ generated by $f_1,\dots,f_p$ and let $f'_1,\dots,f'_p \in \mathfrak{g}$ such that $e^{\Psi(f'_1)} = \Phi(f_1),\dots,e^{\Psi(f'_p)} = \Phi(f_p)$. Then the following are equivalent:*
(i) $\mathcal{G}$ has a dense orbit in $\mathbb{C}^n$
(ii) the maps $f_1,\dots,f_p$ satisfy property $\mathcal{D}$
(iii) $\mathfrak{g}_{w_0} = \sum_{k=1}^{p} \mathbb{Z} f'_k(w_0) + 2i\pi \sum_{k=1}^{r} \mathbb{Z}(p_2 \circ Pe^{(k)})$ is an additive group dense in $\mathbb{C}^n$.

Corollary 1.4. *If $\mathcal{G}$ is of finite type p with $p \leq 2n - r$, then it has no dense orbit.*

Corollary 1.5. *If $\mathcal{G}$ is of finite type p with $p \leq n$, then it has no dense orbit.*

2. Notations and Lemmas

Denote by $\mathcal{L}_{\mathcal{G}}$ the set of the linear parts of all elements of $\mathcal{G}$ and $vect(F)$ is the vector space generated by a subset $F \subset \mathbb{C}^{n+1}$.

Lemma 2.1. *Let $P \in \Phi(GA(n,\mathbb{C}))$ and $h : M_{n+1}(\mathbb{C}) \longrightarrow M_{n+1}(\mathbb{C})$ the isomorphism defined by $h(A) = PAP^{-1}$. Then $h(\mathcal{F}_{n+1}) = \mathcal{F}_{n+1}$ and $h(\Phi(GA(n,\mathbb{C}))) = \Phi(GA(n,\mathbb{C}))$.*

Proof. Let $f = (A,a) \in MA(n,\mathbb{C})$. As $P \in \Phi(GA(n,\mathbb{C}))$, write

$$P = \begin{bmatrix} 1 & 0 \\ d & C \end{bmatrix},\ C \in GL(n,\mathbb{C}),\ d \in \mathbb{C}^n.$$

Then

$$\begin{aligned} h(\Psi(f)) &= \begin{bmatrix} 1 & 0 \\ d & C \end{bmatrix} \begin{bmatrix} 0 & 0 \\ a & A \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -C^{-1}d & C^{-1} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ d + Ca - CAC^{-1}d & CAC^{-1} \end{bmatrix}, \end{aligned}$$

so $h(\Psi(f)) \in \mathcal{F}_{n+1}$. It follows that $h(\mathcal{F}_{n+1}) \subset \mathcal{F}_{n+1}$. Conversely, let $h' = h^{-1}$, so $h' : M_{n+1}(\mathbb{C}) \longrightarrow M_{n+1}(\mathbb{C})$ is the isomorphism defined by $h(B) = P^{-1}BP$. By replacing h by h' in above, we have $h'(\mathcal{F}_{n+1}) \subset \mathcal{F}_{n+1}$. It follows that

$$\mathcal{F}_{n+1} = h \circ h'(\mathcal{F}_{n+1}) \subset h(\mathcal{F}_{n+1}).$$

The same proof is used to show $h(\Phi(GA(n,\mathbb{C}))) = \Phi(GA(n,\mathbb{C}))$. □

Proposition 2.2. ([1], *Proposition 2.3*) *Let G be an abelian subgroup of $GL(n,\mathbb{C})$. Then there exists $P \in GL(n,\mathbb{C})$ such that $P^{-1}GP$ is a subgroup of $\mathcal{K}_{\eta',r'}(\mathbb{C})$, for some $r' \leq n$ and $\eta' = (n'_1, \ldots, n'_{r'}) \in \mathbb{N}_0^{r'}$.*

Proposition 2.3. *Let $\mathcal{G}$ be an abelian subgroup of $GA(n,\mathbb{C})$ and $G = \Phi(\mathcal{G})$. Then there exists $P \in \Phi(GA(n,\mathbb{C}))$ such that $P^{-1}GP$ is a subgroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C}) \cap \Phi(GA(n,\mathbb{C}))$, for some $r \leq n+1$ and $\eta' = (n_1, \ldots, n_r) \in \mathbb{N}_0^r$. In particular, $Pu_0 \in \{1\} \times \mathbb{C}^n$.*

Proof. We have $\mathcal{L}_{\mathcal{G}}$ is an abelian subgroup of $GL(n,\mathbb{C})$. By Proposition 2.2, there exists $P_1 \in GL(n,\mathbb{C})$ such that $P_1^{-1}GP_1$ is a subgroup of $\mathcal{K}^*_{\eta',r'}(\mathbb{C})$ for some $r' \leq n$ and $\eta' = (n'_1, \ldots, n'_{r'}) \in \mathbb{N}_0^{r'}$ such that $n'_1 + \ldots, n'_{r'} = n$. Write for every $A \in \mathcal{L}_{\mathcal{G}}$, $P_1^{-1}AP_1 = \mathrm{diag}(A_1, \ldots, A'_{r'})$ with $A_k \in \mathbb{T}^*_{n'_k}$ and μ_{A_k} is the only eigenvalue of A_k, $k = 1, \ldots, r'$. Let $J = \{k \in \{1, \ldots, r'\},\ \ \mu_{A_k} = 1,\ \forall\ A \in \mathcal{L}_{\mathcal{G}}\}$. There are two cases:

- *Case1:* Suppose that $J = \{k_1, \ldots, k_s\}$ for some $s \leq r'$. We can take $J = \{1, \ldots, s\}$, otherwise, we replace P_1 by RP_1 for some permutation matrix R of $GL(n,\mathbb{C})$. Let $P = \mathrm{diag}(1, P_1)$, so $P \in \Phi(GA(n,\mathbb{C}))$ and

$$P^{-1}\Phi(f)P = \begin{bmatrix} 1 & 0 \\ P_1^{-1}a & P_1^{-1}AP_1 \end{bmatrix}.$$

Write $P(\mathcal{B}_0) = (e_1, \mathcal{C}_1, \ldots, \mathcal{C}_{r'})$, where $\mathcal{C}_k = (e'_{k,1}, \ldots, e'_{k,n'_k})$ is a basis of $\mathbb{C}^{n'_k}$, $k = 1, \ldots, r'$. Then $H = \mathrm{vect}(e_1, \mathcal{C}_1, \ldots, \mathcal{C}_s)$ is G-invariant and the restriction $\Phi(f)_{/H}$ has 1 as only eigenvalue for every $f \in \mathcal{G}$. Therefore for every $f = (A, a) \in \mathcal{G}$ we have

$$P^{-1}\Phi(f)P = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix},$$

where $A_1 \in \mathbb{T}^*_q(\mathbb{C})$, $q = 1 + n'_1 + \cdots + n'_s$ and $A_2 \in \mathcal{K}_{\eta'',r'-s}(\mathbb{C})$ with $\eta'' = (n'_{s+1}, \ldots, n'_{r'})$. As $\mu_{A_1} = 1$ is the only eigenvalue of A_1, so $P^{-1}\Phi(f)P \in \Phi(GA(n,\mathbb{C}))$. It follows that $P^{-1}GP$ is a subgroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C}) \cap \Phi(GA(n,\mathbb{C}))$, where $r = r' - s + 1$ and $\eta = (q, n'_{s+1}, \ldots, n'_{r'})$.

- *Case:2* Suppose that $J = \emptyset$, then 1 is the commune eigenvalue of all element of G and let v be the commune eigenvector of all element of G associated to 1. Since $\{0\} \times \mathbb{C}^n$ is G-invariant, one has $v \in \mathbb{C}^* \times \mathbb{C}^n$, so we can take $v = (1, v_1)$, $v_1 \in \mathbb{C}^n$, otherwise, if $v = (\lambda, v'_1)$ we replace v by $\frac{1}{\lambda}v$. As $\Phi(f)(1, v_1) = (1, f(v_1)) = (1, v_1)$

so $f(v_1) = \pi^{-1}(\Phi(f)(1, v_1)) = \pi^{-1}(1, v_1) = v_1$, hence $f(v_1) = Av_1 + a = v_1$. Let $P = \begin{bmatrix} 1 & 0 \\ v_1 & P_1 \end{bmatrix} \in \Phi(GA(n, \mathbb{C}))$, so for every $f = (A, a) \in \mathcal{G}$

$$\begin{aligned} P^{-1}\Phi(f)P &= \begin{bmatrix} 1 & 0 \\ -P_1^{-1}v_1 & P_1^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ a & A \end{bmatrix} \begin{bmatrix} 1 & 0 \\ v_1 & P_1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ P_1^{-1}(Av_1 + a - v_1) & P_1^{-1}AP_1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & P_1^{-1}AP_1 \end{bmatrix}. \end{aligned}$$

It follows that $P^{-1}GP$ is a subgroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C}))$, where $r = r' + 1$ and $\eta = (1, n'_1, \ldots, n'_{r'})$.
Since $u_0 \in \{1\} \times \mathbb{C}^n$ and $P \in \Phi(GA(n, \mathbb{C}))$, so $Pu_0 \in \{1\} \times \mathbb{C}^n$. □

Denote by:

- $G' = P^{-1}GP$.
- $\mathrm{g}' = exp^{-1}(G') \cap \mathcal{K}_{\eta,r}(\mathbb{C}) \cap \mathcal{F}_{n+1}$.
- $\mathcal{C}(G') = \{B \in \mathcal{K}_{\eta,r}(\mathbb{C}),\ A'B = BA',\ A' \in G'\}$.
- $\mathcal{C}(G) = P\mathcal{C}(G')P^{-1}$.

Proposition 2.4. (*Under above notation*), *we have* $exp(\mathrm{g}) = G$. *In particular,* $exp(\mathrm{g}') = G'$.

The proof of Proposition 2.4 uses the following Lemmas:

Lemma 2.5. ([1], *Lemma 4.2*) *Let* G *be an abelian subgroup of* $GL(n+1, \mathbb{C})$ *and* $P \in GL(n+1, \mathbb{C})$ *such that* $P^{-1}GP$ *is a subgroup of* $\mathcal{K}_{\eta,r}(\mathbb{C})$. *If* $\mathrm{g}_1 = exp^{-1}(G) \cap \left(P\mathcal{K}_{\eta,r}(\mathbb{C})P^{-1}\right)$ *then:*
(i) $exp(\mathrm{g}_1) = G$.
(ii) $\mathrm{g}_1 \subset \mathcal{C}(G)$.

Lemma 2.6. ([1], *Proposition 3.2*) $exp(\mathcal{K}_{\eta,r}(\mathbb{C})) = \mathcal{K}^*_{\eta,r}(\mathbb{C})$.

Lemma 2.7. *(i) If* $N \in P\mathcal{K}_{\eta,r}(\mathbb{C})P^{-1}$ *such that* $e^N \in \Phi(GA(n, \mathbb{C}))$, *so* $N - 2ik\pi I_{n+1} \in \mathcal{F}_{n+1}$, *for some* $k \in \mathbb{Z}$.
(ii) $exp(\mathcal{F}_{n+1}) = \Phi(GA(n, \mathbb{C}))$.
(iii) $exp(\mathcal{F}_{n+1} \cap \mathcal{K}_{\eta,r}(\mathbb{C})) = \Phi(GA(n, \mathbb{C})) \cap \mathcal{K}^*_{\eta,r}(\mathbb{C})$.

Proof. (i) Let $N' = P^{-1}NP \in \mathcal{K}_{\eta,r}(\mathbb{C})$, $M = e^N$ and $M' = P^{-1}MP$. We have $e^{N'} = M'$ and by Lemma 2.6, $M' \in \mathcal{K}^*_{\eta,r}(\mathbb{C})$. Write $M' = \mathrm{diag}(M'_1, \ldots, M'_r)$ and $N' = \mathrm{diag}(N'_1, \ldots, N'_r)$, $M'_k, N'_k \in \mathbb{T}_{n_k}(\mathbb{C})$, $k = 1, \ldots, r$. Then $e^{N'} = \mathrm{diag}(e^{N'_1}, \ldots, e^{N'_r})$, so $e^{N'_1} = M'_1$. As 1 is the only eigenvalue of M'_1, N'_1 has an eigenvalue $\mu \in \mathbb{C}$ such that $e^\mu = 1$. Thus $\mu = 2ik\pi$ for some $k \in \mathbb{Z}$. Therefore, $N'' = N' - 2ik\pi I_{n+1} \in \mathcal{F}_{n+1}$ and satisfying $e^{N''} = e^{-2ik\pi}e^{N'} = M'$. It follows by Lemma 2.1, that $N - 2ik\pi I_{n+1} = PN''P^{-1} \in P\mathcal{F}_{n+1}P^{-1} = \mathcal{F}_{n+1}$, since $P \in \Phi(GA(n, \mathbb{C}))$.

(ii) let $N = \begin{bmatrix} 0 & 0 \\ b & B \end{bmatrix} \in \mathcal{F}_{n+1}$, we have $N^k = \begin{bmatrix} 0 & 0 \\ B^{k-1}b & B^k \end{bmatrix} \in \mathcal{F}_{n+1}$ for every $k \in \mathbb{N}_0$, so

$$\begin{aligned} e^N &= I_{n+1} + \sum_{k\in\mathbb{N}_0} \frac{1}{k!} N^k \\ &= \begin{bmatrix} 1 & 0 \\ \sum\limits_{k\in\mathbb{N}_0} \frac{1}{k!} B^{k-1}b & I_n + \sum\limits_{k\in\mathbb{N}_0} \frac{1}{k!} B^k \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ \sum\limits_{k\in\mathbb{N}_0} \frac{1}{k!} B^{k-1}b & e^B \end{bmatrix}. \end{aligned}$$

Then $e^N \in \Phi(GA(n,\mathbb{C}))$. It follows that $exp(\mathcal{F}_{n+1}) \subset \Phi(GA(n,\mathbb{C}))$. Conversely, let $M = \begin{bmatrix} 1 & 0 \\ a & A \end{bmatrix} \in \Phi(GA(n,\mathbb{C}))$. By applying Proposition 2.3 on the group generated by M, there exists $P \in \Phi(GA(n,\mathbb{C}))$ such that $M' = P^{-1}MP \in \mathcal{K}^*_{\eta,r}(\mathbb{C}) \cap \Phi(GA(n,\mathbb{C}))$. By Lemma 2.6, there exists $N' \in \mathcal{K}_{\eta,r}(\mathbb{C})$ such that $e^{N'} = M'$. By (i), $N'' = N' - 2ik\pi I_{n+1} \in \mathcal{F}_{n+1}$ for some $k \in \mathbb{Z}$ and satisfying $e^{N''} = M'$. Hence , $N = PN'P^{-1} \in P\mathcal{F}_{n+1}P^{-1}$ and satisfying $e^N = Pe^{N''}P^{-1} = PM'P^{-1} = M$. By Lemma 2.1, $N \in \mathcal{F}_{n+1}$. It follows that $\Phi(GA(n,\mathbb{C})) \subset exp(\mathcal{F}_{n+1})$.

(iii) By Lemma 2.6 and (ii) we have $exp(\mathcal{F}_{n+1} \cap \mathcal{K}_{\eta,r}(\mathbb{C})) \subset \Phi(GA(n,\mathbb{C})) \cap \mathcal{K}^*_{\eta,r}(\mathbb{C})$. Conversely, let $M = \begin{bmatrix} 1 & 0 \\ a & A \end{bmatrix} \in \Phi(GA(n,\mathbb{C})) \cap \mathcal{K}^*_{\eta,r}(\mathbb{C})$. By Lemma 2.6, there exists $N' \in \mathcal{K}_{\eta,r}(\mathbb{C})$ such that $e^{N'} = M$. By (i), $N = N' - 2ik\pi I_{n+1} \in \mathcal{F}_{n+1}$ for some $k \in \mathbb{Z}$ and satisfying $e^N = e^{-2ik\pi}e^{N'} = M$. It follows that $\Phi(GA(n,\mathbb{C})) \cap \mathcal{K}^*_{\eta,r}(\mathbb{C}) \subset exp(\mathcal{F}_{n+1})$. The proof is complete. □

Proof of Proposition 2.4. By Lemma 2.5.(i), we have $exp(\mathrm{g}_1) = G$. Since $\mathrm{g} = \mathrm{g}_1 \cap \mathcal{F}_{n+1}$ then

$$exp(\mathrm{g}) \subset exp(\mathrm{g}_1) = G.$$

Conversely, let $M \in G$. By lemma 2.5.(i) we have $exp(\mathrm{g}_1) = G$ then there exists $N \in \mathrm{g}_1 \subset \mathcal{K}_{\eta,r}(\mathbb{C})$ such that $e^N = M$. By Lemma 2.7.(i), there exists $k \in \mathbb{Z}$ such that $N' = N - 2ik\pi I_{n+1} \in \mathcal{F}_{n+1}$. Since N and $2ik\pi I_{n+1} \in P\mathcal{K}_{\eta,r}(\mathbb{C})P^{-1}$, $N' \in \mathrm{g}_1 \cap \mathcal{F}_{n+1} = \mathrm{g}$. It follows that $M = e^{N'} \in exp(\mathrm{g})$. The proof is complete.
In particular if $P = I_{n+1}$, we have $exp(\mathrm{g}') = G'$. □

3. **Parametrization**

Let $\mathcal{G}$ be an abelian subgroup of $GA(n,\mathbb{C})$ and $G=\Phi(\mathcal{G})$. In all the following, P denotes the matrix defined in Proposition 2.3 (i.e. $P\in\Phi(GA(n,\mathbb{C}))$ such that $P^{-1}GP$ is a subgroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C})\cap\Phi(GA(n,\mathbb{C}))$). Recall that $G'=P^{-1}GP$ and $\mathrm{g}'=\exp^{-1}(G')\cap\mathcal{K}_{\eta,r}(\mathbb{C})\cap\mathcal{F}_{n+1}$, so every $A'\in G'$ is in the form

$$A'=\mathrm{diag}(A'_1,\dots,A'_r)\in\mathcal{K}^*_{\eta,r}(\mathbb{C}),\quad A'_k\in\mathbb{T}^*_{n_k}(\mathbb{C}),\ k=1,\dots,r$$

and every $B'\in\mathrm{g}'$ is in the form

$$B'=\mathrm{diag}(B'_1,\dots,B'_r)\in\mathcal{K}_{\eta,r}(\mathbb{C}),\quad B'_k\in\mathbb{T}_{n_k}(\mathbb{C}),\ k=1,\dots,r$$

For every $k=1,\dots,r$, denote by:
- $G'_k=\{A'_k,\quad A'\in G'\}$.
- $\mathrm{g}'_k=\{B'_k,\quad B'\in\mathrm{g}'\}$.
- $F_{G'_k}=\left\{(A'_k-\mu_{A'_k}I_{n_k})e_{k,i},\quad A'\in G',\ 1\le i\le n_k-1\right\}$.
- $F_{\mathrm{g}'_k}=\left\{(B'_k-\mu_{B'_k}I_{n_k})e_{k,i},\quad B'\in\mathrm{g}',\ 1\le i\le n_k-1\right\}$.

where $\mu_{A'_k}$ (resp. $\mu_{B'_k}$) is the only eigenvalue of A'_k. (resp. B'_k) and $(e_{k,1},\dots,e_{k,n_k})$ is the canonical basis of $\mathbb{C}^{n_k}$.
- Let $h:\ M_{n+1}(\mathbb{C})\longrightarrow M_{n+1}(\mathbb{C})$ be the isomorphism defined by $h(B)=PBP^{-1}$. one has $h(G')=G$.

See that π is an affine map, because $\pi=(D_0,e_1)$, where $D_0=\mathrm{diag}(0,I_n)$. Remark that $\pi\notin GA(n,\mathbb{C})$, however $\pi:\mathbb{C}^n\longrightarrow\{1\}\times\mathbb{C}^n$ is invertible.
Denote by :
- $U=\prod_{k=1}^{r}\mathbb{C}^*\times\mathbb{C}^{n_k-1}$, we have U is a G'-invariant open set.
- $V=\pi^{-1}\circ P\left(U\cap(\{1\}\times\mathbb{C}^n)\right)$.

We have V is an open subset of $\mathbb{C}^n$, because $V=\pi^{-1}\left(P(U)\cap P(\{1\}\times\mathbb{C}^n)\right)$, π is continuous and $P(U)\cap P(\{1\}\times\mathbb{C}^n)=P(U)\cap(\{1\}\times\mathbb{C}^n)$ is an open set of $\{1\}\times\mathbb{C}^n$ (since $P\in\Phi(GA(n,\mathbb{C}))$ and so $P(\{1\}\times\mathbb{C}^n)=\{1\}\times\mathbb{C}^n$).
- $\mathcal{C}(\mathrm{g}')=\{B\in\mathcal{K}_{\eta,r}(\mathbb{C}),\ A'B=BA',\ A'\in\mathrm{g}'\}$.
- $\mathcal{C}(\mathrm{g})=P\mathcal{C}(\mathrm{g}')P^{-1}$ (i.e. $\mathcal{C}(\mathrm{g})=h(\mathcal{C}(\mathrm{g}'))$).
- $\mathcal{C}(\mathcal{G})=\Phi^{-1}(\mathcal{C}(G)\cap\Phi(GA(n,\mathbb{C})))$.
- $\mathcal{C}(\mathfrak{g})=\Psi^{-1}(\mathcal{C}(\mathrm{g})\cap\mathcal{F}_{n+1})$.

Notice that by construct, $\mathcal{C}(\mathcal{G})$ is an affine subspace of $MA(n,\mathbb{C})$, since Φ is affine map. However, $\mathcal{C}(\mathfrak{g})$ is a vector subspace of $MA(n,\mathbb{C})$, since Ψ is a linear map.
Recall the following results proved in [1] for an abelian subgroup G' of $\mathcal{K}^*_{\eta,r}(\mathbb{C})$.

Lemma 3.1. ([1], *Lemma 4.2*) *We have* $\mathcal{C}(G')=\mathrm{C}(\mathrm{g}')$.

By Lemma 3.1 we have $\mathcal{C}(G')=\mathcal{C}(\mathrm{g}')$ and so $\mathcal{C}(G)=\mathcal{C}(\mathrm{g})$. By construct, this identification can not be true in the affine case (i.e. $\mathcal{C}(\mathcal{G})\neq\mathcal{C}(\mathfrak{g})$). Because $f_0=(0,0)\notin\mathcal{C}(\mathcal{G})$ and $\mathcal{C}(\mathfrak{g})$ is a vector space.

Proposition 3.2. ([1], *Theorem 5.2 and Corollary 5.4*) *Let G' be an abelian subgroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C})$. If $rank(F_{G'_k}) = n_k - 1$ (resp. $rank(F_{\mathrm{g}'_k}) = n_k - 1$), $k = 1, \ldots, r$, then there exists a linear injective map $\varphi : \mathbb{C}^{n+1} \longrightarrow \mathcal{K}_{\eta,r}(\mathbb{C})$ (resp. $\psi : \mathbb{C}^{n+1} \longrightarrow \mathcal{K}_{\eta,r}(\mathbb{C})$) such that:*
(i) $\mathcal{C}(G') \subset \varphi(\mathbb{C}^{n+1})$, (resp. $\mathcal{C}(G') \subset \psi(\mathbb{C}^{n+1})$).
(ii) For every $v \in \mathbb{C}^{n+1}$, $\varphi(v)u_0 = v$, (resp. $\psi(v)u_0 = v$). In particular $\varphi(G'(u_0)) = G'$ and $\varphi(\mathrm{g}'_{u_0}) = \mathrm{g}'$ (resp. $\psi(G'(u_0)) = G'$ and $\psi(\mathrm{g}'_{u_0}) = \mathrm{g}'$).

Lemma 3.3. ([1], *Corollaries 2.7 and 3.2.(ii)*) *The linear injective map $\psi : \mathbb{C}^{n+1} \longrightarrow \mathcal{K}_{\eta,r}(\mathbb{C}))$ defined in Proposition 3.2 satisfies $exp(\mathcal{C}(\mathrm{g}')) = \mathcal{C}(G') \cap \mathcal{K}^*_{\eta,r}(\mathbb{C})$.*

Corollary 3.4. ([1], *Proposition 6.2*) *If $\overline{G'(u_0)} = \mathbb{C}^{n+1}$(resp. $\overline{\mathrm{g}'_{u_0}} = \mathbb{C}^{n+1}$) then φ (resp. ψ) is an isomorphism of $\mathbb{C}^{n+1}$ on $\mathcal{C}(G')$.*

Corollary 3.5. ([1], *Corollary 3.7 and Propositions 3.6, 3.2*) *The restriction $exp_{/\mathcal{K}_{\eta,r}(\mathbb{C})} : \mathcal{K}_{\eta,r}(\mathbb{C}) \longrightarrow \mathcal{K}^*_{\eta,r}(\mathbb{C})$ is a local diffeomorphism.*

Lemma 3.6. *The restriction $exp_{/(\mathcal{K}_{\eta,r}(\mathbb{C})\cap\mathcal{F}_{n+1})} : \mathcal{K}_{\eta,r}(\mathbb{C})\cap\mathcal{F}_{n+1} \longrightarrow \Phi(GA(n,\mathbb{C}))\cap \mathcal{K}^*_{\eta,r}(\mathbb{C})$ is a local diffeomorphism.*

Proof. The proof results from Corollary 3.5 and Lemma 2.7.(iii). □

Recall the following Proposition:

Proposition 3.7. ([1], *Proposition 6.6*) *U is G'-invariant dense open set in $\mathbb{C}^{n+1}$ and all orbits of G' in U are minimal in U.*

Lemma 3.8. *Under notation of Proposition 3.2, we have:*
(i) $h(\mathrm{g}') = \mathrm{g}$ and $P^{-1}(\mathrm{g}_{(1,w_0)}) = \mathrm{g}'_{u_0}$.
(ii) $\varphi(\mathbb{C}^{n+1}) \cap \Phi(GA(n,\mathbb{C})) \subset \varphi(\{1\} \times \mathbb{C}^n)$ and $\varphi(\mathbb{C}^{n+1}) \cap \mathcal{F}_{n+1} \subset \varphi(\{0\} \times \mathbb{C}^n)$.
(iii) $h(\mathcal{C}(G')) = \mathcal{C}(G)$.
(iv) $\widetilde{\pi}(f'(u)) = \Psi(f')(1,u)$, for every $f' \in \mathfrak{g}$ and $u \in \mathbb{C}^n$. In particular, $\widetilde{\pi}(\mathfrak{g}_u) = \mathrm{g}_{(1,u)}$.

Proof. (i) Let $B' \in \mathrm{g}' = exp^{-1}(G) \cap \mathcal{K}_{\eta,r}(\mathbb{C}) \cap \mathcal{F}_{n+1}$, so $e^{h(B)} = h(e^B) \in h(G') = G$ and since $h(\mathcal{F}_{n+1}) = \mathcal{F}_{n+1}$ so $h(B) \in \mathcal{F}_{n+1}$, then

$$h(B) \in exp^{-1}(G) \cap h(\mathcal{K}_{\eta,r}(\mathbb{C})) \cap \mathcal{F}_{n+1} = \mathrm{g}.$$

Conversely, we use the same proof by replacing h by h^{-1}.
Since $h(\mathrm{g}') = \mathrm{g}$, so $\mathrm{g} = h^{-1}(\mathrm{g}') = P^{-1}\mathrm{g}'P$. Then

$$\begin{aligned} P^{-1}(\mathrm{g}_{(1,w_0)}) &= \{P^{-1}B(1,w_0) : \quad B \in \mathrm{g}\} \\ &= \{P^{-1}BPu_0 : \quad B \in \mathrm{g}\} \\ &= \{B'u_0 : \quad B' \in \mathrm{g}'\} \\ &= \mathrm{g}'_{u_0} \end{aligned}$$

(ii) Let $v = [x_1, \ldots, x_{n+1}]^T \in \mathbb{C}^{n+1}$ such that $\varphi(v) \in \Phi(GA(n,\mathbb{C}))$, so by Proposition 3.2.(ii), $\varphi(v)u_0 = v$. It follows that $x_1 = 1$ and so $v \in \{1\} \times \mathbb{C}^n$.

(iii) Since $h(G') = G$ and $AB = BA$ if and only if $h(A)h(B) = h(B)h(A)$, for every $A, B \in M_{n+1}(\mathbb{C})$, it follows that $h(\mathcal{C}(G')) = \mathcal{C}(G)$.

(iv) Let $f' = (B, b) \in \mathfrak{g}$ and $u \in \mathbb{C}^n$, then

$$\Psi(f')(1,u) = \begin{bmatrix} 0 & 0 \\ b & B \end{bmatrix} \begin{bmatrix} 1 \\ u \end{bmatrix} = (0, f'(u)) = \widetilde{\pi}(f'(u)),$$

so

$$\widetilde{\pi}(\mathfrak{g}_u) = \Psi(\mathfrak{g})_{(1,u)} = \mathrm{g}_{(1,u)} = \{0\} \times \mathfrak{g}_u \subset \{0\} \times \mathbb{C}^n.$$

□

Proposition 3.9. *Let $\mathcal{G}$ be an abelian subgroup of $GA(n,\mathbb{C})$, $G = \Phi(\mathcal{G})$ and $P \in \Phi(GA(n,\mathbb{C}))$ so that $G' = P^{-1}GP$ is a subgroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C}) \cap \Phi(GA(n,\mathbb{C}))$. If $rank(F_{G'_k}) = n_k - 1$ (resp. $rank(F_{\mathrm{g}'_k}) = n_k - 1$), $k = 1, \ldots, r-1$, then there exists two injective affine maps $\gamma : \mathbb{C}^n \longrightarrow GA(n,\mathbb{C})$ and $\widetilde{\gamma} : \mathbb{C}^n \longrightarrow MA(n,\mathbb{C})$ (resp. $\delta : \mathbb{C}^n \longrightarrow GA(n,\mathbb{C})$ and $\widetilde{\delta} : \mathbb{C}^n \longrightarrow MA(n,\mathbb{C})$) such that :*
(i) $\mathcal{C}(\mathcal{G}) \subset \gamma(\mathbb{C}^n)$ and $\mathcal{C}(\mathfrak{g}) \subset \widetilde{\gamma}(\mathbb{C}^n)$ (resp. $\mathcal{C}(\mathcal{G}) \subset \delta(\mathbb{C}^n)$ and $\mathcal{C}(\mathfrak{g}) \subset \widetilde{\delta}(\mathbb{C}^n)$).
(ii) $\gamma(\mathcal{G}(w_0)) = \mathcal{G}$ and $\widetilde{\gamma}(\mathfrak{g}_{w_0}) = \mathfrak{g}$. (resp. $\delta(\mathcal{G}(w_0)) = \mathcal{G}$ and $\widetilde{\delta}(\mathfrak{g}_{w_0}) = \mathfrak{g}$).

Proof. Let's prove the Proposition for $\mathrm{rank}(F_{G_k}) = n_k - 1$. The proof for $\mathrm{rank}(F_{\mathrm{g}_k}) = n_k - 1$, $k = 1, \ldots, r$ is analogous.

(ii) As $G(1,w_0) = \{1\} \times \mathcal{G}(w_0)$ we have $\pi(\mathcal{G}(w_0)) = G(1,w_0)$. Then

$$P^{-1} \circ \pi(\mathcal{G}(w_0)) = P^{-1}G(1,w_0) = P^{-1}GP(u_0) = G'(u_0) \qquad (2)$$

By proposition 3.2, there exists a linear injective map $\varphi : \mathbb{C}^{n+1} \longrightarrow \mathcal{K}_{\eta,r}(\mathbb{C})$ such that $\varphi(G'(u_0)) = G'$. Then by (2) we have

$$h \circ \varphi(P^{-1} \circ \pi(\mathcal{G}(w_0))) = h \circ \varphi(G'(u_0)) = h(G') = G. \qquad (3)$$

Since $\Phi^{-1}(G) = \mathcal{G}$ then by (3), we have

$$\mathcal{G} = \Phi^{-1} \circ h \circ \varphi \circ P^{-1} \circ \pi(\mathcal{G}(w_0)).$$

It follows that $\gamma = \Phi^{-1} \circ h \circ \varphi \circ P^{-1} \circ \pi$ is an injective affine map from $\mathbb{C}^n$ to $GA(n,\mathbb{C})$ and satisfying $\gamma(\mathcal{G}(w_0)) = \mathcal{G}$. (Because π and Φ are invertible affine maps; h, P are isomorphisms and φ is an injective linear map)

Let $\widetilde{\gamma} = \Psi^{-1} \circ h \circ \varphi \circ P^{-1} \circ \widetilde{\pi}$, so $\widetilde{\gamma}$ is an injective affine map from $\mathbb{C}^n$ to $MA(n,\mathbb{C})$ and satisfying:

$$\begin{aligned}
\widetilde{\gamma}(\mathfrak{g}_{w_0}) &= \Psi^{-1} \circ h \circ \varphi \circ P^{-1} \circ \widetilde{\pi}\,(\mathfrak{g}_{w_0}) \\
&= \Psi^{-1} \circ h \circ \varphi \circ P^{-1}\left(\mathrm{g}_{(1,w_0)}\right) && \text{(by Lemma 3.8.}(iv)\text{)} \\
&= \Psi^{-1} \circ h \circ \varphi(\mathrm{g}'_{u_0}) && \text{(by Lemma 3.8.}(i)\text{)} \\
&= \Psi^{-1} \circ h(\mathrm{g}') \\
&= \Psi^{-1}(\mathrm{g}) && \text{(by Lemma 3.8.}(i)\text{)} \\
&= \mathfrak{g}
\end{aligned}$$

By analogous proof, we replace φ by ψ in the construction of γ and $\widetilde{\gamma}$ and we find

$$\delta = \Phi^{-1} \circ h \circ \psi \circ P^{-1} \circ \pi \qquad \text{and} \qquad \widetilde{\delta} = \Psi^{-1} \circ h \circ \psi \circ P^{-1} \circ \widetilde{\pi} \tag{4}$$

(i) By Lemma 3.8.(iii), $h(\mathcal{C}(G')) = \mathcal{C}(G)$ and by proposition 3.2, $\mathcal{C}(G') \subset \varphi(\mathbb{C}^{n+1})$. Then we have:

$$\begin{aligned}
\mathcal{C}(\mathcal{G}) &= \Phi^{-1}(\mathcal{C}(G) \cap \Phi(GA(n,\mathbb{C}))) \\
&= \Phi^{-1}\left(h(\mathcal{C}(G')) \cap \Phi(GA(n,\mathbb{C}))\right) \\
&\subset \Phi^{-1} \circ h\left(\varphi(\mathbb{C}^{n+1}) \cap h^{-1}(\Phi(GA(n,\mathbb{C})))\right) \\
&\subset \Phi^{-1} \circ h\left(\varphi(\mathbb{C}^{n+1}) \cap \Phi(GA(n,\mathbb{C}))\right) && \text{(by Lemma 2.1)} \\
&\subset \Phi^{-1} \circ h \circ \varphi\left(\{1\} \times \mathbb{C}^n\right) && \text{(by Lemma 3.8.}(ii)\text{)} \qquad (5)
\end{aligned}$$

On the other hand, we have $P^{-1}(\{1\}\times\mathbb{C}^n) = \{1\}\times\mathbb{C}^n$, because $P \in \Phi(GA(n,\mathbb{C}))$. As $\{1\} \times \mathbb{C}^n = \pi(\mathbb{C}^n)$, so $\{1\} \times \mathbb{C}^n = P^{-1}(\{1\} \times \mathbb{C}^n) = P^{-1} \circ \pi(\mathbb{C}^n)$. By (5), we have

$$\mathcal{C}(\mathcal{G}) \subset \Phi^{-1} \circ h \circ \varphi \circ P^{-1} \circ \pi(\mathbb{C}^n) = \gamma(\mathbb{C}^n).$$

By Lemma 3.1 and Proposition 3.2, $\mathcal{C}(\mathrm{g}') = \mathcal{C}(G') \subset \varphi(\mathbb{C}^{n+1})$. By Lemma 3.8.(iii), $h(\mathcal{C}(\mathrm{g}')) = \mathcal{C}(\mathrm{g})$, then we have:

$$\begin{aligned}
\mathcal{C}(\mathfrak{g}) &= \Psi^{-1}(\mathcal{C}(\mathrm{g}) \cap \mathcal{F}_{n+1}) \\
&= \Psi^{-1}\left(h(\mathcal{C}(\mathrm{g}')) \cap h(\mathcal{F}_{n+1})\right) && \text{(by Lemma 2.1)} \\
&\subset \Psi^{-1} \circ h\left(\varphi(\mathbb{C}^{n+1}) \cap \mathcal{F}_{n+1}\right) \\
&\subset \Psi^{-1} \circ h \circ \varphi\left(\{0\} \times \mathbb{C}^n\right) && \text{(by Lemma 3.8.}(i)\text{)} \qquad (6)
\end{aligned}$$

On the other hand, we have $P^{-1}(\{0\}\times\mathbb{C}^n) = \{0\}\times\mathbb{C}^n$, because $P \in \Phi(GA(n,\mathbb{C}))$. As $\{0\} \times \mathbb{C}^n = \widetilde{\pi}(\mathbb{C}^n)$, so $\{0\} \times \mathbb{C}^n = P^{-1}(\{0\} \times \mathbb{C}^n) = P^{-1} \circ \widetilde{\pi}(\mathbb{C}^n)$. By (6), we have

$$\mathcal{C}(\mathfrak{g}) \subset \Psi^{-1} \circ h \circ \varphi \circ P^{-1} \circ \widetilde{\pi}(\mathbb{C}^n) = \widetilde{\gamma}(\mathbb{C}^n).$$

This completes the proof. □

Corollary 3.10. *If* $\overline{\mathcal{G}(w_0)} = \mathbb{C}^n$ *(resp.* $\overline{\mathfrak{g}_{w_0}} = \mathbb{C}^n$*) then* $\gamma : \mathbb{C}^n \longrightarrow \mathcal{C}(\mathcal{G})$ *and* $\widetilde{\gamma} : \mathbb{C}^n \longrightarrow \mathcal{C}(\mathfrak{g})$ *(resp.* $\delta : \mathbb{C}^n \longrightarrow \mathcal{C}(\mathcal{G})$ *and* $\widetilde{\delta} : \mathbb{C}^n \longrightarrow \mathcal{C}(\mathfrak{g})$*) are homeomorphisms.*

Proof. Suppose that $\overline{\mathcal{G}(w_0)} = \mathbb{C}^n$. Then by Proposition 3.9 there exists an injective affine map $\gamma : \mathbb{C}^n \longrightarrow GA(n, \mathbb{C})$ such that $\mathcal{C}(\mathcal{G}) \subset \gamma(\mathbb{C}^n)$. Let's prove that $\gamma(\mathbb{C}^n) \subset \mathcal{C}(\mathcal{G})$:
By proposition 3.9.(ii), we have $\gamma(\mathcal{G}(w_0)) = \mathcal{G}$. Then since γ is continuous, we have:

$$\gamma(\mathbb{C}^n) = \gamma(\overline{\mathcal{G}(w_0)}) \subset \overline{\gamma(\mathcal{G}(w_0))} = \overline{\mathcal{G}}.$$

By construction, $\mathcal{C}(\mathcal{G})$ is an affine subspace of $\mathbb{C}^n$ then it is closed and so $\overline{\mathcal{G}} \subset \mathcal{C}(\mathcal{G})$. Hence $\gamma(\mathbb{C}^n) \subset \mathcal{C}(\mathcal{G})$. We conclude that γ is an homeomorphism of $\mathbb{C}^n$ on $\mathcal{C}(\mathcal{G})$.

The same proof is given for $\widetilde{\gamma}$, δ and $\widetilde{\delta}$. □

Corollary 3.11. *If* $\overline{\mathcal{G}(w_0)} = \mathbb{C}^n$ *then the map* $\alpha : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ *defined by* $\alpha = \gamma^{-1} \circ \Phi^{-1} \circ h \circ \left(exp_{/\mathcal{K}_{\eta,r}(\mathbb{C}) \cap \mathcal{F}_{n+1}}\right) \circ h^{-1} \circ \Psi \circ \widetilde{\gamma}$ *satisfies:*
(i) α *is an open map*
(ii) $\alpha^{-1}(\mathcal{G}(w_0)) = \mathfrak{g}_{w_0}$.

Proof. If $\overline{\mathcal{G}(w_0)} = \mathbb{C}^n$ then φ is invertible so are γ and $\widetilde{\gamma}$. Hence α is well defined.
(i) Since $exp_{/\mathcal{K}_{\eta,r}(\mathbb{C})}$ is a local diffeomorphism (Corollary 3.5) then α is open.
(ii) By Proposition 3.9.(ii), we have $\widetilde{\gamma}^{-1}(\mathfrak{g}) = \mathfrak{g}_{u_0}$ and $\gamma(\mathcal{G}(w_0)) = \mathcal{G}$. Then

$$\begin{aligned}
\alpha^{-1}(\mathcal{G}(w_0)) &= \widetilde{\gamma}^{-1} \circ \Psi^{-1} \circ h \circ \left(exp_{/(\mathcal{K}_{\eta,r}(\mathbb{C}) \cap \mathcal{F}_{n+1})}\right)^{-1} \circ h^{-1} \circ \Phi \circ \gamma\left(\mathcal{G}(w_0)\right) \\
&= \widetilde{\gamma}^{-1} \circ \Psi^{-1} \circ h \circ \left(exp_{/(\mathcal{K}_{\eta,r}(\mathbb{C}) \cap \mathcal{F}_{n+1})}\right)^{-1} \circ h^{-1} \circ \Phi(\mathcal{G}) \\
&= \widetilde{\gamma}^{-1} \circ \Psi^{-1} \circ h \circ \left(exp_{/(\mathcal{K}_{\eta,r}(\mathbb{C}) \cap \mathcal{F}_{n+1})}\right)^{-1} \circ h^{-1}(G) \\
&= \widetilde{\gamma}^{-1} \circ \Psi^{-1} \circ h \circ \left(exp_{/(\mathcal{K}_{\eta,r}(\mathbb{C}) \cap \mathcal{F}_{n+1})}\right)^{-1} (G') \\
&= \widetilde{\gamma}^{-1} \circ \Psi^{-1} \circ h(\mathrm{g}') \qquad \text{(by definition of g}') \\
&= \widetilde{\gamma}^{-1} \circ \Psi^{-1}(\mathrm{g}) \qquad \text{(by Lemma 3.8.}(i)\text{)} \\
&= \widetilde{\gamma}^{-1}(\mathfrak{g}) \\
&= \mathfrak{g}_{w_0}.
\end{aligned}$$

□

Lemma 3.12. *Let* $v \in \mathbb{C}^{n+1}$. *Then* $v \in U \cap (\{1\} \times \mathbb{C}^n)$ *if and only if* $\psi(v) \in \mathcal{K}^*_{\eta,r}(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C}))$. *Moreover,* $\psi^{-1}(\mathcal{K}^*_{\eta,r}(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C})) = U \cap (\{1\} \times \mathbb{C}^n)$.

Proof. Let $x = [x_{1,1}, \ldots, x_{1,n_1}; \ldots; x_{r,1}, \ldots, x_{r,n_r}]^T \in \mathbb{C}^{n+1}$. By Proposition 3.2.(i), $\psi(x) = \mathrm{diag}(B_1, \ldots, B_r) \in \mathcal{K}_{\eta,r}(\mathbb{C})$ with $B_k \in \mathbb{T}_{n_k}(\mathbb{C})$, $k = 1, \ldots, r$. By Proposition 3.2.(ii), $x = \psi(x)u_0 = [B_1 e_{1,1}, \ldots B_r e_{r,1}]^T$. Therefore $B_1 e_{1,1} = [1, x_{1,2}, \ldots, x_{1,n_1}]^T$ and $B_k e_{k,1} = [x_{k,1}, \ldots, x_{k,n_k}]^T$, $k = 2, \ldots, r$. The proof of Lemma 3.12 follows, since B_k has $x_{k,1}$ as only eigenvalue, for every $k = 1, \ldots, r$. □

Lemma 3.13. ([1], *Proposition 2.3*) *If $A, B \in \mathcal{K}_{\eta,r}(\mathbb{C})$ satisfy $e^A e^B = e^B e^A$ then $AB = BA$.*

Lemma 3.14. *Under notation of Proposition 3.9. If $\overline{\mathfrak{g}_{w_0}} = \mathbb{C}^n$ then:*
(i) $U \cap (\{1\} \times \mathbb{C}^n) = \psi^{-1}(\mathcal{C}(G') \cap \Phi(GA(n,\mathbb{C})) \cap \mathcal{K}^*_{\eta,r}(\mathbb{C}))$.
(ii) $V = \delta^{-1}\left(\mathcal{C}(\mathcal{G}) \cap \Phi^{-1} \circ h(\mathcal{K}^*_{\eta,r}(\mathbb{C}))\right)$.
(iii) $exp\left(\mathcal{C}(\mathrm{g}') \cap \mathcal{F}_{n+1}\right) = \mathcal{C}(G') \cap \mathcal{K}^*_{\eta,r}(\mathbb{C}) \cap \Phi(GA(n,\mathbb{C}))$.

Proof. (i) By Lemma 3.12, we have

$$\psi^{-1}(\mathcal{C}(G') \cap \Phi(GA(n,\mathbb{C})) \cap \mathcal{K}^*_{\eta,r}(\mathbb{C})) \subset \psi^{-1}\left(\mathcal{K}^*_{\eta,r}(\mathbb{C}) \cap \Phi(GA(n,\mathbb{C}))\right) = U \cap (\{1\} \times \mathbb{C}^n).$$

Conversely, let $x \in U \cap (\{1\} \times \mathbb{C}^n)$, so $x_{k,1} \neq 0$, $k = 2, \ldots, r$. Then $\psi(x) \in \psi(\mathbb{C}^n) = \mathcal{C}(G')$ (Corollary 3.4). By Lemma 3.12, $\psi(x) \in \mathcal{K}^*_{\eta,r}(\mathbb{C}) \cap \Phi(GA(n,\mathbb{C}))$. It follows that $\psi(x) \in \mathcal{C}(G') \cap \Phi(GA(n,\mathbb{C})) \cap \mathcal{K}^*_{\eta,r}(\mathbb{C})$.

(ii) Recall that $\mathcal{C}(\mathcal{G}) = \Phi^{-1}(\mathcal{C}(G) \cap \Phi(GA(n,\mathbb{C})))$, by Lemma 2.1, $h(\Phi(GA(n,\mathbb{C}))) = \Phi(GA(n,\mathbb{C}))$ and by (4), $\delta = \Phi^{-1} \circ h \circ \psi \circ P^{-1} \circ \pi$. Then we have

$$\begin{aligned}
V &= \pi^{-1} \circ P(U \cap (\{1\} \times \mathbb{C}^n)) \\
&= \pi^{-1} \circ P[\psi^{-1}(\mathcal{C}(G') \cap \Phi(GA(n,\mathbb{C})) \cap \mathcal{K}^*_{\eta,r}(\mathbb{C}))] && \text{(by } (i)\text{)} \\
&= \pi^{-1} \circ P \circ \psi^{-1} \circ h^{-1}[\mathcal{C}(G) \cap \Phi(GA(n,\mathbb{C})) \cap h(\mathcal{K}^*_{\eta,r}(\mathbb{C}))] && \text{(by Lemma 3.8.}(iii)\text{)} \\
&= \left(\pi^{-1} \circ P \circ \psi^{-1} \circ h^{-1} \circ \Phi\right)\left(\Phi^{-1}\left[\mathcal{C}(G) \cap \Phi(GA(n,\mathbb{C}))\right] \cap \Phi^{-1} \circ h(\mathcal{K}^*_{\eta,r}(\mathbb{C}))\right) \\
&= \delta^{-1}(\mathcal{C}(\mathcal{G}) \cap \Phi^{-1} \circ h(\mathcal{K}^*_{\eta,r}(\mathbb{C})))
\end{aligned}$$

(iii) By Lemma 3.3.(i), $exp(\mathcal{C}(\mathrm{g}')) = \mathcal{C}(G') \cap \mathcal{K}^*_{\eta,r}(\mathbb{C})$ and by Lemma 2.7.(i), $exp(\mathcal{F}_{n+1}) = \Phi(GA(n,\mathbb{C}))$. So $exp\left(\mathcal{C}(\mathrm{g}') \cap \mathcal{F}_{n+1}\right) \subset \mathcal{C}(G') \cap \mathcal{K}^*_{\eta,r}(\mathbb{C}) \cap \Phi(GA(n,\mathbb{C}))$. Conversely, let $M \in \mathcal{C}(G') \cap \mathcal{K}^*_{\eta,r}(\mathbb{C}) \cap \Phi(GA(n,\mathbb{C}))$. By Lemma 2.7.(iii). there exists $N \in \mathcal{F}_{n+1} \cap \mathcal{K}_{\eta,r}(\mathbb{C})$ such that $e^N = M$. As $M \in \mathcal{C}(G') = \mathcal{C}(\mathrm{g}')$ (Lemma 3.1), so $MA = AM$ for every $A \in \mathrm{g}'$ thus $Me^A = e^A M$ then $e^N e^A = e^A e^N$. Since $A, N \in \mathcal{K}_{\eta,r}(\mathbb{C})$, then by Lemma 3.13, $NA = AN$, so $N \in \mathcal{C}(\mathrm{g}')$. It follows that $M = e^N \in exp(\mathcal{C}(\mathrm{g}') \cap \mathcal{F}_{n+1})$. □

Corollary 3.15. *If $\overline{\mathfrak{g}_{w_0}} = \mathbb{C}^n$ then the map $\beta : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ defined by $\beta = \delta^{-1} \circ \Phi^{-1} \circ h \circ \left(exp_{/\mathcal{K}_{\eta,r}(\mathbb{C})}\right) \circ h^{-1} \circ \Psi \circ \widetilde{\delta}$ satisfies:*
(i) $\beta(\mathfrak{g}_{w_0}) = \mathcal{G}(w_0)$.
(ii) $\beta(\mathbb{C}^n) = V$.

Proof. If $\overline{\mathfrak{g}_{w_0}} = \mathbb{C}^n$ then by Corollary 3.10, $\delta : \mathbb{C}^n \longrightarrow \mathcal{C}(\mathcal{G})$ and $\widetilde{\delta} : \mathbb{C}^n \longrightarrow \mathcal{C}(\mathfrak{g})$ are homeomorphisms, so $\widetilde{\delta}(\mathbb{C}^n) = \mathcal{C}(\mathfrak{g})$.

(i) By proposition 2.4, we have $exp(\mathrm{g}') = G'$. By Proposition 3.9, we have $\widetilde{\delta}(\mathfrak{g}_{w_0}) = \mathfrak{g}$ and $\delta(\mathcal{G}(w_0)) = \mathcal{G}$. As δ is invertible, then $\delta^{-1}(\mathcal{G}) = \mathcal{G}(w_0)$. Therefore :

$$\begin{aligned}
\beta(\mathfrak{g}_{w_0}) &= \delta^{-1} \circ \Phi^{-1} \circ h \circ \left(exp_{/\mathcal{K}_{\eta,r}(\mathbb{C})}\right) \circ h^{-1} \circ \Psi \circ \widetilde{\delta}(\mathfrak{g}_{w_0}) \\
&= \delta^{-1} \circ \Phi^{-1} \circ h \circ \left(exp_{/\mathcal{K}_{\eta,r}(\mathbb{C})}\right) \circ h^{-1} \circ \Psi(\mathfrak{g}) \\
&= \delta^{-1} \circ \Phi^{-1} \circ h \circ \left(exp_{/\mathcal{K}_{\eta,r}(\mathbb{C})}\right) \circ h^{-1}(\mathrm{g}) \\
&= \delta^{-1} \circ \Phi^{-1} \circ h \circ \left(exp_{/\mathcal{K}_{\eta,r}(\mathbb{C})}\right) (\mathrm{g}') \\
&= \delta^{-1} \circ \Phi^{-1} \circ h(G') \\
&= \delta^{-1} \circ \Phi^{-1}(G) \\
&= \delta^{-1}(\mathcal{G}) \\
&= \mathcal{G}(w_0).
\end{aligned}$$

(ii) By Lemma 2.1, $h(\mathcal{F}_{n+1}) = \mathcal{F}_{n+1}$ and $h(\Phi(GA(n,\mathbb{C}))) = \Phi(GA(n,\mathbb{C}))$ and by Lemma 3.14, we have:

$$\begin{aligned}
\beta(\mathbb{C}^n) &= \delta^{-1} \circ \Phi^{-1} \circ h \circ \left(exp_{/\mathcal{K}_{\eta,r}(\mathbb{C})}\right) \circ h^{-1} \circ \Psi \circ \widetilde{\delta}(\mathbb{C}^n) \\
&= \delta^{-1} \circ \Phi^{-1} \circ h \circ \left(exp_{/\mathcal{K}_{\eta,r}(\mathbb{C})}\right) \circ h^{-1} \circ \Psi(\mathcal{C}(\mathfrak{g})) \\
&= \delta^{-1} \circ \Phi^{-1} \circ h \circ \left(exp_{/\mathcal{K}_{\eta,r}(\mathbb{C})}\right) \circ h^{-1}(\mathcal{C}(\mathrm{g}) \cap \mathcal{F}_{n+1}) \\
&= \delta^{-1} \circ \Phi^{-1} \circ h \circ \left(exp_{/\mathcal{K}_{\eta,r}(\mathbb{C})}\right) (\mathcal{C}(\mathrm{g}') \cap h^{-1}(\mathcal{F}_{n+1})) \\
&= \delta^{-1} \circ \Phi^{-1} \circ h \circ \left(exp_{/\mathcal{K}_{\eta,r}(\mathbb{C})}\right) (\mathcal{C}(\mathrm{g}') \cap \mathcal{F}_{n+1}) \\
&= \delta^{-1} \circ \Phi^{-1} \circ h(\mathcal{C}(G') \cap \mathcal{K}^*_{\eta,r}(\mathbb{C}) \cap \Phi(GA(n,\mathbb{C}))) \\
&= \delta^{-1} \circ \Phi^{-1}(\mathcal{C}(G) \cap \Phi(GA(n,\mathbb{C})) \cap h(\mathcal{K}^*_{\eta,r}(\mathbb{C})))) \\
&= \delta^{-1} \left(\mathcal{C}(\mathcal{G}) \cap \Phi^{-1} \circ h(\mathcal{K}^*_{\eta,r}(\mathbb{C}))\right) \\
&= V.
\end{aligned}$$

□

Proposition 3.16. *V is $\mathcal{G}$-invariant dense open set of $\mathbb{C}^n$ and every orbit of $\mathcal{G}$, contained in V is minimal in V.*

Proof. Let $x \in V$ and $f = (A, a) \in \mathcal{G}$. Then $\pi(x) = (1, x) \in P(U)$ and $\Phi(f) \in G$. By Proposition 3.7, U is G'-invariant so $P(U)$ is G-invariant. Then $\Phi(f)(1, x) = (1, f(x)) \in P(U)$, so $f(x) = \pi^{-1}(1, f(x)) \in \pi^{-1} \circ P(U) = V$. It follows that V is $\mathcal{G}$-invariant.
By proposition 3.7, we have all orbits of $G' = P^{-1}GP$ and contained in U are minimal in U. Let $x \in V$ and $y \in \overline{\mathcal{G}(x)} \cap V$, so $P^{-1} \circ \pi(x) = P^{-1}(1, x) \in U$ and also $P^{-1}(1, y) \in U$. We have

$$\{1\} \times \overline{\mathcal{G}(x)} = \overline{G(1, x)} = P\left(\overline{G'(P^{-1}(1, x))}\right). \tag{7}$$

As $P^{-1}(1,x),\ P^{-1}(1,y) \in U$, then

$$\overline{G'(P^{-1}(1,x))} \cap U = \overline{G'(P^{-1}(1,y))} \cap U.$$

Therefore

$$\begin{aligned}\left(\{1\} \times \overline{\mathcal{G}(x)}\right) \cap P(U) &= \overline{G(1,x)} \cap P(U) \\ &= P\left(\overline{G'\left(P^{-1}(1,x)\right)} \cap U\right) \\ &= P\left(\overline{G'\left(P^{-1}(1,y)\right)} \cap U\right) \\ &= \overline{G(1,y)} \cap P(U) \\ &= \left(\{1\} \times \overline{\mathcal{G}(y)}\right) \cap P(U) \qquad \text{(by (7))}\end{aligned}$$

Thus

$$\left(\{1\} \times \overline{\mathcal{G}(x)}\right) \cap P\left(U \cap (\{1\} \times \mathbb{C}^n)\right) = \left(\{1\} \times \overline{\mathcal{G}(y)}\right) \cap P\left(U \cap (\{1\} \times \mathbb{C}^n)\right)$$

Then

$$\overline{\mathcal{G}(x)} \cap \left[\pi^{-1} \circ P\left(U \cap (\{1\} \times \mathbb{C}^n)\right)\right] = \overline{\mathcal{G}(y)} \cap \left[\pi^{-1} \circ P\left(U \cap (\{1\} \times \mathbb{C}^n)\right)\right].$$

So

$$\overline{\mathcal{G}(x)} \cap V = \overline{\mathcal{G}(y)} \cap V.$$

The proof is complete. □

Lemma 3.17. *$\mathbb{C}^n \backslash V$ is union of at most n, $\mathcal{G}$-invariant affine spaces with dimension less to $n-1$.*

Proof. We have $\mathbb{C}^{n+1} \backslash U = \bigcup_{k=1}^{r} H_k$, where

$$H_k = \{u = (u_1, \ldots, u_r) \in \mathbb{C}^{n+1} :\ u_k \in \{0\} \times \mathbb{C}^{n_k - 1}, u_i \in \mathbb{C}^* \times \mathbb{C}^{n_i - 1},\ 1 \leq i \leq r,\ i \neq k\}$$

is a vector space G'-invariant with dimension n, $k = 1, \ldots, r$. Since $r \leq n+1$, $\mathbb{C}^{n+1} \backslash U$ is a union of at most $n+1$, G'-invariant vector spaces with dimension n. Then $\mathbb{C}^{n+1} \backslash P(U)$ is a union of at most $n+1$, G-invariant vector spaces with dimension n. As $H_1 \cap (\{1\} \times \mathbb{C}^n) = \emptyset$, so

$$\begin{aligned}(\{1\} \times \mathbb{C}^n) \backslash (U \cap (\{1\} \times \mathbb{C}^n)) &= \left(\mathbb{C}^{n+1} \backslash U\right) \cap (\{1\} \times \mathbb{C}^n) \\ &= \bigcup_{k=2}^{r} (H_k \cap (\{1\} \times \mathbb{C}^n)) \qquad (8).\end{aligned}$$

Since $P(\{1\}\times\mathbb{C}^n)=\{1\}\times\mathbb{C}^n$, then by (8), we have

$$\begin{aligned}\mathbb{C}^n\backslash V &= \pi^{-1}(\{1\}\times\mathbb{C}^n)\backslash\pi^{-1}\circ P\left(U\cap(\{1\}\times\mathbb{C}^n)\right)\\ &= \pi^{-1}\circ P((\{1\}\times\mathbb{C}^n)\backslash(U\cap(\{1\}\times\mathbb{C}^n)))\\ &= \pi^{-1}\circ P\left(\bigcup_{k=2}^{r}(H_k\cap(\{1\}\times\mathbb{C}^n))\right)\\ &= \bigcup_{k=2}^{r}E_k,\end{aligned}$$

where $E_k=\pi^{-1}\circ P(H_k\cap(\{1\}\times\mathbb{C}^n))$ is $\mathcal{G}$-invariant affine space for every $2\leq k\leq r$. As $H_k\neq\{1\}\times\mathbb{C}^n$, then $\dim(H_k\cap(\{1\}\times\mathbb{C}^n))\leq n-1$, so $\dim(E_k)\leq n-1$. It follows that $\mathbb{C}^n\backslash V$ is union of at most n spaces E_k, since $r\leq n+1$. □

Corollary 3.18. *If $\mathcal{G}$ has a dense orbit in $\mathbb{C}^n$ then all orbits of V are dense in $\mathbb{C}^n$.*

Proof. Let $u\in\mathbb{C}^n$ such that $\overline{\mathcal{G}(u)}=\mathbb{C}^n$. By Lemma 3.17, $\mathbb{C}^n\backslash V$ is union of at most n, $\mathcal{G}$-invariant affine spaces with dimension less to $n-1$, so $u\in V$. Let $v\in V$. Since $V=\overline{\mathcal{G}(u)}\cap V$ then by Proposition 3.16, $\overline{\mathcal{G}(v)}\cap V=\overline{\mathcal{G}(u)}\cap V=V$. As $\overline{V}=\mathbb{C}^n$, one has $\overline{\mathcal{G}(v)}=\mathbb{C}^n$. □

4. **Proof of Theorem 1.1 and corollary 1.2**

4.1. **Proof of Theorem 1.1.** Let $\mathcal{G}$ be an abelian affine subgroup of $GA(n,\ \mathbb{C})$ and let $P\in\Phi(GA(n,\mathbb{C}))$ defined in Proposition 2.3. Recall that $G=\Phi(\mathcal{G})$ and $\mathrm{g}=exp^{-1}(G)\cap\left(P\mathcal{K}_{\eta,r}(\mathbb{C})P^{-1}\right)\cap\mathcal{F}_{n+1}$.

ii) $\Longrightarrow$ **i)** is clear.

i) $\Longrightarrow$ **ii)** This follows from Corollary 3.18, since $u_0\in U$ and so $w_0=\pi^{-1}\circ P(u_0)\in V$.

iii) $\Rightarrow$ **ii)** : Suppose that $\overline{\mathfrak{g}_{w_0}}=\mathbb{C}^n$. Then by Corollary 3.15 there exists a continuous map $\beta:\mathbb{C}^n\longrightarrow\mathbb{C}^n$ such that $\beta(\mathfrak{g}_{w_0})=\mathcal{G}(w_0)$ and $\beta(\mathbb{C}^n)=V$. By continuity, one has:

$$V=\beta(\mathbb{C}^n)=\beta\left(\overline{\mathfrak{g}_{w_0}}\right)\subset\overline{\beta(\mathfrak{g}_{w_0})}=\overline{\mathcal{G}(w_0)}$$

Since V is a dense open set in $\mathbb{C}^n$ then $\overline{\mathcal{G}(w_0)}=\mathbb{C}^n$.

ii) $\Rightarrow$ **iii)** : Suppose that $\overline{\mathcal{G}(w_0)}=\mathbb{C}^n$.

By Corollary 3.11, there exists an open map $\alpha:\mathbb{C}^n\longrightarrow\mathbb{C}^n$ such that $\mathfrak{g}_{w_0}=\alpha^{-1}(\mathcal{G}(w_0))$. Then:

$$\mathbb{C}^n=\alpha^{-1}(\mathbb{C}^n)=\alpha^{-1}(\overline{G(w_0)})\subset\overline{\alpha^{-1}(G(w_0))}=\overline{\mathfrak{g}_{w_0}}$$

□

4.2. **Proof of Corollary 1.2.** Let $\mathcal{G}$ be an abelian subgroup of $GA(n,\mathbb{C})$ having a dense orbit. By Theorem 1.1, $\overline{\mathcal{G}(w_0)} = \mathbb{C}^n$. Then by Proposition 3.2 and corollary 3.10, there exists an invertible affine map $\gamma : \mathbb{C}^n \longrightarrow \mathcal{C}(\mathcal{G})$ such that $\gamma(\mathcal{G}(w_0)) = \mathcal{G}$. Therefore:

$$\gamma(\mathbb{C}^n) = \gamma(\overline{\mathcal{G}(w_0)}) = \overline{\gamma(\mathcal{G}(w_0))} = \overline{\mathcal{G}}.$$

It follows that $\overline{\mathcal{G}}$ is an affine space of dimension n.

5. **Finitely generated subgroups**

5.1. **Proof of Theorem 1.3.**

Denote by $v_0 = Pu_0 = (1, w_0)$. So $w_0 = p_2(v_0)$.

Lemma 5.1. *We have $p_2(\Psi(f')v_0) = f'(w_0)$, for every $f' \in \mathfrak{g}$. In particular, $\mathfrak{g}_{w_0} = p_2(\mathrm{g}_{v_0})$.*

Proof. Let $f' = (B, b) \in \mathfrak{g}$, then

$$\Psi(f')v_0 = \begin{bmatrix} 0 & 0 \\ b & B \end{bmatrix} \begin{bmatrix} 1 \\ w_0 \end{bmatrix} = \begin{bmatrix} 0 \\ b + Bv_0 \end{bmatrix}.$$

So $p_2(\Psi(f')v_0) = b + Bv_0 = f'(w_0)$. □

Proposition 5.2. ([1], *Proposition 8.1*) *Let G be a subgroup of $GL(n+1,\mathbb{C})$ generated by $A_1, \ldots, A_p$. Let $B_1, \ldots, B_p \in \mathrm{g}$ such that $A_k = e^{B_k}$, $k = 1, \ldots, p$. Then:*

$$\mathrm{g}_{v_0} = \sum_{k=1}^{p} \mathbb{Z} B_k v_0 + \sum_{k=1}^{r} 2i\pi\mathbb{Z} P e^{(k)}.$$

Proposition 5.3. (*Under notations of Proposition 2.3*) *Let $\mathcal{G}$ be an abelian subgroup of $GA(n,\mathbb{C})$ generated by $f_1, \ldots, f_p$ and let $f'_1, \ldots, f'_p \in \mathfrak{g}$ such that $\Phi(f_k) = e^{\Psi(f'_k)}$, $k = 1, .., p$. Then:*

$$\mathfrak{g}_{w_0} = \sum_{k=1}^{p} \mathbb{Z} f'_k(w_0) + \sum_{k=1}^{r} 2i\pi\mathbb{Z}(p_2 \circ P e^{(k)}).$$

Proof. Let $G = \Phi(\mathcal{G})$. Then G is generated by $\Phi(f_1), \ldots, \Phi(f_p)$ and g is the additive group generated by $\Psi(f'_1), \ldots, \Psi(f'_p)$. By proposition 5.2 we have

$$\mathrm{g}_{v_0} = \sum_{k=1}^{p} \mathbb{Z} \Psi(f'_k) v_0 + \sum_{k=1}^{r} 2i\pi\mathbb{Z} P e^{(k)}.$$

By Lemma 5.1, we have $\mathfrak{g}_{w_0} = p_2\left(\mathrm{g}_{v_0}\right)$ and $p_2(\Psi(f_k')v_0) = f_k'(w_0)$. then

$$\mathfrak{g}_{w_0} = p_2\left(\sum_{k=1}^{p}\mathbb{Z}\Psi(f_k')v_0 + \sum_{k=1}^{r}2i\pi\mathbb{Z}Pe^{(k)}\right).$$
$$= \sum_{k=1}^{p}\mathbb{Z}f_k'(w_0) + \sum_{k=1}^{r}2i\pi\mathbb{Z}(p_2\circ Pe^{(k)}).$$

□

Recall the following Proposition which was proven in [2]:

Proposition 5.4. (*cf.* [2], *page* 35). *Let* $H = \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_p$ *with* $u_k = (u_{k,1}, \ldots, u_{k,n}) \in \mathbb{C}^n$ *and* $u_{k,i} = Re(u_{k,i}) + iIm(u_{k,i})$, $k = 1, \ldots, p$, $i = 1, \ldots, n$. *Then* H *is dense in* $\mathbb{C}^n$ *if and only if for every* $(s_1, \ldots, s_p) \in \mathbb{Z}^p\backslash\{0\}$ *:*

$$rank\begin{bmatrix} Re(u_{1,1}) & \ldots & \ldots & Re(u_{p,1}) \\ \vdots & \vdots & \vdots & \vdots \\ Re(u_{1,n}) & \ldots & \ldots & Re(u_{p,n}) \\ Im(u_{1,1}) & \ldots & \ldots & Im(u_{p,1}) \\ \vdots & \vdots & \vdots & \vdots \\ Im(u_{1,n}) & \ldots & \ldots & Im(u_{p,n}) \\ s_1 & \ldots & \ldots & s_p \end{bmatrix} = 2n+1.$$

Proof of Theorem 1.3: This follows directly from Theorem 1.1, Propositions 5.3 and 5.4.

5.2. **Proof of Corollaries 1.4 and 1.5.**
Proof of Corollary 1.4: We show first that if $H = \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_m$, $u_k \in \mathbb{C}^n$ with $m \leq 2n$, then H can not be dense: Write $u_k \in \mathbb{C}^n$, $u_k = Re(u_k) + iIm(u_k)$ and $v_k = [Re(u_k); Im(u_k); s_k]^T \in \mathbb{R}^{2n+1}$, $1 \leq k \leq m$. Since $m \leq 2n$, it follows that $rank(v_1, \ldots, v_m) \leq 2n$, and so H is not dense in $\mathbb{C}^n$ by Proposition 5.4.
Now, by applying Theorem 1.3 and the fact that $m = p + r \leq 2n$, the Corollary 1.4 follows. □

Proof of Corollary 1.5: Since $p \leq n$ and $r \leq n$ then $p + r \leq 2n$. Corollary 1.5 follows from Corollary 1.4. □

5.3. **The case: n = 1.**

Let $\mathcal{G}$ be a subgroup of $GA(1, \mathbb{C})$, generated by $f_1 = (a_1, \lambda_1), \ldots, f_p = (a_p, \lambda_p)$, where $a_k = \rho_k e^{i\theta_k}$, $\lambda_k = r_k e^{i\alpha_k}$, $\rho_k, r_k > 0$, $\theta_k, \alpha_k \in \mathbb{R}$, $1 \leq k \leq p$. Therefore $G = \Phi(\mathcal{G})$ is an abelian subgroup of $GL(2, \mathbb{C})$. Let $f_1', \ldots, f_p' \in \mathfrak{g}$ such that $e^{\Psi(f_k')} = \Phi\left(f_k\right)$, $1 \leq k \leq p$. Since $\mathcal{G}$ is abelian then there are two cases:

◇ *Case 1:* $\mathcal{G}$ *is a group of translations.*

In this case $\mathcal{G}(x) = T_x\left(\sum_{k=1}^{p}\lambda_k\mathbb{Z}\right)$. Then $\mathcal{G}$ has a dense orbit if and only if $\mathcal{G}(0) = \sum_{k=1}^{p}\lambda_k\mathbb{Z}$ has a dense orbit, which is equivalent, by Proposition 5.4, to:

For every $(s_1,\dots,s_p,t)\in\mathbb{Z}^{p+1}\backslash\{0\}$:

$$rank\begin{bmatrix} log(r_1) & \dots & log(r_p) & 0\\ \alpha_1 & \dots & \alpha_p & 2\pi\\ s_1 & \dots & s_p & t\end{bmatrix} = 3.$$

$\diamond$ *Case 2 : $\mathcal{G}$ contains no translation:*
In this case $u_0=(1,1)$, $a_k\neq 1$ for every $k=1,\dots,p$ and $f'_k=(B_k,b_k)$ is defined by:

$$B_k=\begin{bmatrix}0 & 0\\ b_k & \mu_k\end{bmatrix},\quad where\ b_k=log(r_k)+i\theta_k\ \ and\ \ \mu_k=\frac{\lambda_k(log(r_k)+i\theta_k)}{(a_k-1)}.$$

Since $w_0=1$ then

$$\begin{aligned}\mathfrak{g}_{w_0} &= f'_1(w_0)\mathbb{Z}+\dots+f'_p(w_0)\mathbb{Z}+2i\pi\mathbb{Z}\\ &= (b_1+\mu_1)\mathbb{Z}+\dots+(b_p+\mu_p)\mathbb{Z}+2i\pi\mathbb{Z}\end{aligned}$$

Moreover, the property $\mathcal{D}$ in this cases is as follow:
$f_1,\dots,f_p$ satisfy property $\mathcal{D}$ if and only if for every $(s_1,\dots,s_p,t)\in\mathbb{Z}^{p+1}-\{0\}$:

$$rank\begin{bmatrix} log|b_1+\mu_1| & \dots & log|b_p+\mu_p| & 0\\ arg(b_1+\mu_1) & \dots & arg(b_p+\mu_p) & 2\pi\\ s_1 & \dots & s_p & t\end{bmatrix} = 3.$$

where $arg(z)=\theta\in[0,2\pi[$ for every $z=re^{i\theta}\in\mathbb{C}$, $r>0$. Therefore, from Theorem 1.3, we obtain:

Proposition 5.5. *If $\mathcal{G}$ is a subgroup of $GA(1,\mathbb{C})$ generated by $f_1=(a_1,\lambda_1),\dots,f_p=(a_p,\lambda_p)$, where $a_k=\rho_k e^{i\theta_k}\neq 1$, $\lambda_k=r_k e^{i\alpha_k}$, $\rho_k,r_k>0$, $\theta_k,\alpha_k\in\mathbb{R}$, $1\leq k\leq p$. Then the following are equivalent:*
(i) G is minimal (i.e. every orbit of $\mathcal{G}$ is dense in $\mathbb{C}$).
(ii) G has a dense orbit
(iii) $\sum_{k=1}^{p}(b_k+\mu_k)\mathbb{Z}+2i\pi\mathbb{Z}$ is dense in $\mathbb{C}$, where $b_k=log(r_k)+i\theta_k$ and $\mu_k=(log\rho_k+i\theta_k)\left(1+\frac{\lambda_k}{a_k-1}\right)$.
(iv) for every $(s_1,\dots,s_p,t)\in\mathbb{Z}^{p+1}\backslash\{0\}$:

$$rank\begin{bmatrix} log|b_1+\mu_1| & \dots & log|b_p+\mu_p| & 0\\ arg(b_1+\mu_1) & \dots & arg(b_p+\mu_p) & 2\pi\\ s_1 & \dots & s_p & t\end{bmatrix} = 3.$$

6. **Examples**

Example 6.1. Let $a_k = e^{\pi(s_k+it_k)}$, $s_k,\ t_k \in \mathbb{Q}$, $1 \leq k \leq p$ and let $\mathcal{G}$ be the subgroup of translations of $\mathbb{C}$ generated by $T_{a_1}, \ldots, T_{a_p}$. Then $\mathcal{G}$ has no dense orbit.

Proof. We can write $s_k = \frac{m_k}{q}$ and $t_k = \frac{p_k}{q}$, $q \in \mathbb{N}^*$ and $m_k,\ p_k \in \mathbb{Z}$ for every $1 \leq k \leq p$. One has: $\mathfrak{g}_x = T_x(\mathfrak{g}_0)$, where

$$\mathfrak{g}_0 = \frac{\pi}{q}\left(\sum_{k=1}^{p}(m_k + ip_k)\mathbb{Z} \ + \ 2iq\mathbb{Z}\right) \subset \frac{\pi}{q}(\mathbb{Z} + i\mathbb{Z}).$$

So $\overline{\mathfrak{g}_0} \neq \mathbb{C}$. By Theorem 1.1, $\overline{\mathcal{G}(0)} \neq \mathbb{C}$. It follows by Proposition 5.5, that $\overline{\mathcal{G}(x)} \neq \mathbb{C}$. □

Example 6.2. Let $\mathcal{G}$ the group generated by $f_1 = (A_1, a_1)$, $f_2 = (A_2, a_2)$ and $f_3 = (A_3, a_3)$, where:

$$A_1 = \mathrm{diag}(e^i, 1), \quad a_1 = (i(1-e^i), 0), \qquad A_2 = \mathrm{diag}(1, e^i), \quad a_2 = \left(0,\ 2i\left(e^{-i}-1\right)\right).$$

$$A_3 = \mathrm{diag}\left(e^{\frac{-\sqrt{3}}{2\pi}+i\left(\frac{\sqrt{5}}{2}-\frac{\sqrt{3}}{2\pi}\right)},\ e^{\frac{-\sqrt{2}}{\pi}+i\left(\frac{\sqrt{2}}{2\pi}-\frac{\sqrt{7}}{2}\right)}\right) \qquad \text{and} \quad a_3 = (0,0).$$

Then every orbit in $V = \mathbb{C}^* \times \mathbb{C}^*$ is dense in $\mathbb{C}^2$.

Proof. Denote by $G = \Phi(\mathcal{G})$. Then G generated by

$$A_1' = \Phi(f_1) = \begin{bmatrix} 1 & 0 & 0 \\ i(1-e^i) & e^i & 0 \\ 0 & 0 & 1 \end{bmatrix}, A_2' = \Phi(f_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2i(e^{-i}-1) & 0 & e^i \end{bmatrix},$$

and

$$A_3' = \Phi(f_3) = \mathrm{diag}\left(1,\ e^{\frac{-\sqrt{3}}{2\pi}+i\left(\frac{\sqrt{5}}{2}-\frac{\sqrt{3}}{2\pi}\right)},\ e^{\frac{-\sqrt{2}}{\pi}+i\left(\frac{\sqrt{2}}{2\pi}-\frac{\sqrt{7}}{2}\right)}\right).$$

Let $f_1' = (B_1, b_1)$, $f_2' = (B_2, b_2)$ and $f_2' = (B_2, b_2)$ such that $e^{\Psi(f_k')} = A_k'$, $k = 1, 2, 3$. We have

$$B_1 = \mathrm{diag}(i, 0), \quad B_2 = \mathrm{diag}(0, i),$$

$$B_3 = \mathrm{diag}\left(\frac{-\sqrt{3}}{2\pi} + i\left(\frac{\sqrt{5}}{2} - \frac{\sqrt{3}}{2\pi}\right),\quad \frac{-\sqrt{2}}{\pi} + i\left(\frac{\sqrt{2}}{2\pi} - \frac{\sqrt{7}}{2}\right)\right),$$

$$b_1 = (1,0), \quad b_2 = (0,-2), \qquad \text{and} \qquad b_3 = (0,0).$$

Here, we have:
- $P = I_3$, $U = (\mathbb{C}^*)^3$, $u_0 = (1,1,1)$.
- $V = \mathbb{C}^* \times \mathbb{C}^*$, $w_0 = (1,1)$.

The additive group $\mathrm{g} = exp^{-1}(G) \cap \left(\mathcal{K}_{(1,1,1),3}(\mathbb{C})\right) \cap \mathcal{F}_3$ is generated by N_1, N_2 and N_3, where

$$N_k = \Psi(f_k') = \begin{bmatrix} 0 & 0 \\ b_k & B_k \end{bmatrix}, \qquad 1 \leq k \leq 3$$

In the other hand, for every $(s_1, s_2, s_3, t_1, t_2) \in \mathbb{Z}^5 \backslash \{0\}$, one has the determinant:

$$\Delta = \begin{vmatrix} Re(B_1 w_0 + b_1) & Re(B_2 w_0 + b_2) & Re(B_3 w_0 + b_3) & 0 & 0 \\ Im(B_1 w_0 + b_1) & Im(B_2 w_0 + b_2) & Im(B_3 w_0 + b_3) & 2\pi e_1 & 2\pi e_2 \\ s_1 & s_2 & s_3 & t_1 & t_2 \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{\sqrt{3}}{2\pi} & 1 & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{\pi} & 0 & -2 & 0 & 0 \\ \frac{\sqrt{5}}{2} - \frac{\sqrt{3}}{2\pi} & 1 & 0 & 2\pi & 0 \\ \frac{\sqrt{2}}{2\pi} - \frac{\sqrt{7}}{2} & 0 & 1 & 0 & 2\pi \\ s_1 & s_2 & s_3 & t_1 & t_2 \end{vmatrix}$$

$$= -2\pi \left((4s_1)\pi - (2s_3)\sqrt{2} + (2s_2)\sqrt{3} + t_1\sqrt{5} + t_2\sqrt{7} \right).$$

Since π, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ and $\sqrt{7}$ are rationally independent then $\Delta \neq 0$ for every $(s_1, s_2, s_3, t_1, t_2) \in \mathbb{Z}^5 \backslash \{0\}$. It follows that:

$$rank \begin{bmatrix} -\frac{\sqrt{3}}{2\pi} & 1 & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{\pi} & 0 & -2 & 0 & 0 \\ \frac{\sqrt{5}}{2} - \frac{\sqrt{3}}{2\pi} & 1 & 0 & 2\pi & 0 \\ \frac{\sqrt{2}}{2\pi} - \frac{\sqrt{7}}{2} & 0 & 1 & 0 & 2\pi \\ s_1 & s_2 & s_3 & t_1 & t_2 \end{bmatrix} = 5$$

and by Theorem 1.3, $\mathcal{G}$ has a dense orbit and every orbit of V is dense in $\mathbb{C}^2$. □

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