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Wavelet Transforms of Nonstationary Random Processes: Contributing Factors for Stationarity and Decorrelation

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Abstract

The paper presents some statistical properties of the wavelet transforms, in the framework of nonstationary random processes. It investigates the factors that make wavelet subband coefficients behave as stationary discrete sequences. Conditions are given under which stationarization, decorrelation and higher order dependency reduction occur among the wavelet coefficients of a large class of nonstationary random processes. The paper also highlights the presence of singular paths, *i.e.* the paths such that no stationarization occurs and those for which no dependency reduction are expected through successive decompositions. The overall presentation focuses on understanding the role played by the parameters that govern stationarization and dependency reduction in the wavelet domain. This is performed with respect to semi-analytical expansions of cumulants for modeling different types of correlation structures that characterize many random processes.

keywords: Wavelet Transforms ; Nonstationary Random Processes ; Cumulant analysis.

I. INTRODUCTION

Statistical signal processing and time series analysis are substantially simpler, not simplistic, assuming that data collected are issued from independent and identically distributed random vari-

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ables. In this respect, suitable representations for a random process relate to transforms that have stationarization and decorrelating properties and, more generally, transforms that make it possible to reduce higher order dependencies between the random variables describing the process time or spatial evolution. Among transforms that approximately achieve this goal, wavelet decompositions are highly effective because wavelets operate unconditionally with respect to the input process and tend to achieve the desirable stationarization and decorrelating properties for a large class of stochastic processes. This class contains stationary random processes (see for instance [1], [2], [3], [4], [5]) and some nonstationary random processes such as fractional Brownian motions ([6], [7], [8], [9], [10], [11], [12], [13], [14]) and fractionally differenced processes ([15]).

This paper points out the key parameters that contribute to make the wavelet coefficients become stationary and tend to be distributed as independent and identical. These parameters are the null moment condition and the wavelet order in governing the stationarization and dependency reductions induced by wavelet transforms. The paper describes how these parameters impact the statistical properties of wavelet and wavelet packet coefficients of many nonstationary random processes. The class of random processes under consideration is characterized through cumulant expansions. A random process pertaining to this class is such that its cumulant of order N can be expanded in 3 terms: a *projective term* (with dimension $N - 1$), a *stationary term* (which lies along the dimension $N - 1$) and an *N -variate polynomial term*. Furthermore, the paper also points out the singular wavelet paths: the paths associated with nodes (subbands) such that no stationarization can occur and those for which dependency reduction is not expected to hold true.

The paper is organized as follows. In Section II, we present some preliminary notation and assumptions used throughout the paper. Section III deals with the conditions under which stationarization occurs for wavelet packet coefficients. Section IV provides asymptotic results on the decorrelation and the higher order dependency reduction that can be reached for the class of nonstationary processes introduced in Section III. This section also presents an heuristic approach based on paraunitary filtering for understanding the theoretical results obtained. Section V discusses the contribution of such an analysis to characterize ARFIMA and GARMA type random processes in the wavelet domain. This section emphasizes that wavelet packet spectra of many real world texture images behave like ARFIMA and GARMA spectra. Section VI concludes the work and mentions some prospects related to the results given in the paper.

II. PRELIMINARY NOTATION AND ASSUMPTIONS

A. Wavelet subbands

Throughout the paper, H_0 denotes a *scaling filter* and W_0 is the *scaling function* associated with this filter. Similarly, H_1 denotes the *wavelet filter* associated with H_0 and W_1 is the *wavelet function* associated with W_0 , [16], [17]. The functional subspace of $L^2(\mathbb{R})$ generated from the translated versions of W_0 is denoted by $\mathbf{W}_{0,0}$ (input space).

Let $(H_{\epsilon_1}, H_{\epsilon_2}, \dots, H_{\epsilon_j})$ be the sequence of wavelet filters successively applied for decomposing $\mathbf{W}_{0,0}$, with $\epsilon_\ell \in \{0, 1\}$ for every $\ell \in \{1, 2, \dots, j\}$. This sequence defines a wavelet packet subspace $\mathbf{W}_{j,n}$ (subband (j, n)) where the frequency index n at decomposition level j is

$$n = n_{\mathcal{P}(j)} = \sum_{\ell=1}^j \epsilon_\ell 2^{j-\ell}. \quad (1)$$

Sequence $(H_{\epsilon_1}, H_{\epsilon_2}, \dots, H_{\epsilon_j})$ can thus be associated with the path $\mathcal{P}(j) = \{\epsilon_1, \epsilon_2, \dots, \epsilon_j\}$ with root node $\mathbf{W}_{0,0}$ and terminal node $\mathbf{W}_{j,n}$ (see [5] for more details).

Assume that orthonormal wavelet transforms are concerned. Then $\mathbf{W}_{j,n}$ is generated from the sequence of wavelet functions $\{W_{j,n,k} : k \in \mathbb{Z}\}$, where $W_{j,n,k}(\cdot) = W_{j,n}(\cdot - 2^j k)$, with $W_{j,n}$ satisfying, in the Fourier domain¹:

$$\mathcal{F} W_{j,n}(\omega) = 2^{j/2} \left[\prod_{\ell=1}^j H_{\epsilon_\ell}(2^{\ell-1} \omega) \right] \mathcal{F} W_0(\omega), \quad (2)$$

with equality holding in $L^2(\mathbb{R})$ sense. Note that depending on the binary sequence $\{\epsilon_1, \epsilon_2, \dots, \epsilon_j\}$ that characterizes $\mathcal{P}(j)$, then $\mathbf{W}_{j,n}$ is associated with

- a *wavelet decomposition* when $n_{\mathcal{P}(\ell)} \in \{0, 1\}$ for every $\ell \in \{1, 2, \dots, j\}$,
- a *full wavelet packet decomposition* when $n_{\mathcal{P}(\ell)} \in \{0, 1, \dots, 2^\ell - 1\}$ for every $\ell \in \{1, 2, \dots, j\}$.

For the sake of generality, we consider the wavelet packet framework in the following.

B. Vanishing moments

A wavelet function W_1 is said to have r vanishing moments if

$$\int_{\mathbb{R}} t^m W_1(t) dt = 0, \quad \text{for every } m = 0, 1, \dots, r-1. \quad (3)$$

Let $\mathcal{M}_{j,n,k}(m)$ denote the $(m+1)$ -th *moment* of the function $W_{j,n,k}$, $m \in \{0, 1, 2, \dots\}$:

$$\mathcal{M}_{j,n,k}(m) = \int_{\mathbb{R}} t^m W_{j,n,k}(t) dt. \quad (4)$$

¹Fourier transform: $\mathcal{F}f(\omega) = \int_{\mathbb{R}} f(t) e^{-i\omega t} dt$ if $f \in L^1(\mathbb{R})$.

Under the condition given by Eq. (3) and if we assume that the paraunitary filters $(h_\epsilon)_{\epsilon \in \{0,1\}}$ have finite impulse responses, then for $n \neq 0$, we have:

$$\mathcal{M}_{j,n,k}(m) = 0, \quad \text{for every } m = 0, 1, \dots, r-1. \quad (5)$$

It follows that $W_{j,n,k}$ has at least r vanishing moments.

Note that by construction, every wavelet function W_1 is with at least one vanishing moment whereas the contrary holds for the scaling function W_0 (see for instance [16]). Consequently, $\mathcal{M}_{j,0,k}(0) \neq 0$ whereas $\mathcal{M}_{j,n,k}(0) = 0$ for every $n \in \{1, 2, \dots, 2^j - 1\}$.

C. Preliminary assumptions

Let X be a zero-mean second order real random process, continuous in quadratic mean. Let $R(t, s) = \mathbb{E}[X(t)X(s)]$ be the autocorrelation function of X . Assume that

$$\iint_{\mathbb{R}^2} R(t, s) W_{j,n,k}(t) W_{j,n,k}(s) dt ds < \infty. \quad (6)$$

Then, the coefficients of the projection of X on subband $\mathbf{W}_{j,n}$ define a discrete sequence of zero-mean second order real random variables:

$$c_{j,n}[k] = \int_{\mathbb{R}} X(t) W_{j,n,k}(t) dt, \quad k \in \mathbb{Z}, \quad (7)$$

The statistical properties of the wavelet coefficients of X depend on the analytical form of $R(t, s)$. In what follows, we assume that this autocorrelation function admits the following expansion:

Condition (ACF)

The autocorrelation function R can be written in the form:

$$R(t, s) = \underbrace{F(t) + F(s)}_{\text{Projective terms}} + \underbrace{\mathcal{S}(t-s)}_{\text{Stationary term}} + \underbrace{\sum_{1 \leq p, q \leq M} \alpha_{p,q} t^p s^q}_{\text{Bivariate-polynomial}} \quad (8)$$

with

$$F(t) W_{j,n,k}(t) \in L^1(\mathbb{R})$$

and

$$t^p s^q W_{j,n,k}(t) W_{j,n,\ell}(s) \in L^1(\mathbb{R}^2)$$

for every $1 \leq p, q \leq M$.

Due to the symmetry of the autocorrelation function, we can order the terms of the bivariate-polynomial with respect to one of the variables t, s . Therefore, we call M , the *degree of the bivariate-polynomial* involved in Eq. (8). It is worth noticing that in Condition (ACF) above, no restriction

is imposed on M . In practice, when the degree M is fixed (for modeling, for example), then only bivariate monomial terms with high degrees are excluded from the bivariate-polynomial of Condition **(ACF)**. Now, when M tends to infinity, the expansion Eq. (8) of the autocorrelation function involves a Taylor-like expansion when both F and \mathcal{S} are polynomial functions and monomial coefficients are chosen adequately. From these considerations, we have that Condition **(ACF)** is reasonable for modeling to approximate the autocorrelation functions of a wide class of stochastic processes.

Example 1

For stationary processes, we have $F = 0$ and for a fractional Brownian motion, we have $F(x) = \mathcal{S}(x) = |x|^{2H}$ where H is the Hurst parameter. Both processes admit no bivariate polynomial terms.

Now, let $N \in \mathbb{N}$ and denote by

$$\text{cum}(t_0, t_1, t_2, \dots, t_N) = \text{cum}\{X(t_0), X(t_1), X(t_2), \dots, X(t_N)\}$$

the cumulant of order $N+1$ of X . The above cumulant is hereafter assumed to belong to $L^2(\mathbb{R}^{N+1})$ and to be finite.

When higher order properties are concerned, *i.e.* $N \geq 2$, we will also consider the following assumptions:

Condition (C1)

The cumulant $\text{cum}(t_0, t_1, t_2, \dots, t_N)$ can be written in the form

$$\begin{aligned} & \text{cum}(t_0, t_1, t_2, \dots, t_N) \\ &= F^N(t_0, t_1, \dots, t_N) \\ &+ \mathcal{S}^N(t_1 - t_0, t_2 - t_0, \dots, t_N - t_0) \\ &+ \sum_{1 \leq q_0, q_1, \dots, q_N \leq M_N} \alpha_{q_0, q_1, \dots, q_N} t_0^{q_0} t_1^{q_1} \dots t_N^{q_N} \end{aligned} \tag{9}$$

where function F^N , the projective term, is of the form:

$$F^N(t_0, t_1, \dots, t_N) = \sum_{\ell=0}^N F_\ell(t_0, t_1, \dots, t_{\ell-1}, t_{\ell+1}, \dots, t_N)$$

and function \mathcal{S}^N (stationary term) is such that

$$\mathcal{S}^N(t_{k_1} - t_{k_0}, \dots, t_{k_N} - t_{k_0}) = \mathcal{S}^N(t_1 - t_0, \dots, t_N - t_0)$$

for any permutation $\{k_0, k_1, \dots, k_N\}$ of $\{0, 1, \dots, N\}$.

Condition (C2)

We have:

$$F_\ell(t_1, t_2, \dots, t_N) \times W_{j,n,\ell_1}(t_1) \\ \times W_{j,n,\ell_2}(t_2) \times \dots \times W_{j,n,\ell_N}(t_N) \in L^1(\mathbb{R}^N)$$

and

$$t_0^{q_0} \dots t_N^{q_N} W_{j,n,\ell_0}(t_0) \dots W_{j,n,\ell_N}(t_N) \in L^1(\mathbb{R}^{N+1}).$$

III. STATIONARIZATION

Subband wavelet packet (and therefore wavelet) coefficients of stationary random processes are stationary [1], [2], [3], [4], [5]. For the class of nonstationary random processes satisfying Condition **(ACF)**, this section presents theoretical results stating that their subbands wavelet coefficients are wide sense stationary, provided that the wavelet used has at least $r \geq M+1$ vanishing moments, where M is the degree of the bivariate-polynomial involved in Eq. (8). Furthermore, if conditions **(C1)**, **(C2)** are satisfied and the sequence $(M_N)_{N \geq 2}$ of multivariate-polynomial degrees involved in Eq. (9) is bounded, with $M_\infty = \sup\{M_N : N \geq 2\}$, then strict sense stationarity of the subband coefficients of X follows from wavelets having $r \geq \max\{M, M_\infty\} + 1$ vanishing moments. These results are formalized in Theorems 1 and 2. In the rest of the section, an upper index r (notation $c_{j,n}^r$, $R_{j,n}^r$, $\text{cum}_{j,n}^r$ and $\mathcal{M}_{j,n,k}^r$) denotes that wavelet subbands are generated from a wavelet function \mathbf{W}_1 having r vanishing moments. We will also assume that the wavelet filters are with finite impulse response. Specifically, from Section II-B, we have: $\mathcal{M}_{j,n,k}^r(p) = 0$ for every $n \neq 0$ and $p = 0, 1, \dots, r-1$.

A. Wide-sense stationarity

Let $R_{j,n}^r$ be the autocorrelation function of the sequence $c_{j,n}^r$ of subband $\mathbf{W}_{j,n}$ wavelet coefficients. Assume that condition **(ACF)** holds true. Then $F(t)W_{j,n,k}^r(t) \in L^1(\mathbb{R})$, $t^p s^q W_{j,n,k}^r(t)W_{j,n,\ell}^r(s) \in L^1(\mathbb{R}^2)$ and thus, we have:

$$R_{j,n}^r[k, \ell] = \mathcal{M}_{j,n,\ell}^r(0) \times \left(\int_{\mathbb{R}} F(t) W_{j,n,k}^r(t) dt \right) \\ + \mathcal{M}_{j,n,k}^r(0) \times \left(\int_{\mathbb{R}} F(t) W_{j,n,\ell}^r(t) dt \right) \\ + \sum_{1 \leq p, q \leq M} \alpha_{p,q} \mathcal{M}_{j,n,k}^r(p) \times \mathcal{M}_{j,n,\ell}^r(q) \\ + \int_{\mathbb{R}} \int_{\mathbb{R}} \mathcal{S}(t-s) W_{j,n,k}^r(t) W_{j,n,\ell}^r(s) dt ds$$

Taking into account Eq. (5), it follows that $R_{j,n}^r$ only depends on \mathcal{S} when $n \neq 0$ and $r \geq M+1$:

$$R_{j,n}^r[k, \ell] = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathcal{S}(t-s) W_{j,n,k}^r(t) W_{j,n,\ell}^r(s) dt ds. \quad (10)$$

If we assume further that:

$$\iint_{\mathbb{R}^2} \mathcal{S}(t-s) W_{j,n,k}^r(t) W_{j,n,\ell}^r(s) dt ds < \infty.$$

and that \mathcal{S} has a Fourier transform (in $L^1(\mathbb{R})$ or $L^2(\mathbb{R})$ sense), then we have, using the same notation and with the above assumptions:

Theorem 1

Under Condition (ACF), the discrete random sequence $c_{j,n}^r, n \neq 0$, is wide sense stationary for $r \geq M+1$: $R_{j,n}^r[k, \ell] \equiv R_{j,n}^r[k-\ell]$, with

$$R_{j,n}^r[m] = \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F} \mathcal{S}(\omega) \left| \mathcal{F} W_{j,n}^r(\omega) \right|^2 e^{i2jm\omega} d\omega. \quad (11)$$

From Theorem 1, it follows that both projective and bivariate-polynomial terms have no impact on the wavelet packet autocorrelation functions when the wavelet function used has enough vanishing moments. Therefore, stationarity of wavelet packet coefficients follows under an integrability condition on \mathcal{S} .

Example 2

If X is a fractional Brownian motion with Hurst parameter H , then from the formalism given above, applying the Fourier transform to $F(x) = \mathcal{S}(x) = |x|^{2H}$ can be inferred from the distributional sense. This leads to:

$$\mathcal{F} \mathcal{S}(\omega) = \frac{\sigma^2 \Gamma(2H+1) \sin(\pi H)}{|\omega|^{2H+1}}, \quad (12)$$

where Γ is the standard Gamma function. However, note that the result can be proven without referring to the distributions, as performed in [8] (wavelet framework), [14] (wavelet packet framework).

Remark 1

Assume that $\mathcal{P} \neq \mathcal{P}_0$. Then, in absence of a bivariate-polynomial term, wavelet filters have the same stationarization effect. Indeed, only the first moment is required for annihilating the projective terms and all wavelets functions $W_{j,n,k}$, $n \neq 0$, have their first moments that vanish: $\mathcal{M}_{j,n,k}^r(1) = 0$ for every

$n \neq 0$ and every $r \geq 1$, by construction. In contrast, when a bivariate-polynomial term is present in the autocorrelation function of X , then only wavelets with $r \geq M + 1$ vanishing moments have the desirable stationarization property.

B. Higher order stationarity

Let $N \in \mathbb{N}$. The cumulant of order $N + 1$ of the random process $c_{j,n}^r$ is given by

$$\begin{aligned} \text{cum}_{j,n}^r[k, \ell_1, \ell_2, \dots, \ell_N] &= \text{cum}\{c_{j,n}^r[k]c_{j,n}^r[\ell_1]c_{j,n}^r[\ell_2] \dots c_{j,n}^r[\ell_N]\} \\ &= \int_{\mathbb{R}^{N+1}} dt ds_1 ds_2 \dots ds_N \text{cum}(t, s_1, s_2, \dots, s_N) W_{j,n,k}^r(t) \\ &\quad W_{j,n,\ell_1}^r(s_1) W_{j,n,\ell_2}^r(s_2) \dots W_{j,n,\ell_N}^r(s_N). \end{aligned}$$

When $N = 1$, then $\text{cum}_{j,n}^r[k, \ell] = R_{j,n}^r[k, \ell]$ and wavelet packets are wide sense stationary under condition **(ACF)** and further assumptions used in Section III-A. These assumptions are supposed to hold true in this section and we also assume that conditions **(C1)** and **(C2)** are satisfied. Then, if we proceed as in Section III-A and if we take into account the null-moment condition given by Eq. (5), we obtain:

$$\begin{aligned} \text{cum}_{j,n}^r[k, \ell_1, \ell_2, \dots, \ell_N] &= \int_{\mathbb{R}^{N+1}} dt ds_1 ds_2 \dots ds_N \mathcal{S}^N(s_1, s_2, \dots, s_N) W_{j,n,k}^r(t) \\ &\quad W_{j,n,\ell_1}^r(t + s_1) W_{j,n,\ell_2}^r(t + s_2) \dots W_{j,n,\ell_N}^r(t + s_N) \end{aligned}$$

for $r \geq M_N + 1$ that is, $\text{cum}_{j,n}^r[k, \ell_1, \ell_2, \dots, \ell_N]$ depends only on the stationary term \mathcal{S}^N .

Furthermore, let us assume that:

$$\begin{aligned} &\mathcal{S}^N(s_1, s_2, \dots, s_N) W_{j,n,k}^r(t) W_{j,n,\ell_1}^r(t + s_1) \\ &\quad \times W_{j,n,\ell_2}^r(t + s_2) \dots W_{j,n,\ell_N}^r(t + s_N) \in L^1(\mathbb{R}^{N+1}) \end{aligned}$$

and that the Fourier transform of \mathcal{S}^N exists, then, $\text{cum}_{j,n}^r[k, \ell_1, \ell_2, \dots, \ell_N]$ can be written in the following form

$$\begin{aligned} \text{cum}_{j,n}^r[k, \ell_1, \ell_2, \dots, \ell_N] &= \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} d\omega_1 d\omega_2 \dots d\omega_N \\ &\quad e^{-iM^j((\ell_1-k)\omega_1 + \dots + (\ell_N-k)\omega_N)} \\ &\quad \mathcal{F} \mathcal{S}^N(-\omega_1, -\omega_2, \dots, -\omega_N) \\ &\quad \mathcal{F} W_{j,n}^r(-\omega_1 - \omega_2 - \dots - \omega_N) \\ &\quad \mathcal{F} W_{j,n}^r(\omega_1) \mathcal{F} W_{j,n}^r(\omega_2) \dots \mathcal{F} W_{j,n}^r(\omega_N) \end{aligned}$$

so that, with the assumptions encountered above and those used to obtain Theorem 1, we have:

Theorem 2

Under conditions **(ACF)**, **(C1)**, **(C2)**, we have that $c_{j,n}^r$, $n \neq 0$, is strictly stationary for $r \geq \max\{M, M_\infty\} + 1$: for every $N \geq 1$, $\text{cum}_{j,n}^r[k, \ell_1, \ell_2, \dots, \ell_N] \equiv \text{cum}_{j,n}^r[\ell_1 - k, \ell_2 - k, \dots, \ell_N - k]$, with

$$\begin{aligned} & \text{cum}_{j,n}^r[k_1, k_2, \dots, k_N] \\ &= \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} d\omega_1 d\omega_2 \dots d\omega_N \\ & \quad e^{-iM^j(k_1\omega_1 + k_2\omega_2 + \dots + k_N\omega_N)} \\ & \quad \mathcal{F}\mathcal{S}^N(-\omega_1, -\omega_2, \dots, -\omega_N) \\ & \quad \mathcal{F}W_{j,n}^r(-\omega_1 - \omega_2 - \dots - \omega_N) \\ & \quad \mathcal{F}W_{j,n}^r(\omega_1)\mathcal{F}W_{j,n}^r(\omega_2)\dots\mathcal{F}W_{j,n}^r(\omega_N). \end{aligned}$$

Remark 2

Stationarization properties obtained in Theorems 1 and 2 assume $n \neq 0$. When $n = n(j) = 0$ for every j , the path concerned is the *approximation path* denoted by \mathcal{P}_0 (all other paths are said to be *detail paths*). This path is subject to a specific behavior because the subbands $\mathbf{W}_{j,n=0}^r, j \geq 1$, involved in this path are generated from functions $\{W_{j,n=0,k}^r : k \in \mathbb{Z}\}$: these functions are scaling functions and do not have vanishing moments. Thus the contribution of the projective and the multivariate terms in the autocorrelation function of the approximation coefficients do not annihilate: this implies that no stationarization can occur in general for the approximation coefficients. The same remark holds true for higher order cumulants because of the contribution of the nonstationary terms involved in the cumulants of X .

IV. DEPENDENCIES

This section presents some results concerning the capability of wavelets for decorrelating the coefficients of stochastic processes satisfying assumption **(ACF)**. It provides additional results concerning higher order dependency reduction induced by wavelets on stochastic processes satisfying assumptions **(ACF)**, **(C1)** and **(C2)**.

From Theorem 1, the autocorrelation function of the wavelet packet coefficients can be written in the form of Eq. (11), under the assumption that the wavelet function W_1 has $r \geq M + 1$ vanishing

moments. In Eq. (11), $\mathcal{F}\mathcal{S}$ can be seen as the *wavelet spectrum* of the random process X . Distributing $c_{j,n}^r$ as a sequence of decorrelated coefficients involves finding parameters that make $R_{j,n}^r[m]$ vanishes for every $m \in \mathbb{Z} \setminus \{0\}$. Since no restrictions are imposed on $\mathcal{F}\mathcal{S}$, apart those required for integrability condition, then the parameters that govern the behavior of $R_{j,n}^r$ are the shape and the support of $\mathcal{F}W_{j,n}^r$.

One can probably design a wavelet function, depending on the close form of $\mathcal{F}\mathcal{S}$ so as to yield vanishing $R_{j,n}^r[m]$, for $m \neq 0$. In such a scenario, the wavelet function is computed adaptively with respect to the input random process, yielding a Karhuren-Loève-like expansion. The first limitation of this approach is that such an adaptive consideration can be restrictive when a large class of stochastic processes is concerned. In addition, the above consideration is also limited because the spectrum $\mathcal{F}\mathcal{S}$ is usually unknown.

In order to seek wavelet decorrelation capability, unconditionally with respect to the input process, the remaining parameter is the size of the support of $\mathcal{F}W_{j,n}^r$. Indeed, by drastically reducing this size, we can expect to reduce the amplitude $R_{j,n}^r[m]$. Note that support reduction is the trick used to construct the Shannon wavelets: by dividing the support of $\mathcal{F}W_{j,n}^r$ per 2 when j increases, the spectrum $\mathcal{F}\mathcal{S}$ is analysed on a very tight frequency interval when j is large. In this sense, these wavelets provide us with a framework for analyzing the decorrelation induced by wavelets.

Theorem 3 below formalizes the above heuristic considerations. In this theorem, as well as in the rest of the paper, we need to describe $\mathcal{F}W_{j,n}^r$ with an additional parameter that relates to the size of the support of $\mathcal{F}W_{j,n}^r$ or, equivalently, describe how close $\mathcal{F}W_{j,n}^r$ is, with respect to the corresponding Shannon wavelet function. For the sake of generality, this parameter needs to be different with the number r of vanishing moments of the wavelet function W_0 since support size of a function is not necessarily connected with the number of vanishing moments of this function [16]. However, for the standard family of wavelet filters, the support size is linked to the number of wavelet vanishing moments so we can save notation. Section IV-A below provides this connection.

A. Wavelet order: the connection between the wavelet support size and the number of wavelet vanishing moments

Consider a filter with impulse response $h_0 = (h_0[\ell])_{\ell \in \mathbb{Z}}$. Let us define, up to a factor $1/\sqrt{2}$, the Fourier transform of h_0 by:

$$H_0(\omega) = \frac{1}{\sqrt{2}} \sum_{\ell \in \mathbb{Z}} h_0[\ell] e^{-i\ell\omega}.$$

The non-negative integer s such that H_0 admits the polynomial factorization [18]

$$H_0(\omega) = \left(\frac{1 + e^{-i\omega}}{2} \right)^s Q(e^{i\omega}), \quad (13)$$

is called the *filter order* and is also called *s-regularity* in [19]. The factorization given by Eq. (13) assumes that Q has no poles or zeros at $\omega = \pi$. From this factorization, it follows that order s describes the flatness of H_0 at $\omega = 0$ and $\omega = \pi$. It is worth stressing that $H_0(\pi) = 0$ so that increasing the flatness of H_0 at $\omega = \pi$ involves reducing the support size of H_0 .

Assume now that H_0 is a *scaling filter* [16], [17], [19]. Let H_1 be the *wavelet filter* associated with H_0 : (H_0, H_1) is a couple of paraunitary filters. Then the same remark as above holds true for H_1 due to this paraunitary condition: by inverting the role played by $\omega = 0$ and $\omega = \pi$, it follows that increasing the flatness of H_0 at $\omega = \pi$ involves increasing the flatness of H_1 at $\omega = 0$ and, consequently, this implies reducing the support size of H_1 . Furthermore, the wavelet function $\mathcal{F}W_1$ inherits the above properties of H_1 by taking into account that $\mathcal{F}W_1(\omega) = H_1(\frac{\omega}{2})\mathcal{F}W_0(\frac{\omega}{2})$. Moreover, parameter s is exactly the number of vanishing moments (parameter r in the previous sections) of the wavelet function W_1 .

From the above analysis, we decide to recover the upper index r in the notation of wavelet functions (see Section III). However, this index will be written with the following conventional notation: r , meaning that wavelet functions are generated from, or generate paraunitary filters with order r (factorization given by Eq. (13)). These wavelets are also said with order r .

If we consider standard families of wavelet filters such as Daubechies, Symmlet or Battle-Lemarié spline filters, then the filters corresponding to $r = 1$ and $r = +\infty$ are respectively the Haar and the Shannon filters. These filters play an important role in describing the filter families mentioned above: for a given order $1 < r < +\infty$, the shapes of a couple of paraunitary filters $(H_\epsilon^{[r]})_{\epsilon \in \{0,1\}}$ can be inferred by their closeness to the shapes of $(H_\epsilon^{[1]})_{\epsilon \in \{0,1\}}$, Haar paraunitary filters, or $(H_\epsilon^S)_{\epsilon \in \{0,1\}}$, Shannon paraunitary filters. This assumption is reasonable because when the order r increases from 1 to $+\infty$, then the sequence $(H_\epsilon^{[r]})_{[r] \geq 1}$ converges almost everywhere to H_ϵ^S and the shapes of $(H_\epsilon^{[r]})_r$ vary smoothly between the shape of $H_\epsilon^{[1]}$ and that of H_ϵ^S on both frequency intervals $] -\pi/2, \pi/2[$ and $] -\pi, -\pi/2[\cup] \pi/2, \pi[$ (see [14] for details).

B. Correlation structure

This section provides decorrelating properties of wavelets for the coefficients of stochastic processes satisfying **(ACF)**. These properties are consequences of some asymptotic results, depending

on the wavelet decomposition level and the wavelet order. Stating these asymptotic results require a whole wavelet packet path specification. This specification can be achieved by considering the infinite sequence of filter indices defined by $\mathcal{P} = \{\epsilon_j\}_{j \in \mathbb{N}}$ or, equivalently, by indexing the subbands involved by this filtering sequence: $\mathcal{P} = \left(\mathbf{W}_{0,0}^{[r]}, \left\{ \mathbf{W}_{j,n_{\mathcal{P}}(j)}^{[r]} \right\}_{j \in \mathbb{N}} \right)$. We use the above notation in the following. See [5] for more details on such a characterization.

With the same notation and assumptions proposed in Section III-A, we have:

Theorem 3

Let $\mathcal{P} = \{\epsilon_j\}_{j \in \mathbb{N}} = \left(\mathbf{W}_{0,0}^{[r]}, \left\{ \mathbf{W}_{j,n_{\mathcal{P}}(j)}^{[r]} \right\}_{j \in \mathbb{N}} \right)$ be a path in the wavelet packet tree. Assume that $\mathcal{P} \neq \mathcal{P}_0$ where \mathcal{P}_0 is the approximation path.

Assume $\mathcal{F}\mathcal{S}$ is continuous at the frequency $\omega_{\mathcal{P}}$ defined by

$$\omega_{\mathcal{P}} = \lim_{j \rightarrow +\infty} \frac{\mathcal{J}(n_{\mathcal{P}}(j))\pi}{2^j},$$

where \mathcal{J} is a permutation recursively defined by $\mathcal{J}(2\ell + \epsilon) = 3\mathcal{J}(\ell) + \epsilon - 2 \left\lfloor \frac{\mathcal{J}(\ell) + \epsilon}{2} \right\rfloor$.

Then, the autocorrelation $R_{j,n_{\mathcal{P}}(j)}^{[r]}$ of $c_{j,n_{\mathcal{P}}(j)}^{[r]}$ uniformly satisfies:

$$\lim_{j \rightarrow +\infty} \left(\lim_{[r] \rightarrow +\infty} R_{j,n_{\mathcal{P}}(j)}^{[r]}[k] \right) = \mathcal{F}\mathcal{S}(\omega_{\mathcal{P}})\delta[k]. \quad (15)$$

Proof: Since the contribution of the nonstationary terms induced by the projective and the bivariate-polynomial terms annihilates when $\mathcal{P} \neq \mathcal{P}_0$, then the result follows by mimicking the proofs of [4, Proposition 1, Theorem 1]. ■

Let us consider path \mathcal{P}_0 , which is the sole path that is not concerned by Theorem 3:

Proposition 1

$$\omega_{\mathcal{P}} = 0 \iff \mathcal{P} = \mathcal{P}_0.$$

Proof: Proposition 1 follows from that the approximation path is the unique path such that the sequence $(n_{\mathcal{P}}(j))_{j \in \mathbb{N}}$ of frequency indices associated with path \mathcal{P} can be upper bounded by a constant independent with j . ■

Remark 3

From Theorem 3 and Proposition 1, it follows that asymptotic decorrelation cannot be expected in the neighborhood of the null frequency, even when $\omega = 0$ is a continuity point of the spectrum $\mathcal{F}\mathcal{S}$.

C. High order dependencies

As in Section IV-B, the filter order plays an important role for dependency reduction through wavelet packet decomposition levels. When this order is maximal, that is, when the Shannon filters are concerned, we have:

$$\begin{aligned} & \text{cum}_{j,n}^S[k_1, k_2, \dots, k_N] \\ &= \frac{2^{j(N+1)/2}}{(2\pi)^N} \int_{\Delta_{j,\mathcal{F}(n)}^N} \mathbf{d}\omega_1 \mathbf{d}\omega_2 \dots \mathbf{d}\omega_N \\ & \quad e^{-i2^j(k_1\omega_1 + k_2\omega_2 + \dots + k_N\omega_N)} \\ & \quad \mathcal{F}\mathcal{S}^N(-\omega_1, -\omega_2, \dots, -\omega_N) \\ & \quad \mathbf{1}_{\Delta_{j,\mathcal{F}(n)}(\omega_1 + \omega_2 + \dots + \omega_N)}. \end{aligned}$$

where $\Delta_{j,\mathcal{F}(n)}^N = \underbrace{\Delta_{j,\mathcal{F}(n)} \times \Delta_{j,\mathcal{F}(n)} \times \dots \times \Delta_{j,\mathcal{F}(n)}}_{N \text{ times}}$.

The above integral involves computing the integrand in Eq. (16) on the narrow hypercube $\Delta_{j,\mathcal{F}(n)}^N$. If we assume that $\mathcal{S}^q \in L^\infty(\mathbb{R}^N)$, then, $\text{cum}_{j,n}^S[k_1, k_2, \dots, k_N]$ vanishes when j tends to ∞ . More precisely, for any natural number $N \geq 2$, we have:

$$|\text{cum}_{j,n}^S[k_1, k_2, \dots, k_N]| \leq \frac{\|\mathcal{S}^N\|_\infty}{M^{j(N-1)/2}}.$$

Finally, since the sequences of filters considered in this paper (see Eq. (13)) converge to the Shannon filters when their order r increases. Then, it is reasonable to expect that higher order filters will yield cumulant decay with a factor that is close to the decay induced by the Shannon filters.

D. Wavelet filter support sizes

By considering Eq. (2), we can derive the “equivalent” filter applied to obtain the subband (j, n) wavelet coefficients. This equivalent filter is $2^{j/2}\mathbf{H}_{j,n}(\omega)$, with:

$$\mathbf{H}_{j,n}(\omega) = \left[\prod_{\ell=1}^j H_{e_\ell}(2^{\ell-1}\omega) \right]. \quad (17)$$

When $j = 1$, the filter involved in Eq. (17) is either the scaling filter H_0 (low-pass) or the wavelet filter H_1 (high-pass). When $j > 1$, then \mathbf{H} is obtained from a combination of low-pass and high-pass filterings from $\mathcal{P}(j)$.

From the analysis presented in this section, it follows that dependency reductions are strongly linked to the width of the support and the shape of $\mathcal{F}W_{j,n}^{[r]}$ and thus, those for $\mathbf{H}_{j,n}^{[r]}$. It follows that tighter supports for $\mathbf{H}_{j,n}^{[r]}$ would yield more dependency reduction because when observed on a small window size, a spectrum $\mathcal{F}\mathcal{S}$ with regular shape can be seen as approximately constant on this support. This is what actually occurs for the Shannon filters ($r = S \equiv \infty$), in contrast with the Haar filters ($r = 1$). Figure 1 provides the graphs of the Haar and the Shannon filters and Figure 2 provides the graphs of their equivalent filters $\mathbf{H}_{4,0}^{[r]}(\omega)$ and $\mathbf{H}_{4,1}^{[r]}(\omega)$ at decomposition level 4. The latter are obtained from the filtering sequences $(h_0^{[r]}, h_0^{[r]}, h_0^{[r]}, h_0^{[r]})$ and $(h_0^{[r]}, h_0^{[r]}, h_0^{[r]}, h_1^{[r]})$, respectively. We recall that, given $\epsilon \in \{0, 1\}$, the Haar filters are such that:

$$H_\epsilon^{[1]}(\omega) = \frac{1}{2} \left(1 + (1 - 2\epsilon)e^{-i\omega} \right) \quad (18)$$

and that the Shannon filters satisfy:

$$H_\epsilon^S(\omega) = \sum_{\ell \in \mathbb{Z}} \mathbf{1}_{\Delta_\epsilon}(\omega - 2\pi\ell), \quad (19)$$

where $\Delta_0 = [-\pi/2, \pi/2]$ and $\Delta_1 = [-\pi, -\pi/2] \cup [\pi/2, \pi]$.

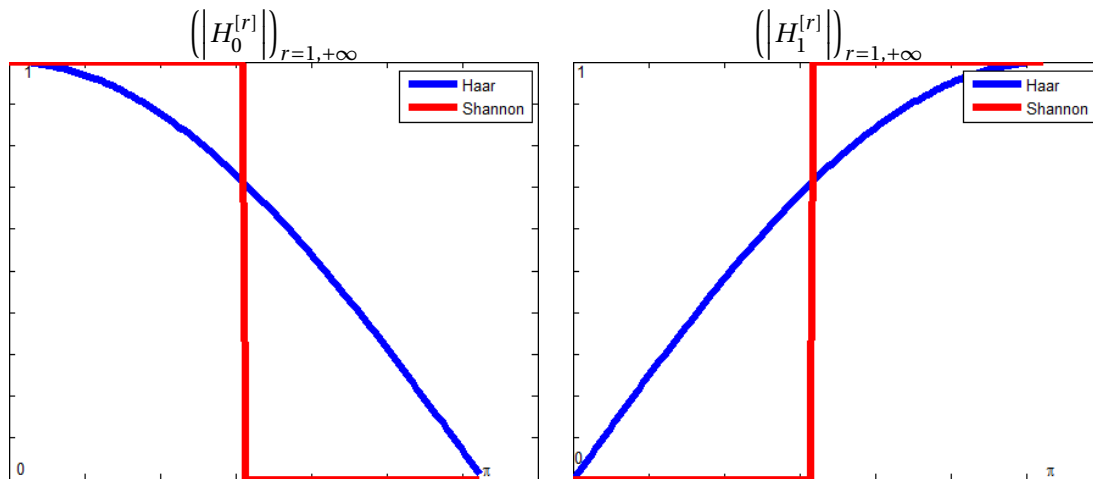


Fig. 1. Haar ($r = 1$) versus Shannon ($r = \infty$) low-pass and high-pass filters.

As can be seen in Figure 2, the frequency support of the equivalent filter of a sequence of Shannon filtering is very tight whereas the contrary holds for the sequence of Haar filters.

V. STOCHASTIC PROCESSES INVOLVING SINGULAR WAVELET PACKET PATHS AND THEIR MODELING

From theorems 1, 2, 3 and the results of Section IV-C, it follows that under the null-moment condition, then detail wavelet packet coefficients become stationary in the wide sense (resp. strict

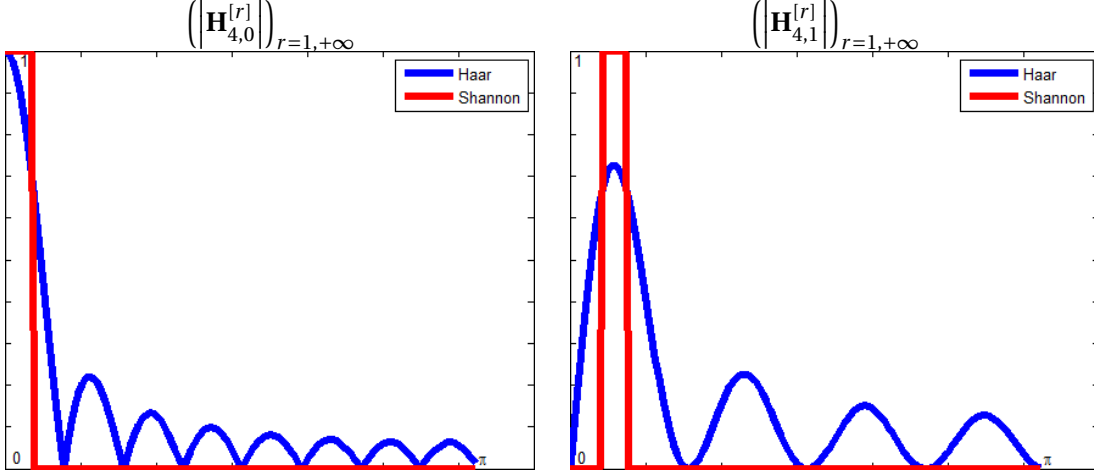


Fig. 2. Normalized equivalent filters $\mathbf{H}_{4,0}^{[r]}(\omega)$ and $\mathbf{H}_{4,1}^{[r]}(\omega)$ for the Haar ($r = 1$) and the Shannon ($r = \infty$) wavelets at decomposition level 4, the filtering sequences used are $(h_0^{[r]}, h_0^{[r]}, h_0^{[r]}, h_0^{[r]})$ and $(h_0^{[r]}, h_0^{[r]}, h_0^{[r]}, h_1^{[r]})$, respectively.

sense) for processes satisfying conditions **(ACF)** (resp. **(C1)**, **(C2)**). Under the same conditions, and provided that the decomposition level and the filter order are large enough, then correlation and higher order dependency reduction follow. The above properties depend on the shape of the input process spectrum: the asymptotic results are attained in any detail path of the wavelet packet tree associated to a frequency $\omega_{\mathcal{P}}$ which is a continuity point of the spectrum $\mathcal{F}\mathcal{S}$. Let us focus on the paths for which no decorrelation can be expected. This is the case for paths associated with poles of $\mathcal{F}\mathcal{S}$. We consider the following examples.

Example 3

[Singular paths for ARFIMA type spectrum]

Assume that $\mathcal{F}\mathcal{S}$ has the following form:

$$\mathcal{F}\mathcal{S}(\omega) = \frac{\sigma^2}{|4 \sin(\frac{\omega}{2})|^\delta} \times B(\omega), \quad (20)$$

with $\omega \in [-\pi, \pi]$. Then, X will be said to be an ARFIMA process in the sense that its spectrum Eq. (20) with respect to the wavelet packet transform is of ARFIMA type.

Let $B(\omega) = 1$, then X is fractionally integrated with no autoregressive and moving average parts. If $\delta = 0$, then X is a white Gaussian process and the wavelet packet coefficients are uncorrelated at every decomposition level. If $\delta \neq 0$, then the unique pole of $\mathcal{F}\mathcal{S}$ is $\omega = 0$ and thus, $c_{j,n_{\mathcal{P}}(j)}^{[r]}$ tends to decorrelate with j, r in every path $\mathcal{P} \neq \mathcal{P}_0$.

Now, consider

$$B(\omega) = \frac{|1 - \sum_{k=1}^p \mu_k e^{-ik\omega}|^2}{|1 - \sum_{\ell=1}^q \phi_\ell e^{-i\ell\omega}|^2},$$

that is, X is fractionally integrated with autoregressive parts “driven by” $(\phi_\ell)_\ell$ and moving average parts driven by $(\mu_\ell)_\ell$. Assume that $B(\omega)$ has K distinct and non-null poles ω_ℓ for $\ell = 1, 2, \dots, K$, with $K \leq q$. Then, $\mathcal{F}\mathcal{S}$ has poles $\omega \in \{0, \omega_1, \omega_2, \dots, \omega_K\}$ and asymptotic decorrelation follows for all wavelet packet paths, except for the paths associated with the above poles.

Example 4

[Singular paths for k -factor Gegenbauer type spectrum]

Assume that $\mathcal{F}\mathcal{S}$ has the following form:

$$\mathcal{F}\mathcal{S}(\omega) = \prod_{i=1}^k \frac{\sigma^2}{\{2|\cos\omega - \psi_i|\}^{2\delta_i}}.$$

$\mathcal{F}\mathcal{S}$ is a k -factor Gegenbauer ARMA (GARMA) type spectrum and, as above, we will say that X is a k -factor GARMA process. The poles of $\mathcal{F}\mathcal{S}$ are the Gegenbauer frequencies

$$\omega_{G_i} = \cos^{-1} \psi_i, \quad i = 1, 2, \dots, k.$$

Thus, the decorrelating properties of the wavelet packet decomposition concerns paths $\mathcal{P} \notin \{\mathcal{P}_0, \mathcal{P}_{\omega_{G_i}} : i = 1, 2, \dots, k\}$.

In order to identify the wavelet packet path associated with a given pole ω^* , we need to find the sequence of frequency indices $(n(j))_{j \in \mathbb{N}}$ such that $n(j) = n_{\mathcal{P}^*(j)}$ is subject to Eq. (1) and satisfies

$$\lim_{j \rightarrow +\infty} \frac{\mathcal{J}(n_{\mathcal{P}^*(j)})\pi}{2^j} = \omega^*.$$

For instance, consider the ARFIMA process with: $\sigma = 1$, $\delta = 1/2$, $\mu_1 = 1$, $\phi_1 = e^{i\omega_1} + e^{i\omega_2}$, $\phi_2 = e^{2i\omega_2} (1 - e^{-i\omega_2} (e^{i\omega_1} + e^{i\omega_2}))$, where $\omega_1 = 3\pi/4 - \Delta\omega$ and $\omega_2 = 3\pi/4 + \Delta\omega$. This process admits 3 poles at frequencies $\omega = 0$, ω_1 and ω_2 . Small values are chosen for $\Delta\omega$ so as to have ω_1 and ω_2 closer to the “central” frequency $3\pi/4$: $\Delta\omega \in \{\pi/48, \pi/72\}$ in Figure 3. Note that the path associated with $3\pi/4$ is characterized by the frequency indices $\mathcal{J}(n(j)) = 2^{j-1} + 2^{j-2}$. Thus, by using the inverse permutation \mathcal{J}^{-1} (see [14]), this path is characterized by the frequency indices $n(j) = 2^{j-1} + 2^{j-3}$ for $j \geq 3$.

We synthesize such an ARFIMA process and compute its spectrum by using the wavelet packet method of [14]. Figure 3 presents the wavelet packet spectra obtained, as well as spectra computed on Welch’s method (averaged and modified periodogram computed on the basis of the discrete

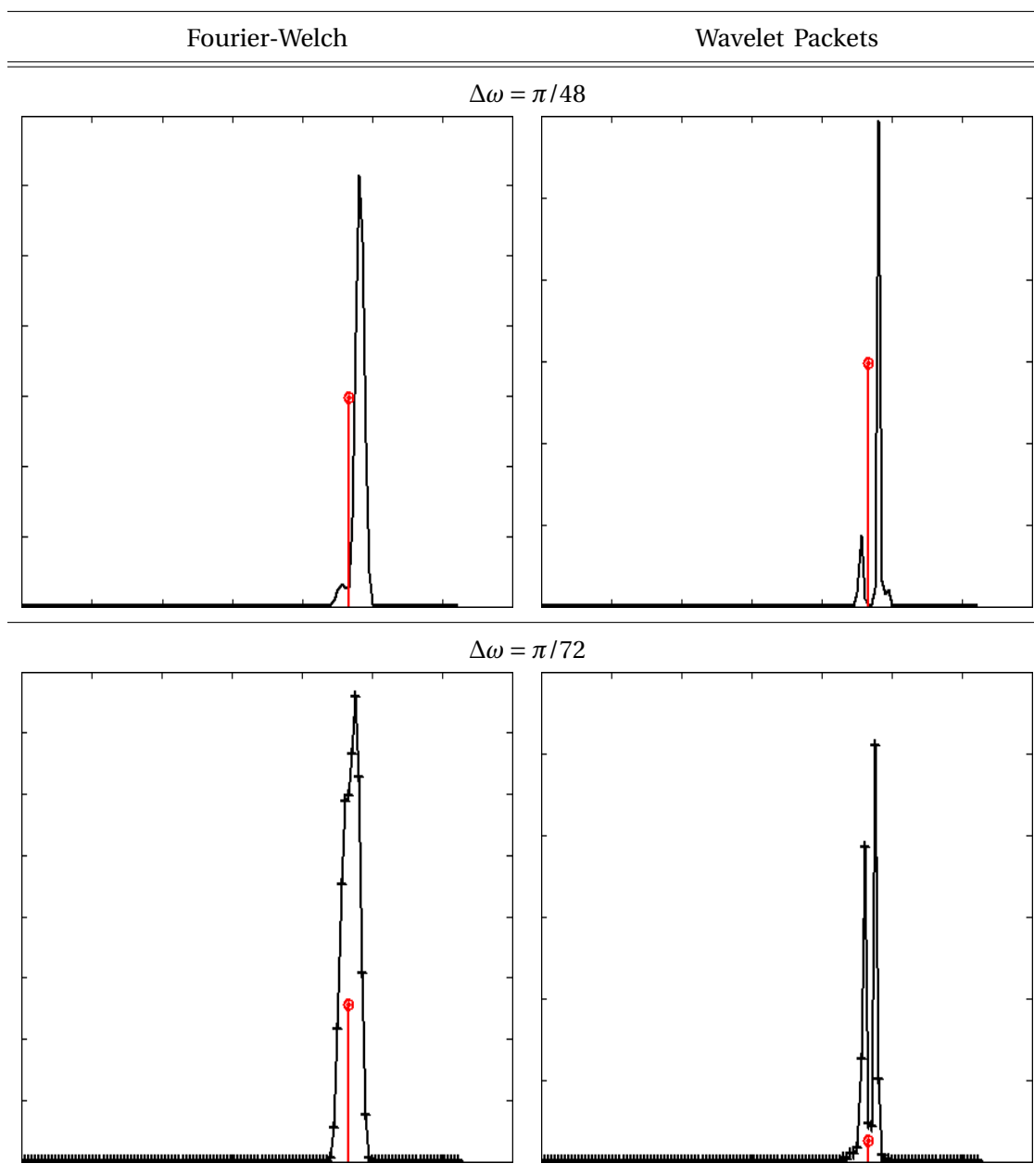


Fig. 3. “Fourier-Welch” and “Wavelet packet” spectra of ARFIMA processes, where any ARFIMA function B used admits two poles at frequencies $\omega = 3\pi/4 \pm \Delta\omega$. The graphs are given for $\Delta\omega = \pi/48$ and $\Delta\omega = \pi/72$. The “central” frequency $3\pi/4$ is marked by a vertical (red) line.

Fourier transform). As can be seen, the two poles of $B(\omega)$ (see Eq. (20)) are well detected by the wavelet packet method in terms of narrow peaks in the neighborhoods of the poles. In contrast, the peaks tend to overlap when using Welch's method, when $\Delta\omega$ is small.

In practice, the presence of singularities, in terms of narrow peaks, within the wavelet packet spectrum can ease model selection from many stochastic processes encountered in real world data, signals and images. Indeed, a path associated with a pole reflects a *long memory* behavior for certain “events” characterizing the process since the wavelet packet autocorrelations are non-summable in this path. In this sense, ARFIMA or GARMA models can be used to represent the input random process, and wavelet packet spectrum makes explicit the form of the denominator of B . Note that long memory property has been established to arise in biomedical and telecommunication signals. The following highlights that it also arises in many real word texture images.

Wavelet packet spectra of some texture images are shown in Figure 4. The wavelet packet spectra have been computed from the two-dimensional extension of the method given in [14]. Spectra computed from the Fourier are also given in this figure, for comparison.

The spectra given in Figure 4 show peaks at zero frequency, reflecting the contribution of fractional terms, as well as peaks at other spatial frequencies, indicating terms induced by ARMA-type poles. As a matter of fact, the long memory hypothesis inducing these peaks can be inferred through a visual analysis of the content of the input images: let us consider, for instance, the image “Fabric.11”. This image is characterized by non-overlapping bands, every band having a specific content. These bands replicate repeatedly, depending on certain spatial frequencies. We can mainly “distinguish” two frequencies if we consider the two coarse bands. Moreover, we can say that these frequencies are close since the occurrences of the bands approximately occupy the same space and interact as an on-and-off system. The graph of the wavelet packet spectrum of this image clearly shows these frequencies, whereas the graph obtained by computing the two dimensional discrete Fourier transform exhibits only one peak (see Figure 4). Furthermore, from the wavelet packet spectrum of the “Fabirc.11” image, we can see that the two frequencies under consideration are close, as stated above by using the visual inspection criterion. It is worth emphasizing that the poorness of the content of Fourier spectra is not due to a lack of resolution in the sampling step of the Fourier transform. This poorness can be explained by noting that Fourier transform is sensitive to global spatial regularity, whereas wavelet packets can capture local spatial regularity and lead to a more informative spectrum.

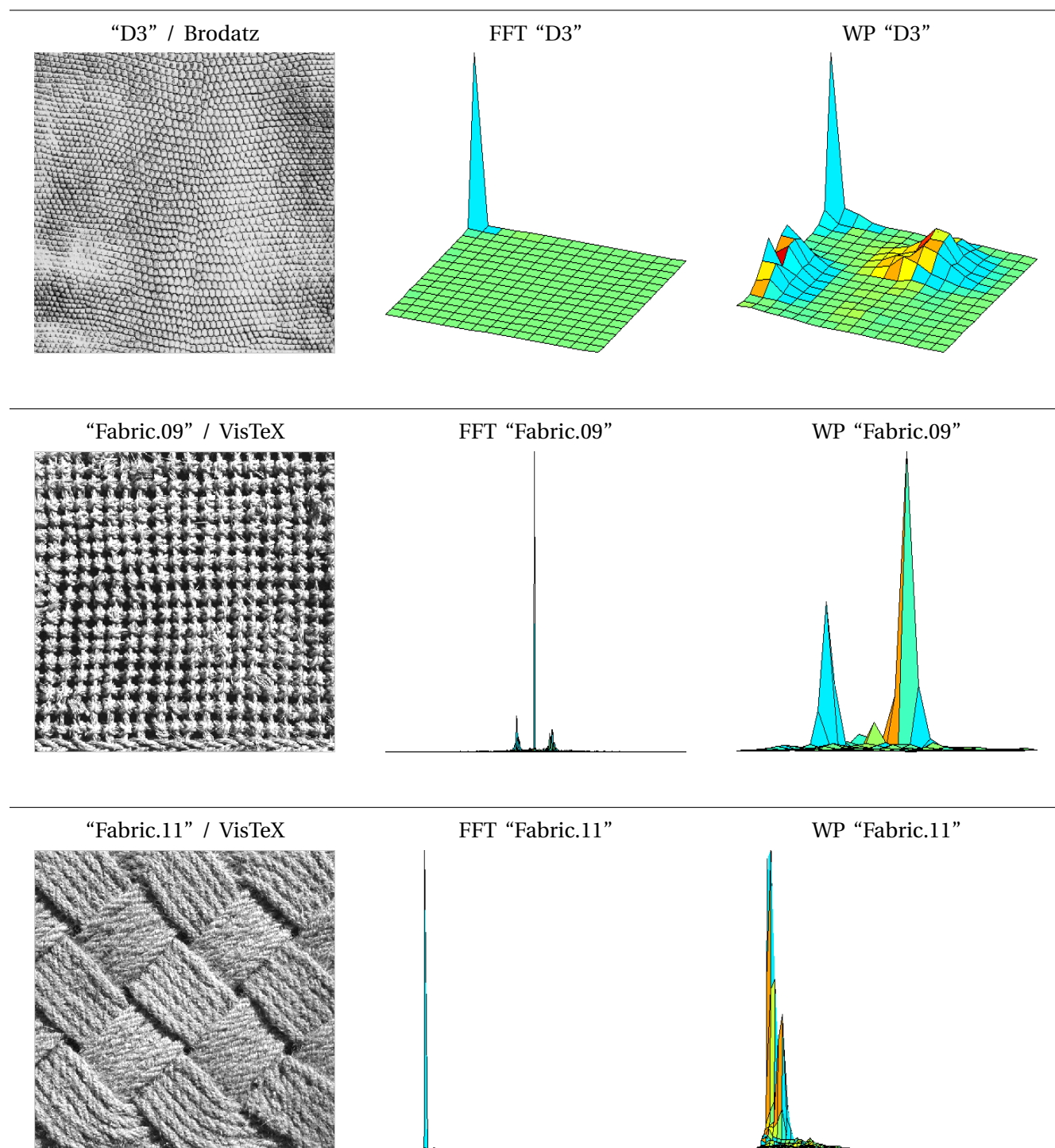


Fig. 4. Textures "Fabrics" from the VisTeX database.

VI. CONCLUSION

In this paper, statistical properties of wavelet packet transform have been investigated when the input process is not necessarily stationary. It has been shown that under some weak assumptions on the autocorrelation function and the cumulants of X , the coefficients associated with the projection of a random process X with respect to a path $\mathcal{P} \neq \mathcal{P}_0$ become stationary and tend to decorrelate when the decomposition level and the filter order increases, provided that the limit frequency $\omega_{\mathcal{P}}$ is not a singular point of the wavelet packet spectrum associated with X . The analysis performed in the paper has highlighted that the null-moment condition on the wavelet function plays an important role in the stationarization induced by wavelet based transforms and the decorrelating properties are strongly linked on the width of the support and the shape of the Fourier transform of the function generating wavelet subbands. By exhibiting singular paths, the paper opens some new prospects regarding the analysis and synthesis of stochastic processes: iid Gaussian modeling is not relevant in these paths, even for large decomposition levels. These paths are associated with non summable wavelet packet autocorrelations and reflect long memory behavior. This long memory property has been shown to occur for specific texture images. In this respect, the analysis performed in the paper opens some new prospects in texture modeling in that random processes with long memory property such as ARFIMA and GARMA processes generalize standard ARMA processes used in Wold's based texture analysis [20].

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