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Anna Kuczmaszewska. On complete convergence for arrays of rowwise dependent random variables. *Statistics and Probability Letters*, 2009, 77 (11), pp.1050. 10.1016/j.spl.2006.12.007 . hal-00547985

HAL Id: hal-00547985

<https://hal.science/hal-00547985>

Submitted on 18 Dec 2010

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Author's Accepted Manuscript

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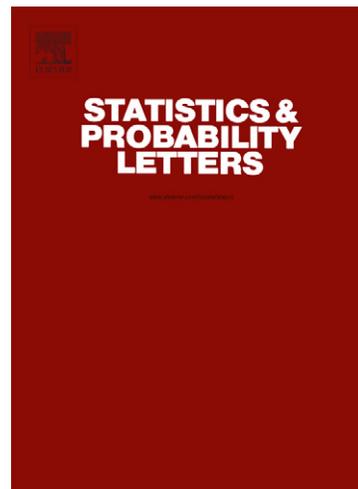
PII: S0167-7152(07)00023-5
DOI: doi:10.1016/j.spl.2006.12.007
Reference: STAPRO 4556

To appear in: *Statistics & Probability Letters*

Received date: 6 January 2006
Revised date: 20 October 2006
Accepted date: 19 December 2006

Cite this article as: Anna Kuczmazewska, On complete convergence for arrays of rowwise dependent random variables, *Statistics & Probability Letters* (2007), doi:10.1016/j.spl.2006.12.007

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On complete convergence for arrays of rowwise dependent random variables

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Abstract

This paper establishes two results for complete convergence in the law of large numbers for arrays under ϱ -mixing and $\tilde{\varrho}$ -mixing association in rows. They extend several known results.

AMS classification: 60F15

Keywords: complete convergence, random variables, ϱ -mixing sequence, $\tilde{\varrho}$ -mixing sequence.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on probability space (Ω, \mathcal{F}, P) . A sequence $\{X_n, n \geq 1\}$ is said to converge completely to a constant C if

$$\sum_{n=1}^{\infty} P[|X_n - C| > \varepsilon] < \infty, \forall \varepsilon > 0.$$

Hsu and Robbins (1947), who introduce this concept, proved that the sequence of arithmetic means of independent identically distributed random variables converges completely to the expected value of the summands, provided the variance is finite. The converse theorem was proved by Erdős.

The extensions of Hsu-Robbins-Erdős's result, due to Katz (1963), Baum and Katz (1965), Chow (1973), form a complete convergence theorem with a Marcinkiewicz-Zygmunt type normalization (see Gut (1983)).

Theorem 1.1. *Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed random variables and let $\alpha r \geq 1$, $\alpha > \frac{1}{2}$. The following statements are equivalent:*

- (i) $E|X_1|^r < \infty$, and, if $r \geq 1$, $EX_1 = 0$,
- (ii) $\sum_{n=1}^{\infty} n^{\alpha r - 2} P\left[\left|\sum_{i=1}^n X_i\right| > n^\alpha \varepsilon\right] < \infty, \forall \varepsilon > 0$,
- (iii) $\sum_{n=1}^{\infty} n^{\alpha r - 2} P\left[\max_{k \leq n} \left|\sum_{i=1}^k X_i\right| > n^\alpha \varepsilon\right] < \infty, \forall \varepsilon > 0$.

If $\alpha r > 1$, $\alpha > \frac{1}{2}$ the above are also equivalent to

- (iv) $\sum_{n=1}^{\infty} n^{\alpha r - 2} P\left[\sup_{k \geq n} k^{-\alpha} \sum_{i=1}^k X_i > \varepsilon\right] < \infty, \forall \varepsilon > 0$.

Many authors generalized and extended this result without assumption of identical distribution in several directions. They studied the cases of independent, stochastically dominated random variables, triangular arrays of rowwise independent, stochastically dominated in the Cesaro sense random variables and sequences of independent random variables taking value in a Banach space (Pruitt (1966), Rohatgi (1971), Hu, Moricz and Taylor (1989), Gut (1992), Wang, Bhaskara Rao

and Yang (1993), Kuczmaszewska and Szynal (1988, 1991, 1994), Sung (1997) Bozorgnia, Patterson and Taylor (1993), Hu, Rosalsky, Szynal and Volodin (1999), Hu, Szynal and Volodin (1998), Hu and Volodin (2000), Ahmed, Antonini and Volodin (2002), Kuczmaszewska (2004), Sung, Volodin and Hu (2005), Tómacs (2005)).

In this paper we consider a complete convergence in the strong law of large numbers for arrays of dependent random variables. We study the complete convergence for ϱ -mixing and $\tilde{\varrho}$ -mixing sequences of random variables. The obtained results extend some previous known. Some results for complete convergence for ϱ -mixing and $\tilde{\varrho}$ -mixing random variables one can find in Zhengyan and Chuanrong (1996) (Shao, Kong and Zang, Section 8.4), Shao (1995), Shixin (2004).

Definition 1.1. A sequence of random variables $\{X_n, n \geq 1\}$ is said to be a ϱ -mixing sequence if the maximal correlation coefficient

$$\varrho(n) = \sup_{k \geq 1, X \in L^2(\mathcal{F}_1^k), Y \in L^2(\mathcal{F}_{k+n}^\infty)} \frac{|\text{cov}(X, Y)|}{\sqrt{\text{Var} X \cdot \text{Var} Y}} \rightarrow 0, \quad (1.1)$$

as $n \rightarrow \infty$, where \mathcal{F}_n^m is the σ -field generated by random variables X_n, X_{n+1}, \dots, X_m .

Definition 1.2. A sequence of random variable $\{X_n, n \geq 1\}$ is said to be a $\tilde{\varrho}$ -mixing sequence if there exists $k \in \mathbb{N}$ such that

$$\tilde{\varrho}(k) = \sup_{S, T} \left(\sup_{X \in L^2(\mathcal{F}_S), Y \in L^2(\mathcal{F}_T)} \frac{\text{cov}(X, Y)}{\sqrt{\text{Var} X \cdot \text{Var} Y}} \right) < 1, \quad (1.2)$$

where S, T are the finite subsets of positive integers such that $\text{dist}(S, T) \geq k$ and \mathcal{F}_W is the σ -field generated by the random variable $\{X_i, i \in W \subset \mathbb{N}\}$.

The $\tilde{\varrho}$ -mixing conception is similar to the ϱ -mixing, but they are quite different from each other.

In our further consideration we need the following definition and lemmas.

Definition 1.3. A real valued function $l(x)$, positive and measurable on $[A, \infty)$ for some $A > 0$, is said to be slowly varying if

$$\lim_{x \rightarrow \infty} \frac{l(x \cdot \lambda)}{l(x)} = 1, \quad \text{for each } \lambda > 0.$$

Lemma 1.1. (Shao, (1995)). Let $\{X_n, n \geq 1\}$ be a ϱ -mixing sequence of random variables such that $EX_n = 0, n \geq 1$. Then for any $q \geq 2$, there exists a constant $K = K(q, \varrho(\cdot))$ depending only on q and $\varrho(\cdot)$ such that

$$\begin{aligned} E \left[\max_{k \leq n} |S_k| \right]^q &\leq K \left(n^{q/2} \exp \left\{ K \sum_{i=0}^{[\ln n]} \varrho(2^i) \right\} \max_{k \leq n} (EX_k^2)^{q/2} \right. \\ &\quad \left. + n \exp \left\{ K \sum_{i=0}^{[\ln n]} \varrho^{2/q}(2^i) \right\} \max_{k \leq n} E|X_k|^q \right). \end{aligned} \quad (1.3)$$

Lemma 1.2. (Yang Shanchao, (1998) and Peligrad and Gut, (1999)). Let $\{\xi_n, n \geq 1\}$ be a $\tilde{\varrho}$ -mixing sequence with $\tilde{\varrho}(1) < 1$. Let $X_n \in \sigma(\xi_i, i \geq 1)$, $EX_n = 0$, $E|X_n|^p < \infty, n \geq 1, p > 1$. Then there exists a positive constant C such that

$$E \left| \sum_{i=1}^n X_i \right|^q \leq C \sum_{i=1}^n E|X_i|^q, \quad \forall n \geq 1, \quad 1 < q \leq 2, \quad (1.4)$$

$$E \left| \sum_{i=1}^n X_i \right|^q \leq C \left[\sum_{i=1}^n E|X_i|^q + \left(\sum_{i=1}^n EX_i^2 \right)^{\frac{q}{2}} \right], \quad \forall n \geq 1, \quad q > 2. \quad (1.5)$$

We also assume that in our consideration constant C isn't the same constant in each case.

2. The main results

In this paper we consider arrays of random variables.

Let $\varrho_n(i)$ denotes the maximal correlation coefficient defined in (1.1) for the n -th row of an array $\{X_{ni}, i \geq 1, n \geq 1\}$ i.e for the sequence $X_{n1}, X_{n2}, X_{n3}, \dots, n \geq 1$.

Similarly, we will use the notation $\tilde{\varrho}_n(k)$ for denoting the coefficient defined in (1.2) for the sequence $X_{n1}, X_{n2}, X_{n3}, \dots, n \geq 1$.

Moreover, let $\{\varphi_n, n \geq 1\}$ be a sequence of nonnegative, even, continuous and nondecreasing on $(0, \infty)$ functions $\varphi_n: \mathbb{R} \rightarrow \mathbb{R}^+$ with $\lim_{x \rightarrow \infty} \varphi_n(x) = \infty$ and such that

$$\varphi_n(x)/x \nearrow \quad \text{and} \quad \varphi_n(x)/x^2 \searrow, \quad \text{as } x \rightarrow \infty. \quad (2.1)$$

Theorem 2.1. *Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of random variables, which are ϱ -mixing sequences in each row, and such that $E\varphi_n(|X_{ni}|) < \infty, i \geq 1, n \geq 1$, where φ_n is defined in (2.1). Assume that for the constant K defined in (1.3), $0 < t < 2, q \geq 2$, some sequence $\{c_n, n \geq 1\}$ of positive real numbers and some strictly increasing sequence $\{b_n, n \geq 1\}$ of positive integers the following conditions are fulfilled*

- (i) $\sum_{n=1}^{\infty} c_n \sum_{i=1}^{b_n} P[|X_{ni}| \geq b_n^{\frac{1}{t}}] < \infty,$
- (ii) $\sum_{n=1}^{\infty} c_n \exp \left\{ K \sum_{i=1}^{[\log b_n]} \varrho_n(2^i) \right\} b_n^{\frac{q}{2}} \frac{\max_{i \leq b_n} [E\varphi_n(X_{ni}) I[|X_{ni}| < b_n^{\frac{1}{t}}]]^{\frac{q}{2}}}{[\varphi_n(b_n^{\frac{1}{t}})]^{\frac{q}{2}}} < \infty,$
- (iii) $\sum_{n=1}^{\infty} c_n \exp \left\{ K \sum_{i=1}^{[\log b_n]} \varrho_n^{\frac{2}{t}}(2^i) \right\} b_n^{1-\frac{q}{t}} \max_{i \leq b_n} E|X_{ni}|^q I[|X_{ni}| < b_n^{\frac{1}{t}}] < \infty.$

Then for $S_{nk} = \sum_{i=1}^k X_{ni}$ and any $\varepsilon > 0$

$$\sum_{n=1}^{\infty} c_n P \left[\max_{i \leq b_n} |S_{ni} - \sum_{j=1}^i EX_{nj} I[|X_{nj}| < b_n^{\frac{1}{t}}]| \geq b_n^{\frac{1}{t}} \varepsilon \right] < \infty. \quad (2.2)$$

Proof. Let $X'_{ni} = X_{ni} I[|X_{ni}| < b_n^{\frac{1}{t}}], Y_{ni} = X'_{ni} - EX'_{ni}$ and $S'_{nk} = \sum_{i=1}^k Y_{ni}$. Using Lemma 1.1 we obtain

$$\begin{aligned} & P \left[\max_{i \leq b_n} \left| S_{ni} - \sum_{j=1}^i EX_{nj} I[|X_{nj}| < b_n^{\frac{1}{t}}] \right| \geq b_n^{\frac{1}{t}} \varepsilon \right] \\ & \leq \sum_{i=1}^{b_n} P[|X_{ni}| \geq b_n^{\frac{1}{t}}] + P \left[\max_{i \leq b_n} |S'_{ni}| \geq b_n^{\frac{1}{t}} \varepsilon \right] \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i=1}^{b_n} P[|X_{ni}| \geq b_n^{\frac{1}{i}}] + 2^q K b_n^{-\frac{q}{i}} \{b_n^{\frac{q}{2}} \exp(K \sum_{i=0}^{\lfloor \log b_n \rfloor} \varrho_n(2^i)) \max_{i \leq b_n} (E|X_{ni}|^2 I[|X_{ni}| < b_n^{\frac{1}{i}}])^{\frac{q}{2}} \\
&\quad + b_n \exp(K \sum_{i=0}^{\lfloor \log b_n \rfloor} \varrho_n^{\frac{2}{q}}(2^i)) \max_{i \leq b_n} E|X_{ni}|^q I[|X_{ni}| < b_n^{\frac{1}{i}}]\}. \tag{2.3}
\end{aligned}$$

Therefore, by (2.3) and the assumptions (i) and (iii), it is enough to show that

$$\sum_{n=1}^{\infty} c_n b_n^{-\frac{q}{i}} \{b_n^{\frac{q}{2}} \exp(K \sum_{i=0}^{\lfloor \log b_n \rfloor} \varrho_n(2^i)) \max_{i \leq b_n} (E|X_{ni}|^2 I[|X_{ni}| < b_n^{\frac{1}{i}}])^{\frac{q}{2}}\} < \infty.$$

Indeed we see

$$\begin{aligned}
&\sum_{n=1}^{\infty} c_n b_n^{-\frac{q}{i}} \{b_n^{\frac{q}{2}} \exp(K \sum_{i=0}^{\lfloor \log b_n \rfloor} \varrho_n(2^i)) \max_{i \leq b_n} (E|X_{ni}|^2 I[|X_{ni}| < b_n^{\frac{1}{i}}])^{\frac{q}{2}}\} \\
&\leq \sum_{n=1}^{\infty} c_n b_n^{\frac{q}{2}} \exp(K \sum_{i=0}^{\lfloor \log b_n \rfloor} \varrho_n(2^i)) \frac{\max_{i \leq b_n} (E|X_{ni}|^2 I[|X_{ni}| < b_n^{\frac{1}{i}}])^{\frac{q}{2}}}{(b_n^{\frac{1}{i}})^{\frac{q}{2}}} \\
&\leq \sum_{n=1}^{\infty} c_n b_n^{\frac{q}{2}} \exp(K \sum_{i=0}^{\lfloor \log b_n \rfloor} \varrho_n(2^i)) \frac{\max_{i \leq b_n} (E\varphi_n(|X_{ni}|) I[|X_{ni}| < b_n^{\frac{1}{i}}])^{\frac{q}{2}}}{(\varphi_n(b_n^{\frac{1}{i}}))^{\frac{q}{2}}} < \infty.
\end{aligned}$$

This completes the proof.

From Theorem 2.1 we get the following result.

Corollary 2.1. *Let $\alpha p > 1$ and in the case (a) $r = 2$ if $1 \leq p < 2$ or in the case (b) $r > 2$ if $r > p \geq 2$ and $\alpha > \frac{r-1}{r-p}$. Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of identically distributed random variables, which are q -mixing sequences in each row, and such that $EX_{11} = 0$ and $E|X_{11}|^p < \infty$. Assume that*

$$\sum_{i=1}^{\infty} \varrho_1^{\frac{2}{q}}(2^i) < \infty. \tag{2.4}$$

Then

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} P[\max_{i \leq n} |S_{ni}| \geq \varepsilon n^{\alpha}] < \infty, \tag{2.5}$$

for any $\varepsilon > 0$.

Proof. Let $q := r, \frac{1}{i} := \alpha, c_n := n^{\alpha p - 2}, b_n = n$ and $\varphi_n(x) := x^2$. Then, by the assumption $E|X_{11}|^p < \infty$, we obtain

$$\begin{aligned}
(i) \quad &\sum_{n=1}^{\infty} c_n \sum_{i=1}^n P[|X_{ni}| \geq n^{\frac{1}{i}}] = \sum_{n=1}^{\infty} n^{\alpha p - 1} P[|X_{11}| \geq n^{\alpha}] \\
&= \sum_{n=1}^{\infty} n^{\alpha p - 1} \sum_{j=n}^{\infty} P[j^{\alpha} \leq |X_{11}| < (j+1)^{\alpha}] = \sum_{n=1}^{\infty} P[n^{\alpha} \leq |X_{11}| < (n+1)^{\alpha}] \sum_{i=1}^n i^{\alpha p - 1} \\
&\leq C \sum_{n=1}^{\infty} n^{\alpha p} P[n^{\alpha} \leq |X_{11}| < (n+1)^{\alpha}] = CE|X_{11}|^p < \infty.
\end{aligned}$$

Moreover, we note that in the case (a) ($r = 2, 1 \leq p < 2$) the conditions (ii) and (iii) from Theorem 2.1 are identical. So, by (2.4) we get

$$\begin{aligned} & \sum_{n=1}^{\infty} n^{\alpha p-2} \exp\left\{K \sum_{i=1}^{[\log n]} \varrho_n(2^i)\right\} n^{1-2\alpha} \max_{i \leq n} [E|X_{ni}|^2 I[|X_{ni}| < n^\alpha]] \\ &= C \sum_{n=1}^{\infty} n^{\alpha(p-2)-1} E|X_{11}|^{p+(2-p)} I[|X_{11}| < n^\alpha] < \infty. \end{aligned}$$

In the case (b) ($r > p \geq 2, \alpha > \frac{r-1}{r-p}$), by (2.4) and the assumption $E|X_{11}|^p < \infty$, we have

$$\begin{aligned} (ii) \quad & \sum_{n=1}^{\infty} c_n \exp\left\{K \sum_{i=1}^{[\log n]} \varrho_n(2^i)\right\} n^{\frac{\alpha}{2}} \frac{\max_{i \leq n} [E\varphi_n(X_{ni}) I[|X_{ni}| < n^{\frac{1}{t}}]]^{\frac{q}{2}}}{[\varphi_n(n^{\frac{1}{t}})]^{\frac{q}{2}}} \\ &= C \sum_{n=1}^{\infty} n^{\alpha p-2+\frac{r}{2}-\alpha r} (E|X_{11}|^2)^{\frac{r}{2}} < \infty \end{aligned}$$

and

$$\begin{aligned} (iii) \quad & \sum_{n=1}^{\infty} c_n \exp\left\{K \sum_{i=1}^{[\log n]} \varrho_n^{\frac{2}{t}}(2^i)\right\} n^{1-\frac{q}{t}} \max_{i \leq n} E|X_{ni}|^q I[|X_{ni}| < n^{\frac{1}{t}}] \\ &= C \sum_{n=1}^{\infty} n^{\alpha(p-r)-1} E|X_{11}|^{p+(r-p)} I[|X_{11}| < n^\alpha] < \infty. \end{aligned}$$

To complete this proof, it enough to show that

$$n^{-\alpha} \sum_{n=j}^i E X_{11} I[|X_{11}| < n^\alpha] \longrightarrow 0, \quad n \longrightarrow \infty$$

for each $1 \leq i \leq n$.

This fact immediately follows from the assumptions $E X_{11} = 0$ and $E|X_{11}|^p < \infty$.

Indeed we see

$$\begin{aligned} & \left| n^{-\alpha} \sum_{j=1}^i E X_{11} I[|X_{11}| < n^\alpha] \right| \leq n^{-\alpha} \sum_{j=1}^i E|X_{11}| I[|X_{11}| \geq n^\alpha] \\ & \leq n^{-\alpha} \sum_{j=1}^i \frac{E|X_{11}|^p}{(n^\alpha)^{p-1}} \leq n^{-\alpha} n^{-\alpha p+\alpha} i E|X_{11}| = C \frac{i}{n^{\alpha p}} \longrightarrow 0, \quad n \longrightarrow \infty. \end{aligned}$$

This proves that (2.5) holds.

This corollary generalizes Shao's result (1995) obtained for ϱ -mixing sequences of identically distributed random variables. As a consequence of the above result we can get the following Marcinkiewicz- Zygmunt law of large numbers.

Corollary 2.2. *Let $1 \leq p < 2, \alpha p > 1$ and $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of identically distributed random variables, which are ϱ -mixing sequences in each row, and such that $E X_{11} = 0$ and $E|X_{11}|^p < \infty$. Assume that*

$$\sum_{i=1}^{\infty} \varrho_1(2^i) < \infty.$$

Then

$$n^{-\alpha} \sum_{i=1}^n X_{ni} \rightarrow 0, \quad a.s., \quad n \rightarrow \infty.$$

The next corollary we get for identically distributed random variables putting in Theorem 2.1 $c_n = n^{\alpha p - 2} l(n)$, where $l(n)$ is a slowly varying function.

Corollary 2.3. *Let $\alpha p > 1$ and in the case (a) $r = 2$ if $1 \leq p < 2$ or in the case (b) $r > 2$ if $r > p \geq 2$ and $\alpha > \frac{r-1}{r-p}$. Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of identically distributed random variables, which are q -mixing sequences in each row, and such that $EX_{11} = 0$ and $E|X_{11}|^p l(|X_{11}|^{\frac{1}{\alpha}}) < \infty$. Assume that (2.4) holds*

Then

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} l(n) P[\max_{i \leq n} |S_{ni}| \geq \varepsilon n^{\alpha}] < \infty, \quad (2.6)$$

for any $\varepsilon > 0$.

Proof. Putting $q := r$, $\frac{1}{t} := \alpha$, $c_n := n^{\alpha p - 2} l(n)$, $b_n := n$ and $\varphi_n(x) := x^2$, by the assumption $E|X_{11}|^p l(|X_{11}|^{\frac{1}{\alpha}}) < \infty$, we obtain

$$(i) \quad \sum_{n=1}^{\infty} n^{\alpha p - 2} l(n) \sum_{i=1}^n P[|X_{ni}| \geq n^{\alpha}] \leq \sum_{k=1}^{\infty} (2^k)^{\alpha p} l(2^k) \sum_{j=k}^{\infty} P[(2^j)^{\alpha} \leq |X_{11}| < (2^{j+1})^{\alpha}]$$

$$\leq C E|X_{11}|^p l(|X_{11}|^{\frac{1}{\alpha}}) < \infty$$

In the case (a) ($r = 2, 1 \leq p < 2$) we have

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} l(n) \exp\left\{K \sum_{i=1}^{[\log n]} \varrho_n(2^i)\right\} n^{1-2\alpha} \max_{i \leq n} [E|X_{ni}|^2 I[|X_{ni}| < n^{\alpha}]]$$

$$\leq C \sum_{k=1}^{\infty} (2^k)^{\alpha(p-2)} l(2^k) E|X_{11}|^2 I[|X_{11}| < (2^k)^{\alpha}]$$

$$\leq C \sum_{k=1}^{\infty} (2^k)^{\alpha(p-2)} l(2^k) \sum_{i=1}^k \int_{(2^{i-1})^{\alpha}}^{(2^i)^{\alpha}} x^2 dF(x)$$

$$\leq C \sum_{m=1}^{\infty} (2^m)^{\alpha(p-2)} \int_{(2^{m-1})^{\alpha}}^{(2^m)^{\alpha}} l(|x|^{\frac{1}{\alpha}}) x^2 dF(x) \leq C E|X_{11}|^p l(|X_{11}|^{\frac{1}{\alpha}}) < \infty,$$

which proves that (ii) and (iii) hold.

In case (b) ($r > p \geq 2$) we have

$$(ii) \quad \sum_{n=1}^{\infty} n^{\alpha p - 2} l(n) \exp\left\{K \sum_{i=1}^{[\log n]} \varrho_n(2^i)\right\} n^{\frac{r}{2}} \frac{\max_{i \leq n} [E\varphi_n(X_{ni}) I[|X_{ni}| < n^{\alpha}]]^{\frac{r}{2}}}{[\varphi_n(n^{\alpha})]^{\frac{r}{2}}}$$

$$= C \sum_{n=1}^{\infty} n^{\alpha p - 2 + \frac{r}{2} - \alpha r} l(n) (E|X_{11}|^2)^{\frac{r}{2}} \leq C \sum_{k=1}^{\infty} (2^k)^{\alpha(p-r) - 1 + \frac{r}{2}} l(2^k) < \infty$$

and

$$(iii) \quad \sum_{n=1}^{\infty} n^{\alpha p - 2} l(n) \exp\left\{K \sum_{i=1}^{[\log n]} \varrho_n^{\frac{2}{r}}(2^i)\right\} n^{1-\alpha r} \max_{i \leq n} E|X_{ni}|^r I[|X_{ni}| < n^{\alpha}]$$

$$= C \sum_{k=1}^{\infty} (2^k)^{\alpha(p-r)} l(2^k) \sum_{i=1}^k \int_{(2^{i-1})^{\alpha}}^{(2^i)^{\alpha}} |x|^r dF(x)$$

$$\leq C \sum_{m=1}^{\infty} (2^m)^{\alpha(p-r)} \int_{(2^{m-1})^\alpha}^{(2^m)^\alpha} l(|x|^{\frac{1}{\alpha}}) x^r dF(x) \leq CE|X_{11}|^p l(|X_{11}|^{\frac{1}{\alpha}}) < \infty.$$

Moreover, we see that $EX_{11} = 0$ and $E|X_{11}|^p l(|X_{11}|^{\frac{1}{\alpha}}) < \infty$ imply

$$|n^{-\alpha} \sum_{j=1}^i EX_{11} I[|X_{11}| < n^\alpha]| \rightarrow 0, \text{ as } n \rightarrow \infty \text{ for each } 1 \leq i \leq n.$$

This completes the proof.

The case $\alpha p = 1$ was considered by Shao for identically distributed random variables and by Kong and Zhang for non-identically distributed random variables (see Zhang and Chuanrong (1996)).

Let now $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a nonnegative continuous and increasing function, satisfying (2.1) and let φ^{-1} be an inverse of φ . The consideration similar to presented in proofs of Theorem 2.1 and Corollary 2.3 and the fact that

$$P[\max_{i \leq n} \varphi(|S_{ni}|) \geq \varepsilon n] \leq P[\max_{i \leq n} (|S_{ni}|) \geq \varphi^{-1}(\varepsilon n)]$$

allow us to formulate the following result.

Theorem 2.2. *Let $\alpha p > 1$ and $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of identically distributed random variables, which are ρ -mixing sequences in each row, and such that $EX_{11} = 0$ and $E(\varphi(|X_{11}|))^{\alpha p} l(\varphi(|X_{11}|)) < \infty$. Assume that (2.4) with $r = 2$ holds.*

Then

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} l(n) P[\max_{i \leq n} \varphi(|S_{ni}|) \geq \varepsilon n] < \infty,$$

for any $\varepsilon > 0$.

Shao proved the analogous result for $\alpha p = 1$ (Zhengyan and Chuanrong (1996) Section 8.4).

In the next part of this section we consider arrays of random variables, which are $\tilde{\rho}$ -mixing sequences.

Theorem 2.3. *Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of random variables, which are $\tilde{\rho}$ -mixing sequences in each row with $\tilde{\rho}_n(1) < 1, n \geq 1$. Assume that for some sequence $\{c_n, n \geq 1\}$ of positive real numbers, some strictly increasing sequence $\{b_n, n \geq 1\}$ of positive integers and $0 < t < 2$ the following conditions are fulfilled*

$$\begin{aligned} (I) \quad & \sum_{n=1}^{\infty} c_n \sum_{i=1}^{b_n} P[|X_{ni}| \geq b_n^{\frac{1}{t}}] < \infty, \\ (II) \quad & \sum_{n=1}^{\infty} c_n b_n^{-\frac{q}{t}} \sum_{i=1}^{b_n} E|X_{ni}|^q I[|X_{ni}| < b_n^{\frac{1}{t}}] < \infty, \\ (III) \quad & \sum_{n=1}^{\infty} c_n b_n^{-\frac{q}{t}} \left(\sum_{i=1}^{b_n} E|X_{ni}|^2 I[|X_{ni}| < b_n^{\frac{1}{t}}] \right)^{\frac{q}{2}} < \infty, \end{aligned}$$

Then for $S_n = \sum_{i=1}^{b_n} X_{ni}$ and any $\varepsilon > 0$

$$\sum_{n=1}^{\infty} c_n P\left[\left| S_n - \sum_{i=1}^{b_n} EX_{ni} I[|X_{ni}| < b_n^{\frac{1}{t}}] \right| \geq b_n^{\frac{1}{t}} \varepsilon \right] < \infty. \quad (2.7)$$

Proof. Let $X'_{ni} = X_{ni}I[|X_{ni}| < b_n^{\frac{1}{t}}]$, $Y_{ni} = X'_{ni} - EX'_{ni}$ and $S'_n = \sum_{i=1}^{b_n} Y_{ni}$. Using Lemma 1.2 we obtain

$$\begin{aligned} P[|S'_n| \geq b_n^{\frac{1}{t}}\varepsilon] &\leq C\varepsilon^{-q}b_n^{-\frac{q}{t}}\left\{\sum_{i=1}^{b_n} E|Y_{ni}|^q + \left(\sum_{i=1}^{b_n} E|Y_{ni}|^2\right)^{\frac{q}{2}}\right\} \\ &\leq C\varepsilon^{-q}b_n^{-\frac{q}{t}}\left\{\sum_{i=1}^{b_n} E|X'_{ni}|^q + \left(\sum_{i=1}^{b_n} E|X'_{ni}|^2\right)^{\frac{q}{2}}\right\}. \end{aligned} \quad (2.8)$$

Moreover, we see that

$$P\left[\left|S_n - \sum_{i=1}^{b_n} X_{ni}I[|X_{ni}| < b_n^{\frac{1}{t}}]\right| \geq b_n^{\frac{1}{t}}\varepsilon\right] \leq P[|S'_n| \geq b_n^{\frac{1}{t}}\varepsilon] + \sum_{i=1}^{b_n} P[|X_{ni}| \geq b_n^{\frac{1}{t}}]. \quad (2.9)$$

Therefore, by (2.8), (2.9), (I), (II) and (III) we see that (2.7) holds.

Corollary 2.4. Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of random variables, which are \tilde{q} -mixing sequences in each row with $\tilde{q}_n(1) < 1$, $n \geq 1$. Moreover, let $E\varphi_n(|X_{ni}|) < \infty$, $i \geq 1, n \geq 1$, where φ_n is defined in (2.1). Assume, that for some sequence $\{c_n, n \geq 1\}$ of positive real numbers, some strictly increasing sequence $\{b_n, n \geq 1\}$ of positive integers and $0 < t < 2$ (I) holds and the following conditions are fulfilled

$$\begin{aligned} (II') \quad &\sum_{n=1}^{\infty} c_n b_n^{-\frac{4}{t}} \sum_{i=1}^{b_n} E|X_{ni}|^4 I[|X_{ni}| < b_n^{\frac{1}{t}}] < \infty, \\ (III') \quad &\sum_{n=1}^{\infty} c_n \varphi_n^{-2}(b_n^{\frac{1}{t}}) \sum_{i=2}^{b_n} \sum_{j=1}^{i-1} E\varphi_n(|X_{ni}|) E\varphi_n(|X_{nj}|) < \infty. \end{aligned}$$

Then for any $\varepsilon > 0$ (2.9) holds.

Proof. Putting $q = 4$ in Theorem 2.3 we see, that condition (II) is fulfilled and

$$\begin{aligned} &\sum_{n=1}^{\infty} c_n b_n^{-\frac{4}{t}} \left(\sum_{i=1}^{b_n} E|X'_{ni}|^2\right)^2 \\ &\leq C \left\{ \sum_{n=1}^{\infty} c_n b_n^{-\frac{4}{t}} \sum_{i=1}^{b_n} (E|X'_{ni}|^2)^2 + \sum_{n=1}^{\infty} c_n b_n^{-\frac{4}{t}} \sum_{i=2}^{b_n} \sum_{j=1}^{i-1} E|X'_{ni}|^2 E|X'_{nj}|^2 \right\} \\ &\leq C \left\{ \sum_{n=1}^{\infty} c_n b_n^{-\frac{4}{t}} \sum_{i=1}^{b_n} E|X'_{ni}|^4 + \sum_{n=1}^{\infty} c_n \varphi_n^{-2}(b_n^{\frac{1}{t}}) \sum_{i=2}^{b_n} \sum_{j=1}^{i-1} E\varphi_n(|X_{ni}|) E\varphi_n(|X_{nj}|) \right\} < \infty, \end{aligned}$$

by (II') and (III').

Therefore (2.9) holds.

Corollary 2.5. Let $\alpha p > 1$, $q > 2$ and in the case (a) $1 \leq p < 2$ or in the case (b) $2 \leq p < q$ and $\alpha > \frac{q-2}{2(q-p)}$. Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of identically distributed random variables, which are \tilde{q} -mixing sequences in each row with $\tilde{q}_1(1) < 1$ and such that $EX_{11} = 0$ and $E|X_{11}|^p < \infty$. Then for any $\varepsilon > 0$ we have

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} P\left[\left|\sum_{i=1}^n X_{ni}\right| \geq \varepsilon n^{\alpha}\right] < \infty. \quad (2.10)$$

Proof. Put $c_n = n^{\alpha p - 2}$, $\frac{1}{t} := \alpha$ and $b_n = n$ in Theorem 2.3. Then we have

$$\begin{aligned}
 (I) \quad & \sum_{n=1}^{\infty} c_n \sum_{i=1}^n P[|X_{ni}| \geq n^{\frac{1}{t}}] = \sum_{n=1}^{\infty} n^{\alpha p - 1} P[|X_{11}| \geq n^{\alpha}] \\
 & = \sum_{n=1}^{\infty} n^{\alpha p - 1} \sum_{j=n}^{\infty} P[j^{\alpha} \leq |X_{11}| < (j+1)^{\alpha}] = \sum_{n=1}^{\infty} P[n^{\alpha} \leq |X_{11}| < (n+1)^{\alpha}] \sum_{i=1}^n i^{\alpha p - 1} \\
 & < C \sum_{n=1}^{\infty} n^{\alpha p} P[n^{\alpha} \leq |X_{11}| < (n+1)^{\alpha}] = CE|X_{11}|^p < \infty \text{ and} \\
 (II) \quad & \sum_{n=1}^{\infty} c_n (b_n)^{-\frac{q}{t}} \sum_{i=1}^{b_n} E|X_{ni}|^q I[|X_{ni}| < b_n^{\frac{1}{t}}] \\
 & \leq \sum_{n=1}^{\infty} n^{\alpha(p-q)-1} E|X_{11}|^{p+(q-p)} I[|X_{11}| < n^{\alpha}] < \infty.
 \end{aligned}$$

To show that condition (III) is fulfilled we must consider the cases (a) and (b) separately. In the case (a) we have

$$\begin{aligned}
 (III) \quad & \sum_{n=1}^{\infty} c_n b_n^{-\frac{q}{t}} \left(\sum_{i=1}^{b_n} E|X'_{ni}|^2 \right)^{\frac{q}{2}} \\
 & \leq \sum_{n=1}^{\infty} n^{\alpha p - 2} n^{-\alpha q} n^{\frac{q}{2}} n^{(2-p)\alpha \frac{q}{2}} (E|X_{11}|^p)^{\frac{q}{2}} = C \sum_{n=1}^{\infty} n^{(\alpha p - 1)(1 - \frac{q}{2}) - 1} < \infty
 \end{aligned}$$

and in the case (b)

$$\begin{aligned}
 (III) \quad & \sum_{n=1}^{\infty} c_n b_n^{-\frac{q}{t}} \left(\sum_{i=1}^{b_n} E|X'_{ni}|^2 \right)^{\frac{q}{2}} \\
 & \leq \sum_{n=1}^{\infty} n^{\alpha p - 2} n^{-\alpha q} n^{\frac{q}{2}} (E|X_{11}|^2)^{\frac{q}{2}} = C \sum_{n=1}^{\infty} n^{\alpha(p-q) - 2 + \frac{q}{2}} < \infty.
 \end{aligned}$$

Thus we have established that all assumptions from Theorem 2.3 are fulfilled. Moreover, we see

$$\begin{aligned}
 & \left| n^{-\alpha} \sum_{i=1}^n EX_{ni} I[|X_{ni}| < n^{\alpha}] \right| \leq n^{-\alpha} \sum_{i=1}^n |EX_{ni}| I[|X_{ni}| \geq n^{\alpha}] \\
 & = n^{-\alpha + 1} \frac{E|X_{11}|^p}{(n^{\alpha})^{p-1}} = n^{1-\alpha p} E|X_{11}|^p \longrightarrow 0, \text{ as } n \longrightarrow \infty,
 \end{aligned}$$

because $\alpha p > 1$. Therefore (2.10) holds.

Corollary 2.5 is more general than result obtained by G. Shixin (2004). The condition (2.10) is stronger than this presented by Shixin.

Corollary 2.6. Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of identically distributed random variables, which are \tilde{q} -mixing sequences in each row with $\tilde{q}_1(1) < 1$ and such that $EX_{11} = 0$ and $E|X_{11}|^2 < \infty$. Then for any $\varepsilon > 0$ we have

$$\sum_{n=1}^{\infty} n^{-1} P\left[\left| \sum_{i=1}^n X_{ni} \right| \geq \varepsilon n^{\alpha} \right] < \infty. \quad (2.11)$$

Proof. Putting $c_n = n^{-1}$, $t = 1$, $b_n = n$, $p = 2$ and $\varphi_n(x) = x^2$ in Corollary 2.4 we get

$$\begin{aligned}
(I) \quad & \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^n P[|X_{ni}| \geq n] = \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} P[j \leq |X_{11}| < j+1] \\
& = \sum_{n=1}^{\infty} P[n \leq |X_{11}| < n+1] \sum_{i=1}^n i < CE|X_{11}|^2 < \infty, \\
(II') \quad & \sum_{n=1}^{\infty} n^{-1} n^{-4} \sum_{i=1}^n E|X_{11}|^4 I[|X_{11}| < n] \leq \sum_{n=1}^{\infty} n^{-2} E|X_{11}|^2 = C \sum_{n=1}^{\infty} n^{-2} < \infty \\
& \text{and} \\
(III') \quad & \sum_{n=1}^{\infty} n^{-1} n^{-4} \sum_{i=2}^n \sum_{j=1}^{i-1} (E|X_{11}|^2)^2 = C \sum_{n=1}^{\infty} n^{-3} < \infty.
\end{aligned}$$

Moreover

$$\begin{aligned}
& \left| n^{-1} \sum_{i=1}^n EX_{11} I[|X_{11}| < n] \right| \\
& \leq n^{-1} \sum_{i=1}^n E|X_{11}| I[|X_{11}| \geq n] = \frac{E|X_{11}|^2}{n} \rightarrow 0, \text{ as } n \rightarrow \infty.
\end{aligned}$$

Corollary 2.7. Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of random variables, which are \tilde{q} -mixing sequences in each row with $\tilde{q}_n(1) < 1, n \geq 1$ and such that $EX_{ni} = 0, i \geq 1, n \geq 1$. Assume that for some sequence $\{\lambda_n, n \geq 1\}$ with $0 < \lambda_n \leq 1$ we have $E|X_{ni}|^{1+\lambda_n} < \infty, 1 \leq i \leq b_n, n \geq 1$, where $\lambda = \sup_n \lambda_n$ and $\{b_n, n \geq 1\}$ is a strictly increasing sequence of positive integers. If for some sequence $\{c_n, n \geq 1\}$ of positive real numbers and $0 < t < 2$ the following condition is fulfilled

$$\sum_{n=1}^{\infty} c_n (b_n^{\frac{1}{t}})^{-1-\lambda_n} \sum_{i=1}^{b_n} E|X_{ni}|^{1+\lambda_n} < \infty, \quad (2.12)$$

then

$$\sum_{n=1}^{\infty} c_n P\left[\left|\sum_{i=1}^{b_n} X_{ni}\right| \geq b_n^{\frac{1}{t}} \varepsilon\right] < \infty, \quad \forall \varepsilon > 0. \quad (2.13)$$

Proof. Note, that if $\{c_n, n \geq 1\}$ is such that the series $\sum_{n=1}^{\infty} c_n$ converges then (2.13) holds. Therefore it is enough to consider only such sequences $\{c_n, n \geq 1\}$ which satisfy the condition $\sum_{n=1}^{\infty} c_n = \infty$. Then (2.12) implies

$$(b_n^{\frac{1}{t}})^{-1-\lambda_n} \sum_{i=1}^{b_n} E|X_{ni}|^{1+\lambda_n} < 1.$$

By Corollary 2.4 we see that in order to show (2.13), it is enough to prove that if (2.12) holds and $\varphi_n(x) = |x|^{1+\lambda_n}$ the assumption (I), (II') and (III') are fulfilled.

Indeed we have

$$\begin{aligned}
(I) \quad & \sum_{n=1}^{\infty} c_n \sum_{i=1}^{b_n} P[|X_{ni}| \geq b_n^{\frac{1}{t}}] \leq \sum_{n=1}^{\infty} c_n (b_n^{\frac{1}{t}})^{-1-\lambda_n} \sum_{i=1}^{b_n} E|X_{ni}|^{1+\lambda_n} < \infty, \\
(II') \quad & \sum_{n=1}^{\infty} c_n (b_n^{\frac{1}{t}})^{-\frac{4}{t}} \sum_{i=1}^{b_n} E|X_{ni}|^4 I[|X_{ni}| < b_n^{\frac{1}{t}}] \leq \sum_{n=1}^{\infty} c_n (b_n^{\frac{1}{t}})^{-1-\lambda_n} \sum_{i=1}^{b_n} E|X_{ni}|^{1+\lambda_n} < \infty,
\end{aligned}$$

$$\begin{aligned}
(III') \quad & \sum_{n=1}^{\infty} c_n \varphi_n^{-2} (b_n^{\frac{1}{t}}) \sum_{i=2}^{b_n} \sum_{j=1}^{i-1} E \varphi_n(|X_{ni}|) E \varphi_n(|X_{nj}|) \\
& \leq \sum_{n=1}^{\infty} c_n (b_n^{\frac{1}{t}})^{-2(1+\lambda_n)} \sum_{i=2}^{b_n} \sum_{j=1}^{i-1} E|X_{ni}|^{1+\lambda_n} E|X_{nj}|^{1+\lambda_n} \\
& \leq \sum_{n=1}^{\infty} c_n (b_n^{\frac{1}{t}})^{-2(1+\lambda_n)} \left(\sum_{i=1}^{b_n} E|X_{ni}|^{1+\lambda_n} \right)^2 < \infty.
\end{aligned}$$

Moreover, we see

$$\begin{aligned}
& \left| b_n^{-\frac{1}{t}} \sum_{i=1}^{b_n} E X_{ni} I[|X_{ni}| < b_n^{\frac{1}{t}}] \right| \\
& \leq b_n^{-\frac{1}{t}} \sum_{i=1}^{b_n} E|X_{ni}| I[|X_{ni}| \geq b_n^{\frac{1}{t}}] \\
& = (b_n^{\frac{1}{t}})^{-(1+\lambda_n)} \sum_{i=1}^{b_n} E|X_{ni}|^{1+\lambda_n} \rightarrow 0, \text{ as } n \rightarrow \infty.
\end{aligned}$$

The corollary is proved.

Acknowledgment. The author is very grateful to the referee for the suggestions which allowed her to improve this paper.

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