



HAL
open science

New estimates for the div-curl-grad operators and elliptic problems with L1-data in the half-space

Chérif Amrouche, Huy Hoang Nguyen

► **To cite this version:**

Chérif Amrouche, Huy Hoang Nguyen. New estimates for the div-curl-grad operators and elliptic problems with L1-data in the half-space. *Applied Mathematics Letters*, 2011, 24 (5), pp.697-702. 10.1016/j.aml.2010.12.008 . hal-00549184

HAL Id: hal-00549184

<https://hal.science/hal-00549184>

Submitted on 21 Dec 2010

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

New estimates for the div-curl-grad operators and elliptic problems with L^1 -data in the half-space

Chérif Amrouche ^a, Huy Hoang Nguyen ^b

^a*Laboratoire de Mathématiques Appliquées, Université de Pau et des Pays de l'Adour
IPRA, Avenue de l'Université – 64000 Pau – France*

^b*Departamento de Matematica, IMECC, Universidade Estadual de Campinas
Caixa Postal 6065, Campinas, SP 13083-970, BRAZIL*

Abstract

In this Note, we study some properties of the div-curl-grad operators and elliptic problems in the half-space. We consider data in weighted Sobolev spaces and in L^1 .

Keywords: div-curl-grad operators, elliptic, half-space, weighted Sobolev spaces

1. Introduction

The purpose of this Note is to give some estimates for div-curl-grad operators and elliptic problems with L^1 -data in the half-space. We know that if $f \in L^n(\mathbb{R}_+^n)$, there exists $\mathbf{u} \in \mathring{\mathbf{W}}_0^{1,n}(\mathbb{R}_+^n)$ such that $\operatorname{div} \mathbf{u} = f$ holds, but does $\mathbf{u} \in \mathring{\mathbf{W}}_0^{1,n}(\mathbb{R}_+^n) \cap \mathbf{L}^\infty(\mathbb{R}_+^n)$ hold? This is indeed true. In section 2, we give new estimates for L^1 -vector fields, which improve estimates for the solutions of elliptic systems in \mathbb{R}_+^n with data in L^1 .

In this Note, we use bold type characters to denote vector distributions or spaces of vector distributions with n components and $C > 0$ usually denotes a generic constant (the value of which may change from line to line). For any $q \in \mathbb{N}$, \mathcal{P}_q (respectively, \mathcal{P}_q^Δ) stands for the space of polynomials (respectively harmonic polynomials) of degree $\leq q$. If q is strictly negative integer, we set by convention $\mathcal{P}_q = \{0\}$. Let Ω be an open subset in the n -dimensional real Euclidean space. A typical point in \mathbb{R}^n is denoted by $\mathbf{x} = (x_1, \dots, x_n)$ and its norm is given by $r = |\mathbf{x}| = (x_1^2 + \dots + x_n^2)^{\frac{1}{2}}$. We define the weight function $\rho(\mathbf{x}) = 1 + r$. For each $p \in \mathbb{R}$ and $1 < p < \infty$, the conjugate exponent p' is given by the relation $\frac{1}{p} + \frac{1}{p'} = 1$. We now define the weighted Sobolev space $W_0^{1,p}(\Omega) = \{u \in \mathcal{D}'(\Omega), \frac{u}{w_1} \in L^p(\Omega), \nabla u \in \mathbf{L}^p(\Omega)\}$, where $w_1 = 1 + r$ if $p \neq n$

Email addresses: cherif.amrouche@univ-pau.fr (Chérif Amrouche ^a),
nguyen@ime.unicamp.br (Huy Hoang Nguyen ^b)

and $w_1 = (1+r) \ln(2+r)$ if $p = n$. This space is a reflexive Banach space when endowed with the norm:

$$\|u\|_{W_0^{1,p}(\Omega)} = \left(\left\| \frac{u}{w_1} \right\|_{L^p(\Omega)}^p + \|\nabla u\|_{\mathbf{L}^p(\Omega)}^p \right)^{1/p}.$$

We also introduce the space $W_0^{2,p}(\Omega) = \{u \in \mathcal{D}'(\Omega), \frac{u}{w_2} \in L^p(\Omega), \frac{\nabla u}{w_1} \in \mathbf{L}^p(\Omega), D^2 u \in \mathbf{L}^p(\Omega)\}$, where $w_2 = (1+r)^2$ if $p \notin \{\frac{n}{n-1}, n\}$ and $w_2 = (1+r)^2 \ln(2+r)$ in the remaining case. This space is a reflexive Banach space endowed with its natural norm given by

$$\|u\|_{W_0^{2,p}(\Omega)} = \left(\left\| \frac{u}{w_2} \right\|_{L^p(\Omega)}^p + \left\| \frac{\nabla u}{w_1} \right\|_{\mathbf{L}^p(\Omega)}^p + \|D^2 u\|_{\mathbf{L}^p(\Omega)}^p \right)^{1/p}.$$

We note that the logarithmic weight only appears if $p = n$ or $p = \frac{n}{n-1}$ and all the local properties of $W_0^{1,p}(\Omega)$ (respectively, $W_0^{2,p}(\Omega)$) coincide with those of the corresponding classical Sobolev space $W^{1,p}(\Omega)$ (respectively, $W^{2,p}(\Omega)$). For $m = 1$ or $m = 2$, we set $\overset{\circ}{W}_0^{m,p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{W_0^{m,p}(\Omega)}$ and we denote the dual space of $\overset{\circ}{W}_0^{m,p}(\Omega)$ by $W_0^{-m,p'}(\Omega)$, which is the space of distributions. When $\Omega = \mathbb{R}^n$, we have $W_0^{m,p}(\mathbb{R}^n) = \overset{\circ}{W}_0^{m,p}(\mathbb{R}^n)$ (see [2] for more details).

2. The div-grad-curl operators and elliptic systems

First of all, we set $B_a^+ = \{\mathbf{x} \in \mathbb{R}_+^n; |\mathbf{x}| < a\}$ with $a > 0$. In this section, we consider the case $n \geq 2$. We introduce the following theorem.

Theorem 2.1. *Let $f \in L^n(\mathbb{R}_+^n)$. Then there exists $\mathbf{u} \in \overset{\circ}{\mathbf{W}}_0^{1,n}(\mathbb{R}_+^n) \cap \mathbf{L}^\infty(\mathbb{R}_+^n)$ such that $\operatorname{div} \mathbf{u} = f$. Moreover, we have the following estimate*

$$\|\mathbf{u}\|_{\mathbf{L}^\infty(\mathbb{R}_+^n)} + \|\mathbf{u}\|_{\mathbf{W}_0^{1,n}(\mathbb{R}_+^n)} \leq C \|f\|_{L^n(\mathbb{R}_+^n)}. \quad (1)$$

Proof. Let $(f_k)_{k \in \mathbb{N}} \subset \mathcal{D}(\mathbb{R}_+^n)$ converges towards $f \in L^n(\mathbb{R}_+^n)$ and $B_{r_k}^+ \subset \mathbb{R}_+^n$ such that $\operatorname{supp} f_k \subset B_{r_k}^+$. We set $g(\mathbf{x}) = r_k f(r_k \mathbf{x})$. Thanks to Theorem 3 [4], there exists $\mathbf{v} \in \mathbf{W}_0^{1,n}(B_1^+) \cap \mathbf{L}^\infty(B_1^+)$ satisfying $\operatorname{div} \mathbf{v} = g$ in B_1^+ and the following estimate

$$\|\mathbf{v}\|_{\mathbf{L}^\infty(B_1^+)} + \|\nabla \mathbf{v}\|_{\mathbf{L}^n(B_1^+)} \leq C \|g\|_{L^n(B_1^+)}.$$

We now set $\mathbf{u}_k(\mathbf{x}) = \mathbf{v}(\frac{\mathbf{x}}{r_k})$ with $\mathbf{x} \in B_{r_k}^+$. It is easy to show that $\mathbf{u}_k \in \mathbf{W}_0^{1,n}(B_{r_k}^+) \cap \mathbf{L}^\infty(B_{r_k}^+)$ satisfies $\operatorname{div} \mathbf{u}_k = f_k$ and we have the following estimate

$$\|\mathbf{u}_k\|_{\mathbf{L}^\infty(B_{r_k}^+)} + \|\nabla \mathbf{u}_k\|_{\mathbf{L}^n(B_{r_k}^+)} \leq C \|f_k\|_{L^n(B_{r_k}^+)} \leq C \|f_k\|_{L^n(\mathbb{R}_+^n)}$$

where C depends only on n . By extending \mathbf{u}_k in \mathbb{R}_+^n by zero outside $B_{r_k}^+$ and denoting $\widetilde{\mathbf{u}}_k$ its extended function, we have $\|\widetilde{\mathbf{u}}_k\|_{\mathbf{L}^\infty(\mathbb{R}_+^n)} = \|\mathbf{u}_k\|_{\mathbf{L}^\infty(B_{r_k}^+)}$ and

$\|\widetilde{\mathbf{u}}_k\|_{\mathbf{W}_0^{1,n}(\mathbb{R}_+^n)} = \|\mathbf{u}_k\|_{\mathbf{W}_0^{1,n}(B_{r_k}^+)}$. Then $(\widetilde{\mathbf{u}}_k)_k$ is bounded in $\mathring{\mathbf{W}}_0^{1,n}(\mathbb{R}_+^n) \cap \mathbf{L}^\infty(\mathbb{R}_+^n)$ and we can deduce that there exists a subsequence, again denoted by $(\widetilde{\mathbf{u}}_k)_k$ such that $\widetilde{\mathbf{u}}_k \rightharpoonup \mathbf{u}$ in $\mathring{\mathbf{W}}_0^{1,n}(\mathbb{R}_+^n)$ and $\widetilde{\mathbf{u}}_k \xrightarrow{*} \mathbf{u}$ in $\mathbf{L}^\infty(\mathbb{R}_+^n)$. Hence, we have $\operatorname{div} \mathbf{u} = f$ and the estimate (1). \square

The conclusion of Theorem 2.1 is equivalent to the estimate

$$\forall u \in L^{n/(n-1)}(\mathbb{R}_+^n), \quad \|u\|_{L^{n/(n-1)}(\mathbb{R}_+^n)} \leq C_n \|\nabla u\|_{\mathbf{L}^1(\mathbb{R}_+^n) + \mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}_+^n)}. \quad (2)$$

Indeed, let us consider the following unbounded operator

$$A = -\nabla : L^{n/(n-1)}(\mathbb{R}_+^n) \rightarrow \mathbf{L}^1(\mathbb{R}_+^n) + \mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}_+^n).$$

Then the domain $D(A)$ is dense in $L^{n/(n-1)}(\mathbb{R}_+^n)$ and A is closed. Because the adjoint operator

$$A^* = \operatorname{div} : \mathbf{W}_0^{1,n}(\mathbb{R}_+^n) \cap \mathbf{L}^\infty(\mathbb{R}_+^n) \rightarrow L^n(\mathbb{R}_+^n)$$

is surjective, we deduce the estimate (2).

It follows from (2) and that range A is closed in $\mathbf{L}^1(\mathbb{R}_+^n) + \mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}_+^n)$. Then $(\operatorname{Ker} A^*)^\perp = \operatorname{Im} A$ and as consequence we obtain the following version of De Rham's Theorem: for any \mathbf{f} belonging to the space $\mathbf{L}^1(\mathbb{R}_+^n) + \mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}_+^n)$ and satisfying the following compatibility condition

$$\forall \mathbf{v} \in \mathbf{W}_0^{1,n}(\mathbb{R}_+^n) \cap \mathbf{L}^\infty(\mathbb{R}_+^n), \quad \text{with } \operatorname{div} \mathbf{v} = 0, \quad \langle \mathbf{f}, \mathbf{v} \rangle = 0, \quad (3)$$

then there exists a unique $\pi \in L^{n/(n-1)}(\mathbb{R}_+^n)$ such that $\mathbf{f} = \nabla \pi$.

The following theorem, given without proof, is proved by J. Bourgain - H. Brézis [5] (see also [6]) in the case where \mathbb{R}_+^n is replaced by a bounded smooth domain Ω .

Theorem 2.2. *i) Let $\varphi \in \mathbf{W}_0^{1,n}(\mathbb{R}_+^n)$. Then there exists $\psi = (\psi_1, \dots, \psi_n) \in \mathbf{W}_0^{1,n}(\mathbb{R}_+^n) \cap \mathbf{L}^\infty(\mathbb{R}_+^n)$ and $\eta \in W_0^{2,n}(\mathbb{R}_+^n)$ such that $\psi_n = 0$ on $\Gamma = \mathbb{R}^{n-1} \times \{0\}$ satisfying*

$$\varphi = \psi + \nabla \eta$$

and the following estimate holds

$$\|\psi\|_{\mathbf{L}^\infty(\mathbb{R}_+^n)} + \|\psi\|_{\mathbf{W}_0^{1,n}(\mathbb{R}_+^n)} + \|\eta\|_{W_0^{2,n}(\mathbb{R}_+^n)} \leq C \|\varphi\|_{\mathbf{W}_0^{1,n}(\mathbb{R}_+^n)}. \quad (4)$$

ii) If $\varphi \in \mathring{\mathbf{W}}_0^{1,n}(\mathbb{R}_+^n)$, then the same result holds with $\psi \in \mathring{\mathbf{W}}_0^{1,n}(\mathbb{R}_+^n) \cap \mathbf{L}^\infty(\mathbb{R}_+^n)$ and $\eta \in \mathring{W}_0^{2,n}(\mathbb{R}_+^n)$ and the corresponding estimate.

Define the space $\mathbf{X}(\mathbb{R}_+^n) = \{\mathbf{v} \in \mathbf{L}^1(\mathbb{R}_+^n); \operatorname{div} \mathbf{v} \in W_0^{-2,n/(n-1)}(\mathbb{R}_+^n)\}$. Let $\mathbf{f} \in \mathbf{X}(\mathbb{R}_+^n)$. Thanks to Theorem 2.2, the linear operator $\mathbf{F} : \varphi \mapsto \int_{\mathbb{R}_+^n} \mathbf{f} \cdot \varphi \, dx$ satisfies

$$\forall \varphi \in \mathcal{D}(\mathbb{R}_+^n), \quad |\langle \mathbf{F}, \varphi \rangle| \leq C \|\mathbf{f}\|_{\mathbf{X}} \|\varphi\|_{\mathbf{W}_0^{1,n}}.$$

As $\mathcal{D}(\mathbb{R}_+^n)$ is dense in $\mathring{\mathbf{W}}_0^{1,n}(\mathbb{R}_+^n)$ and by applying Hahn-Banach Theorem, we can uniquely extend \mathbf{F} by an element $\tilde{\mathbf{F}} \in \mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}_+^n)$ satisfying $\|\tilde{\mathbf{F}}\|_{\mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}_+^n)} \leq C \|\mathbf{f}\|_{\mathbf{X}(\mathbb{R}_+^n)}$. Besides, the following linear operator $\mathbf{f} \longrightarrow \tilde{\mathbf{F}}$ from $\mathbf{X}(\mathbb{R}_+^n)$ into $\mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}_+^n)$ is continuous and injective. Therefore, $\mathbf{X}(\mathbb{R}_+^n)$ can be identified to a subspace of $\mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}_+^n)$ with continuous and dense embedding. Thanks to [3], we can then deduce that for any $\mathbf{f} \in \mathbf{X}(\mathbb{R}_+^n)$, the following problem

$$-\Delta \mathbf{u} = \mathbf{f} \text{ in } \mathbb{R}_+^n \text{ and } \mathbf{u} = \mathbf{0} \text{ on } \Gamma, \quad (5)$$

has a unique solution $\mathbf{u} \in \mathbf{W}_0^{1,n/(n-1)}(\mathbb{R}_+^n)$ satisfying the following estimate $\|\mathbf{u}\|_{\mathbf{W}_0^{1,n/(n-1)}(\mathbb{R}_+^n)} \leq C \|\mathbf{f}\|_{\mathbf{X}(\mathbb{R}_+^n)}$.

The first statement of the following corollary is the equivalent of Theorem 2.1 for the curl operator and the second statement gives a Helmholtz decomposition.

Corollary 2.3. *i) Let $\mathbf{f} \in \mathbf{L}^3(\mathbb{R}_+^3)$ such that $\operatorname{div} \mathbf{f} = 0$ in \mathbb{R}_+^3 and $f_3 = 0$ on Γ . Then there exists $\boldsymbol{\psi} \in \mathring{\mathbf{W}}_0^{1,3}(\mathbb{R}_+^3) \cap \mathbf{L}^\infty(\mathbb{R}_+^3)$ such that $\mathbf{f} = \mathbf{curl} \boldsymbol{\psi}$ and we have the following estimate*

$$\|\boldsymbol{\psi}\|_{\mathbf{W}_0^{1,3}(\mathbb{R}_+^3)} + \|\boldsymbol{\psi}\|_{\mathbf{L}^\infty(\mathbb{R}_+^3)} \leq C \|\mathbf{f}\|_{\mathbf{L}^3(\mathbb{R}_+^3)}. \quad (6)$$

ii) Let $\mathbf{f} \in \mathbf{L}^3(\mathbb{R}_+^3)$. Then there exist $\boldsymbol{\varphi} \in \mathring{\mathbf{W}}_0^{1,3}(\mathbb{R}_+^3) \cap \mathbf{L}^\infty(\mathbb{R}_+^3)$ and $\pi \in W_0^{1,3}(\mathbb{R}_+^3)$ unique up to an additive constant and satisfying

$$\mathbf{f} = \mathbf{curl} \boldsymbol{\varphi} + \nabla \pi. \quad (7)$$

Moreover, we have the following estimate

$$\|\boldsymbol{\varphi}\|_{\mathbf{W}_0^{1,3}(\mathbb{R}_+^3)} + \|\boldsymbol{\varphi}\|_{\mathbf{L}^\infty(\mathbb{R}_+^3)} + \|\nabla \pi\|_{\mathbf{L}^3(\mathbb{R}_+^3)} \leq C \|\mathbf{f}\|_{\mathbf{L}^3(\mathbb{R}_+^3)}. \quad (8)$$

In the following proposition, we improve the estimate given by Corollary 1.4 of Van Schaftingen [8].

Proposition 2.4. *Let $\mathbf{f} \in \mathbf{L}^1(\mathbb{R}_+^3)$ such that $\operatorname{div} \mathbf{f} = 0$. Then we have the following estimate*

$$\forall \boldsymbol{\varphi} \in \mathring{\mathbf{W}}_0^{1,3}(\mathbb{R}_+^3), \quad |\langle \mathbf{f}, \boldsymbol{\varphi} \rangle| \leq C \|\mathbf{f}\|_{\mathbf{L}^1(\mathbb{R}_+^3)} \|\mathbf{curl} \boldsymbol{\varphi}\|_{\mathbf{L}^3(\mathbb{R}_+^3)}. \quad (9)$$

Proof. First remark that from the hypothesis, we deduce $\mathbf{f} \in \mathbf{W}_0^{-1,3/2}(\mathbb{R}_+^3)$. Let $\boldsymbol{\varphi} \in \mathring{\mathbf{W}}_0^{1,3}(\mathbb{R}_+^3)$. Then we have $\mathbf{curl} \boldsymbol{\varphi} \in \mathbf{L}^3(\mathbb{R}_+^3)$. Thanks to Corollary 2.3, there exists $\boldsymbol{\psi} \in \mathring{\mathbf{W}}_0^{1,3}(\mathbb{R}_+^3) \cap \mathbf{L}^\infty(\mathbb{R}_+^3)$ such that $\mathbf{curl} \boldsymbol{\psi} = \mathbf{curl} \boldsymbol{\varphi}$ with the following estimate

$$\|\boldsymbol{\psi}\|_{\mathbf{W}_0^{1,3}(\mathbb{R}_+^3)} + \|\boldsymbol{\psi}\|_{\mathbf{L}^\infty(\mathbb{R}_+^3)} \leq C \|\mathbf{curl} \boldsymbol{\varphi}\|_{\mathbf{L}^3(\mathbb{R}_+^3)}. \quad (10)$$

Besides, there exists $\eta \in W_0^{2,3}(\mathbb{R}_+^3)$ such that $\varphi = \psi + \nabla\eta$ in \mathbb{R}_+^3 . Then we have

$$\begin{aligned} \langle \mathbf{f}, \varphi \rangle_{\mathbf{W}_0^{-1,3/2}(\mathbb{R}_+^3) \times \mathring{\mathbf{W}}_0^{1,3}(\mathbb{R}_+^3)} &= \int_{\mathbb{R}_+^3} \mathbf{f} \cdot \psi + \langle \mathbf{f}, \nabla\eta \rangle_{\mathbf{W}_0^{-1,3/2}(\mathbb{R}_+^3) \times \mathring{\mathbf{W}}_0^{1,3}(\mathbb{R}_+^3)} \\ &= \int_{\mathbb{R}_+^3} \mathbf{f} \cdot \psi. \end{aligned}$$

Therefore, the estimate (9) is deduced from the estimate (10). \square

Here is a variant of Theorem 2.2.

Theorem 2.5. *Let $\varphi \in \mathring{\mathbf{W}}_0^{1,3}(\mathbb{R}_+^3)$. Then there exist $\psi \in \mathring{\mathbf{W}}_0^{1,3}(\mathbb{R}_+^3) \cap \mathbf{L}^\infty(\mathbb{R}_+^3)$ and $\eta \in \mathring{\mathbf{W}}_0^{2,3}(\mathbb{R}_+^3)$ such that*

$$\varphi = \psi + \mathbf{curl} \eta \quad \text{with} \quad \operatorname{div} \Delta^2 \eta = 0 \quad \text{in} \quad \mathbb{R}_+^3.$$

However, we have the following estimate

$$\|\psi\|_{\mathbf{W}_0^{1,3}(\mathbb{R}_+^3)} + \|\psi\|_{\mathbf{L}^\infty(\mathbb{R}_+^3)} + \|\eta\|_{\mathbf{W}_0^{2,3}(\mathbb{R}_+^3)} \leq C \|\varphi\|_{\mathbf{W}_0^{1,3}(\mathbb{R}_+^3)}.$$

Proof. From the hypothesis, we have $\operatorname{div} \varphi \in L^3(\mathbb{R}_+^3)$. Thanks to Theorem 2.1, there exists ψ in $\mathring{\mathbf{W}}_0^{1,3}(\mathbb{R}_+^3) \cap \mathbf{L}^\infty(\mathbb{R}_+^3)$ such that $\operatorname{div} \psi = \operatorname{div} \varphi$ and we have the following estimate

$$\|\psi\|_{\mathbf{L}^\infty(\mathbb{R}_+^3)} + \|\psi\|_{\mathbf{W}_0^{1,3}(\mathbb{R}_+^3)} \leq C \|\operatorname{div} \varphi\|_{L^3(\mathbb{R}_+^3)}.$$

We set $\mathbf{f} = \varphi - \psi$, then $\mathbf{f} \in \mathring{\mathbf{W}}_0^{1,3}(\mathbb{R}_+^3)$ and $\operatorname{div} \mathbf{f} = 0$. We extend \mathbf{f} to \mathbb{R}^3 as follows

$$\tilde{\mathbf{f}}(\mathbf{x}', x_3) = \begin{cases} \mathbf{f}(\mathbf{x}', x_3) & \text{if } x_3 > 0, \\ (f_1(\mathbf{x}', -x_3), f_2(\mathbf{x}', -x_3), -f_3(\mathbf{x}', -x_3)) & \text{if } x_3 < 0. \end{cases}$$

Then we have $\tilde{\mathbf{f}} \in \mathbf{W}_0^{1,3}(\mathbb{R}^3)$ and $\operatorname{div} \tilde{\mathbf{f}} = 0$ in \mathbb{R}^3 . Thanks to [2], there exists $\mathbf{y} \in \mathbf{W}_0^{2,3}(\mathbb{R}^3)$ such that $-\Delta \mathbf{y} = \mathbf{curl} \tilde{\mathbf{f}}$ in \mathbb{R}^3 . As $\operatorname{div} \mathbf{y} \in W_0^{1,3}(\mathbb{R}^3)$ and is harmonic, then $\operatorname{div} \mathbf{y} = a$ where a is a constant. Thus, we have $\mathbf{curl}(\mathbf{curl} \mathbf{y} - \tilde{\mathbf{f}}) = \mathbf{0}$. Therefore, we deduce $-\Delta(\mathbf{curl} \mathbf{y}) = \mathbf{curl} \mathbf{curl} \tilde{\mathbf{f}} = -\Delta \tilde{\mathbf{f}}$, it means that $\tilde{\mathbf{f}} - \mathbf{curl} \mathbf{y} \in \mathbf{W}_0^{1,3}(\mathbb{R}^3)$ and again, $\tilde{\mathbf{f}} - \mathbf{curl} \mathbf{y} = \mathbf{b}$ where \mathbf{b} is a constant vector in \mathbb{R}^3 . Thanks to Lemma 3.1 [7], there exists a polynomial $\mathbf{s} \in \mathcal{P}_1$ such that $\mathbf{curl} \mathbf{s} = \mathbf{b}$ and $\operatorname{div} \mathbf{s} = -a$. The function $\mathbf{z} := \mathbf{y} + \mathbf{s}$ belongs to $\mathbf{W}_0^{2,3}(\mathbb{R}^3)$ and $\tilde{\mathbf{f}} = \mathbf{curl} \mathbf{z}$ in \mathbb{R}^3 with $\operatorname{div} \mathbf{z} = 0$ in \mathbb{R}^3 . Let \mathbf{w} be the vector field defined on \mathbb{R}_-^3 by $\mathbf{w}(\mathbf{x}', x_3) = (-z_1(\mathbf{x}', x_3), -z_2(\mathbf{x}', x_3), z_3(\mathbf{x}', x_3))$ with $x_3 < 0$. It is easy to show $\mathbf{curl} \mathbf{w} = \mathbf{curl} \mathbf{z}$ in \mathbb{R}_-^3 . Then we have

$$\mathbf{w} = \mathbf{z} + \nabla \theta \quad \text{in} \quad \mathbb{R}_-^3 \tag{11}$$

with $\theta \in \mathcal{D}'(\mathbb{R}_-^3)$. As $\nabla \theta$ belongs to $\mathbf{W}_0^{1,3}(\mathbb{R}_-^3)$, we can show $\theta \in W_0^{2,3}(\mathbb{R}_-^3)$. Let $\theta_0 \in W_0^{2,3}(\mathbb{R}_+^3)$ (cf. [3]) such that $\Delta \theta_0 = 0$ in \mathbb{R}_+^3 and $\theta_0 = \theta$ on Γ . We

set $\zeta = z - \frac{1}{2}\nabla\theta_0$. Then we have $\mathbf{curl}\zeta = \mathbf{f}$ and $\operatorname{div}\zeta = 0$ in \mathbb{R}_+^3 . Remark now that from (11), we deduce $2z' = \nabla'\theta = \nabla'\theta_0$ on Γ , *i.e.*, $\zeta' = \mathbf{0}$ on Γ . Let $\chi \in W_0^{3,3}(\mathbb{R}_+^3)$ be a solution of the system: $\Delta^2\chi = 0$ in \mathbb{R}_+^3 , $\chi = 0$ and $\frac{\partial\chi}{\partial x_3} = \zeta_3$ on Γ . We set $\mathbf{h} = \zeta - \nabla\chi$. Then $\mathbf{h} \in \mathbf{W}_0^{2,3}(\mathbb{R}_+^3)$ such that $\mathbf{f} = \mathbf{curl}\mathbf{h}$ and $\operatorname{div}\Delta\mathbf{h} = 0$ in \mathbb{R}_+^3 , $\mathbf{h} = \mathbf{0}$ on Γ . We know that there exists $\mu \in W_0^{3,3}(\mathbb{R}_+^3)$ satisfying $\Delta^3\mu = 0$ in \mathbb{R}_+^3 , $\frac{\partial\mu}{\partial x_3} = 0$ and $\frac{\partial^2\mu}{\partial x_3^2} = \frac{\partial h_3}{\partial x_3}$ on Γ . We set $\boldsymbol{\eta} = \mathbf{h} - \nabla\mu$. Then $\boldsymbol{\eta} \in \mathbf{W}_0^{2,3}(\mathbb{R}_+^3)$, $\mathbf{curl}\boldsymbol{\eta} = \mathbf{f}$ in \mathbb{R}_+^3 and $\boldsymbol{\eta} = \mathbf{0}$ on Γ . It is easy to show that $\frac{\partial\boldsymbol{\eta}}{\partial x_3} = 0$ on Γ and then $\boldsymbol{\eta} \in \overset{\circ}{\mathbf{W}}_0^{2,3}(\mathbb{R}_+^3)$. The proof is finished. \square

We introduce the following proposition.

Proposition 2.6. *Let $\mathbf{f} \in \mathbf{L}^1(\mathbb{R}_+^3)$ such that $\operatorname{div}\mathbf{f} = 0$ in \mathbb{R}_+^3 . Then there exists a unique $\boldsymbol{\varphi} \in \mathbf{L}^{3/2}(\mathbb{R}_+^3)$ such that $\mathbf{curl}\boldsymbol{\varphi} = \mathbf{f}$, $\operatorname{div}\boldsymbol{\varphi} = 0$ in \mathbb{R}_+^3 and $\varphi_3 = 0$ on Γ satisfying the following estimate*

$$\|\boldsymbol{\varphi}\|_{\mathbf{L}^{3/2}(\mathbb{R}_+^3)} \leq C\|\mathbf{f}\|_{\mathbf{L}^1(\mathbb{R}_+^3)}.$$

Proof. It is easy to show the uniqueness of $\boldsymbol{\varphi}$. We now try to prove the existence of $\boldsymbol{\varphi}$. We know that $\mathbf{f} \in \mathbf{W}_0^{-1,3/2}(\mathbb{R}_+^3)$. Then, there exists a unique $\mathbf{z} \in \overset{\circ}{\mathbf{W}}_0^{1,3/2}(\mathbb{R}_+^3)$ satisfying $-\Delta\mathbf{z} = \mathbf{f}$, with $\operatorname{div}\mathbf{z} = 0$ in \mathbb{R}_+^3 . The function $\boldsymbol{\varphi} = \mathbf{curl}\mathbf{z}$ is the required function. \square

A variant of this result can be obtained. If $\mathbf{f} \in \mathbf{L}^1(\mathbb{R}_+^3)$ such that $\operatorname{div}\mathbf{f} = 0$ in \mathbb{R}_+^3 and $f_3 = 0$ on Γ , then we can prove the existence of a unique $\boldsymbol{\varphi} \in \mathbf{L}^{3/2}(\mathbb{R}_+^3)$ such that $\mathbf{f} = \mathbf{curl}\boldsymbol{\varphi}$ with $\operatorname{div}\boldsymbol{\varphi} = 0$ in \mathbb{R}_+^3 and $\boldsymbol{\varphi}' = \mathbf{0}$ on Γ .

More generally, we can prove the following result.

Theorem 2.7. *i) Let $\mathbf{f} \in \mathbf{L}^1(\mathbb{R}_+^3) + \mathbf{W}_0^{-1,3/2}(\mathbb{R}_+^3)$ such that $\operatorname{div}\mathbf{f} = 0$. Then there exists a unique $\boldsymbol{\varphi} \in \mathbf{L}^{3/2}(\mathbb{R}_+^3)$ such that $\mathbf{curl}\boldsymbol{\varphi} = \mathbf{f}$ with $\operatorname{div}\boldsymbol{\varphi} = 0$ in \mathbb{R}_+^3 and $\varphi_3 = 0$ on Γ satisfying the following estimate*

$$\|\boldsymbol{\varphi}\|_{\mathbf{L}^{3/2}(\mathbb{R}_+^3)} \leq C\|\mathbf{f}\|_{\mathbf{L}^1(\mathbb{R}_+^3) + \mathbf{W}_0^{-1,3/2}(\mathbb{R}_+^3)}.$$

ii) Let $\mathbf{f} \in \mathbf{X}(\mathbb{R}_+^3)$. Then there exists a unique $\boldsymbol{\varphi} \in \mathbf{L}^{3/2}(\mathbb{R}_+^3)$ such that $\operatorname{div}\boldsymbol{\varphi} = 0$ with $\varphi_3 = 0$ on Γ and a unique $\pi \in L^{3/2}(\mathbb{R}_+^3)$ satisfying $\mathbf{f} = \mathbf{curl}\boldsymbol{\varphi} + \nabla\pi$ with the corresponding estimate.

Combining the above results with [1] and [3], we can solve the follow elliptical systems.

Theorem 2.8. *i) Let $\mathbf{g}' \in \mathbf{L}^1(\Gamma)$ and $g_n \in W_0^{-1+\frac{1}{n}, \frac{n}{n-1}}(\Gamma)$ satisfying the compatibility condition $\int_\Gamma \mathbf{g}' = \mathbf{0}$ and $\langle g_n, 1 \rangle = 0$. If $\operatorname{div}'\mathbf{g}' \in W_0^{-2+\frac{1}{n}, \frac{n}{n-1}}(\Gamma)$, then the system*

$$-\Delta\mathbf{u} = \mathbf{0} \text{ in } \mathbb{R}_+^n \quad \text{and} \quad \mathbf{u} = \mathbf{g} \text{ on } \Gamma \quad (12)$$

has a unique very weak solution $\mathbf{u} \in \mathbf{L}^{n/(n-1)}(\mathbb{R}_+^n)$.

ii) Let $\mathbf{f} \in \mathbf{L}^1(\mathbb{R}_+^n)$, $\mathbf{g}' \in \mathbf{L}^1(\Gamma)$ and $g_n \in W_0^{-1+\frac{1}{n}, \frac{n}{n-1}}(\Gamma)$ satisfying the compatibility condition $\int_{\mathbb{R}_+^n} \mathbf{f} + \int_{\Gamma} \mathbf{g}' = \mathbf{0}$ and $\int_{\mathbb{R}_+^n} f_n + \langle g_n, 1 \rangle = 0$. If

$$[\mathbf{f}, \mathbf{g}'] = \sup_{\xi \in W_0^{2,n}(\mathbb{R}_+^n), \xi \neq 0} \frac{|\int_{\mathbb{R}_+^n} \mathbf{f} \cdot \nabla \xi + \int_{\Gamma} \mathbf{g}' \cdot \nabla' \xi|}{\|\xi\|_{W_0^{2,n}(\mathbb{R}_+^n)}} < \infty$$

then the system

$$-\Delta \mathbf{u} = \mathbf{f} \text{ in } \mathbb{R}_+^n \quad \text{and} \quad \frac{\partial \mathbf{u}}{\partial x_n} = \mathbf{g} \text{ on } \Gamma \quad (13)$$

has a unique solution $\mathbf{u} \in \mathbf{W}_0^{1,n/(n-1)}(\mathbb{R}_+^n)$.

iii) Let $\mathbf{f} \in \mathbf{L}^1(\mathbb{R}_+^n)$ such that $\text{div } \mathbf{f} \in [W_0^{2,n}(\mathbb{R}_+^n) \cap \overset{\circ}{W}_{-1}^{1,n}(\mathbb{R}_+^n)]'$ and $\int_{\mathbb{R}_+^n} f_n = 0$.

Then $\mathbf{f} \in \mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}_+^n)$ and the system

$$-\Delta \mathbf{u} = \mathbf{f} \text{ in } \mathbb{R}_+^n; \quad \mathbf{u}' = \mathbf{0} \quad \text{and} \quad \frac{\partial u_n}{\partial x_n} = 0 \quad \text{on } \Gamma \quad (14)$$

has a unique solution $\mathbf{u} \in \mathbf{W}_0^{1,n/(n-1)}(\mathbb{R}_+^n)$.

iv) Let $\mathbf{f} \in \mathbf{L}^1(\mathbb{R}_+^n)$ such that $\int_{\mathbb{R}_+^n} \mathbf{f}' = \mathbf{0}$. If

$$[\mathbf{f}] = \sup_{\xi \in \mathcal{D}(\overline{\mathbb{R}_+^n}), \frac{\partial \xi}{\partial x_n} = 0 \text{ on } \Gamma} \frac{|\int_{\mathbb{R}_+^n} \mathbf{f} \cdot \nabla \xi|}{\|\xi\|_{W_0^{2,n}(\mathbb{R}_+^n)}} < \infty$$

holds, then the system

$$-\Delta \mathbf{u} = \mathbf{f} \text{ in } \mathbb{R}_+^n, \quad u_n = 0 \quad \text{and} \quad \frac{\partial \mathbf{u}'}{\partial x_n} = \mathbf{0} \quad \text{on } \Gamma$$

has a unique solution $\mathbf{u} \in \mathbf{W}_0^{1,n/(n-1)}(\mathbb{R}_+^n)$.

Proof. We will prove only the first point i).

Step 1: The proof will be started by showing $\mathbf{g}' \in \mathbf{W}_0^{-1+\frac{1}{n}, \frac{n}{n-1}}(\Gamma)$. Let $\boldsymbol{\mu} \in \mathcal{D}(\Gamma)$ and $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,n}(\mathbb{R}_+^n)$ such that $\boldsymbol{\varphi} = \boldsymbol{\mu}$ on Γ . Thanks to Theorem 2.2, there exist $\boldsymbol{\psi} \in \mathbf{W}_0^{1,n}(\mathbb{R}_+^n) \cap \mathbf{L}^\infty(\mathbb{R}_+^n)$ and $\eta \in W_0^{2,n}(\mathbb{R}_+^n)$ such that $\psi_n = 0$ on Γ satisfying $\boldsymbol{\varphi} = \boldsymbol{\psi} + \nabla \eta$ and the estimate

$$\|\boldsymbol{\psi}\|_{\mathbf{L}^\infty(\mathbb{R}_+^n)} + \|\boldsymbol{\psi}\|_{\mathbf{W}_0^{1,n}(\mathbb{R}_+^n)} + \|\eta\|_{W_0^{2,n}(\mathbb{R}_+^n)} \leq C \|\boldsymbol{\varphi}\|_{\mathbf{W}_0^{1,n}(\mathbb{R}_+^n)}. \quad (15)$$

Then

$$\begin{aligned} \langle \mathbf{g}, \boldsymbol{\mu} \rangle_{\mathcal{D}'(\Gamma) \times \mathcal{D}(\Gamma)} &= \int_{\Gamma} \mathbf{g}' \cdot \boldsymbol{\psi}' - \langle \text{div}' \mathbf{g}', \eta \rangle_{W_0^{-2+\frac{1}{n}, \frac{n}{n-1}}(\Gamma) \times W_0^{2-\frac{1}{n}, n}(\Gamma)} \\ &\quad + \langle g_n, \varphi_n \rangle_{W_0^{-1+\frac{1}{n}, \frac{n}{n-1}}(\Gamma) \times W_0^{1-\frac{1}{n}, n}(\Gamma)} \end{aligned}$$

and

$$\begin{aligned} | \langle \mathbf{g}, \boldsymbol{\mu} \rangle_{\mathcal{D}'(\Gamma) \times \mathcal{D}(\Gamma)} | &\leq \| \mathbf{g}' \|_{\mathbf{L}^1(\Gamma)} \| \boldsymbol{\psi}' \|_{\mathbf{L}^\infty(\mathbb{R}_+^n)} + \| \operatorname{div}' \mathbf{g}' \|_{\mathbf{W}_0^{-2+\frac{1}{n}, \frac{n}{n-1}}(\Gamma)} \times \\ &\quad \times \| \eta \|_{\mathbf{W}_0^{2,n}(\mathbb{R}_+^n)} + \| g_n \|_{\mathbf{W}_0^{-1+\frac{1}{n}, \frac{n}{n-1}}(\Gamma)} \| \varphi_n \|_{\mathbf{W}_0^{1-\frac{1}{n}, n}(\Gamma)}. \end{aligned}$$

Thanks to the density of $\mathcal{D}(\Gamma)$ in $\mathbf{W}_0^{1-\frac{1}{n}, n}(\Gamma)$ and (15), we can deduce $\mathbf{g}' \in \mathbf{W}_0^{-1+\frac{1}{n}, \frac{n}{n-1}}(\Gamma)$.

Step 2: The system (13) is equivalent to the following one: Find \mathbf{u} belonging $\mathbf{L}^{n/(n-1)}(\mathbb{R}_+^n)$ such that for all $\mathbf{v} \in \mathbf{W}_0^{2,n}(\mathbb{R}_+^n) \cap \overset{\circ}{\mathbf{W}}_{-1}^{1,n}(\mathbb{R}_+^n)$,

$$\int_{\mathbb{R}_+^n} \mathbf{u} \cdot \Delta \mathbf{v} = - \langle \mathbf{g}, \frac{\partial \mathbf{v}}{\partial \mathbf{n}} \rangle_{\mathbf{W}_0^{-1+\frac{1}{n}, \frac{n}{n-1}}(\Gamma) \times \mathbf{W}_0^{1-\frac{1}{n}, n}(\Gamma)}.$$

But for all $\mathbf{F} \in \mathbf{L}^n(\mathbb{R}_+^n)$, there exists $\mathbf{v} \in \mathbf{W}_0^{2,n}(\mathbb{R}_+^n) \cap \overset{\circ}{\mathbf{W}}_{-1}^{1,n}(\mathbb{R}_+^n)$, unique up to an element of $x_n \mathbb{R}^n$, such that $-\Delta \mathbf{v} = \mathbf{F}$ in \mathbb{R}_+^n , $\mathbf{v} = \mathbf{0}$ on Γ and the following estimate holds

$$\| \mathbf{v} \|_{\mathbf{W}_0^{2,n}(\mathbb{R}_+^n)/x_n \mathbb{R}^n} \leq C \| \mathbf{F} \|_{\mathbf{L}^n(\mathbb{R}_+^n)}.$$

Then, we have for all $\mathbf{a} \in \mathbb{R}^n$,

$$\begin{aligned} &| \langle \mathbf{g}, \frac{\partial \mathbf{v}}{\partial x_n} \rangle_{\mathbf{W}_0^{-1+\frac{1}{n}, \frac{n}{n-1}}(\Gamma) \times \mathbf{W}_0^{1-\frac{1}{n}, n}(\Gamma)} | \\ &= | \langle \mathbf{g}, \frac{\partial(\mathbf{v} + \mathbf{a}x_n)}{\partial x_n} \rangle_{\mathbf{W}_0^{-1+\frac{1}{n}, \frac{n}{n-1}}(\Gamma) \times \mathbf{W}_0^{1-\frac{1}{n}, n}(\Gamma)} | \\ &\leq C \| \mathbf{g} \|_{\mathbf{W}_0^{-1+\frac{1}{n}, \frac{n}{n-1}}(\Gamma)} \| \mathbf{v} + \mathbf{a}x_n \|_{\mathbf{W}_0^{2,n}(\mathbb{R}_+^n)}. \end{aligned}$$

Consequently, taking the infimum, we have

$$| \langle \mathbf{g}, \frac{\partial \mathbf{v}}{\partial x_n} \rangle_{\mathbf{W}_0^{-1+\frac{1}{n}, \frac{n}{n-1}}(\Gamma) \times \mathbf{W}_0^{1-\frac{1}{n}, n}(\Gamma)} | \leq C \| \mathbf{g} \|_{\mathbf{W}_0^{-1+\frac{1}{n}, \frac{n}{n-1}}(\Gamma)} \| \mathbf{F} \|_{\mathbf{L}^n(\mathbb{R}_+^n)}.$$

Then, the linear operator

$$T : \mathbf{F} \longrightarrow \langle \mathbf{g}, \frac{\partial \mathbf{v}}{\partial x_n} \rangle_{\mathbf{W}_0^{-1+\frac{1}{n}, \frac{n}{n-1}}(\Gamma) \times \mathbf{W}_0^{1-\frac{1}{n}, n}(\Gamma)}$$

is continuous on $\mathbf{L}^n(\mathbb{R}_+^n)$ and thanks to the Riesz representation theorem, there exists a unique $\mathbf{u} \in \mathbf{L}^{n/(n-1)}(\mathbb{R}_+^n)$ such that $T(\mathbf{F}) = \int_{\mathbb{R}_+^n} \mathbf{u} \cdot \mathbf{F}$, *i.e.*, \mathbf{u} is the solution of (13). \square

Acknowledgments. We would like to thank Jean Van Schaftingen for many helpful comments and discussions. HHN's research has been supported in part by FAPESP grant #2009/11017-6 and #2007/51490-7.

References

- [1] C. AMROUCHE, The Neumann problem in the half-space, *Comptes Rendus de l'Académie des Sciences* **335**, Série I, 151-156, (2002).
- [2] C. AMROUCHE, V. GIRAULT, J. GIROIRE, Weighted Sobolev spaces for the Laplace equation in \mathbb{R}^n , *Journal de Mathématiques Pures et Appliquées*, **73** (6), 579-606, (1994).
- [3] C. AMROUCHE, S. NEČASOVÁ, Laplace equation in the half-space with a nonhomogeneous Dirichlet boundary condition, *Mathematica Bohemica*, **126** (2), 265-274, (2001).
- [4] J. BOURGAIN, H. BRÉZIS, On the equation $\operatorname{div} Y = f$ and application to control of phases, *Journal of the American Mathematical Society*, **16** (2), 393-426, (2002).
- [5] J. BOURGAIN, H. BRÉZIS, New estimates for elliptic equations and Hodge type systems, *Journal of the European Mathematical Society*, **9** (2), 277-315, (2007).
- [6] H. BRÉZIS, J. V. SCHAFTINGEN, Boundary estimates for elliptic systems with L^1 -data, *Calculus of Variations and Partial Differential Equations*, **30** (3), 369-388, (2007).
- [7] V. GIRAULT, The gradient, divergence, curl and Stokes operators in weighted Sobolev spaces of \mathbb{R}^3 , *Journal of the faculty of science, The University of Tokyo*, **39** (Sec. IA) (2), 279-307, (1992).
- [8] J. VAN SCHAFTINGEN, Estimates for L^1 -vector fields, *Comptes Rendus de l'Académie des Sciences* **339**, Série I, 181-186, (2004).