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New estimates for the div-curl-grad operators and elliptic problems with L^1 -data in the half-space

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Abstract

In this Note, we study some properties of the div-curl-grad operators and elliptic problems in the half-space. We consider data in weighted Sobolev spaces and in L^1 .

Keywords: div-curl-grad operators, elliptic, half-space, weighted Sobolev spaces

1. Introduction

The purpose of this Note is to give some estimates for div-curl-grad operators and elliptic problems with L^1 -data in the half-space. We know that if $f \in L^n(\mathbb{R}_+^n)$, there exists $\mathbf{u} \in \mathring{\mathbf{W}}_0^{1,n}(\mathbb{R}_+^n)$ such that $\operatorname{div} \mathbf{u} = f$ holds, but does $\mathbf{u} \in \mathring{\mathbf{W}}_0^{1,n}(\mathbb{R}_+^n) \cap \mathbf{L}^\infty(\mathbb{R}_+^n)$ hold? This is indeed true. In section 2, we give new estimates for L^1 -vector fields, which improve estimates for the solutions of elliptic systems in \mathbb{R}_+^n with data in L^1 .

In this Note, we use bold type characters to denote vector distributions or spaces of vector distributions with n components and $C > 0$ usually denotes a generic constant (the value of which may change from line to line). For any $q \in \mathbb{N}$, \mathcal{P}_q (respectively, \mathcal{P}_q^Δ) stands for the space of polynomials (respectively harmonic polynomials) of degree $\leq q$. If q is strictly negative integer, we set by convention $\mathcal{P}_q = \{0\}$. Let Ω be an open subset in the n -dimensional real Euclidean space. A typical point in \mathbb{R}^n is denoted by $\mathbf{x} = (x_1, \dots, x_n)$ and its norm is given by $r = |\mathbf{x}| = (x_1^2 + \dots + x_n^2)^{\frac{1}{2}}$. We define the weight function $\rho(\mathbf{x}) = 1 + r$. For each $p \in \mathbb{R}$ and $1 < p < \infty$, the conjugate exponent p' is given by the relation $\frac{1}{p} + \frac{1}{p'} = 1$. We now define the weighted Sobolev space $W_0^{1,p}(\Omega) = \{u \in \mathcal{D}'(\Omega), \frac{u}{w_1} \in L^p(\Omega), \nabla u \in \mathbf{L}^p(\Omega)\}$, where $w_1 = 1 + r$ if $p \neq n$

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and $w_1 = (1+r) \ln(2+r)$ if $p = n$. This space is a reflexive Banach space when endowed with the norm:

$$\|u\|_{W_0^{1,p}(\Omega)} = \left(\left\| \frac{u}{w_1} \right\|_{L^p(\Omega)}^p + \|\nabla u\|_{\mathbf{L}^p(\Omega)}^p \right)^{1/p}.$$

We also introduce the space $W_0^{2,p}(\Omega) = \{u \in \mathcal{D}'(\Omega), \frac{u}{w_2} \in L^p(\Omega), \frac{\nabla u}{w_1} \in \mathbf{L}^p(\Omega), D^2 u \in \mathbf{L}^p(\Omega)\}$, where $w_2 = (1+r)^2$ if $p \notin \{\frac{n}{n-1}, n\}$ and $w_2 = (1+r)^2 \ln(2+r)$ in the remaining case. This space is a reflexive Banach space endowed with its natural norm given by

$$\|u\|_{W_0^{2,p}(\Omega)} = \left(\left\| \frac{u}{w_2} \right\|_{L^p(\Omega)}^p + \left\| \frac{\nabla u}{w_1} \right\|_{\mathbf{L}^p(\Omega)}^p + \|D^2 u\|_{\mathbf{L}^p(\Omega)}^p \right)^{1/p}.$$

We note that the logarithmic weight only appears if $p = n$ or $p = \frac{n}{n-1}$ and all the local properties of $W_0^{1,p}(\Omega)$ (respectively, $W_0^{2,p}(\Omega)$) coincide with those of the corresponding classical Sobolev space $W^{1,p}(\Omega)$ (respectively, $W^{2,p}(\Omega)$). For $m = 1$ or $m = 2$, we set $\overset{\circ}{W}_0^{m,p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{W_0^{m,p}(\Omega)}$ and we denote the dual space of $\overset{\circ}{W}_0^{m,p}(\Omega)$ by $W_0^{-m,p'}(\Omega)$, which is the space of distributions. When $\Omega = \mathbb{R}^n$, we have $W_0^{m,p}(\mathbb{R}^n) = \overset{\circ}{W}_0^{m,p}(\mathbb{R}^n)$ (see [2] for more details).

2. The div-grad-curl operators and elliptic systems

First of all, we set $B_a^+ = \{\mathbf{x} \in \mathbb{R}_+^n; |\mathbf{x}| < a\}$ with $a > 0$. In this section, we consider the case $n \geq 2$. We introduce the following theorem.

Theorem 2.1. *Let $f \in L^n(\mathbb{R}_+^n)$. Then there exists $\mathbf{u} \in \overset{\circ}{\mathbf{W}}_0^{1,n}(\mathbb{R}_+^n) \cap \mathbf{L}^\infty(\mathbb{R}_+^n)$ such that $\operatorname{div} \mathbf{u} = f$. Moreover, we have the following estimate*

$$\|\mathbf{u}\|_{\mathbf{L}^\infty(\mathbb{R}_+^n)} + \|\mathbf{u}\|_{\mathbf{W}_0^{1,n}(\mathbb{R}_+^n)} \leq C \|f\|_{L^n(\mathbb{R}_+^n)}. \quad (1)$$

Proof. Let $(f_k)_{k \in \mathbb{N}} \subset \mathcal{D}(\mathbb{R}_+^n)$ converges towards $f \in L^n(\mathbb{R}_+^n)$ and $B_{r_k}^+ \subset \mathbb{R}_+^n$ such that $\operatorname{supp} f_k \subset B_{r_k}^+$. We set $g(\mathbf{x}) = r_k f(r_k \mathbf{x})$. Thanks to Theorem 3 [4], there exists $\mathbf{v} \in \mathbf{W}_0^{1,n}(B_1^+) \cap \mathbf{L}^\infty(B_1^+)$ satisfying $\operatorname{div} \mathbf{v} = g$ in B_1^+ and the following estimate

$$\|\mathbf{v}\|_{\mathbf{L}^\infty(B_1^+)} + \|\nabla \mathbf{v}\|_{\mathbf{L}^n(B_1^+)} \leq C \|g\|_{L^n(B_1^+)}.$$

We now set $\mathbf{u}_k(\mathbf{x}) = \mathbf{v}(\frac{\mathbf{x}}{r_k})$ with $\mathbf{x} \in B_{r_k}^+$. It is easy to show that $\mathbf{u}_k \in \mathbf{W}_0^{1,n}(B_{r_k}^+) \cap \mathbf{L}^\infty(B_{r_k}^+)$ satisfies $\operatorname{div} \mathbf{u}_k = f_k$ and we have the following estimate

$$\|\mathbf{u}_k\|_{\mathbf{L}^\infty(B_{r_k}^+)} + \|\nabla \mathbf{u}_k\|_{\mathbf{L}^n(B_{r_k}^+)} \leq C \|f_k\|_{L^n(B_{r_k}^+)} \leq C \|f_k\|_{L^n(\mathbb{R}_+^n)}$$

where C depends only on n . By extending \mathbf{u}_k in \mathbb{R}_+^n by zero outside $B_{r_k}^+$ and denoting $\widetilde{\mathbf{u}}_k$ its extended function, we have $\|\widetilde{\mathbf{u}}_k\|_{\mathbf{L}^\infty(\mathbb{R}_+^n)} = \|\mathbf{u}_k\|_{\mathbf{L}^\infty(B_{r_k}^+)}$ and

$\|\widetilde{\mathbf{u}}_k\|_{\mathbf{W}_0^{1,n}(\mathbb{R}_+^n)} = \|\mathbf{u}_k\|_{\mathbf{W}_0^{1,n}(B_{r_k}^+)}$. Then $(\widetilde{\mathbf{u}}_k)_k$ is bounded in $\mathring{\mathbf{W}}_0^{1,n}(\mathbb{R}_+^n) \cap \mathbf{L}^\infty(\mathbb{R}_+^n)$ and we can deduce that there exists a subsequence, again denoted by $(\widetilde{\mathbf{u}}_k)_k$ such that $\widetilde{\mathbf{u}}_k \rightharpoonup \mathbf{u}$ in $\mathring{\mathbf{W}}_0^{1,n}(\mathbb{R}_+^n)$ and $\widetilde{\mathbf{u}}_k \xrightarrow{*} \mathbf{u}$ in $\mathbf{L}^\infty(\mathbb{R}_+^n)$. Hence, we have $\operatorname{div} \mathbf{u} = \mathbf{f}$ and the estimate (1). \square

The conclusion of Theorem 2.1 is equivalent to the estimate

$$\forall u \in L^{n/(n-1)}(\mathbb{R}_+^n), \quad \|u\|_{L^{n/(n-1)}(\mathbb{R}_+^n)} \leq C_n \|\nabla u\|_{\mathbf{L}^1(\mathbb{R}_+^n) + \mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}_+^n)}. \quad (2)$$

Indeed, let us consider the following unbounded operator

$$A = -\nabla : L^{n/(n-1)}(\mathbb{R}_+^n) \rightarrow \mathbf{L}^1(\mathbb{R}_+^n) + \mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}_+^n).$$

Then the domain $D(A)$ is dense in $L^{n/(n-1)}(\mathbb{R}_+^n)$ and A is closed. Because the adjoint operator

$$A^* = \operatorname{div} : \mathbf{W}_0^{1,n}(\mathbb{R}_+^n) \cap \mathbf{L}^\infty(\mathbb{R}_+^n) \rightarrow L^n(\mathbb{R}_+^n)$$

is surjective, we deduce the estimate (2).

It follows from (2) and that range A is closed in $\mathbf{L}^1(\mathbb{R}_+^n) + \mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}_+^n)$. Then $(\operatorname{Ker} A^*)^\perp = \operatorname{Im} A$ and as consequence we obtain the following version of De Rham's Theorem: for any \mathbf{f} belonging to the space $\mathbf{L}^1(\mathbb{R}_+^n) + \mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}_+^n)$ and satisfying the following compatibility condition

$$\forall \mathbf{v} \in \mathbf{W}_0^{1,n}(\mathbb{R}_+^n) \cap \mathbf{L}^\infty(\mathbb{R}_+^n), \quad \text{with } \operatorname{div} \mathbf{v} = 0, \quad \langle \mathbf{f}, \mathbf{v} \rangle = 0, \quad (3)$$

then there exists a unique $\pi \in L^{n/(n-1)}(\mathbb{R}_+^n)$ such that $\mathbf{f} = \nabla \pi$.

The following theorem, given without proof, is proved by J. Bourgain - H. Brézis [5] (see also [6]) in the case where \mathbb{R}_+^n is replaced by a bounded smooth domain Ω .

Theorem 2.2. *i) Let $\varphi \in \mathbf{W}_0^{1,n}(\mathbb{R}_+^n)$. Then there exists $\psi = (\psi_1, \dots, \psi_n) \in \mathbf{W}_0^{1,n}(\mathbb{R}_+^n) \cap \mathbf{L}^\infty(\mathbb{R}_+^n)$ and $\eta \in W_0^{2,n}(\mathbb{R}_+^n)$ such that $\psi_n = 0$ on $\Gamma = \mathbb{R}^{n-1} \times \{0\}$ satisfying*

$$\varphi = \psi + \nabla \eta$$

and the following estimate holds

$$\|\psi\|_{\mathbf{L}^\infty(\mathbb{R}_+^n)} + \|\psi\|_{\mathbf{W}_0^{1,n}(\mathbb{R}_+^n)} + \|\eta\|_{W_0^{2,n}(\mathbb{R}_+^n)} \leq C \|\varphi\|_{\mathbf{W}_0^{1,n}(\mathbb{R}_+^n)}. \quad (4)$$

ii) If $\varphi \in \mathring{\mathbf{W}}_0^{1,n}(\mathbb{R}_+^n)$, then the same result holds with $\psi \in \mathring{\mathbf{W}}_0^{1,n}(\mathbb{R}_+^n) \cap \mathbf{L}^\infty(\mathbb{R}_+^n)$ and $\eta \in \mathring{W}_0^{2,n}(\mathbb{R}_+^n)$ and the corresponding estimate.

Define the space $\mathbf{X}(\mathbb{R}_+^n) = \{\mathbf{v} \in \mathbf{L}^1(\mathbb{R}_+^n); \operatorname{div} \mathbf{v} \in W_0^{-2,n/(n-1)}(\mathbb{R}_+^n)\}$. Let $\mathbf{f} \in \mathbf{X}(\mathbb{R}_+^n)$. Thanks to Theorem 2.2, the linear operator $\mathbf{F} : \varphi \mapsto \int_{\mathbb{R}_+^n} \mathbf{f} \cdot \varphi \, dx$ satisfies

$$\forall \varphi \in \mathcal{D}(\mathbb{R}_+^n), \quad |\langle \mathbf{F}, \varphi \rangle| \leq C \|\mathbf{f}\|_{\mathbf{X}} \|\varphi\|_{\mathbf{W}_0^{1,n}}.$$

As $\mathcal{D}(\mathbb{R}_+^n)$ is dense in $\mathring{\mathbf{W}}_0^{1,n}(\mathbb{R}_+^n)$ and by applying Hahn-Banach Theorem, we can uniquely extend \mathbf{F} by an element $\tilde{\mathbf{F}} \in \mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}_+^n)$ satisfying $\|\tilde{\mathbf{F}}\|_{\mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}_+^n)} \leq C \|\mathbf{f}\|_{\mathbf{X}(\mathbb{R}_+^n)}$. Besides, the following linear operator $\mathbf{f} \longrightarrow \tilde{\mathbf{F}}$ from $\mathbf{X}(\mathbb{R}_+^n)$ into $\mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}_+^n)$ is continuous and injective. Therefore, $\mathbf{X}(\mathbb{R}_+^n)$ can be identified to a subspace of $\mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}_+^n)$ with continuous and dense embedding. Thanks to [3], we can then deduce that for any $\mathbf{f} \in \mathbf{X}(\mathbb{R}_+^n)$, the following problem

$$-\Delta \mathbf{u} = \mathbf{f} \text{ in } \mathbb{R}_+^n \text{ and } \mathbf{u} = \mathbf{0} \text{ on } \Gamma, \quad (5)$$

has a unique solution $\mathbf{u} \in \mathbf{W}_0^{1,n/(n-1)}(\mathbb{R}_+^n)$ satisfying the following estimate $\|\mathbf{u}\|_{\mathbf{W}_0^{1,n/(n-1)}(\mathbb{R}_+^n)} \leq C \|\mathbf{f}\|_{\mathbf{X}(\mathbb{R}_+^n)}$.

The first statement of the following corollary is the equivalent of Theorem 2.1 for the curl operator and the second statement gives a Helmholtz decomposition.

Corollary 2.3. *i) Let $\mathbf{f} \in \mathbf{L}^3(\mathbb{R}_+^3)$ such that $\operatorname{div} \mathbf{f} = 0$ in \mathbb{R}_+^3 and $f_3 = 0$ on Γ . Then there exists $\boldsymbol{\psi} \in \mathring{\mathbf{W}}_0^{1,3}(\mathbb{R}_+^3) \cap \mathbf{L}^\infty(\mathbb{R}_+^3)$ such that $\mathbf{f} = \mathbf{curl} \boldsymbol{\psi}$ and we have the following estimate*

$$\|\boldsymbol{\psi}\|_{\mathbf{W}_0^{1,3}(\mathbb{R}_+^3)} + \|\boldsymbol{\psi}\|_{\mathbf{L}^\infty(\mathbb{R}_+^3)} \leq C \|\mathbf{f}\|_{\mathbf{L}^3(\mathbb{R}_+^3)}. \quad (6)$$

ii) Let $\mathbf{f} \in \mathbf{L}^3(\mathbb{R}_+^3)$. Then there exist $\boldsymbol{\varphi} \in \mathring{\mathbf{W}}_0^{1,3}(\mathbb{R}_+^3) \cap \mathbf{L}^\infty(\mathbb{R}_+^3)$ and $\pi \in W_0^{1,3}(\mathbb{R}_+^3)$ unique up to an additive constant and satisfying

$$\mathbf{f} = \mathbf{curl} \boldsymbol{\varphi} + \nabla \pi. \quad (7)$$

Moreover, we have the following estimate

$$\|\boldsymbol{\varphi}\|_{\mathbf{W}_0^{1,3}(\mathbb{R}_+^3)} + \|\boldsymbol{\varphi}\|_{\mathbf{L}^\infty(\mathbb{R}_+^3)} + \|\nabla \pi\|_{\mathbf{L}^3(\mathbb{R}_+^3)} \leq C \|\mathbf{f}\|_{\mathbf{L}^3(\mathbb{R}_+^3)}. \quad (8)$$

In the following proposition, we improve the estimate given by Corollary 1.4 of Van Schaftingen [8].

Proposition 2.4. *Let $\mathbf{f} \in \mathbf{L}^1(\mathbb{R}_+^3)$ such that $\operatorname{div} \mathbf{f} = 0$. Then we have the following estimate*

$$\forall \boldsymbol{\varphi} \in \mathring{\mathbf{W}}_0^{1,3}(\mathbb{R}_+^3), \quad |\langle \mathbf{f}, \boldsymbol{\varphi} \rangle| \leq C \|\mathbf{f}\|_{\mathbf{L}^1(\mathbb{R}_+^3)} \|\mathbf{curl} \boldsymbol{\varphi}\|_{\mathbf{L}^3(\mathbb{R}_+^3)}. \quad (9)$$

Proof. First remark that from the hypothesis, we deduce $\mathbf{f} \in \mathbf{W}_0^{-1,3/2}(\mathbb{R}_+^3)$. Let $\boldsymbol{\varphi} \in \mathring{\mathbf{W}}_0^{1,3}(\mathbb{R}_+^3)$. Then we have $\mathbf{curl} \boldsymbol{\varphi} \in \mathbf{L}^3(\mathbb{R}_+^3)$. Thanks to Corollary 2.3, there exists $\boldsymbol{\psi} \in \mathring{\mathbf{W}}_0^{1,3}(\mathbb{R}_+^3) \cap \mathbf{L}^\infty(\mathbb{R}_+^3)$ such that $\mathbf{curl} \boldsymbol{\psi} = \mathbf{curl} \boldsymbol{\varphi}$ with the following estimate

$$\|\boldsymbol{\psi}\|_{\mathbf{W}_0^{1,3}(\mathbb{R}_+^3)} + \|\boldsymbol{\psi}\|_{\mathbf{L}^\infty(\mathbb{R}_+^3)} \leq C \|\mathbf{curl} \boldsymbol{\varphi}\|_{\mathbf{L}^3(\mathbb{R}_+^3)}. \quad (10)$$

Besides, there exists $\eta \in W_0^{2,3}(\mathbb{R}_+^3)$ such that $\varphi = \psi + \nabla\eta$ in \mathbb{R}_+^3 . Then we have

$$\begin{aligned} \langle \mathbf{f}, \varphi \rangle_{\mathbf{W}_0^{-1,3/2}(\mathbb{R}_+^3) \times \mathring{\mathbf{W}}_0^{1,3}(\mathbb{R}_+^3)} &= \int_{\mathbb{R}_+^3} \mathbf{f} \cdot \psi + \langle \mathbf{f}, \nabla\eta \rangle_{\mathbf{W}_0^{-1,3/2}(\mathbb{R}_+^3) \times \mathring{\mathbf{W}}_0^{1,3}(\mathbb{R}_+^3)} \\ &= \int_{\mathbb{R}_+^3} \mathbf{f} \cdot \psi. \end{aligned}$$

Therefore, the estimate (9) is deduced from the estimate (10). \square

Here is a variant of Theorem 2.2.

Theorem 2.5. *Let $\varphi \in \mathring{\mathbf{W}}_0^{1,3}(\mathbb{R}_+^3)$. Then there exist $\psi \in \mathring{\mathbf{W}}_0^{1,3}(\mathbb{R}_+^3) \cap \mathbf{L}^\infty(\mathbb{R}_+^3)$ and $\eta \in \mathring{\mathbf{W}}_0^{2,3}(\mathbb{R}_+^3)$ such that*

$$\varphi = \psi + \mathbf{curl} \eta \quad \text{with} \quad \operatorname{div} \Delta^2 \eta = 0 \quad \text{in} \quad \mathbb{R}_+^3.$$

However, we have the following estimate

$$\|\psi\|_{\mathbf{W}_0^{1,3}(\mathbb{R}_+^3)} + \|\psi\|_{\mathbf{L}^\infty(\mathbb{R}_+^3)} + \|\eta\|_{\mathbf{W}_0^{2,3}(\mathbb{R}_+^3)} \leq C \|\varphi\|_{\mathbf{W}_0^{1,3}(\mathbb{R}_+^3)}.$$

Proof. From the hypothesis, we have $\operatorname{div} \varphi \in L^3(\mathbb{R}_+^3)$. Thanks to Theorem 2.1, there exists ψ in $\mathring{\mathbf{W}}_0^{1,3}(\mathbb{R}_+^3) \cap \mathbf{L}^\infty(\mathbb{R}_+^3)$ such that $\operatorname{div} \psi = \operatorname{div} \varphi$ and we have the following estimate

$$\|\psi\|_{\mathbf{L}^\infty(\mathbb{R}_+^3)} + \|\psi\|_{\mathbf{W}_0^{1,3}(\mathbb{R}_+^3)} \leq C \|\operatorname{div} \varphi\|_{L^3(\mathbb{R}_+^3)}.$$

We set $\mathbf{f} = \varphi - \psi$, then $\mathbf{f} \in \mathring{\mathbf{W}}_0^{1,3}(\mathbb{R}_+^3)$ and $\operatorname{div} \mathbf{f} = 0$. We extend \mathbf{f} to \mathbb{R}^3 as follows

$$\tilde{\mathbf{f}}(\mathbf{x}', x_3) = \begin{cases} \mathbf{f}(\mathbf{x}', x_3) & \text{if } x_3 > 0, \\ (f_1(\mathbf{x}', -x_3), f_2(\mathbf{x}', -x_3), -f_3(\mathbf{x}', -x_3)) & \text{if } x_3 < 0. \end{cases}$$

Then we have $\tilde{\mathbf{f}} \in \mathbf{W}_0^{1,3}(\mathbb{R}^3)$ and $\operatorname{div} \tilde{\mathbf{f}} = 0$ in \mathbb{R}^3 . Thanks to [2], there exists $\mathbf{y} \in \mathbf{W}_0^{2,3}(\mathbb{R}^3)$ such that $-\Delta \mathbf{y} = \mathbf{curl} \tilde{\mathbf{f}}$ in \mathbb{R}^3 . As $\operatorname{div} \mathbf{y} \in W_0^{1,3}(\mathbb{R}^3)$ and is harmonic, then $\operatorname{div} \mathbf{y} = a$ where a is a constant. Thus, we have $\mathbf{curl}(\mathbf{curl} \mathbf{y} - \tilde{\mathbf{f}}) = \mathbf{0}$. Therefore, we deduce $-\Delta(\mathbf{curl} \mathbf{y}) = \mathbf{curl} \mathbf{curl} \tilde{\mathbf{f}} = -\Delta \tilde{\mathbf{f}}$, it means that $\tilde{\mathbf{f}} - \mathbf{curl} \mathbf{y} \in \mathbf{W}_0^{1,3}(\mathbb{R}^3)$ and again, $\tilde{\mathbf{f}} - \mathbf{curl} \mathbf{y} = \mathbf{b}$ where \mathbf{b} is a constant vector in \mathbb{R}^3 . Thanks to Lemma 3.1 [7], there exists a polynomial $\mathbf{s} \in \mathcal{P}_1$ such that $\mathbf{curl} \mathbf{s} = \mathbf{b}$ and $\operatorname{div} \mathbf{s} = -a$. The function $\mathbf{z} := \mathbf{y} + \mathbf{s}$ belongs to $\mathbf{W}_0^{2,3}(\mathbb{R}^3)$ and $\tilde{\mathbf{f}} = \mathbf{curl} \mathbf{z}$ in \mathbb{R}^3 with $\operatorname{div} \mathbf{z} = 0$ in \mathbb{R}^3 . Let \mathbf{w} be the vector field defined on \mathbb{R}_-^3 by $\mathbf{w}(\mathbf{x}', x_3) = (-z_1(\mathbf{x}', x_3), -z_2(\mathbf{x}', x_3), z_3(\mathbf{x}', x_3))$ with $x_3 < 0$. It is easy to show $\mathbf{curl} \mathbf{w} = \mathbf{curl} \mathbf{z}$ in \mathbb{R}_-^3 . Then we have

$$\mathbf{w} = \mathbf{z} + \nabla\theta \quad \text{in} \quad \mathbb{R}_-^3 \tag{11}$$

with $\theta \in \mathcal{D}'(\mathbb{R}_-^3)$. As $\nabla\theta$ belongs to $\mathbf{W}_0^{1,3}(\mathbb{R}_-^3)$, we can show $\theta \in W_0^{2,3}(\mathbb{R}_-^3)$. Let $\theta_0 \in W_0^{2,3}(\mathbb{R}_+^3)$ (cf. [3]) such that $\Delta\theta_0 = 0$ in \mathbb{R}_+^3 and $\theta_0 = \theta$ on Γ . We

set $\zeta = z - \frac{1}{2}\nabla\theta_0$. Then we have $\mathbf{curl}\zeta = \mathbf{f}$ and $\operatorname{div}\zeta = 0$ in \mathbb{R}_+^3 . Remark now that from (11), we deduce $2z' = \nabla'\theta = \nabla'\theta_0$ on Γ , *i.e.*, $\zeta' = \mathbf{0}$ on Γ . Let $\chi \in W_0^{3,3}(\mathbb{R}_+^3)$ be a solution of the system: $\Delta^2\chi = 0$ in \mathbb{R}_+^3 , $\chi = 0$ and $\frac{\partial\chi}{\partial x_3} = \zeta_3$ on Γ . We set $\mathbf{h} = \zeta - \nabla\chi$. Then $\mathbf{h} \in \mathbf{W}_0^{2,3}(\mathbb{R}_+^3)$ such that $\mathbf{f} = \mathbf{curl}\mathbf{h}$ and $\operatorname{div}\Delta\mathbf{h} = 0$ in \mathbb{R}_+^3 , $\mathbf{h} = \mathbf{0}$ on Γ . We know that there exists $\mu \in W_0^{3,3}(\mathbb{R}_+^3)$ satisfying $\Delta^3\mu = 0$ in \mathbb{R}_+^3 , $\frac{\partial\mu}{\partial x_3} = 0$ and $\frac{\partial^2\mu}{\partial x_3^2} = \frac{\partial h_3}{\partial x_3}$ on Γ . We set $\boldsymbol{\eta} = \mathbf{h} - \nabla\mu$. Then $\boldsymbol{\eta} \in \mathbf{W}_0^{2,3}(\mathbb{R}_+^3)$, $\mathbf{curl}\boldsymbol{\eta} = \mathbf{f}$ in \mathbb{R}_+^3 and $\boldsymbol{\eta} = \mathbf{0}$ on Γ . It is easy to show that $\frac{\partial\boldsymbol{\eta}}{\partial x_3} = 0$ on Γ and then $\boldsymbol{\eta} \in \overset{\circ}{\mathbf{W}}_0^{2,3}(\mathbb{R}_+^3)$. The proof is finished. \square

We introduce the following proposition.

Proposition 2.6. *Let $\mathbf{f} \in \mathbf{L}^1(\mathbb{R}_+^3)$ such that $\operatorname{div}\mathbf{f} = 0$ in \mathbb{R}_+^3 . Then there exists a unique $\boldsymbol{\varphi} \in \mathbf{L}^{3/2}(\mathbb{R}_+^3)$ such that $\mathbf{curl}\boldsymbol{\varphi} = \mathbf{f}$, $\operatorname{div}\boldsymbol{\varphi} = 0$ in \mathbb{R}_+^3 and $\varphi_3 = 0$ on Γ satisfying the following estimate*

$$\|\boldsymbol{\varphi}\|_{\mathbf{L}^{3/2}(\mathbb{R}_+^3)} \leq C\|\mathbf{f}\|_{\mathbf{L}^1(\mathbb{R}_+^3)}.$$

Proof. It is easy to show the uniqueness of $\boldsymbol{\varphi}$. We now try to prove the existence of $\boldsymbol{\varphi}$. We know that $\mathbf{f} \in \mathbf{W}_0^{-1,3/2}(\mathbb{R}_+^3)$. Then, there exists a unique $\mathbf{z} \in \overset{\circ}{\mathbf{W}}_0^{1,3/2}(\mathbb{R}_+^3)$ satisfying $-\Delta\mathbf{z} = \mathbf{f}$, with $\operatorname{div}\mathbf{z} = 0$ in \mathbb{R}_+^3 . The function $\boldsymbol{\varphi} = \mathbf{curl}\mathbf{z}$ is the required function. \square

A variant of this result can be obtained. If $\mathbf{f} \in \mathbf{L}^1(\mathbb{R}_+^3)$ such that $\operatorname{div}\mathbf{f} = 0$ in \mathbb{R}_+^3 and $f_3 = 0$ on Γ , then we can prove the existence of a unique $\boldsymbol{\varphi} \in \mathbf{L}^{3/2}(\mathbb{R}_+^3)$ such that $\mathbf{f} = \mathbf{curl}\boldsymbol{\varphi}$ with $\operatorname{div}\boldsymbol{\varphi} = 0$ in \mathbb{R}_+^3 and $\boldsymbol{\varphi}' = \mathbf{0}$ on Γ .

More generally, we can prove the following result.

Theorem 2.7. *i) Let $\mathbf{f} \in \mathbf{L}^1(\mathbb{R}_+^3) + \mathbf{W}_0^{-1,3/2}(\mathbb{R}_+^3)$ such that $\operatorname{div}\mathbf{f} = 0$. Then there exists a unique $\boldsymbol{\varphi} \in \mathbf{L}^{3/2}(\mathbb{R}_+^3)$ such that $\mathbf{curl}\boldsymbol{\varphi} = \mathbf{f}$ with $\operatorname{div}\boldsymbol{\varphi} = 0$ in \mathbb{R}_+^3 and $\varphi_3 = 0$ on Γ satisfying the following estimate*

$$\|\boldsymbol{\varphi}\|_{\mathbf{L}^{3/2}(\mathbb{R}_+^3)} \leq C\|\mathbf{f}\|_{\mathbf{L}^1(\mathbb{R}_+^3) + \mathbf{W}_0^{-1,3/2}(\mathbb{R}_+^3)}.$$

ii) Let $\mathbf{f} \in \mathbf{X}(\mathbb{R}_+^3)$. Then there exists a unique $\boldsymbol{\varphi} \in \mathbf{L}^{3/2}(\mathbb{R}_+^3)$ such that $\operatorname{div}\boldsymbol{\varphi} = 0$ with $\varphi_3 = 0$ on Γ and a unique $\pi \in L^{3/2}(\mathbb{R}_+^3)$ satisfying $\mathbf{f} = \mathbf{curl}\boldsymbol{\varphi} + \nabla\pi$ with the corresponding estimate.

Combining the above results with [1] and [3], we can solve the follow elliptical systems.

Theorem 2.8. *i) Let $\mathbf{g}' \in \mathbf{L}^1(\Gamma)$ and $g_n \in W_0^{-1+\frac{1}{n}, \frac{n}{n-1}}(\Gamma)$ satisfying the compatibility condition $\int_\Gamma \mathbf{g}' = \mathbf{0}$ and $\langle g_n, 1 \rangle = 0$. If $\operatorname{div}'\mathbf{g}' \in W_0^{-2+\frac{1}{n}, \frac{n}{n-1}}(\Gamma)$, then the system*

$$-\Delta\mathbf{u} = \mathbf{0} \text{ in } \mathbb{R}_+^n \quad \text{and} \quad \mathbf{u} = \mathbf{g} \text{ on } \Gamma \quad (12)$$

has a unique very weak solution $\mathbf{u} \in \mathbf{L}^{n/(n-1)}(\mathbb{R}_+^n)$.

ii) Let $\mathbf{f} \in \mathbf{L}^1(\mathbb{R}_+^n)$, $\mathbf{g}' \in \mathbf{L}^1(\Gamma)$ and $g_n \in W_0^{-1+\frac{1}{n}, \frac{n}{n-1}}(\Gamma)$ satisfying the compatibility condition $\int_{\mathbb{R}_+^n} \mathbf{f} + \int_{\Gamma} \mathbf{g}' = \mathbf{0}$ and $\int_{\mathbb{R}_+^n} f_n + \langle g_n, 1 \rangle = 0$. If

$$[\mathbf{f}, \mathbf{g}'] = \sup_{\xi \in W_0^{2,n}(\mathbb{R}_+^n), \xi \neq 0} \frac{|\int_{\mathbb{R}_+^n} \mathbf{f} \cdot \nabla \xi + \int_{\Gamma} \mathbf{g}' \cdot \nabla' \xi|}{\|\xi\|_{W_0^{2,n}(\mathbb{R}_+^n)}} < \infty$$

then the system

$$-\Delta \mathbf{u} = \mathbf{f} \text{ in } \mathbb{R}_+^n \quad \text{and} \quad \frac{\partial \mathbf{u}}{\partial x_n} = \mathbf{g} \text{ on } \Gamma \quad (13)$$

has a unique solution $\mathbf{u} \in \mathbf{W}_0^{1,n/(n-1)}(\mathbb{R}_+^n)$.

iii) Let $\mathbf{f} \in \mathbf{L}^1(\mathbb{R}_+^n)$ such that $\text{div } \mathbf{f} \in [W_0^{2,n}(\mathbb{R}_+^n) \cap \overset{\circ}{W}_{-1}^{1,n}(\mathbb{R}_+^n)]'$ and $\int_{\mathbb{R}_+^n} f_n = 0$.

Then $\mathbf{f} \in \mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}_+^n)$ and the system

$$-\Delta \mathbf{u} = \mathbf{f} \text{ in } \mathbb{R}_+^n; \quad \mathbf{u}' = \mathbf{0} \quad \text{and} \quad \frac{\partial u_n}{\partial x_n} = 0 \quad \text{on } \Gamma \quad (14)$$

has a unique solution $\mathbf{u} \in \mathbf{W}_0^{1,n/(n-1)}(\mathbb{R}_+^n)$.

iv) Let $\mathbf{f} \in \mathbf{L}^1(\mathbb{R}_+^n)$ such that $\int_{\mathbb{R}_+^n} \mathbf{f}' = \mathbf{0}$. If

$$[\mathbf{f}] = \sup_{\xi \in \mathcal{D}(\overline{\mathbb{R}_+^n}), \frac{\partial \xi}{\partial x_n} = 0 \text{ on } \Gamma} \frac{|\int_{\mathbb{R}_+^n} \mathbf{f} \cdot \nabla \xi|}{\|\xi\|_{W_0^{2,n}(\mathbb{R}_+^n)}} < \infty$$

holds, then the system

$$-\Delta \mathbf{u} = \mathbf{f} \text{ in } \mathbb{R}_+^n, \quad u_n = 0 \quad \text{and} \quad \frac{\partial \mathbf{u}'}{\partial x_n} = \mathbf{0} \quad \text{on } \Gamma$$

has a unique solution $\mathbf{u} \in \mathbf{W}_0^{1,n/(n-1)}(\mathbb{R}_+^n)$.

Proof. We will prove only the first point i).

Step 1: The proof will be started by showing $\mathbf{g}' \in \mathbf{W}_0^{-1+\frac{1}{n}, \frac{n}{n-1}}(\Gamma)$. Let $\boldsymbol{\mu} \in \mathcal{D}(\Gamma)$ and $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,n}(\mathbb{R}_+^n)$ such that $\boldsymbol{\varphi} = \boldsymbol{\mu}$ on Γ . Thanks to Theorem 2.2, there exist $\boldsymbol{\psi} \in \mathbf{W}_0^{1,n}(\mathbb{R}_+^n) \cap \mathbf{L}^\infty(\mathbb{R}_+^n)$ and $\eta \in W_0^{2,n}(\mathbb{R}_+^n)$ such that $\psi_n = 0$ on Γ satisfying $\boldsymbol{\varphi} = \boldsymbol{\psi} + \nabla \eta$ and the estimate

$$\|\boldsymbol{\psi}\|_{\mathbf{L}^\infty(\mathbb{R}_+^n)} + \|\boldsymbol{\psi}\|_{\mathbf{W}_0^{1,n}(\mathbb{R}_+^n)} + \|\eta\|_{W_0^{2,n}(\mathbb{R}_+^n)} \leq C \|\boldsymbol{\varphi}\|_{\mathbf{W}_0^{1,n}(\mathbb{R}_+^n)}. \quad (15)$$

Then

$$\begin{aligned} \langle \mathbf{g}, \boldsymbol{\mu} \rangle_{\mathcal{D}'(\Gamma) \times \mathcal{D}(\Gamma)} &= \int_{\Gamma} \mathbf{g}' \cdot \boldsymbol{\psi}' - \langle \text{div}' \mathbf{g}', \eta \rangle_{W_0^{-2+\frac{1}{n}, \frac{n}{n-1}}(\Gamma) \times W_0^{2-\frac{1}{n}, n}(\Gamma)} \\ &\quad + \langle g_n, \varphi_n \rangle_{W_0^{-1+\frac{1}{n}, \frac{n}{n-1}}(\Gamma) \times W_0^{1-\frac{1}{n}, n}(\Gamma)} \end{aligned}$$

and

$$\begin{aligned} | \langle \mathbf{g}, \boldsymbol{\mu} \rangle_{\mathcal{D}'(\Gamma) \times \mathcal{D}(\Gamma)} | &\leq \| \mathbf{g}' \|_{\mathbf{L}^1(\Gamma)} \| \boldsymbol{\psi}' \|_{\mathbf{L}^\infty(\mathbb{R}_+^n)} + \| \operatorname{div}' \mathbf{g}' \|_{\mathbf{W}_0^{-2+\frac{1}{n}, \frac{n}{n-1}}(\Gamma)} \times \\ &\quad \times \| \eta \|_{\mathbf{W}_0^{2,n}(\mathbb{R}_+^n)} + \| g_n \|_{\mathbf{W}_0^{-1+\frac{1}{n}, \frac{n}{n-1}}(\Gamma)} \| \varphi_n \|_{\mathbf{W}_0^{1-\frac{1}{n}, n}(\Gamma)}. \end{aligned}$$

Thanks to the density of $\mathcal{D}(\Gamma)$ in $\mathbf{W}_0^{1-\frac{1}{n}, n}(\Gamma)$ and (15), we can deduce $\mathbf{g}' \in \mathbf{W}_0^{-1+\frac{1}{n}, \frac{n}{n-1}}(\Gamma)$.

Step 2: The system (13) is equivalent to the following one: Find \mathbf{u} belonging $\mathbf{L}^{n/(n-1)}(\mathbb{R}_+^n)$ such that for all $\mathbf{v} \in \mathbf{W}_0^{2,n}(\mathbb{R}_+^n) \cap \overset{\circ}{\mathbf{W}}_{-1}^{1,n}(\mathbb{R}_+^n)$,

$$\int_{\mathbb{R}_+^n} \mathbf{u} \cdot \Delta \mathbf{v} = - \langle \mathbf{g}, \frac{\partial \mathbf{v}}{\partial \mathbf{n}} \rangle_{\mathbf{W}_0^{-1+\frac{1}{n}, \frac{n}{n-1}}(\Gamma) \times \mathbf{W}_0^{1-\frac{1}{n}, n}(\Gamma)}.$$

But for all $\mathbf{F} \in \mathbf{L}^n(\mathbb{R}_+^n)$, there exists $\mathbf{v} \in \mathbf{W}_0^{2,n}(\mathbb{R}_+^n) \cap \overset{\circ}{\mathbf{W}}_{-1}^{1,n}(\mathbb{R}_+^n)$, unique up to an element of $x_n \mathbb{R}^n$, such that $-\Delta \mathbf{v} = \mathbf{F}$ in \mathbb{R}_+^n , $\mathbf{v} = \mathbf{0}$ on Γ and the following estimate holds

$$\| \mathbf{v} \|_{\mathbf{W}_0^{2,n}(\mathbb{R}_+^n)/x_n \mathbb{R}^n} \leq C \| \mathbf{F} \|_{\mathbf{L}^n(\mathbb{R}_+^n)}.$$

Then, we have for all $\mathbf{a} \in \mathbb{R}^n$,

$$\begin{aligned} &| \langle \mathbf{g}, \frac{\partial \mathbf{v}}{\partial x_n} \rangle_{\mathbf{W}_0^{-1+\frac{1}{n}, \frac{n}{n-1}}(\Gamma) \times \mathbf{W}_0^{1-\frac{1}{n}, n}(\Gamma)} | \\ &= | \langle \mathbf{g}, \frac{\partial(\mathbf{v} + \mathbf{a}x_n)}{\partial x_n} \rangle_{\mathbf{W}_0^{-1+\frac{1}{n}, \frac{n}{n-1}}(\Gamma) \times \mathbf{W}_0^{1-\frac{1}{n}, n}(\Gamma)} | \\ &\leq C \| \mathbf{g} \|_{\mathbf{W}_0^{-1+\frac{1}{n}, \frac{n}{n-1}}(\Gamma)} \| \mathbf{v} + \mathbf{a}x_n \|_{\mathbf{W}_0^{2,n}(\mathbb{R}_+^n)}. \end{aligned}$$

Consequently, taking the infimum, we have

$$| \langle \mathbf{g}, \frac{\partial \mathbf{v}}{\partial x_n} \rangle_{\mathbf{W}_0^{-1+\frac{1}{n}, \frac{n}{n-1}}(\Gamma) \times \mathbf{W}_0^{1-\frac{1}{n}, n}(\Gamma)} | \leq C \| \mathbf{g} \|_{\mathbf{W}_0^{-1+\frac{1}{n}, \frac{n}{n-1}}(\Gamma)} \| \mathbf{F} \|_{\mathbf{L}^n(\mathbb{R}_+^n)}.$$

Then, the linear operator

$$T : \mathbf{F} \longrightarrow \langle \mathbf{g}, \frac{\partial \mathbf{v}}{\partial x_n} \rangle_{\mathbf{W}_0^{-1+\frac{1}{n}, \frac{n}{n-1}}(\Gamma) \times \mathbf{W}_0^{1-\frac{1}{n}, n}(\Gamma)}$$

is continuous on $\mathbf{L}^n(\mathbb{R}_+^n)$ and thanks to the Riesz representation theorem, there exists a unique $\mathbf{u} \in \mathbf{L}^{n/(n-1)}(\mathbb{R}_+^n)$ such that $T(\mathbf{F}) = \int_{\mathbb{R}_+^n} \mathbf{u} \cdot \mathbf{F}$, *i.e.*, \mathbf{u} is the solution of (13). \square

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