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INTERACTION BETWEEN PERIODIC ELASTIC WAVES AND TWO CONTACT NONLINEARITIES

STÉPHANE JUNCA

*Laboratoire J.A. Dieudonné, UMR 6621 CNRS, Université de Nice Sophia-Antipolis,
Parc Valrose, 06108 Nice Cedex 02, France
junca@math.unice.fr*

BRUNO LOMBARD

*Laboratoire de Mécanique et d'Acoustique, UPR 7051 CNRS,
31 chemin Joseph Aiguier, 13402 Marseille, France
lombard@lma.cnrs-mrs.fr*

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Propagation of elastic waves is studied in a 1D medium containing two cracks. The latter are modeled by smooth nonlinear jump conditions accounting for the finite, non-null compressibility of real cracks. The evolution equations are written in the form of a system of two nonlinear neutral delay differential equations. A typically nonlinear phenomenon is found to occur: under periodic excitation, the periodic solutions oscillate around positive mean values, which increase with the forcing level. Perturbation analysis is performed to quantify the effect of this process, which offers non-destructive means to evaluate the cracks. The rest of the paper focuses on the existence, uniqueness and attractivity of periodic solutions. At some particular values of the ratio between the wave travel time and the period of the source, results are obtained whatever the forcing level. With a much larger set of ratios but at small forcing levels, results are obtained under a Diophantine condition.

Keywords: elastic waves; nonlinear jump conditions; neutral delay differential equations; periodic solutions.

AMS Subject Classification: 34A12, 34C60, 35L04, 74J20

1. Introduction

1.1. *Physical motivation*

Studies on the interactions between ultrasonic waves and contact defects have crucial applications in the field of mechanics, especially as far as the nondestructive testing of materials is concerned. When the cracks are much smaller than the wavelengths, they are usually replaced by interfaces with appropriate jump conditions. Linear models for contact have been widely developed.²⁹ However, there is a cur-

rent tendency to develop nonlinear model in order to characterize the cracks more closely.

From the physical point of view, the simplest nonlinear model for crack is that involving unilateral contact.^{30,35} In this non-smooth model, it is assumed that the region of imperfect contact cannot be compressed. A similar approach with a simplified piecewise linear model is studied in Ref. 19. More sophisticated smooth models, involving finite non-null crack compressibility, have been developed for applications to engineering² and geomechanical problems.⁴

The interaction of elastic waves with a contact nonlinearity has been addressed numerically and theoretically by various authors.^{30,7,22,23} In Ref. 18, a detailed analysis was performed in the case of a single crack. It was established that a periodic excitation generates periodic diffracted waves; in addition, the jump in the elastic displacement across the crack has a positive mean value, contrary to what occurs in the case of linear models for contact, where the mean value is null. This jump, which increases strictly with the forcing level, amounts to a mean dilatation of the crack. It was quantified in terms of the parameters involved in the problem, which leads to potential acoustical means of determining the nonlinear properties of the crack.

The physical motivation of the present article is to examine whether the properties found to exist in the case of a single crack also apply to cases involving two cracks. Is there a couple of periodic solution with positive mean values; in the affirmative case, how to estimate these quantities? It is a first step in the direction of a larger number of cracks, which frequently occurs in practical situations.

1.2. *Mathematical motivation*

Even under this simple generalization, the mathematical analysis is much more intricate than in Ref. 18, where one tackled with a ordinary differential equation. The successive reflections of waves between the cracks are described mathematically by a system of two nonlinear neutral-delay differential equation (NDDE) with periodic forcing.¹⁴ The main features of this system are already contained in the following scalar NDDE:

$$x'(t) + ax'(t-1) + bf(x(t)) = s(t), \quad (1.1)$$

where f is a smooth increasing nonlinear function, $f(0) = 0$, $f'(0) = -1$, $|a| \leq 1$, $b > 0$ and s is a periodic excitation. The existence and uniqueness of periodic solutions to (1.1) is not a trivial question when $|a| = 1$, which is the case encountered here. To our knowledge, articles and reference books dealing with forced NDDE and oscillation theory of delay equations, such as Refs. 24, 10, 8, 28, 12, 11, always consider the case $|a| < 1$. In the critical case $|a| = 1$ indeed, the difficulties follow from the very weak stability of periodic solutions: if we consider a null forcing $s = 0$ and a linear function $f(x) = x$, then elementary calculations show that the null solution is asymptotically stable but not exponentially stable.

The mathematical motivation of the present article is to analyze the existence and uniqueness of periodic solutions in this critical case, where the standard techniques cannot be applied.^{6,9,24} We prove that periodic solutions of the linearized NDDE always exist, but that they may lose one order of regularity in some cases described by a Diophantine condition. In these cases, the existence and uniqueness of solutions to the original nonlinear NDDE may be lost. The results obtained here may be of interest in other situations, where coupled systems are described by associating boundary conditions with finite-velocity traveling waves.¹⁴

1.3. *Sketch of the study*

The present paper is organized as follows:

- In section 2, the physical problem is stated in terms of linear hyperbolic partial differential equations and nonlinear jump conditions. Numerical simulations focus on some properties of the diffracted waves. Lastly, the problem is transformed into a system of neutral delay differential equations, which can be studied more easily;
- In section 3, the neutral Cauchy problem is adimensionalized. Existence and uniqueness of global solutions are proved. A qualitative result is presented about the mean values of periodic solutions, or equivalently the mean dilatations of the cracks. Lastly, the harmonic balance method used to compute periodic solutions is briefly introduced;
- In section 4, perturbation analysis is performed. Analytic expressions are obtained for the mean dilatations of each cracks, which makes it possible to use an acoustic approach to estimate the nonlinear parameters of the cracks. The quantitative findings obtained on a single crack in Ref. 18 are thus extended to two cracks;
- In section 5, the existence and uniqueness of periodic solutions are proved whatever the amplitude of the source, but only at specific values of the ratio between the wave propagation time and the period of the source. Upper bounds of the solution are determined, and the geometrical properties of the configuration space are analyzed;
- In section 6, existence and uniqueness of periodic solutions are proved in the case of a larger set of periods, assuming a small source and a Diophantine condition on the ratio travel time / period of the source. Contrary to what occurs with a single crack, the periodic solutions are not exponentially stable in this case. The main part of this study deals with the spectrum of the linearized system and the localization of the small divisors.¹⁷ The results obtained are then transferred to the nonlinear system;
- In section 7, future lines of research are suggested: in one hand, the general case of N cracks could be investigated, and on the other hand, other strategies could be developed for proving the existence and uniqueness of global solutions for a full set of periods and large forcing levels.

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2. Problem statement

2.1. Physical modeling

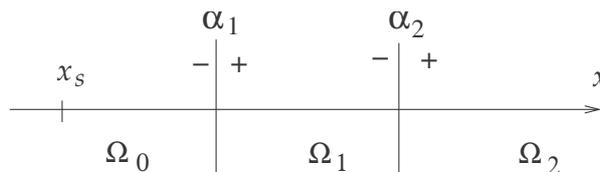


Fig. 1. Cracks at α_1 and α_2 in elastic media Ω_i ; source at $x_s < \alpha_1$.

Let us take two cracks at α_1 and α_2 in the linearly elastic media Ω_0 , Ω_1 and Ω_2 (figure 1). The density ρ and the elastic speed of the compressional waves c are positive, piecewise constant, and they may be discontinuous across the cracks.

Elastic compressional waves are emitted by a source at $x = x_s < \alpha_1$ in Ω_0 , and then diffracted by the cracks. These waves are modeled in terms of the 1-D linear elastodynamics as follows¹

$$\rho \frac{\partial v}{\partial t} = \frac{\partial \sigma}{\partial x}, \quad \frac{\partial \sigma}{\partial t} = \rho c^2 \frac{\partial v}{\partial x} + S(t) \delta(x - x_s), \quad (2.1)$$

where $v = \frac{\partial u}{\partial t}$ is the elastic velocity, u is the elastic displacement, and σ is the elastic stress. The source $S(t)$ is causal, T -periodic, and oscillates around a null mean value: otherwise, the incident elastic displacement would increase linearly with t , which is physically meaningless. The magnitude of the source is described by the amplitude v_0 of the elastic velocity emitted.

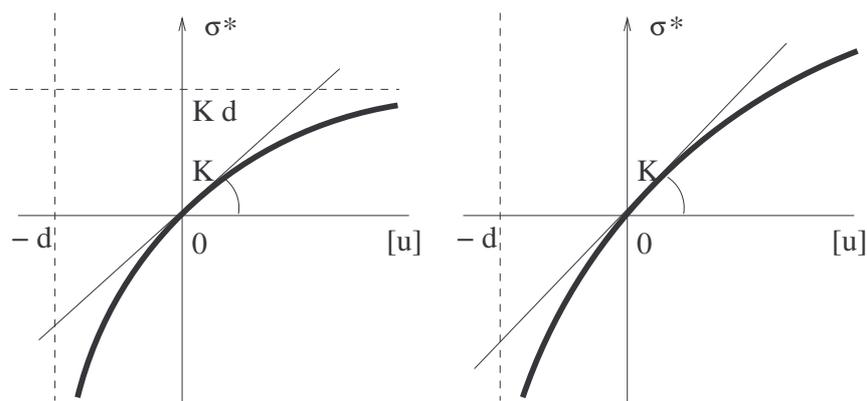


Fig. 2. Sketch of the nonlinear relation between the stress and the jump of the elastic displacement (2.5). Left row: model 1 (2.7), right row: model 2 (2.8).

Two independent jump conditions are required around each crack α_k ($k = 1, 2$) to obtain a well-posed problem. First, the stress is assumed to be continuous across each crack²⁹:

$$[\sigma(\alpha_k, t)] = 0 \Rightarrow \sigma(\alpha_k^+, t) = \sigma(\alpha_k^-, t) = \sigma_k^*(t). \quad (2.2)$$

Secondly, experimental and theoretical studies have yielded the following conclusions:

- the elastic displacement can be discontinuous across the cracks, depending on the stress applied;
- at small stress levels, a linear model is relevant

$$\sigma_k^*(t) = K_k [u(\alpha_k, t)], \quad (2.3)$$

where $K_k > 0$ is the *interfacial stiffness*^{29,32};

- the jump in elastic displacement satisfies the inequality

$$[u(\alpha_k, t)] \geq -d_k, \quad (2.4)$$

where $d_k > 0$ is the *maximum allowable closure*⁴. As the loading increases, the crack tends to become completely closed: $[u(\alpha_k, t)] \rightarrow -d_k^+$ when $\sigma_k^* \rightarrow -\infty$;

- concave stress-closure laws are measured.²⁵

The relation

$$\sigma_k^*(t) = K_k d_k \mathcal{F}_k([u(\alpha_k, t)]/d_k) \quad (2.5)$$

satisfies these requirements, where \mathcal{F}_k is a smoothly increasing concave function

$$\begin{aligned} \mathcal{F}_k :]-1, +\infty[\rightarrow]-\infty, \mathcal{F}_{k \max}[, \quad \lim_{X \rightarrow -1} \mathcal{F}_k(X) = -\infty, \quad 0 < \mathcal{F}_{k \max} \leq +\infty, \\ \mathcal{F}_k(0) = 0, \quad \mathcal{F}'_k(0) = 1, \quad \mathcal{F}''_k < 0 < \mathcal{F}'_k. \end{aligned} \quad (2.6)$$

Two models illustrate the nonlinear relation (2.5): the so-called *model 1* presented in Refs. 2, 4

$$\sigma_k^*(t) = \frac{K_k [u(\alpha_k, t)]}{1 + [u(\alpha_k, t)]/d_k} \Leftrightarrow \mathcal{F}_k(X) = \frac{X}{1 + X}, \quad \mathcal{F}_{k \max} = 1, \quad (2.7)$$

and the so-called *model 2* presented in Ref. 25

$$\sigma_k^*(t) = K_k d_k \ln(1 + [u(\alpha_k, t)]/d_k) \Leftrightarrow \mathcal{F}_k(X) = \ln(1 + X), \quad \mathcal{F}_{k \max} = +\infty. \quad (2.8)$$

These two models are sketched in figure 2. The straight line with a slope K tangential to the curves at the origin gives the linear jump conditions (2.3).

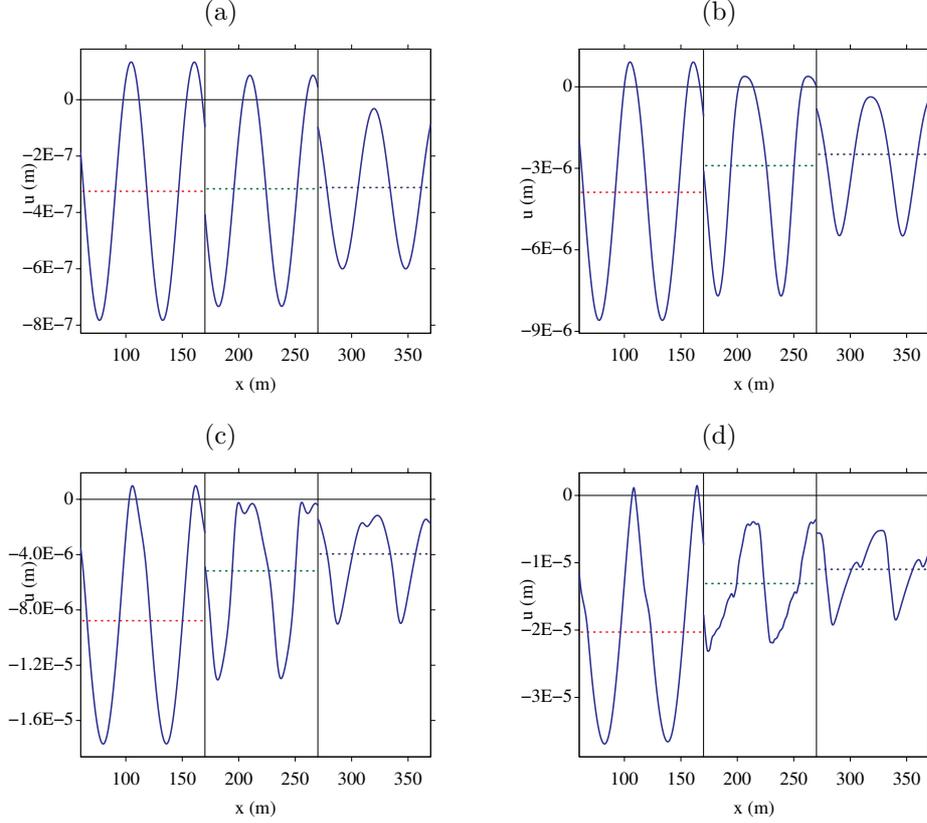
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Fig. 3. Snapshots of the elastic displacement u obtained with model 1 (2.7) and various amplitudes v_0 of the incident elastic velocity: 10^{-4} m/s (a), 10^{-3} m/s (b), $2 \cdot 10^{-3}$ m/s (c) and $5 \cdot 10^{-3}$ m/s (d). The vertical solid lines denote the locations of the cracks. The red, green and navy dotted horizontal lines denote the mean spatial value \bar{u} of the elastic displacement in each subdomain.

2.2. Numerical findings

The influence of the nonlinear jump condition (2.5) on the wave scattering is illustrated numerically. For this purpose, let us consider a 400 m domain with two cracks described by model 1 (2.7) at $\alpha_1 = 170.1$ m and $\alpha_2 = 270.1$ m. The physical parameters are

$$\begin{cases} \rho_0 = \rho_1 = \rho_2 = 1200 \text{ kg.m}^{-3}, & K_1 = K_2 = 1.3 \cdot 10^9 \text{ kg.m}^{-1}.\text{s}^{-2}, \\ c_0 = c_1 = c_2 = 2800 \text{ m.s}^{-1}, & d_1 = d_2 = 6.1 \cdot 10^{-6} \text{ m}. \end{cases} \quad (2.9)$$

A source at $x_s = 50$ m emits a purely sinusoidal wave with a frequency of 50 Hz and an amplitude of elastic velocity v_0 . Four values of v_0 are used, from 10^{-4} m/s to $5 \cdot 10^{-3}$ m/s. The latter value corresponds to a maximum strain $\varepsilon = v_0/c_0 \approx 10^{-6}$, so that the linear elastodynamics models (2.1) is still valid.¹ The linear first-

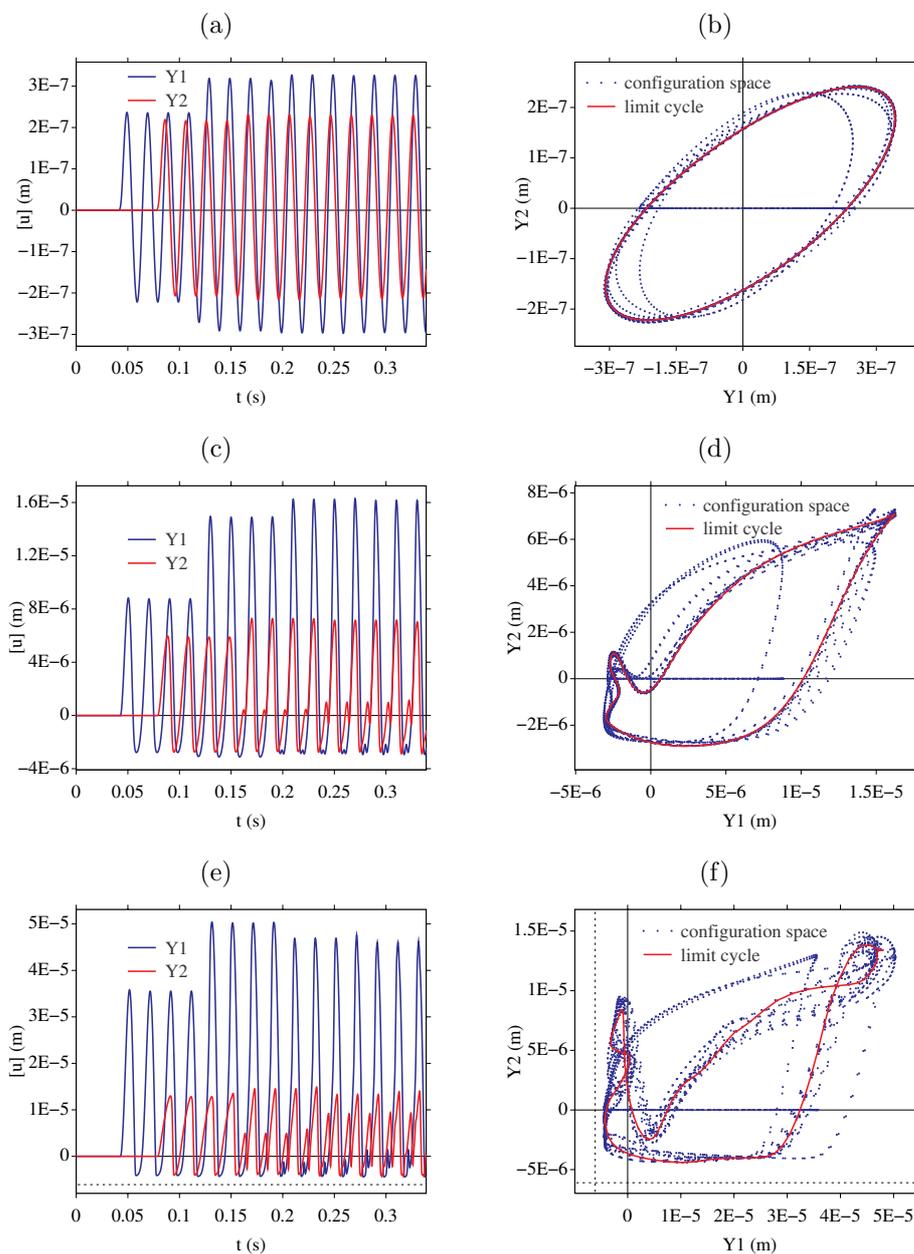


Fig. 4. Time histories of $Y_k = [u(\alpha_k, t)]$ (left row) and configuration space (Y_1, Y_2) (right row) in the case of model 1 (2.7) and various amplitudes v_0 of the incident elastic velocity: 10^{-4} m/s (a-b), $2 \cdot 10^{-3}$ m/s (c-d) and $5 \cdot 10^{-3}$ m/s (e-f). In (e-f), the dotted lines denotes $-d_1 = -d_2$.

order hyperbolic system (2.1) and the jump conditions (2.2) and (2.7) are solved numerically on a (x, t) grid by combining a fourth-order finite-difference ADER scheme with an immersed interface method to discretize the jump conditions.²² At each time step, the scheme gives v , σ , and $Y_k(t) = [u(\alpha_k, t)]$; numerical integration of v also gives u . A sufficiently fine mesh is used to ensure that the numerical solutions can be taken to be the exact values.

Snapshots of u at $t_f = 0.34$ s are shown in figure 3. The mean spatial values \bar{u}_k of the displacements in each of the subdomains are denoted by horizontal dotted lines. Three following three observations are reached:

- a periodic solution is found to be reached in each subdomain;
- the diffracted displacements are distorted when the forcing level increases;
- null or positive jumps in \bar{u}_k occur across the cracks; null jumps occur at small forcing levels v_0 (a), whereas the jumps increase with the forcing level (b,c,d). They amount to a mean dilatation of the crack at α_k .

Based on the method of characteristics, all the scattered fields can be deduced from the known 1-D Green's function and from $Y_k(t)$, with $k = 1, 2$. Since it is simpler to study two functions of t than the solution of a PDE, this study will therefore focus entirely on $Y_1(t)$ and $Y_2(t)$. The positive jumps in \bar{u}_k mentioned above are equal to the mean temporal value of Y_k .

The time histories of Y_k are presented in the left row in figure 4. Logically, Y_k are null until the wave emitted by the source has reached the first crack at α_1 , and subsequently, at α_2 . At small forcing levels (a), Y_k are sinusoids centered around a null mean value. As v_0 increases, Y_k are distorted and centered around an increasing positive mean value. The periodic regime also takes longer to be established. Lastly, it can be noted that the minimum values of Y_k decrease and are bounded below by $-d_1 = -d_2$, as observed in (e) and as required by the model (2.4).

The configuration space (Y_1, Y_2) is shown in the right row in figure 4 shows. The data between $[t_f - T, t_f]$ have been extracted and presented in the form of a continuous red line: at the forcing levels tested, this line is closed, which indicates that a periodic limit cycle has been reached. At small forcing levels (b), a centered elliptic limit cycle is observed, as predicted by the linear theory of oscillators. As v_0 increases, the limit cycle becomes more complex, and even crosses itself (f).

2.3. Reduction of the model

To analyze the numerical findings, we look for evolution equations satisfied by the jumps in the elastic displacements across the cracks. For that purpose, the original problem - a linear PDE with nonlinear jump conditions - is reduced to a system of nonlinear *neutral delay differential equations* (NDDE), i.e. differential equations where the delay is included in the derivatives.¹⁴

Proposition 2.1. *Let us define the travel time $\tau > 0$ and the constants $\beta_k > 0$ and $|\gamma_k| < \beta_k$ ($k = 1, 2$)*

$$\begin{aligned} \tau &= \frac{\alpha_2 - \alpha_1}{c_1}, \\ \beta_1 &= K_1 \left(\frac{1}{\rho_0 c_0} + \frac{1}{\rho_1 c_1} \right), \quad \beta_2 = K_2 \left(\frac{1}{\rho_2 c_2} + \frac{1}{\rho_1 c_1} \right), \\ \gamma_1 &= K_1 \left(\frac{1}{\rho_0 c_0} - \frac{1}{\rho_1 c_1} \right), \quad \gamma_2 = K_2 \left(\frac{1}{\rho_2 c_2} - \frac{1}{\rho_1 c_1} \right). \end{aligned} \quad (2.10)$$

Then the jumps $Y_k(t) = [u(\alpha_k, t)]$ satisfy the system of NDDE

$$\begin{cases} \frac{dY_1}{dt}(t) + \frac{dY_2}{dt}(t - \tau) = -\beta_1 d_1 \mathcal{F}_1 \left(\frac{Y_1(t)}{d_1} \right) - \gamma_2 d_2 \mathcal{F}_2 \left(\frac{Y_2(t - \tau)}{d_2} \right) + \frac{1}{\rho_0 c_0^2} S(t), \\ \frac{dY_2}{dt}(t) + \frac{dY_1}{dt}(t - \tau) = -\beta_2 d_2 \mathcal{F}_2 \left(\frac{Y_2(t)}{d_2} \right) - \gamma_1 d_1 \mathcal{F}_1 \left(\frac{Y_1(t - \tau)}{d_1} \right) + \frac{1}{\rho_0 c_0^2} S(t - \tau), \\ Y_1(t) = Y_2(t) = 0 \text{ for } -\tau \leq t \leq 0. \end{cases} \quad (2.11)$$

Proof. The proof is mainly based on the method of characteristics. In the first part, we collect results based on elastodynamics¹ and on the jump conditions (2.2) and (2.5). In each subdomain Ω_k , the elastic displacement can be split into rightward moving (R) and leftward moving (L) waves

$$u(x, t) = u_{Rk} \left(t - \frac{x}{c_k} \right) + u_{Lk} \left(t + \frac{x}{c_k} \right),$$

where $u_{L2} = 0$: no leftward moving wave comes from $+\infty$. It follows that

$$\frac{\partial u}{\partial t}(x, t) = \begin{cases} +c_k \frac{\partial u}{\partial x}(x, t) + 2u'_{Rk} \left(t - \frac{x}{c_k} \right), \\ -c_k \frac{\partial u}{\partial x}(x, t) + 2u'_{Lk} \left(t + \frac{x}{c_k} \right), \end{cases} \quad (2.12)$$

where $u'_{L2} = 0$. The elastic stresses are

$$\sigma(x, t) = \rho_k c_k^2 \frac{\partial u}{\partial x}(x, t). \quad (2.13)$$

The first jump condition (2.2) and equation (2.13) yield

$$\rho_{k-1} c_{k-1}^2 \frac{\partial u}{\partial x}(\alpha_k^-, t) = \rho_k c_k^2 \frac{\partial u}{\partial x}(\alpha_k^+, t) = \sigma_k^*(t). \quad (2.14)$$

The second jump condition (2.5) and equation (2.14) yield

$$\begin{cases} \frac{\partial u}{\partial x}(\alpha_k^-, t) = \frac{K_k d_k}{\rho_{k-1} c_{k-1}^2} \mathcal{F}_k \left(\frac{Y_k(t)}{d_k} \right), \\ \frac{\partial u}{\partial x}(\alpha_k^+, t) = \frac{K_k d_k}{\rho_k c_k^2} \mathcal{F}_k \left(\frac{Y_k(t)}{d_k} \right). \end{cases} \quad (2.15)$$

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The rightward moving wave emitted by the source (2.1), which impacts the first crack at α_1 is

$$u'_{R0} \left(t - \frac{x}{c_0} \right) = -\frac{1}{2\rho_0 c_0^2} S(t - t_s), \quad (2.16)$$

where $t_s = (\alpha_1 - x_s)/c_0$. From (2.12), (2.15) and (2.16), the traces of $\frac{\partial u}{\partial t}$ can be deduced:

$$\begin{aligned} \frac{\partial u}{\partial t}(\alpha_1^-, t) &= \frac{K_1 d_1}{\rho_0 c_0} \mathcal{F}_1 \left(\frac{Y_1(t)}{d_1} \right) - \frac{1}{\rho_0 c_0^2} S(t - t_s), \\ \frac{\partial u}{\partial t}(\alpha_1^+, t) &= \frac{dY_1}{dt}(t) + \frac{K_1 d_1}{\rho_0 c_0} \mathcal{F}_1 \left(\frac{Y_1(t)}{d_1} \right) - \frac{1}{\rho_0 c_0^2} S(t - t_s), \\ \frac{\partial u}{\partial t}(\alpha_2^-, t) &= -\frac{dY_2}{dt}(t) - \frac{K_2 d_2}{\rho_2 c_2} \mathcal{F}_2 \left(\frac{Y_2(t)}{d_2} \right). \end{aligned} \quad (2.17)$$

In the second part of the proof, we write the traces of rightward and leftward moving waves at α_1^+ and α_2^- . From (2.10), (2.12), (2.15) and (2.17), it follows that

$$2u'_{R1} \left(t - \frac{\alpha_1}{c_1} \right) = \frac{dY_1}{dt}(t) + \gamma_1 d_1 \mathcal{F}_1 \left(\frac{Y_1(t)}{d_1} \right) - \frac{1}{\rho_0 c_0^2} S(t - t_s) \quad (2.18)$$

and

$$2u'_{R1} \left(t - \frac{\alpha_2}{c_1} \right) = -\frac{dY_2}{dt}(t) - \beta_2 d_2 \mathcal{F}_2 \left(\frac{Y_2(t)}{d_2} \right). \quad (2.19)$$

The equality $u'_{R1}(t - \alpha_1/c_1) = u'_{R1}(t + \tau - \alpha_2/c_1)$ along with (2.18) and (2.19) yields

$$\frac{dY_1}{dt}(t) + \frac{dY_2}{dt}(t + \tau) = -\gamma_1 d_1 \mathcal{F}_1 \left(\frac{Y_1(t)}{d_1} \right) - \beta_2 d_2 \mathcal{F}_2 \left(\frac{Y_2(t + \tau)}{d_2} \right) + \frac{1}{\rho_0 c_0^2} S(t - t_s). \quad (2.20)$$

Likewise, it follows from (2.10), (2.12), (2.15) and (2.17) that

$$2u'_{L1} \left(t + \frac{\alpha_1}{c_1} \right) = \frac{dY_1}{dt}(t) + \beta_1 d_1 \mathcal{F}_1 \left(\frac{Y_1(t)}{d_1} \right) - \frac{1}{\rho_0 c_0^2} S \left(t - \frac{\alpha_1 - x_s}{c_0} \right) \quad (2.21)$$

and

$$2u'_{L1} \left(t + \frac{\alpha_2}{c_1} \right) = -\frac{dY_2}{dt}(t) - \gamma_2 d_2 \mathcal{F}_2 \left(\frac{Y_2(t)}{d_2} \right). \quad (2.22)$$

The equality $u'_{L1}(t + \alpha_1/c_1) = u'_{L1}(t - \tau + \alpha_2/c_1)$ along with (2.21) and (2.22) yields

$$\frac{dY_1}{dt}(t) + \frac{dY_2}{dt}(t - \tau) = -\beta_1 d_1 \mathcal{F}_1 \left(\frac{Y_1(t)}{d_1} \right) - \gamma_2 d_2 \mathcal{F}_2 \left(\frac{Y_2(t - \tau)}{d_2} \right) + \frac{1}{\rho_0 c_0^2} S(t - t_s). \quad (2.23)$$

Constants (2.10) are injected into (2.20) and (2.23); the time is shifted: $t - t_s \rightarrow t$; a time shift $t + \tau \rightarrow t$ is also applied to (2.20), which results in the system of NDDE (2.11). Lastly, the initial conditions follow from the causality of the source and the finite propagation time between x_s and α_1 . \square

3. Neutral delay differential equations

3.1. Cauchy problem

Setting

$$\begin{aligned}
 y_1(t) &= \frac{Y_1(t)}{d_1} = \frac{[u(\alpha_1, t)]}{d_1}, & y_2(t) &= \frac{Y_2(t)}{d_1} = \frac{[u(\alpha_2, t)]}{d_1}, & r &= \frac{d_2}{d_1} > 0, \\
 v_0 &= \frac{1}{2\rho_0 c_0^2} \max_{[0, T]} S(t), & S(t) &= 2v_0 \rho_0 c_0^2 \sum_{n=-\infty}^{+\infty} s_n e^{i n \omega t}, & s_0 &= 0, & s(t) &= \frac{1}{d_1 \rho_0 c_0^2} S(t), \\
 f_1(y) &= -\mathcal{F}_1(y), & f_2(y) &= -r \mathcal{F}_2\left(\frac{y}{r}\right), & f_{k \min} &= -\mathcal{F}_{i \max} < 0, & y_{1 \min} &= -1, & y_{2 \min} &= -r,
 \end{aligned} \tag{3.1}$$

the system (2.10)-(2.11) leads to the Cauchy problem

$$\begin{cases}
 y_1'(t) + y_2'(t - \tau) = \beta_1 f_1(y_1(t)) + \gamma_2 f_2(y_2(t - \tau)) + s(t), & t > 0, \\
 y_2'(t) + y_1'(t - \tau) = \beta_2 f_2(y_2(t)) + \gamma_1 f_1(y_1(t - \tau)) + s(t - \tau), & t > 0, \\
 y_k(t) = \phi_k(t) \in C^1([-\tau, 0],]y_{k \min}, +\infty[), & -\tau \leq t \leq 0.
 \end{cases} \tag{3.2}$$

Contrary to what occurred in (2.11), the initial data ϕ_k may differ from 0. Assumptions and new notations are defined as follows ($k = 1, 2$):

$$\begin{aligned}
 \beta_k &> 0, & 0 &\leq |\gamma_k| < \beta_k, & y_{k \min} &< 0, \\
 f_k &\in C^2(]y_{k \min}, +\infty[\rightarrow]f_{k \min}, +\infty[), & \lim_{y \rightarrow y_{k \min}} f_k(y) &= +\infty, & -\infty &\leq f_{k \min} < 0, \\
 f_k(0) &= 0, & f_k'(0) &= -1, & q_k &= \frac{f_k''(0)}{2} > 0, & f_k'(y) &< 0 < f_k''(y), \\
 s &\in C^0(\mathbb{R}), & s(t + T) &= s(t), & t > 0, & T > 0, & \omega &= 2\pi/T, & \varphi &= \omega\tau.
 \end{aligned} \tag{3.3}$$

The inequality $0 \leq |\gamma_k| < \beta_k$ in (3.3) follows from (2.10) and is of crucial importance in following analysis. The reciprocal functions f_k^{-1} satisfy

$$\begin{aligned}
 f_k^{-1} &\in C^2(]f_{k \min}, +\infty[\rightarrow]y_{k \min}, +\infty[), & \lim_{y \rightarrow +\infty} f_k^{-1}(y) &= y_{k \min}, \\
 f_k^{-1}(0) &= 0, & (f_k^{-1})' &< 0 < (f_k^{-1})''.
 \end{aligned} \tag{3.4}$$

In the case of model 1 (2.7), we obtain

$$f_k(y) = f_k^{-1}(y) = -\frac{y}{1 - \frac{y}{y_{k \min}}}, \quad f_{k \min} = y_{k \min}. \tag{3.5}$$

In the case of model 2 (2.8), we obtain

$$f_k(y) = y_{k \min} \ln\left(1 - \frac{y}{y_{k \min}}\right), \quad f_k^{-1}(y) = -y_{k \min} \left(\exp\left(\frac{y}{y_{k \min}}\right) - 1\right), \quad f_{k \min} = -\infty. \tag{3.6}$$

Lastly, a parameter is introduced which plays a key role in our study, namely the ratio between the travel time and the period of the source, or equivalently between

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the distance $\alpha_2 - \alpha_1$ and the wavelength λ_1 in medium Ω_1 , which is denoted by

$$\theta = \frac{\tau}{T} = \frac{\alpha_2 - \alpha_1}{c_1 T} = \frac{\alpha_2 - \alpha_1}{\lambda_1} > 0. \quad (3.7)$$

3.2. Global solutions

In this section, we prove the existence and uniqueness of solutions to the Cauchy problem (3.2)-(3.3). We begin with an elementary result about global solutions for ODE.

Lemma 3.1. *Let $\tau > 0$, $g \in C^0([0, \tau], \mathbb{R})$, and the scalar non-autonomous ODE*

$$\begin{cases} y'(t) = f(y(t)) + g(t), \\ y(0) = y_0, \end{cases} \quad (3.8)$$

with $f \in C^1(]y_{\min}, +\infty[,]f_{\min}, +\infty[)$, $f(0) = 0$, $f' < 0$ and $y_0 > y_{\min}$. The unique maximal solution is therefore also a global solution: $y \in C^1([0, \tau],]y_{\min}, +\infty[)$.

Proof. Two cases are distinguished.

Case 1: $0 \leq \sup g < |f_{\min}|$. The constants $a < b$ are taken to be such that

$$f(a) = -\inf g, \quad f(b) = -\sup g.$$

If $y_{\min} < y < a$, then $f(y) > -\inf g$, hence $y' > 0$, whereas $y > b$ means that $y' < 0$. Consequently, $[a, b]$ is a *funnel*¹⁶: if y_0 belongs to this compact set, then y remains trapped inside. Otherwise, the solution will be found in $[a, b]$ in finite time. In both configurations, the solution remains bounded, and hence it is a global solution.

Case 2: $\sup g \geq |f_{\min}|$. If $y > 0$, then $y' < g(t)$, hence

$$y(t) < y_0 + \int_0^t g(\xi) d\xi < y_0 + t \sup g.$$

In $[0, \tau]$, y is therefore bounded by the upper solution $y_0 + \tau \sup g$, which concludes the proof. \square

Existence of global solutions y_1 and y_2 to the system (3.2)-(3.3) is now proved, together with the inequality $y_k > y_{k \min}$ induced by the model of contact: see (2.4) and (3.1).

Proposition 3.1. *There exists a unique solution $\mathbf{y} = (y_1, y_2)^T$ to (3.2)-(3.3), with $y_k \in C^1([0, +\infty[,]y_{k \min}, +\infty[)$, except at instants $t = k\tau$, $k \in \mathbb{N}$, where the derivatives may be discontinuous.*

Proof. This proof is obtained by induction, according to the *step method*: see for instance the proof of theorems 2-1 or 7-1 in Ref. 14. Intervals $I_k = [k\tau, (k+1)\tau]$, $k = -1, 0, 1, \dots$ are defined. In $I(-1)$, the following explicit solutions are known to exist: $y_k(t) = \phi_k(t) > y_{k \min}$. The couple of solutions is therefore assumed to be

known in I_k , $k \geq 0$, and to satisfy the statements made in proposition 3.1. In I_{k+1} , the system (3.2) is written

$$\begin{cases} y_1'(t) = \beta_1 f_1(y_1(t)) - y_2'(t - \tau) + \gamma_2 f_2(y_2(t - \tau)) + s(t), & t \in I_{k+1}, \\ y_2'(t) = \beta_2 f_2(y_2(t)) - y_1'(t - \tau) + \gamma_1 f_1(y_1(t - \tau)) + s(t - \tau), & t \in I_{k+1}. \end{cases}$$

In the latter system, the right-hand terms $\beta_k f(y_k(t))$ satisfy the assumptions made in lemma 3.1. Since $y_k'(t - \tau)$ in I_{k+1} are equal to $y_k'(t)$ on I_k , the other terms on the right-hand side are continuous, which concludes the proof. \square

3.3. Mean values of periodic solutions

No general proof of the existence and uniqueness of periodic solutions to (3.2)-(3.3) have been obtained, whatever the amplitude of the periodic source s and the ratio θ in (3.7). In section 5, we give results which are valid for any source, but are restricted to specific values of θ . In section 6, a larger set of θ is studied, where the source is small.

Remark 3.1. If there exists a T -periodic solution $\mathbf{y} = (y_1, y_2)^T$, then replacing τ by $\tau + T$, i.e. θ by $\theta + 1$, does not affect the system (3.2). The study of periodic solutions can therefore be restricted to $\theta \in]0, 1]$.

It is assumed here that a periodic regime has been reached, with period T . The solution can be written as Fourier series

$$y_k(t) = \sum_{n=-\infty}^{+\infty} y_k^{(n)} e^{i n \omega t}, \quad k = 1, 2. \quad (3.9)$$

Since the source is real, the coefficients satisfy $y_k^{(-n)} = \widehat{y_k^{(n)}}$, where the hat refers to the conjugate. The following proposition focuses on the mean value of the solution during one period, which is denoted by an overline

$$\overline{y_k} = y_k^{(0)} = \frac{1}{T} \int_0^T y_k(t) dt, \quad k = 1, 2. \quad (3.10)$$

Proposition 3.2. *The mean values of periodic solution $\mathbf{y} = (y_1, y_2)^T$ to (3.2)-(3.3) are strictly positive:*

$$\overline{y_k} > 0, \quad k = 1, 2. \quad (3.11)$$

Proof. T -periodicity of the solution and of the source in (3.2) yields

$$\begin{cases} \beta_1 \overline{f_1(y_1(t))} + \gamma_2 \overline{f_2(y_2(t - \tau))} = 0, \\ \gamma_1 \overline{f_1(y_1(t - \tau))} + \beta_2 \overline{f_2(y_2(t))} = 0. \end{cases}$$

The mean value of a periodic function is invariant by a time shift of the function, hence

$$\begin{cases} \beta_1 \overline{f_1(y_1(t))} + \gamma_2 \overline{f_2(y_2(t))} = 0, \\ \gamma_1 \overline{f_1(y_1(t))} + \beta_2 \overline{f_2(y_2(t))} = 0. \end{cases}$$

The bounds in (3.3) mean that the determinant of the linear system is non-null: $\Delta = \beta_1 \beta_2 - \gamma_1 \gamma_2 > 0$. Therefore,

$$\overline{f_1(y_1(t))} = 0, \quad \overline{f_2(y_2(t))} = 0.$$

Jensen's inequality applied to the convex functions f_1 and f_2 yields

$$f_1(\overline{y_1}) < 0, \quad f_2(\overline{y_2}) < 0.$$

The properties of f_k in (3.3) are used to conclude the proof. \square

Proposition 3.2 means that the jump in the elastic displacement across each crack $Y_k = [u(\alpha_k, t)] = d_1 y_k$ has a positive mean value $\overline{Y_k} > 0$. In other words, there is a mean dilatation of each crack when a periodic source is excited, as in the case of a single crack.¹⁸ This property has been observed both experimentally²⁰ and numerically, in section 2.2. It is analyzed quantitatively in section 4, and the physical implications are addressed in section 7.1.

Remark 3.2. Numerical simulations indicate that $\theta \rightarrow \overline{y_k}(\theta)$ is 0.5-periodic, i.e. that it shows half of the periodicity of the solution y_k (see remark 3.1). No rigorous proof of this statement have been obtained so far, however, except in the case of small forcing levels: see section 4.2 and figure 6.

3.4. Harmonic balance method

Numerical simulations of periodic solutions to (3.2)-(3.3) are performed using the harmonic balance method, where the infinite series (3.9) are truncated to $|n| \leq N$. Injecting the truncated series into (3.2) and (3.5) yields a $(4N + 2) \times (4N + 2)$ nonlinear system

$$\mathbf{F}(\mathbf{X}) = \mathbf{0}, \tag{3.12}$$

where \mathbf{X} is the vector of $4N + 2$ Fourier coefficients.

In the limit case of an infinitesimal forcing level, (3.12) becomes a linear system the solution of which is known analytically: see section 4.1. In the general nonlinear case, since no exact solution of (3.12) is known to exist, numerical methods are required. We have therefore developed computer algebra tools for exactly determining \mathbf{F} and its Jacobian \mathbf{J} . With these programs, high orders (such as $N = 40$) can be reached in a few seconds on a Pentium IV (3 GHz).

Once \mathbf{F} and \mathbf{J} are formally determined, a Newton-Raphson method is applied. Some care must be taken about the initial guess: at large forcing levels, numerical

experiments have shown that multiple roots of (3.12) can occur. To prevent spurious solutions from occurring, a basic continuation method is adopted²⁶:

- a uniform forcing scale is used, ranging from a small $v_0^{(0)}$ up to the forcing level of interest;
- the exact solution $\mathbf{X}^{(0)}$ is computed for $v_0^{(0)}$;
- the forcing level is incremented, and $\mathbf{X}^{(0)}$ is then used as the initial value in the Newton-Raphson algorithm to compute $\mathbf{X}^{(1)}$, and so on.

In each of the simulations presented below, preliminary convergence studies were performed, in order to ensure that a sufficiently large number of Fourier modes were present. Model 1 of contact (3.5) is used here. The source is taken to be monochromatic: $s_{\pm 1} = \mp i / 2$ and $s_{i \neq \pm 1} = 0$ in (3.1), and hence

$$S(t) = 2 v_0 \rho_0 c_0^2 \sin \omega t. \quad (3.13)$$

4. Perturbation analysis

Throughout this section, it will be assumed that there exists a unique periodic solution $\mathbf{y} = (y_1, y_2)^T$ to (3.2)-(3.3), which depends smoothly on the forcing level, and especially on the ratio v_0 / d_1 . In the case of small solutions, a second-order Taylor expansion of the NDDE system gives

$$\left\{ \begin{array}{l} y_1'(t) + y_1'(t - \tau) + \beta_1 y_1(t) - \beta_1 q_1 y_1^2(t) + \gamma_2 y_2(t - \tau) - \gamma_2 q_2 y_2^2(t - \tau) \\ \quad + \mathcal{O}(y_1^3(t) + y_2^3(t - \tau)) = \frac{2 v_0}{d_1} \sum_{n=-\infty}^{+\infty} s_n e^{i n \omega t}, \\ y_2'(t) + y_2'(t - \tau) + \beta_2 y_2(t) - \beta_2 q_2 y_2^2(t) + \gamma_1 y_1(t - \tau) - \gamma_1 q_1 y_1^2(t - \tau) \\ \quad + \mathcal{O}(y_1^3(t - \tau) + y_2^3(t)) = \frac{2 v_0}{d_1} \sum_{n=-\infty}^{+\infty} s_n e^{i n \omega t} e^{-i n \varphi}. \end{array} \right. \quad (4.1)$$

The previous uniqueness assumption ensures that the solution can be sought in the form

$$y_k = y_{k(1)} + y_{k(2)} + \mathcal{O}\left(\left(\frac{v_0}{d_1 \omega}\right)^3\right), \quad |y_{k(2)}| \ll |y_{k(1)}| \ll 1. \quad (4.2)$$

Injecting (4.2) into (4.1) provides a set of recursive linear NDDE systems, which are easy to solve, as done in sections 4.1 and 4.2. Elementary technical result is introduced.

Lemma 4.1. *Let us take the notations defined in section 3.1. Then, for all $n \in \mathbb{N}^*$,*

$$\Delta_n = \left(1 - i \frac{\beta_1}{n \omega}\right) \left(1 - i \frac{\beta_2}{n \omega}\right) - \left(1 - i \frac{\gamma_1}{n \omega}\right) \left(1 - i \frac{\gamma_2}{n \omega}\right) e^{-2 i n \varphi} \neq 0. \quad (4.3)$$

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Proof. Since $\beta_k > |\gamma_k|$ in (3.3), we obtain

$$\left|1 - i \frac{\beta_1}{n\omega}\right| \left|1 - i \frac{\beta_2}{n\omega}\right| > \left|1 - i \frac{\gamma_1}{n\omega}\right| \left|1 - i \frac{\gamma_2}{n\omega}\right|,$$

and hence

$$\left|\left(1 - i \frac{\beta_1}{n\omega}\right) \left(1 - i \frac{\beta_2}{n\omega}\right)\right| > \left|\left(1 - i \frac{\gamma_1}{n\omega}\right) \left(1 - i \frac{\gamma_2}{n\omega}\right) e^{-2in\varphi}\right|,$$

which concludes the proof. \square

4.1. *First-order solution*

The first-order terms in (4.1)-(4.2) are collected, resulting in

$$\begin{cases} y'_{1(1)}(t) + y'_{2(1)}(t - \tau) + \beta_1 y_{1(1)}(t) + \gamma_2 y_{2(1)}(t - \tau) = \frac{2v_0}{d_1} \sum_{n=-\infty}^{+\infty} s_n e^{in\omega t}, \\ y'_{2(1)}(t) + y'_{1(1)}(t - \tau) + \beta_2 y_{2(1)}(t) + \gamma_1 y_{1(1)}(t - \tau) = \frac{2v_0}{d_1} \sum_{n=-\infty}^{+\infty} s_n e^{in\omega t} e^{-in\varphi}. \end{cases} \quad (4.4)$$

Fourier series

$$y_{k(1)} = \sum_{n=-\infty}^{+\infty} y_{k(1)}^{(n)} e^{in\omega t}, \quad k = 1, 2, \quad (4.5)$$

are injected into (4.4), and similar trigonometric arguments are then put together. The constant terms satisfy the linear system

$$\begin{cases} \beta_1 y_{1(1)}^{(0)} + \gamma_2 y_{2(1)}^{(0)} = 0, \\ \gamma_1 y_{1(1)}^{(0)} + \beta_2 y_{2(1)}^{(0)} = 0, \end{cases} \quad (4.6)$$

which involves a non-null determinant $\beta_1 \beta_2 - \gamma_1 \gamma_2 > 0$, and hence

$$y_{1(1)}^{(0)} = y_{2(1)}^{(0)} = 0. \quad (4.7)$$

The first-order solution therefore oscillates around a null-mean value. Collecting the $e^{in\omega t}$ terms gives the linear system

$$\begin{cases} \left(1 - i \frac{\beta_1}{n\omega}\right) y_{1(1)}^{(n)} + \left(1 - i \frac{\gamma_2}{n\omega}\right) e^{-in\varphi} y_{2(1)}^{(n)} = -i \frac{2v_0}{d_1 \omega} \frac{s_n}{n}, \\ \left(1 - i \frac{\gamma_1}{n\omega}\right) e^{-in\varphi} y_{1(1)}^{(n)} + \left(1 - i \frac{\beta_2}{n\omega}\right) y_{2(1)}^{(n)} = -i \frac{2v_0}{d_1 \omega} \frac{s_n}{n} e^{-in\varphi}, \end{cases} \quad (4.8)$$

the determinant of which is $\Delta_n \neq 0$ (see (4.3)). Setting

$$X_{1(1)}^{(n)} = \frac{\left(1 - i \frac{\gamma_2}{n\omega}\right) e^{-2in\varphi} - \left(1 - i \frac{\beta_2}{n\omega}\right)}{\left(1 - i \frac{\beta_1}{n\omega}\right) \left(1 - i \frac{\beta_2}{n\omega}\right) - \left(1 - i \frac{\gamma_1}{n\omega}\right) \left(1 - i \frac{\gamma_2}{n\omega}\right) e^{-2in\varphi}}, \quad (4.9)$$

$$X_{2(1)}^{(n)} = \frac{i \frac{\beta_1 + \gamma_1}{n\omega} e^{-in\varphi}}{\left(1 - i \frac{\beta_1}{n\omega}\right) \left(1 - i \frac{\beta_2}{n\omega}\right) - \left(1 - i \frac{\gamma_1}{n\omega}\right) \left(1 - i \frac{\gamma_2}{n\omega}\right) e^{-2in\varphi}},$$

the non-constant first-order solutions are ($n \neq 0$)

$$y_{1(1)}^{(n)} = i \frac{2v_0}{d_1\omega} X_{1(1)}^{(n)} \frac{s_n}{n}, \quad y_{2(1)}^{(n)} = i \frac{2v_0}{d_1\omega} X_{2(1)}^{(n)} \frac{s_n}{n}. \quad (4.10)$$

4.2. Second-order solution

The second-order terms in (4.1)-(4.2) are collected, resulting in

$$\begin{cases} y'_{1(2)}(t) + y'_{2(2)}(t - \tau) + \beta_1 y_{1(2)}(t) + \gamma_2 y_{2(2)}(t - \tau) = \beta_1 q_1 y_{1(1)}^2(t) + \gamma_2 q_2 y_{2(1)}^2(t - \tau), \\ y'_{2(2)}(t) + y'_{1(2)}(t - \tau) + \beta_2 y_{2(2)}(t) + \gamma_1 y_{1(2)}(t - \tau) = \beta_2 q_2 y_{2(1)}^2(t) + \gamma_1 q_1 y_{1(1)}^2(t - \tau). \end{cases} \quad (4.11)$$

Fourier series

$$y_{k(2)} = \sum_{n=-\infty}^{+\infty} y_{k(2)}^{(n)} e^{in\omega t}, \quad k = 1, 2, \quad (4.12)$$

are injected into (4.11), and similar trigonometric arguments are then put together. From (4.7), the following linear system satisfied by the constant terms

$$\begin{cases} \beta_1 y_{1(2)}^{(0)} + \gamma_2 y_{2(2)}^{(0)} = 2\beta_1 q_1 \sum_{n=1}^{+\infty} |y_{1(1)}^{(n)}|^2 + 2\gamma_2 q_2 \sum_{n=1}^{+\infty} |y_{2(1)}^{(n)}|^2, \\ \gamma_1 y_{1(2)}^{(0)} + \beta_2 y_{2(2)}^{(0)} = 2\gamma_1 q_1 \sum_{n=1}^{+\infty} |y_{1(1)}^{(n)}|^2 + 2\beta_2 q_2 \sum_{n=1}^{+\infty} |y_{2(1)}^{(n)}|^2 \end{cases} \quad (4.13)$$

is obtained, the determinant of which is $\beta_1 \beta_2 - \gamma_1 \gamma_2 > 0$. The system is solved by using first-order solutions (4.10), which gives the constant second-order terms

$$y_{k(2)}^{(0)} = 8q_k \left(\frac{v_0}{d_1\omega}\right)^2 \sum_{n=1}^{+\infty} |X_{k(1)}^{(n)}|^2 \left|\frac{s_n}{n}\right|^2, \quad k = 1, 2. \quad (4.14)$$

Unlike first-order solutions, second-order solutions oscillate around non-null mean values. The effect of this situation on the jumps in the elastic displacements $Y_k = [u(\alpha_k, t)] = d_1 y_k$ are stated in the next proposition.

Proposition 4.1. *At small forcing level, the mean values of the T -periodic jumps Y_k are*

$$\overline{Y}_k = \frac{\mathcal{F}_k''(0)}{d_k} \left(\frac{v_0}{\omega}\right)^2 \sum_{n=1}^{+\infty} |X_{k(1)}^{(n)}|^2 \left|\frac{s_n}{n}\right|^2 + \mathcal{O}\left(\frac{v_0^3}{d_1^2 \omega^3}\right), \quad k = 1, 2, \quad (4.15)$$

where $|X_{k(1)}^{(n)}|$ are defined in (4.9).

Proof. From (3.1) and (3.3), it follows

$$q_1 = \frac{f_1''(0)}{2} = \frac{|\mathcal{F}_1''(0)|}{2}, \quad q_2 = \frac{f_2''(0)}{2} = \frac{d_1}{d_2} \frac{|\mathcal{F}_2''(0)|}{2}. \quad (4.16)$$

On the other hand, putting together (3.1), (4.2), (4.5) and (4.12) yields second-order estimates of \overline{Y}_k

$$\overline{Y}_k = d_1 \left(y_{k(1)}^{(0)} + y_{k(2)}^{(0)} + \mathcal{O}\left(\left(\frac{v_0}{d_1 \omega}\right)^3\right) \right), \quad k = 1, 2. \quad (4.17)$$

Injecting (4.7), (4.14) and (4.16) into (4.17) gives an estimate value of \overline{Y}_k . \square

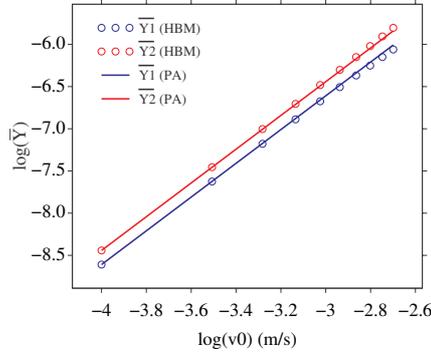


Fig. 5. Log-log evolution of \overline{Y}_k in terms of the forcing amplitude v_0 . Circle: harmonic balance method (HBM). Solid line: perturbation analysis (PA), Eq. (4.17).

Proposition 4.1 is illustrated in figures 5 and 6. The parameters are the same here as in (2.9). The reference solutions are obtained using the harmonic balance method (section 3.4), with $N = 20$ Fourier modes. In figure 5, the forcing level varies from $v_0 = 10^{-4}$ m/s to $v_0 = 2 \cdot 10^{-3}$ m/s, where strong nonlinear effects exist. The frequency and the distance between the cracks studied in section 2.2 give $\theta = \tau/T \approx 0.535$. The approximate solution (4.15) corresponds to a straight line with slope +2, and provides an excellent estimate of \overline{Y}_k at small forcing levels. At

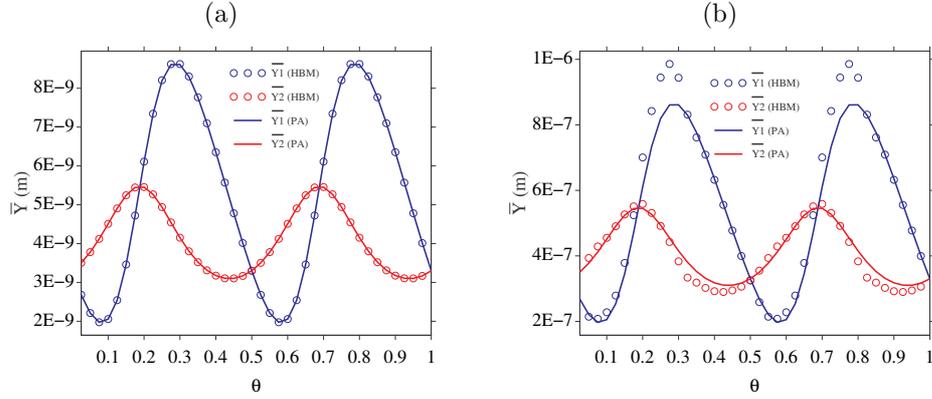


Fig. 6. Evolution of \bar{Y}_k in terms of the ratio $\theta = \tau/T$. (a): $v_0 = 10^{-4}$ m/s; (b): $v_0 = 10^{-3}$ m/s. Circle: harmonic balance method (HBM). Solid line: perturbation analysis (PA), Eq. (4.17).

higher values of v_0 , a slight shift between \bar{Y}_k and (4.15) occurs, but good agreement is still observed.

In figure 6, the parametric study is performed in terms of θ , with $\varphi = 2\pi\theta$ in (4.9) and $\theta \in]0, 1]$. At small forcing level (a), the approximate solution (4.15) provides an excellent estimate of \bar{Y}_k . At large forcing levels, the agreement is naturally less accurate. In both cases, it is observed that \bar{Y}_k are 0.5-periodic in θ , as stated in remark 3.2. This does not mean that Y_k are 0.5-periodic: in practice, the Y_k are only 1-periodic, as stated in remark 3.1.

This property can be proved at small forcing levels. From (4.15) and (4.9), it follows that only $2n\varphi$ is involved in \bar{Y}_k , and not $n\varphi$: the perturbation analysis therefore confirms the 0.5-periodicity of \bar{Y}_k in terms of θ . This property seems to also hold at large forcing levels, as indicated by many other simulations; however, no rigorous proof of this assumption has been obtained so far.

A final (erroneous) property might seem to be suggested by figure 6: it can be observed that \bar{Y}_1 and \bar{Y}_2 intersect at $\theta = 0.5$ and $\theta = 1$. In general, this property is untrue. It is satisfied iff the media are identical ($\rho_0 = \rho_1 = \rho_2$, $c_0 = c_1 = c_2$) and if the cracks are identical ($K_1 = K_2$, $d_1 = d_2$). This can easily be checked, taking theorem 5.1, theorem 5.2 and lemma 5.1 proposed in the next section.

To conclude the perturbation analysis, higher-order expansions than (4.2) could be investigated. However, the contributions of third order and higher-order might yield divergent estimates of \bar{Y}_k in the case of large forcing levels.²² In practice, the second-order expansion (4.2) suffices for this purpose and yields highly accurate estimates of the mean dilatations \bar{Y}_k . The physical relevance of these quantities is discussed in section 7.1

5. Periodic solutions: special cases

5.1. Existence and uniqueness

In this section, the analytical results will be expressed successively in terms of the coefficients of the model problem (3.2), and in terms of the physical parameters (preceded by the sign \equiv).

To begin with, we assume that τ is a period of s : $\tau = nT$, and hence that $\theta = n \in \mathbb{N}^*$ in (3.7). This amounts to saying that Ω_1 contains n wavelengths of the source.

Theorem 5.1. *If $\theta = n \in \mathbb{N}^*$, then there exists a unique periodic solution $\mathbf{y} = (y_1, y_2)^T$ to (3.2)-(3.3) having the same period T as the source. In addition, the increasing diffeomorphism*

$$G(y) = f_2^{-1} \left(\frac{\beta_1 - \gamma_1}{\beta_2 - \gamma_2} f_1(y) \right) \equiv f_2^{-1} \left(\frac{K_1}{K_2} f_1(y) \right), \quad G' > 0, \quad G(0) = 0, \quad (5.1)$$

ensures that

$$y_2(t) = G(y_1(t)). \quad (5.2)$$

Lastly, y_1 and y_2 satisfy a scalar non autonomous ODE without any delay.

Proof. Injecting τ -periodic solutions y_k into (3.2) yields a system of ODE

$$\begin{cases} y_1'(t) + y_2'(t) = \beta_1 f_1(y_1(t)) + \gamma_2 f_2(y_2(t)) + s(t), \\ y_2'(t) + y_1'(t) = \beta_2 f_2(y_2(t)) + \gamma_1 f_1(y_1(t)) + s(t), \end{cases} \quad (5.3)$$

and therefore

$$(\beta_2 - \gamma_2) f_2(y_2(t)) = (\beta_1 - \gamma_1) f_1(y_1(t)).$$

From (2.10), we obtain

$$f_2(y_2(t)) = \frac{\beta_1 - \gamma_1}{\beta_2 - \gamma_2} f_1(y_1(t)) \equiv \frac{K_1}{K_2} f_1(y_1(t)), \quad (5.4)$$

which gives (5.2). The ranges of the left-hand and right-hand sides in (5.4) may be different. To obtain the bijective relation (5.2), the domain of definition of G in (5.1) has to be examined carefully, as done in the corollary 5.1. Equation (5.2) is then injected into (5.3), which gives the ODE

$$H(y_1)' = K(y_1) + s(t), \quad (5.5)$$

where H is an increasing diffeomorphism

$$H(y) = y + G(y), \quad H(0) = 0, \quad H' > 0,$$

and K is a decreasing diffeomorphism (see (3.3), (2.10)):

$$K(y) = k f_1(y),$$

$$k = \beta_1 + \gamma_2 \frac{\beta_1 - \gamma_1}{\beta_2 - \gamma_2} = \gamma_1 + \beta_2 \frac{\beta_1 - \gamma_1}{\beta_2 - \gamma_2} \equiv K_1 \left(\frac{1}{\rho_0 c_0} + \frac{1}{\rho_2 c_2} \right) > 0.$$

Taking $z_1 = H(y_1)$ and (5.5), we obtain the non autonomous ODE

$$z_1' = L(z_1) + s(t), \quad (5.6)$$

where L is a decreasing diffeomorphism

$$L = K \circ H^{-1}, \quad L(0) = 0.$$

The existence and uniqueness of a T -periodic solution to (5.6) follows from the global asymptotic stability of the solution $z_1 = 0$ at null forcing level, which is induced by the properties of L . A similar case has been investigated in propositions 4-1 and 6-1 of Ref. 18. By construction, $y_1 = H^{-1}(z_1)$ and $y_2 = G \circ H^{-1}(z_1)$ are also T -periodic solutions to (3.2). \square

The case where 2τ is an odd period of s can be treated in a similar way: $2\tau = (2n + 1)T$, $n \in \mathbb{N}$. This amounts to saying that Ω_1 contains n and a half wavelengths of the source.

Theorem 5.2. *If $\theta = n + 1/2$ with $n \in \mathbb{N}$, then there exists a unique periodic solution $\mathbf{y} = (y_1, y_2)^T$ to (3.2)-(3.3) having the same period T as the source. In addition, the diffeomorphism (5.1) shows that*

$$y_2(t) = G(y_1(t - T/2)). \quad (5.7)$$

Lastly, y_1 and y_2 satisfy a scalar non autonomous ODE without any delay.

Proof. A time shift $t \rightarrow t + \tau$ is applied to the first equation of (3.2)

$$\begin{cases} y_1'(t + \tau) + y_2'(t) = \beta_1 f_1(y_1(t + \tau)) + \gamma_2 f_2(y_2(t)) + s(t + \tau), \\ y_2'(t) + y_1'(t - \tau) = \beta_2 f_2(y_2(t)) + \gamma_1 f_1(y_1(t - \tau)) + s(t - \tau). \end{cases}$$

The 2τ -periodic solutions to (3.2) therefore satisfy

$$(\beta_2 - \gamma_2) f_2(y_2(t)) = (\beta_1 - \gamma_1) f_1(y_1(t - \tau)),$$

hence

$$y_2(t) = G(y_1(t - \tau)). \quad (5.8)$$

The rest of the proof follows on exactly the same lines as the proof of theorem 5.1, giving the proof of the existence and uniqueness of T -periodic solutions (y_1, y_2) . We therefore obtain

$$\begin{aligned} y_1(t - \tau) &= y_1(t - (n + 1/2)T), \\ &= y_1(t - T/2), \end{aligned}$$

which simplifies (5.8) into (5.7). \square

Some comments are required about theorems 5.1 and 5.2:

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- The relations (5.2) and (5.7) can be expressed in a single formula: if $\theta = n/2$ with $n \in \mathbb{N}$, then

$$y_2(t) = G(y_1(t - t_0)), \quad (5.9)$$

where $t_0 = 0$ if n is even, $t_0 = T/2$ otherwise.

- With model 1 (3.5) and model 2 (3.6), G can be determined exactly

$$\begin{aligned} \text{model 1: } G(y) &= \frac{\beta_1 - \gamma_1}{\beta_2 - \gamma_2} \frac{y}{1 + \left(1 - \frac{1}{r} \frac{\beta_1 - \gamma_1}{\beta_2 - \gamma_2}\right) y} \equiv \frac{K_1}{K_2} \frac{y}{1 + \left(1 - \frac{K_1 d_1}{K_2 d_2}\right) y}, \\ \text{model 2: } G(y) &= r \left((1+y)^r \frac{1}{\beta_2 - \gamma_2} \frac{\beta_1 - \gamma_1}{-1} \right) \equiv \frac{d_2}{d_1} \left((1+y) \frac{K_1 d_1}{K_2 d_2} - 1 \right). \end{aligned} \quad (5.10)$$

- G has a crucial effect on the qualitative properties of (y_1, y_2) , which makes it necessary to analyze this function more closely, as done in the next lemma.

Lemma 5.1. *The diffeomorphism G (5.1) is the identity I if and only if the cracks are identical:*

$$\begin{aligned} G = I &\Leftrightarrow \beta_1 - \gamma_1 = \beta_2 - \gamma_2, \quad r = 1, \quad f_1 = f_2, \\ &\Leftrightarrow K_1 = K_2, \quad d_1 = d_2, \quad \mathcal{F}_1 = \mathcal{F}_2. \end{aligned} \quad (5.11)$$

Proof. Assuming $G = I$ amounts to

$$(\beta_1 - \gamma_1) f_1(y) = (\beta_2 - \gamma_2) f_2(y),$$

for all y in the domain in which G is valid. By differentiation, we obtain

$$(\beta_1 - \gamma_1) f_1'(y) = (\beta_2 - \gamma_2) f_2'(y) \Rightarrow (\beta_1 - \gamma_1) f_1'(0) = (\beta_2 - \gamma_2) f_2'(0).$$

Since $f_i'(0) = -1$, it follows that $\beta_1 - \gamma_1 = \beta_2 - \gamma_2$, or equivalently $K_1 = K_2$. The injectivity of f_1 and f_2 means that $f_1 = f_2$, and hence (3.1) yields

$$\mathcal{F}_1(y) = r \mathcal{F}_1\left(\frac{y}{r}\right).$$

The latter equation is differentiated twice, and then evaluated in 0:

$$\mathcal{F}_1''(0) = \frac{1}{r} \mathcal{F}_1''(0).$$

Since the concavity of the contact laws are non null at the origin (2.6), $r = d_2/d_1 = 1$. The reciprocal of the proof is trivial. \square

Theorems 5.1 and 5.2 are illustrated in figure 7. The densities and celerities are the same as in the tests described in section 2.2; the parameters of the cracks are

$$\begin{aligned} K_1 &= 10^9 \text{ kg.m}^{-1}.\text{s}^{-2}, & d_1 &= 6 \cdot 10^{-6} \text{ m}, \\ K_2 &= 2 \cdot 10^9 \text{ kg.m}^{-1}.\text{s}^{-2}, & d_2 &= 3 \cdot 10^{-6} \text{ m}. \end{aligned} \quad (5.12)$$

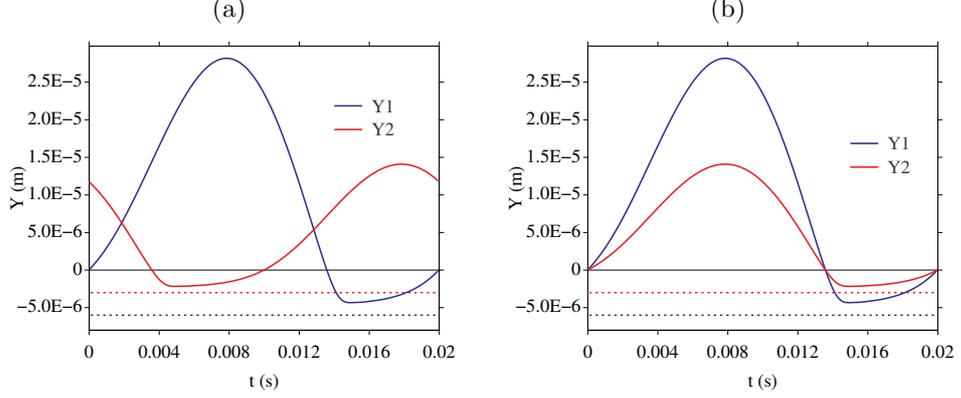


Fig. 7. Time history of y_1 and y_2 during one period, with $\theta = 0.5$ (a) and $\theta = 1$ (b): see theorems 5.1 and 5.2. Parameters: model 1 (3.5), sinusoidal forcing (3.13) with $v_0 = 5 \cdot 10^{-3}$ m/s. The horizontal blue and red dotted lines denote the lower bounds $-d_1$ and $-d_2$, respectively.

Since the cracks are not identical, lemma 5.1 shows that $G \neq I$: the shapes of y_2 and y_1 are therefore not the same. If $\theta = 1$ (b), the respective minima and maxima of y_1 and y_2 are located at the same places, whereas a time shift $T/2$ occurs if $\theta = 0.5$ (a).

5.2. Upper bounds

At some values of $f_{k \min}$ in (3.3), upper bounds $y_k \leq y_{i \max}$ can be deduced from (5.9), as shown by the following corollary.

Corollary 5.1. *Let $\theta = n/2$ with $n \in \mathbb{N}$, and*

$$\xi = \frac{\beta_2 - \gamma_2}{\beta_1 - \gamma_1} \left| \frac{f_{2 \min}}{f_{1 \min}} \right| \equiv \frac{K_2}{K_1} \left| \frac{f_{2 \min}}{f_{1 \min}} \right|.$$

Four cases are distinguished:

(1) $f_{1 \min} > -\infty$, $f_{2 \min} > -\infty$: *one solution may be bounded*

$$* \quad \xi < 1 \Rightarrow y_{1 \max} = f_1^{-1} \left(-\frac{\beta_2 - \gamma_2}{\beta_1 - \gamma_1} |f_{2 \min}| \right) \equiv f_1^{-1} \left(-\frac{K_2}{K_1} |f_{2 \min}| \right), \quad y_{2 \max} = +\infty,$$

$$* \quad \xi = 1 \Rightarrow y_{1 \max} = y_{2 \max} = +\infty,$$

$$* \quad \xi > 1 \Rightarrow y_{2 \max} = f_2^{-1} \left(-\frac{\beta_1 - \gamma_1}{\beta_2 - \gamma_2} |f_{1 \min}| \right) \equiv f_2^{-1} \left(-\frac{K_1}{K_2} |f_{1 \min}| \right), \quad y_{1 \max} = +\infty,$$

(2) $f_{1 \min} > -\infty$, $f_{2 \min} = -\infty$: *one upper bound exists*

$$y_{2 \max} = f_2^{-1} \left(-\frac{\beta_1 - \gamma_1}{\beta_2 - \gamma_2} |f_{1 \min}| \right) \equiv f_2^{-1} \left(-\frac{K_1}{K_2} |f_{1 \min}| \right), \quad y_{1 \max} = +\infty,$$

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(3) $f_{1 \min} = -\infty$, $f_{2 \min} > -\infty$: one upper bound exists

$$y_{1 \max} = f_1^{-1} \left(-\frac{\beta_2 - \gamma_2}{\beta_1 - \gamma_1} |f_{2 \min}| \right) \equiv f_1^{-1} \left(-\frac{K_2}{K_1} |f_{2 \min}| \right), \quad y_{2 \max} = +\infty,$$

(4) $f_{1 \min} = -\infty$, $f_{2 \min} > -\infty$: no upper bound exists.

$$y_{1 \max} = y_{2 \max} = +\infty.$$

Proof. In (5.9), the diffeomorphism $G :]y_{1 \min}, y_{2 \max}[\rightarrow]y_{2 \min}, y_{2 \max}[$ satisfies

$$\begin{aligned} \frac{\beta_1 - \gamma_1}{\beta_2 - \gamma_2} f_1(y_{1 \max}) = f_2(y_{2 \max}) &= \max \left(\frac{\beta_1 - \gamma_1}{\beta_2 - \gamma_2} f_{1 \min}, f_{2 \min} \right), \\ &= -\min \left(\frac{\beta_1 - \gamma_1}{\beta_2 - \gamma_2} |f_{1 \min}|, |f_{2 \min}| \right). \end{aligned} \quad (5.13)$$

Knowing whether (5.13) can be solved or not leads to the four cases distinguished in corollary 5.1. Here, we deal only with case 1: the proof of the other cases can be obtained on exactly the same lines. Solving (5.13) requires to solve

$$f_1(y_{1 \max}) = -|f_{1 \min}| \min \left(1, \frac{\beta_2 - \gamma_2}{\beta_1 - \gamma_1} \left| \frac{f_{2 \min}}{f_{1 \min}} \right| \right) = \Delta_1. \quad (5.14)$$

If $\Delta_1 < f_{1 \min}$, then (5.14) cannot be solved, and $y_{1 \max} = +\infty$. On the contrary,

$$\Delta_1 > f_{1 \min} \Leftrightarrow \frac{\beta_2 - \gamma_2}{\beta_1 - \gamma_1} \left| \frac{f_{2 \min}}{f_{1 \min}} \right| < 1$$

leads to the bounded value $y_{1 \max}$ given in case 1. \square

Model 2 (3.6) corresponds to case 4, where there exists no upper bound, whereas model 1 (3.5) corresponds to case 1, where upper bounds may exist. From (3.1), we obtain

$$\begin{aligned} \xi &= r \frac{\beta_2 - \gamma_2}{\beta_1 - \gamma_1} \equiv \frac{K_2 d_2}{K_1 d_1}, \\ * \quad \xi < 1 &\Rightarrow y_{1 \max} = \frac{1}{1/\xi - 1}, \quad y_{2 \max} = +\infty, \\ * \quad \xi = 1 &\Rightarrow y_{1 \max} = y_{2 \max} = +\infty \\ * \quad \xi > 1 &\Rightarrow y_{2 \max} = \frac{r}{\xi - 1}, \quad y_{1 \max} = +\infty. \end{aligned}$$

Corollary 5.1 is illustrated in figure 5.2, where $\theta = 0.5$, model 1, and a sinusoidal forcing with amplitude $v_0 = 5 \cdot 10^{-3}$ m/s (3.13). The densities and celerities are the same as in the tests presented in section 2.2; the parameters of the cracks are

$$\begin{aligned} K_1 &= 1.3 \cdot 10^9 \text{ kg.m}^{-1}.\text{s}^{-2}, \quad d_1 = 6.1 \cdot 10^{-6} \text{ m}, \\ K_2 &= 2.5 \cdot 10^9 \text{ kg.m}^{-1}.\text{s}^{-2}, \quad d_2 = 8.1 \cdot 10^{-6} \text{ m}, \end{aligned} \quad (5.15)$$

such that $K_1 d_1 < K_2 d_2$ (a). Another set of parameters is also obtained by permuting indexes 1 and 2, hence $K_1 d_1 > K_2 d_2$ (b). In both cases, a positive horizontal

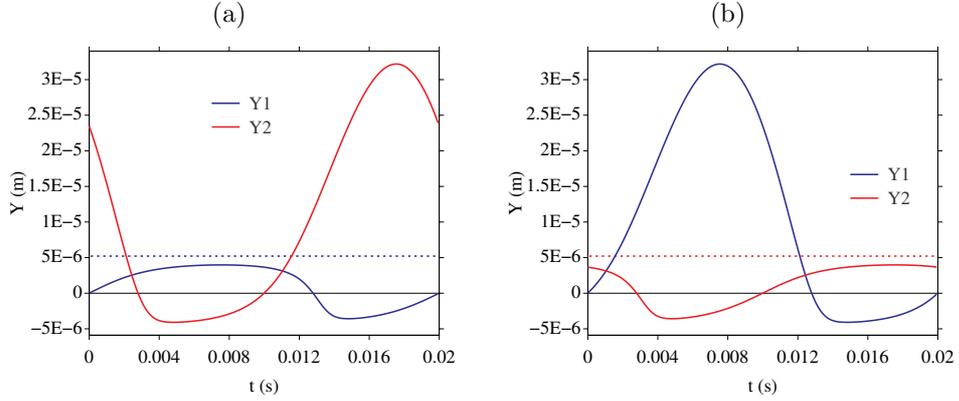


Fig. 8. Upper bounds if $\theta = 0.5$: see corollary 5.1. (a): $K_2 d_2 < K_1 d_1 \Rightarrow y_{1 \max} < +\infty$; (b): $K_2 d_2 > K_1 d_1 \Rightarrow y_{2 \max} < +\infty$. The horizontal blue or red dotted lines denote the upper bounds of the solutions $y_{i \max}$.

asymptote is observed in y_1 (a) and y_2 (b), as predicted by corollary 5.1. Similar figures (not shown here) are observed when $\theta = 1$. Lastly, the previously used parameters (5.12) yield $K_1 d_1 = K_2 d_2$: as predicted by corollary 5.1, no upper bound is observed in figure 7.

5.3. Configuration space

Lastly, we deal with the geometrical features of the configuration space. For this purpose, we take any increasing diffeomorphism f and define the closed curve

$$\Gamma_f : \begin{cases} [0, T] \rightarrow \mathbb{R}^2 \\ t \mapsto (f(y_1(t)), y_2(t)). \end{cases} \quad (5.16)$$

Proposition 5.1. *Let us take the increasing diffeomorphism (5.1). If $\theta = n + 1/2$, $n \in \mathbb{N}$, then*

- Γ_G (5.16) is symmetrical with respect to the first bisecting line;
- if the source is monochromatic (for instance (3.13)), then Γ_G contains a unique double point on this bisecting line.

If $\theta = n \in \mathbb{N}^*$, then Γ_G is a segment on the first bisecting line.

Proof. The case $\theta = n$ is a straightforward consequence of (5.2) and (5.16). In the case $\theta = n + 1/2$, theorem 5.2 yields

$$\begin{cases} G(y_1(t + T/2)) = G(y_1(t - T/2)) = y_2(t), \\ y_2(t + T/2) = G(y_1(t)). \end{cases}$$

The study of Γ_G can therefore be restricted to $[0, T/2]$. On $[T/2, T]$, Γ_G is obtained by permuting $G(y_1)$ and y_2 , which amounts to performing a symmetry with respect to the line $y_2 = G(y_1)$, i.e. the first bisecting line.

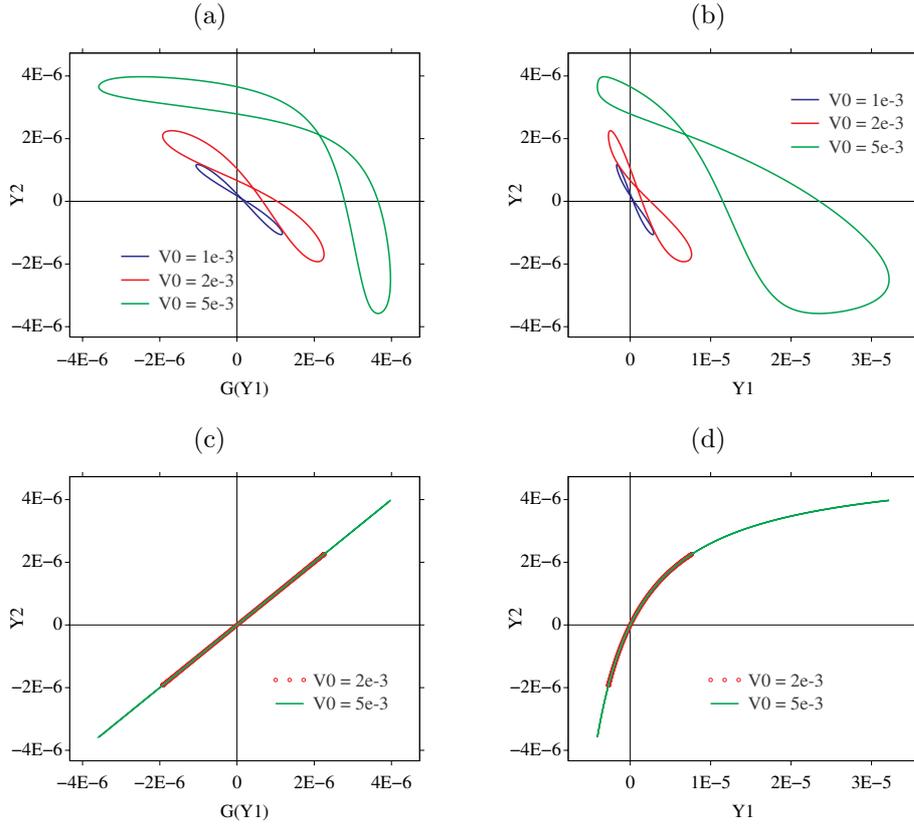


Fig. 9. Configuration space Γ_G (a-c) and Γ_I (b-d) with $\theta = 0.5$ (a-b) and $\theta = 1$ (c-d): see proposition 5.1. Amplitude v_0 of the sinusoidal forcing (3.13): 10^{-3} m/s, $2 \cdot 10^{-3}$ m/s, $5 \cdot 10^{-3}$ m/s.

On the other hand, the phase portrait of the solution to (5.6) in the case of a monochromatic source has been studied extensively in section 4 of ¹⁸: the solution behaves here like a distorted sinus. Since H and $G \circ H^{-1}$ are diffeomorphisms, y_1 and y_2 show the same pattern of evolution. Theorem 5.2 ensures that $G(y_1(t))$ and $y_2(t)$ are T -periodic and equal, with a time shift $T/2$. They therefore cross twice at the same value, which concludes the proof. \square

Proposition 5.1 is illustrated in figure 9, at various levels of v_0 in (3.13). The physical parameters are given in (5.15), and hence lemma 5.1 shows that $G \neq I$.

If $\theta = 0.5$ (a-b), the symmetry properties of Γ_G observed in (a) are lost by Γ_I (b), where the configuration space is (y_1, y_2) . If $\theta = 1$, Γ_G will be a straight line (c), whereas Γ_I is shown on the graph of G (d).

6. Periodic solution: small forcing

6.1. Main result

We take H_T^p to denote the Sobolev space of T -periodic functions of square integrable on a period, with their derivatives up to order p ; in particular, $H_T^0 = L_T^2$.

Unlike the special cases investigated in section 5, the main result obtained in this section involves a much larger set of values of ratios $\theta = \tau / T$, where τ is the travel time between the cracks and T is the period of the source. To quantify this set, the following definition is introduced.

Definition 6.1 (Diophantine condition \mathcal{D}). *Let*

$$\zeta = \frac{\beta_1 - \gamma_1 + \beta_2 - \gamma_2}{4\pi^2} \tau > 0. \quad (6.1)$$

The real number $w \in \mathbb{R}^+$ does not satisfy the condition \mathcal{D} if and only if there exists an infinite number of integers (n, k) such that

$$2w = \frac{n}{k} + \frac{\zeta}{w} \frac{1}{k^2} + o\left(\frac{1}{k^2}\right). \quad (6.2)$$

Condition \mathcal{D} in definition 6.1 is stated as a negative statement. An equivalent positive statement is given by condition \mathcal{P} in proposition 6.3. Condition \mathcal{D} is optimal to avoid a small divisor problem.^{27,17} Assuming that \mathcal{D} is true and considering a small source s , the following local existence and uniqueness results are obtained.

Theorem 6.1. *Let $\theta = \tau / T$ satisfy the Diophantine condition \mathcal{D} . Then there exists a neighborhood $V_s \times V_1 \times V_2$ of the origin in $(H_T^1)^3$ such that for any s in V_s , there exists a unique periodic solution $\mathbf{y} = (y_1, y_2)^T = \Psi(s)$ to (3.2)-(3.3) on $V_1 \times V_2$. This solution has the same period T as the source, and $\Psi \in C^1(V_s, V_1 \times V_2)$.*

Theorem 6.1 is proved in section 6.5. An example of a dense set satisfying \mathcal{D} is given in the next proposition and proved in section 6.4.

Proposition 6.1. *If θ is rational, then \mathcal{D} is satisfied. In this case, the conclusions of theorem 6.1 hold.*

6.2. Spectrum of the linearized system

In the case of small solutions, the system (3.2)-(3.3) is linearized, which leads to

$$\begin{cases} y_1'(t) + y_2'(t - \tau) + \beta_1 y_1(t) + \gamma_2 y_2(t - \tau) = s(t), \\ y_2'(t) + y_1'(t - \tau) + \beta_2 y_2(t) + \gamma_1 y_1(t - \tau) = s(t - \tau). \end{cases} \quad (6.3)$$

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Taking

$$\begin{aligned} \mathbf{y}(t) &= \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \mathbf{s}(t) = \begin{pmatrix} s(t) \\ s(t - \tau) \end{pmatrix}, \\ \mathbf{A} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 0 & \gamma_2 \\ \gamma_1 & 0 \end{pmatrix}, \end{aligned} \quad (6.4)$$

the system (6.3) can be written in the following matrix form:

$$\mathcal{L} \mathbf{y}(t) = \mathbf{y}'(t) + \mathbf{A} \mathbf{y}'(t - \tau) + \mathbf{B} \mathbf{y}(t) + \mathbf{C} \mathbf{y}(t - \tau) = \mathbf{s}(t). \quad (6.5)$$

We are looking for solutions of the homogeneous system obtained by taking $\mathbf{s} = \mathbf{0}$ in (6.5). Injecting $\mathbf{y}(t) = \mathbf{y}_0 e^{\Lambda t}$, where \mathbf{y}_0 is a constant vector, leads to $\mathbf{H}(\Lambda) \mathbf{y}_0 = \mathbf{0}$, where \mathbf{H} is the matrix

$$\mathbf{H}(\Lambda) = \begin{pmatrix} \Lambda + \beta_1 & e^{-\Lambda \tau} (\Lambda + \gamma_2) \\ e^{-\Lambda \tau} (\Lambda + \gamma_1) & \Lambda + \beta_2 \end{pmatrix}. \quad (6.6)$$

Nontrivial solutions are obtained iff $\det \mathbf{H} = 0$, and hence Λ is a root of the characteristic equation

$$h(\Lambda) = (\Lambda + \beta_1)(\Lambda + \beta_2) - e^{-2\Lambda \tau} (\Lambda + \gamma_1)(\Lambda + \gamma_2) = 0. \quad (6.7)$$

The position of characteristic roots of (6.7) plays a decisive role on the stability of the neutral system. As shown in the next proposition, (6.5) with coefficients (6.4) is a critical case not investigated in Refs. 6, 14, which makes the trivial solution poorly attractive.

Proposition 6.2. *The following results hold for the characteristic roots of (6.7):*

- (1) *the roots are located in a vertical strip $\Lambda_{\text{inf}} < \Re \Lambda < 0$ in the complex plane;*
- (2) *a real negative root Λ_0 exists;*
- (3) *the set of roots of h is countable infinite;*
- (4) *the roots Λ tend towards the imaginary axis: $\Re \Lambda \rightarrow 0$ as $|\Lambda| \rightarrow +\infty$.*

The neutral system (6.5) is therefore stable but not exponentially stable.

Proof. Part 1. As $\Re \Lambda \rightarrow +\infty$, $h(\Lambda) \sim \Lambda^2$ which does not vanish. Likewise, $\Re \Lambda \rightarrow -\infty$ means that $e^{2\Lambda \tau} h(\Lambda) \sim -\Lambda^2$, and $h(\Lambda)$ therefore does not vanish, and we obtain

$$-\infty < \Lambda_{\text{inf}} < \Re \Lambda < \Lambda_{\text{sup}} < +\infty.$$

In addition, the characteristic roots Λ satisfy

$$|\Lambda + \beta_1| |\Lambda + \beta_2| = e^{-2\Re \Lambda} |\Lambda + \gamma_1| |\Lambda + \gamma_2|. \quad (6.8)$$

If $\Re \Lambda \geq 0$, then $e^{-2\Re \Lambda} \leq 1$, hence

$$|\Lambda + \beta_1| |\Lambda + \beta_2| \leq |\Lambda + \gamma_1| |\Lambda + \gamma_2|,$$

which is impossible, because of (3.3): and therefore $\Re \Lambda < 0$.

Part 2. The properties

$$h(0) = \beta_1 \beta_2 - \gamma_1 \gamma_2 > 0, \quad \lim_{x \rightarrow -\infty} h(x) = -\infty$$

show the existence of a negative real solution Λ_0 .

Part 3. Let $z = 1 / (2 \Lambda \tau)$. Since $h(0) \neq 0$, $h(\Lambda) = 0$ is equivalent to $g(z) = 0$, where

$$g(z) = e^{\frac{1}{z}} r(z) - 1, \quad r(z) = \frac{(1 + 2\tau\beta_1 z)(1 + 2\tau\beta_2 z)}{(1 + 2\tau\gamma_1 z)(1 + 2\tau\gamma_2 z)}.$$

Based on part 2, $z_0 = 1 / (2 \Lambda_0 \tau)$ is a root of $g: 0$ belongs to the image of g . Lastly, g is holomorphic on \mathbb{C} , except at the essential singularity 0. The Great Picard theorem therefore states that g takes the value 0 infinitely often.

Part 4. Since h is holomorphic, it has a finite number of roots on each compact. Points 1 and 3 therefore mean that $|\Im \Lambda| \rightarrow +\infty$. On the other hand, it follows from (6.8) that $e^{-2\Re \Lambda} \sim 1$ when $|\Im \Lambda| \rightarrow +\infty$, and hence $\Re \Lambda \rightarrow 0$: $\Lambda_{\sup} = 0$ is therefore an accumulation point, and the system (6.5) is stable but not exponentially stable.³⁶ \square

Based on proposition 6.2 (first part), there exists a unique periodic solution to the linear system of NDDE (6.3) whatever the ratio $\theta = \tau / T$. A finer analysis of the characteristic roots Λ of (6.7) is now needed to determine the smoothness of this solution. It is crucial indeed for obtaining existence and uniqueness results about the original nonlinear NDDE (3.2), as seen in section 6.5. The next lemma constitutes a first step in this direction.

Lemma 6.1. *For large values of $|\Lambda|$, there exists $n \in \mathbb{Z}^*$ such that Λ is close to $i \frac{n\pi}{\tau}$. More precisely, this characteristic roots denoted by Λ_n satisfies the following asymptotic expansion:*

$$\Lambda_n = i \left(\frac{n\pi}{\tau} + \frac{\beta_1 - \gamma_1 + \beta_2 - \gamma_2}{2n\pi} \right) - \frac{\tau}{(2n\pi)^2} (\beta_1^2 - \gamma_1^2 + \beta_2^2 - \gamma_2^2) + \mathcal{O} \left(\frac{1}{n^3} \right). \quad (6.9)$$

At sufficiently large values of $n > 0$, the real and imaginary parts of Λ_n increase strictly with n (if $n < 0$, the imaginary parts of Λ_n decrease strictly with n).

Proof. If Λ is a root of (6.7), and $|\Lambda| \rightarrow +\infty$, then

$$e^{2\Lambda\tau} = \frac{(\Lambda + \gamma_1)(\Lambda + \gamma_2)}{(\Lambda + \beta_1)(\Lambda + \beta_2)} \sim 1. \quad (6.10)$$

Therefore there exists z_n tending towards 0 as n increases, such that

$$\Lambda \equiv \Lambda_n = i \frac{n\pi}{\tau} + z_n. \quad (6.11)$$

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To estimate z_n , (6.11) is injected into the characteristic equation (6.7), which gives

$$\begin{aligned} e^{2z_n \tau} \left(i + \frac{\tau}{n\pi} (z_n + \beta_1) \right) \left(i + \frac{\tau}{n\pi} (z_n + \beta_2) \right) \\ - \left(i + \frac{\tau}{n\pi} (z_n + \gamma_1) \right) \left(i + \frac{\tau}{n\pi} (z_n + \gamma_2) \right) = 0. \end{aligned} \quad (6.12)$$

The equation (6.12) is satisfied with the discrete variable n ; it is now extended to the continuous variable $u = \tau / (n\pi)$, and we take $z := z_n$. This yields $F(z, u) = 0$, where $F : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ is given by

$$F(z, u) = e^{2z\tau} (i + uz + u\beta_1) (i + uz + u\beta_2) - (i + uz + u\gamma_1) (i + uz + u\gamma_2). \quad (6.13)$$

A second-order Taylor expansion of F gives

$$\begin{aligned} F(0, 0) = 0, \quad \frac{\partial F}{\partial z}(0, 0) = -2\tau, \quad \frac{\partial F}{\partial u}(0, 0) = i(\beta_1 - \gamma_1 + \beta_2 - \gamma_2), \\ \frac{\partial^2 F}{\partial z^2}(0, 0) = -(2\tau)^2, \quad \frac{\partial^2 F}{\partial z \partial u}(0, 0) = 2i\tau(\beta_1 + \beta_2), \quad \frac{\partial^2 F}{\partial u^2}(0, 0) = 2(\beta_1\beta_2 - \gamma_1\gamma_2). \end{aligned} \quad (6.14)$$

Since $\frac{\partial F}{\partial z}(0, 0) \neq 0$, the implicit function theorem states that $(u, z(u))$ is a graph in a neighborhood of the origin, where $u \rightarrow z(u)$ is a holomorphic function. Differentiating $F(z(u), u) = 0$ yields

$$\frac{\partial F}{\partial z} z' + \frac{\partial F}{\partial u} = 0, \quad (6.15)$$

and hence

$$z'(0) = i \frac{\beta_1 - \gamma_1 + \beta_2 - \gamma_2}{2\tau}. \quad (6.16)$$

Since $z'(0)$ is purely imaginary, one still cannot reach any conclusions about the monotonicity of $\Re(z)$. We therefore differentiate (6.15), which yields

$$\frac{\partial^2 F}{\partial z^2} (z')^2 + 2 \frac{\partial^2 F}{\partial z \partial u} z' + \frac{\partial^2 F}{\partial u^2} + \frac{\partial F}{\partial z} z'' = 0. \quad (6.17)$$

Taking (6.14), (6.16) and (6.17), we obtain

$$z''(0) = -\frac{\beta_1^2 - \gamma_1^2 + \beta_2^2 - \gamma_2^2}{2\tau} < 0. \quad (6.18)$$

Derivatives (6.16) and (6.18) are injected into the second-order Taylor series of z :

$$z(u) = i \frac{\beta_1 - \gamma_1 + \beta_2 - \gamma_2}{2\tau} u - \frac{1}{2} \frac{\beta_1^2 - \gamma_1^2 + \beta_2^2 - \gamma_2^2}{2\tau} u^2 + \mathcal{O}(u^3). \quad (6.19)$$

Based on $u = \tau / (n\pi)$, (6.11) and (6.19), the asymptotic expansion (6.9) is proved.

To prove the monotonicity of the real part of the asymptotic roots, we take $z = x + iy$. Taylor series (6.19) yields $x(0) = 0$, $x'(0) = 0$ and $x''(0) = z''(0) < 0$. As a result, $\Re(z)$ decreases with u and $\Re(\Lambda_n) = \Re(z_n)$ increases with n when $1/n \rightarrow 0$. Lastly, the monotonicity of $\Im(\Lambda_n)$ follows from (6.9). \square

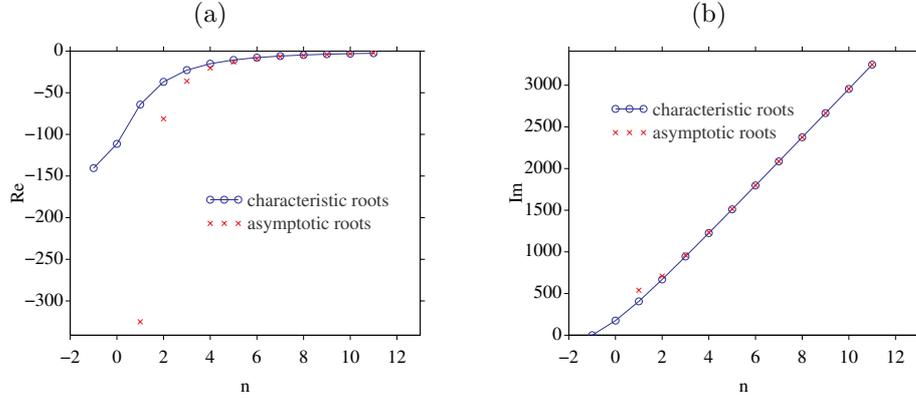


Fig. 10. Characteristic roots of (6.7): real parts (a) and imaginary parts (b). Blue circles and red crosses denote exact values and asymptotic values (6.9), respectively.

Two remarks about lemma 6.1:

- The holomorphic implicit function theorem used in the proof confirm that $z_n = \Lambda_n - i \frac{n\pi}{\tau}$ can be expressed in terms of arbitrary powers of $\frac{\tau}{n\pi}$.
- The monotonic behavior of characteristic roots (6.7) is valid only when sufficiently high values of n are considered. No theoretical result is known in the case of small values of n . However, the results of numerical experiments have indicated that the monotonicity property is satisfied whatever the value of n .

In figure 10, asymptotic values (6.9) for $n = 1, \dots, 10$ are compared with the characteristic roots in the upper part of the complex plane. The roots of (6.7) are computed numerically, with the physical parameters (2.9) and $\alpha_2 - \alpha_1 = 30$ m. The predicted convergence properties are obtained. It is also worth noting that the first characteristic root (on the real line) and the second characteristic root are not included in the asymptotic expansion.

6.3. Inverse linear operator

Let us consider a monochromatic source $s(t) = s_1 e^{i\omega t}$ in the linearized system (6.3). The source in (6.4) becomes

$$\mathbf{s}(t) = s_1 \begin{pmatrix} 1 \\ e^{-i\omega\tau} \end{pmatrix} e^{i\omega t} = \mathbf{s}_1 e^{i\omega t}. \quad (6.20)$$

The solution of (6.5) with the source (6.20) is sought in the form of $\mathbf{y}(t) = \mathbf{y}_1 e^{i\omega t}$, where \mathbf{y}_1 is a constant vector. Straightforward calculations give $\mathbf{y}_1 = \mathcal{L}^{-1}(\mathbf{s}(t)) =$

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$\mathbf{H}^{-1}(i\omega) \mathbf{s}_1$, where \mathcal{L}^{-1} is the inverse operator of \mathcal{L} in (6.5),

$$\mathbf{H}^{-1}(i\omega) = \frac{1}{h(i\omega)} \begin{pmatrix} i\omega + \beta_2 & -e^{-i\omega\tau}(i\omega + \gamma_2) \\ -e^{-i\omega\tau}(i\omega + \gamma_1) & i\omega + \beta_1 \end{pmatrix}, \quad (6.21)$$

and h is the characteristic function in (6.7). The continuity of \mathcal{L}^{-1} from H_T^1 into H_T^1 is a key step to study the nonlinear NDDE (3.2). Based on (6.21), the effect of \mathcal{L}^{-1} on general sources in H_T^p with an infinite number of harmonics is characterized in the next lemma.

Lemma 6.2. *The linear application \mathcal{L}^{-1} is continuous from H_T^p into H_T^p if and only if*

$$\inf \left| \frac{h(ik\omega)}{k} \right| > 0, \quad k \in \mathbb{Z}, \quad k \neq 0. \quad (6.22)$$

Proof. The continuity of \mathcal{L}^{-1} is proved in L_T^2 : since \mathcal{L}^{-1} is a Fourier multiplier, continuity in L_T^2 entrains continuity in H_T^p . It remains to prove that $\sup \|\mathbf{H}^{-1}(ik\omega)\|$ is bounded. The matrix norm induced by the maximum norm in \mathbb{C}^2 is used. The notations $\lambda = ik\omega$, $\beta_{\max} = \max(\beta_1, \beta_2)$, and $\gamma_{\min} = \min(|\gamma_1|, |\gamma_2|)$ are introduced. The property $\beta_k > |\gamma_k|$, along with (6.21), yields

$$\begin{aligned} \|\mathbf{H}^{-1}(\lambda)\| &= \frac{1}{|h(\lambda)|} \max(|\lambda + \beta_1| + |\lambda + \gamma_1|, |\lambda + \beta_2| + |\lambda + \gamma_2|), \\ &\leq \frac{2}{|h(\lambda)|} \max(|\lambda + \beta_1|, |\lambda + \beta_2|), \\ &\leq \frac{2}{|h(\lambda)|} \max(|\lambda| + \beta_{\max}), \\ &\leq 2 \left(1 + \frac{\beta_{\max}}{|\lambda|} \right) / \left| \frac{h(\lambda)}{\lambda} \right|. \end{aligned} \quad (6.23)$$

Likewise, we obtain

$$\|\mathbf{H}^{-1}(\lambda)\| \geq 2 \left(1 + \frac{\gamma_{\min}}{|\lambda|} \right) / \left| \frac{h(\lambda)}{\lambda} \right|, \quad (6.24)$$

and hence $\|\mathbf{H}^{-1}(\lambda)\| \sim 2 \left| \frac{h(\lambda)}{\lambda} \right|$ at high values of $|\lambda|$. Proving the continuity of \mathcal{L}^{-1} therefore amounts to satisfying (6.22). \square

Instead of studying the condition (6.22) directly, we first examine the lower bounds of $|h(\lambda)/\lambda^2|$, $\lambda \in i\mathbb{R}$, $\lambda \neq 0$. Based on (6.9), $|h(\lambda)/\lambda^2|$ can be expected to be very close to zero near the projection λ_n of the characteristic roots Λ_n onto the imaginary axis, as seen in figure 11-(a). Using the asymptotic expansions given in lemma 6.1, we obtain $\lambda_n = i \left(\frac{n\pi}{\tau} + \frac{\beta_1 - \gamma_1 + \beta_2 - \gamma_2}{2n\pi} \right)$. A careful analysis of $|h(\lambda)/\lambda^2|$

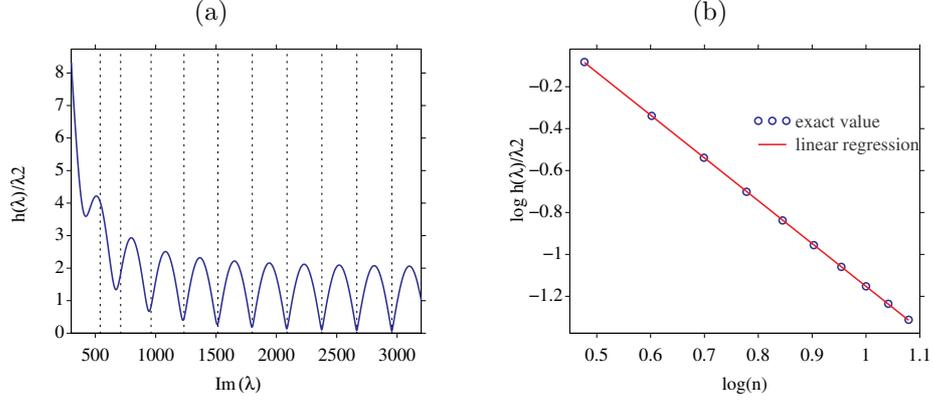


Fig. 11. (a): graph of $|h(\lambda)/\lambda^2|$ along the imaginary axis, where h is the characteristic function (6.7); the dotted vertical lines denote the imaginary part λ_n of the asymptotic expansion of Λ_n : see (6.9) and (6.25). (b): $|h(\lambda_n)/\lambda_n^2|$, where $n = 3, \dots, 13$; the straight red line denotes the slope - 2 deduced from (6.27). In both (a) and (b), the physical parameters are based on (2.9), where $\alpha_2 - \alpha_1 = 30$ m.

along the imaginary axis and especially near λ_n is therefore required, as investigated in the next lemma.

Lemma 6.3. *Some notations are introduced ($n \in \mathbb{Z}$):*

$$\begin{aligned} \lambda &= i(m+r), & m &= \frac{n\pi}{\tau}, & r &\in \left[-\frac{\pi}{2\tau}, +\frac{\pi}{2\tau}\right], \\ \lambda_n &= i(m+r_n), & r_n &= \frac{\beta_1 - \gamma_1 + \beta_2 - \gamma_2}{2n\pi}, \end{aligned} \quad (6.25)$$

and we assume $|\lambda| \gg 1$ (i.e. $|n| \gg 1$). The following estimates hold:

- for all $\varepsilon > 0$, there exists $C_0 > 0$ such that

$$|\lambda - \lambda_n| > \frac{\varepsilon}{|m|} \Rightarrow \left| \frac{h(\lambda)}{\lambda^2} \right| > \frac{C_0}{|m|}. \quad (6.26)$$

- if $\lambda = \lambda_n$, then there exists $C_1 > 0$ such that

$$\left| \frac{h(\lambda_n)}{\lambda_n^2} \right| = \frac{\beta_1^2 - \gamma_1^2 + \beta_2^2 - \gamma_2^2}{2m^2} + \mathcal{O}\left(\frac{1}{m^3}\right) \sim \frac{C_1}{m^2}. \quad (6.27)$$

Proof. We take positive values of m , i.e. positive imaginary values of λ ; the case $m < 0$ can be treated in a similar way. Based on (6.25), it follows

$$\frac{1}{\lambda} = -\frac{i}{m} + \frac{ir}{m^2} + \mathcal{O}\left(\frac{1}{m^3}\right), \quad \frac{1}{\lambda^2} = -\frac{1}{m^2} + \mathcal{O}\left(\frac{1}{m^3}\right). \quad (6.28)$$

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From (6.7) and (6.28), we obtain Taylor series of an increasing order

$$\begin{aligned}
\frac{h(\lambda)}{\lambda^2} &= 1 + \frac{\beta_1 + \beta_2}{\lambda} + \frac{\beta_1 \beta_2}{\lambda^2} - e^{-2i\tau r} \left(1 + \frac{\gamma_1 + \gamma_2}{\lambda} + \frac{\gamma_1 \gamma_2}{\lambda^2} \right), \\
&= 1 - e^{-2i\tau r} + \mathcal{O}\left(\frac{1}{m}\right), \\
&= 1 - e^{-2i\tau r} - i \frac{\beta_1 + \beta_2 - e^{-2i\tau r}(\gamma_1 + \gamma_2)}{m} + \mathcal{O}\left(\frac{1}{m^2}\right), \\
&= 1 - i \frac{\beta_1 + \beta_2}{m} - \frac{\beta_1 \beta_2}{m^2} + i \frac{(\beta_1 + \beta_2)r}{m^2} \\
&\quad - e^{-2i\tau r} \left(1 - i \frac{\gamma_1 + \gamma_2}{m} - \frac{\gamma_1 \gamma_2}{m^2} + i \frac{(\gamma_1 + \gamma_2)r}{m^2} \right) + \mathcal{O}\left(\frac{1}{m^3}\right).
\end{aligned} \tag{6.29}$$

Three cases are then be distinguished.

Case 1: $\delta < |r| \leq \frac{\pi}{2\tau}$, where δ is an arbitrary positive real number. The $\mathcal{O}(1/m)$ series in (6.29) is used together with a concave inequality of sinus function, which yields

$$\left| \frac{h(\lambda)}{\lambda^2} \right| > |1 - e^{2i\tau r}| > 2d\tau|r|,$$

where $d = 2/\pi > 0$. This gives the lower bound

$$\inf \left| \frac{h(\lambda)}{\lambda^2} \right| > 0, \tag{6.30}$$

independently of m .

Case 2: $\frac{\kappa}{m} \leq |r| \leq \delta$, where κ is sufficiently large, for instance $\kappa = \frac{3}{2} \frac{\beta_1 + |\gamma_1| + \beta_2 + |\gamma_2|}{d\tau}$. Triangle inequality gives

$$\left| i \frac{\beta_1 + \beta_2 - e^{-2i\tau r}(\gamma_1 + \gamma_2)}{m} \right| \leq \frac{\beta_1 + |\gamma_1| + \beta_2 + |\gamma_2|}{m}. \tag{6.31}$$

Using successively the concave inequality on sinus function, the assumption about $|r|$, the value of κ , and the inequality (6.31), results in

$$\begin{aligned}
|1 - e^{-2i\tau r}| &> 2d\tau|r|, \\
&> \frac{2d\tau\kappa}{m}, \\
&> 3 \frac{\beta_1 + |\gamma_1| + \beta_2 + |\gamma_2|}{m}, \\
&\geq 2 \frac{\beta_1 + |\gamma_1| + \beta_2 + |\gamma_2|}{m} + \left| i \frac{\beta_1 + \beta_2 - e^{-2i\tau r}(\gamma_1 + \gamma_2)}{m} \right|.
\end{aligned} \tag{6.32}$$

Equation (6.32) is then injected into the $\mathcal{O}(1/m^2)$ series of (6.29), which gives

$$\left| \frac{h(\lambda)}{\lambda^2} \right| > \frac{\beta_1 + |\gamma_1| + \beta_2 + |\gamma_2|}{m}. \quad (6.33)$$

Case 3: $|r| < \frac{\kappa}{m}$. Noting that $r = \mathcal{O}(1/m)$, it follows from (6.29) that

$$\begin{aligned} \frac{h(\lambda)}{\lambda^2} &= i \left(2\tau r - \frac{\beta_1 - \gamma_1 + \beta_2 - \gamma_2}{m} \right) \\ &\quad + \left(2\tau^2 r^2 - \frac{\beta_1 \beta_2 - \gamma_1 \gamma_2}{m^2} + 2\tau r \frac{\gamma_1 + \gamma_2}{m} \right) + \mathcal{O}\left(\frac{1}{m^3}\right), \\ &= i \left(2\tau r - \frac{\beta_1 - \gamma_1 + \beta_2 - \gamma_2}{m} \right) + \mathcal{O}\left(\frac{1}{m^2}\right). \end{aligned} \quad (6.34)$$

Based on (6.34), it can be deduced that the value of $|h(\lambda)/\lambda^2|$ is minimum at λ_n defined in (6.25). Injecting λ_n into the $\mathcal{O}(1/m^3)$ series of (6.34) proves the estimate (6.27). On the other hand, let us take an arbitrarily small $\varepsilon > 0$, smaller than κ . The $\mathcal{O}(1/m^2)$ series given in (6.34) and the definition of r_m given in (6.25) show that

$$\frac{h(\lambda)}{\lambda^2} = 2i\tau(r - r_m) + \mathcal{O}\left(\frac{1}{m^2}\right). \quad (6.35)$$

We therefore obtain

$$|r - r_m| > \frac{\varepsilon}{m} \Rightarrow \left| \frac{h(\lambda)}{\lambda^2} \right| > \frac{2\tau\varepsilon}{m}, \quad (6.36)$$

which proves (6.26). \square

The estimate (6.27) is illustrated in figure 11-(b). A log-log scale is used to see the $\mathcal{O}(1/m^2)$ behavior. As was to be expected, the linear regression curve fitting $|h(\lambda_n)/\lambda_n^2|$ has a slope of - 2.

We can now address the main question of section 6: is the solution to the linear system of NDDE (6.3) as smooth as the source, at least? In the affirmative case, the existence and uniqueness of periodic solutions to the nonlinear system of NDDE (3.2) can be proven by applying standard tools, such as the implicit function theorem or a fixed point method; see section 6.5 for this topic. The following proposition shows that the answer is yes, under a suitable condition

Proposition 6.3 (Condition \mathcal{P}). *Let us take the source (6.20), m and r_n defined in (6.25), and the interval*

$$E_\varepsilon = \bigcup_{n \in \mathbb{Z}} \left] m + r_n - \frac{\varepsilon}{m}, m + r_n + \frac{\varepsilon}{m} \right[, \quad \varepsilon > 0. \quad (6.37)$$

The condition \mathcal{P} is introduced: there exists $\varepsilon > 0$ such that for $k \in \mathbb{Z}$, $k\omega \notin E_\varepsilon$ except for a finite set of integers. The condition \mathcal{P} is true if and only if \mathcal{L}^{-1} is continuous from H_T^p into H_T^p , whatever p . If \mathcal{P} is false, then \mathcal{L}^{-1} is continuous from H_T^p into H_T^{p-1} .

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Proof. It is noticed that $|h(\lambda)/\lambda| \sim |h(\lambda)/\lambda^2||m|$ when λ is large. If \mathcal{P} is true, then there exists $\varepsilon > 0$ such that $|\lambda - \lambda_n| > \frac{\varepsilon}{|m|}$, where $\lambda = ik\omega$. In this case, (6.26) shows that $|h(\lambda)/\lambda| \sim C_1$. The bound (6.22) is therefore satisfied, which proves the case \mathcal{P} true. If \mathcal{P} is false, then it follows from (6.27) that $|h(\lambda)/\lambda| \sim C_0/m$, and hence \mathcal{L}^{-1} is not bounded in H_T^p and one degree of regularity is lost: a small divisor problem arises here.^{15,17} \square

6.4. Diophantine condition

In the following lemma, the condition \mathcal{P} used in proposition 6.3 is expressed differently. This new formulation will be easier to check.

Lemma 6.4. *Condition \mathcal{P} is equivalent to the condition \mathcal{D} in definition 6.1.*

Proof. The notations (2.10), (3.7), (6.25) and (6.37) are used. Condition \mathcal{P} is untrue iff there exists infinite sequences of integers (n, k) satisfying

$$\left| k\omega - \left(\frac{n\pi}{\tau} + \frac{\beta_1 - \gamma_1 + \beta_2 - \gamma_2}{2n\pi} \right) \right| < \frac{\varepsilon\tau}{|n|\pi}. \quad (6.38)$$

Taking $\delta = \left(\frac{\tau}{\pi}\right)^2 \varepsilon$, equation (6.38) yields

$$\left| 2\theta - \left(\frac{n}{k} + \frac{2\zeta}{nk} \right) \right| < \frac{\delta}{|nk|}. \quad (6.39)$$

It follows that $\frac{1}{n} = \frac{1}{2\theta k} + o\left(\frac{1}{k}\right)$, which is injected into (6.39). Since the inequality thus-obtained holds true with all $\delta > 0$, we obtain

$$\left| 2\theta - \left(\frac{n}{k} + \frac{\zeta}{\theta} \frac{1}{k^2} \right) \right| = o\left(\frac{1}{k^2}\right), \quad (6.40)$$

which is in line with (6.2) and proves the lemma. \square

We now prove the proposition 6.1, where it was stated that if $\theta = \tau/T \in \mathbb{Q}$, then condition \mathcal{D} is true.

Proof of proposition 6.1. Let us assume that \mathcal{D} is untrue. Since τ and T are commensurable, there exist integers p and q such that $\theta = p/q$. Injecting this ratio into (6.2) proves that an infinite number of (n, k) satisfy

$$2kp - nq = \frac{\zeta q^2}{kp} + o\left(\frac{1}{k}\right). \quad (6.41)$$

The left-hand side of (6.41) is a sequence of integers, whereas the right-hand side tends towards 0. Therefore there exists k_0 such that if $|k| > k_0$, then $2kp - nq = 0$. In this case,

$$\frac{\zeta q^2}{kp} + o\left(\frac{1}{k}\right) = 0 \iff \frac{\zeta q^2}{p} + o(1) = 0, \quad (6.42)$$

and hence $\frac{\zeta q^2}{p} = 0$, which is impossible. Consequently, \mathcal{D} is true. \square

Proposition 6.1 suffices to show that \mathcal{D} is true, and thus to prove the existence and uniqueness of a T -periodic solution in H_T^1 to (6.5): see proposition 6.3 and lemma 6.4. However, $\theta = \tau/T \in \mathbb{Q}$ is not necessary. The possible existence of a larger set of admissible θ is discussed in section 7.2.

6.5. Nonlinear problem

In section 6.4, theorem 6.1 was proved in the linear case (6.3). The aim of this section is to extend the proof to include the original nonlinear delay system (3.2), under the assumption that the source is small. Standard nonlinear analysis tools are used for this purpose.³ Note that proposition 6.1, which was proved in section 6.4, is valid regardless of the size of the source.

Proof of theorem 6.1. With notations (6.4), the nonlinear NDDE system (3.2) is written

$$\mathbf{f}(\mathbf{y}(t)) = \mathbf{s}(t), \quad (6.43)$$

where the nonlinear function \mathbf{f} depends on f_1 and f_2 . Equation (6.43) is put in the form

$$\mathcal{L}\mathbf{y}(t) = \mathbf{g}(\mathbf{y}(t)) + \mathbf{s}(t), \quad (6.44)$$

where \mathcal{L} is the linearized operator (6.5), and \mathbf{g} is the Taylor remainder that satisfies $\mathbf{g}'(\mathbf{0}) = \mathbf{0}$. It follows that $\mathcal{H}(\mathbf{y}, \mathbf{s}) = \mathbf{0}$, where

$$\mathcal{H}(\mathbf{y}, \mathbf{s}) = \mathbf{y}(t) - \mathcal{L}^{-1}(\mathbf{g}(\mathbf{y}(t)) + \mathbf{s}(t)). \quad (6.45)$$

Under the Diophantine condition \mathcal{D} , proposition 6.3 and lemma 6.4 show that \mathcal{L}^{-1} is continuous. Since $\frac{\partial \mathcal{H}}{\partial \mathbf{y}}(\mathbf{0}, \mathbf{0}) = \mathbf{I} \neq \mathbf{0}$, the implicit function theorem can be applied, which concludes the proof. \square

7. Conclusion

7.1. Physical implications

The property 3.2 and the proposition 4.1 extend to two cracks the results obtained on one crack, in particular in theorem 6-3 presented in Ref. 18. It is proved in the present study that the jump in elastic displacement across each crack \bar{Y}_k oscillates around a positive mean value, which increases quadratically with the forcing level v_0 , and is estimated within the limits of small ratio $v_0 / (\omega d_k)$. This nonlinear process corresponds to the dilatation of each crack. Each term in (4.15) has a clear and separate physical significance:

- $\mathcal{F}_k''(0) / d_k$ is a feature of the nonlinear crack, involving the local concavity of the contact law at the origin $\mathcal{F}_k''(0)$ and the maximum allowable closure d_k ;
- $\frac{v_0}{\omega}$ denotes the amplitude of the source;
- $X_{k(1)}^{(n)}$ corresponds to the mechanical oscillation of the crack in the linear regime;
- $\frac{s_n}{n}$ involves the Fourier spectrum of the source.

The mean dilatation \overline{Y}_k of each crack can be measured experimentally using strain gauges or acoustical methods.²⁰ Eq. (4.15) therefore provides a non-destructive means of determining the ratio between the nonlinear parameters $\mathcal{F}_k''(0) / d_k$.

7.2. *Future lines of investigation*

The physical observables mentioned in section 7.1 depend on the existence and uniqueness of periodic solutions to the nonlinear NDDE (3.2)-(3.3). The results on that topic, which were obtained along section 6, suffer from two limitations:

- in theorem 6.1, existence and uniqueness have been proven for small source under a diophantine condition;
- the sufficient condition in proposition 6.1 holds in the case of a dense set of $\theta = \tau / T$, where τ is the travel time between the cracks, and T is the period of the source. The size of the subset of \mathbb{R} satisfying the necessary and sufficient condition \mathcal{D} is not known.

We do not know whether these restrictions reflect reality (for instance, a loss of uniqueness in the case of large sources), or whether they are simply induced by the technical tools used in the proofs. As an argument in favour of the latter "technical" hypothesis, let us consider the special case $\theta = n / 2$ ($n \in \mathbb{N}$) investigated in section 5. In this case, theorems 5.1 and 5.2 prove the existence and uniqueness of the solution whatever the amplitude of the source, unlike theorem 6.1. As a second argument, many numerical experiments have been performed with large forcing and various values of θ (in practice, rational values of θ are considered numerically): existence of a periodic regime has always been observed.

To be able to clear up this point, other technical tools are probably required. On the one hand, Schauder's fixed point method could be used to prove the existence of a solution whatever the amplitude of the source, but the uniqueness would be lost. Another possible strategy could consist in applying the Nash-Moser theorem,^{15,17} to prove the existence and uniqueness with a full set of ratios θ , but this method would still be restricted to small sources.

Another future line of study could be the investigation of $N \gg 1$ nonlinear cracks. Configurations frequently occur in practical applications, and can generate physically relevant processes: wave localization, band-pass behavior, chaos, etc.³¹ The strategy developed here could also be used to study a system of N nonlinear NDDE but this is probably a very intricate task. Other approaches may be more fruitful, based for instance on the conservation of energy of the mechanical system.²¹

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