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Spectral decay of the sinc kernel operator and approximations by Prolate Spheroidal Wave Functions.

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Abstract— For fixed c , the Prolate Spheroidal Wave Functions (PSWFs) $\psi_{n,c}$ form a basis with remarkable properties for the space of band-limited functions with bandwidth c . They have been largely studied and used after the seminal work of D. Slepian, H. Landau and H. Pollack. Recently, they have been used for the approximation of functions in the Sobolev space $H^s([-1, 1])$. In view of this, we give new estimates on the decay rate of eigenvalues of the Sinc kernel integral operators. This is one of the main issues of this work. A second one is the choice of the parameter c when approximating a function in $H^s([-1, 1])$ by its truncated PSWFs series expansion. Such functions may be seen as the restriction to $[-1, 1]$ of almost time-limited and band-limited functions, for which PSWFs expansions are still well adapted. Finally, we provide the reader with some numerical examples that illustrate the different results of this work.

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1 Introduction

For a given value $c > 0$, called the bandwidth, PSWFs $(\psi_{n,c}(\cdot))_{n \geq 0}$ constitute an orthonormal basis of $L^2([-1, +1])$, an orthogonal system of $L^2(\mathbf{R})$ and an orthogonal basis of the Paley-Wiener space B_c , given by $B_c = \left\{ f \in L^2(\mathbf{R}), \text{Support } \hat{f} \subset [-c, c] \right\}$. Here, \hat{f} denotes the Fourier transform of f . They are eigenfunctions of the compact integral operators \mathcal{F}_c and $\mathcal{Q}_c = \frac{c}{2\pi} \mathcal{F}_c^* \mathcal{F}_c$, defined on $L^2([-1, 1])$ by

$$\mathcal{F}_c(f)(x) = \int_{-1}^1 e^{icxy} f(y) dy, \quad \mathcal{Q}_c(f)(x) = \int_{-1}^1 \frac{\sin c(x-y)}{\pi(x-y)} f(y) dy. \quad (1)$$

Since the operator \mathcal{F}_c commutes with the Sturm-Liouville operator \mathcal{L}_c ,

$$\mathcal{L}_c(\psi) = -\frac{d}{dx} \left[(1-x^2) \frac{d\psi}{dx} \right] + c^2 x^2 \psi, \quad (2)$$

PSWFs $(\psi_{n,c}(\cdot))_{n \geq 0}$ are also eigenfunctions of \mathcal{L}_c . They are ordered in such a way that the corresponding eigenvalues of \mathcal{L}_c , called $\chi_n(c)$, are strictly increasing. Functions $\psi_{n,c}$ are restrictions to the interval $[-1, +1]$ of real analytic functions on the whole real line and eigenvalues $\chi_n(c)$ are values of λ such that the equation $\mathcal{L}_c \psi = \lambda \psi$ has a bounded solution.

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PSWFs have been introduced by D. Slepian, H. Landau and H. Pollak [15, 27, 28, 29] in relation with signal processing. For a detailed review on the properties, the numerical computations, asymptotic results and first applications of the PSWFs, the reader is referred to the recent books on the subject, [18], [24].

By Plancherel identity, PSWFs are normalized so that

$$\int_{-1}^1 |\psi_{n,c}(x)|^2 dx = 1, \quad \int_{\mathbb{R}} |\psi_{n,c}(x)|^2 dx = \frac{1}{\lambda_n(c)}, \quad n \geq 0. \quad (3)$$

Here, $(\lambda_n(c))_n$ is the infinite sequence of the eigenvalues of \mathcal{Q}_c , also arranged in the decreasing order $1 > \lambda_0(c) > \lambda_1(c) > \dots > \lambda_n(c) > \dots$. We call $\mu_n(c)$ the eigenvalues of \mathcal{F}_c . They are given by

$$\mu_n(c) = i^n \sqrt{\frac{2\pi}{c} \lambda_n(c)}.$$

Also, we will adopt the following sign normalization of the PSWFs, given by

$$\psi_{n,c}(0) > 0 \quad \text{for even } n, \quad \psi'_{n,c}(0) > 0, \quad \text{for odd } n. \quad (4)$$

One of the main issues that we discuss here is the decay rate of the eigenvalues $\lambda_n(c)$. This decay rate plays a crucial role in most of the various concrete applications of the PSWFs. In this direction, one knows their asymptotic behaviour for c fixed, which has been given in 1964 by Widom, see [32].

$$\lambda_n(c) \sim \left(\frac{ec}{4(n + \frac{1}{2})} \right)^{2n+1}. \quad (5)$$

This gives the exact decay for n large enough, but one would like to have a more precise information in terms of uniformity of this behaviour, both in n and c . On the other hand, Landau has considered the value of the smallest integer n such that $\lambda_n(c) \leq 1/2$ in [14]. More precisely, if we note c_n^* the unique value of c such that $\lambda_n(c) = 1/2$, then he proves that

$$\frac{\pi}{2}(n-1) \leq c_n^* \leq \frac{\pi}{2}(n+1) \quad \lambda_n(c_n^*) = \frac{1}{2}. \quad (6)$$

So, for c fixed, we almost know when $\lambda_n(c)$ passes through the value $1/2$. Landau and Widom have also described the asymptotic behaviour, when c tends to ∞ , of the distribution of the eigenvalues $\lambda_n(c)$.

The search for more precise estimates has attracted a considerable interest, both in numerical and theoretical studies. We try here to give approximate values for $\lambda_n(c)$ for $c \leq c_n^*$, with some uniformity in the quality of approximation. We rely on the exact formula

$$\lambda_n(c) = \frac{1}{2} \exp \left(-2 \int_c^{c_n^*} \frac{(\psi_{n,\tau}(1))^2}{\tau} d\tau \right). \quad (7)$$

We use our recent work [4, 5] to estimate the value $\psi_{n,\tau}(1)$. In the first paper it is proved that $|\psi_{n,\tau}(1)| \leq 2\chi_n(\tau)^{1/4}$, which is not sufficient to find a sharp estimate for all values c . The approximation given in the second paper leads to a second estimate, valid for $\frac{\pi n}{2} - c$ larger than some multiple of $\ln n$. We finally find an explicit expression $\widetilde{\lambda}_n(c)$, and prove that it is comparable with $\lambda_n(c)$ up to some power of n . This is given by

$$\widetilde{\lambda}_n(c) = \frac{1}{2} \exp \left(-\frac{\pi^2(n + \frac{1}{2})}{2} \int_{\Phi\left(\frac{2c}{\pi(n + \frac{1}{2})}\right)}^1 \frac{1}{t(\mathbf{E}(t))^2} dt \right) \quad (8)$$

Here \mathbf{E} is the elliptic Legendre integral of the second kind. The function Φ is the inverse of the function $t \mapsto \frac{t}{\mathbf{E}(t)}$.

When n tends to ∞ with c fixed, we recover the asymptotic behavior given by Widom, which is already a good test of validity. Numerical experiments prove that this approximation is surprisingly accurate.

As a corollary we have the following, which may be seen as a kind of quantitative Widom's Theorem.

Theorem 1. *Let $m > 0$ be a positive real number and let $M > m$, $\varepsilon > 0$ be given. Then there exists a constant $A(\varepsilon, m, M)$ such that, for all $m \leq c \leq M\sqrt{n}$ and all n , we have the inequality*

$$\lambda_n(c) \leq A(\varepsilon, m, M)e^{\varepsilon n} \left(\frac{ec}{4(n + \frac{1}{2})} \right)^{2n+1}. \quad (9)$$

One can give an explicit constant $A(\varepsilon, m, M)$. When c may take larger values, there is another statement, where the equivalent found by Widom is replaced by $\widetilde{\lambda_n(c)}$.

Let us mention that another method to approximate the values $\lambda_n(c)$ has been used by Osipov in [23]. The estimates given in his paper are of different nature and do not propose such a simple formula. In addition, he mainly considers values of n such that $\frac{\pi n}{2} - c$ is smaller than some multiple of $\ln c$. At this moment both works may be seen complementary. But we underline the fact that numerical tests validate the accuracy of the approximant (8) even when c is close to the critical value, while our theoretical approach is not yet sufficient to do it.

Our second contribution is related to the quality of approximation in Sobolev spaces when a function is replaced by the partial sum of its expansion in some PSWF basis. This question has attracted a growing interest while, at the same time, were built PSWFs based numerical schemes for solving various problems from numerical analysis, see [6, 7, 8, 30, 31]. In particular, in [6], the author has shown that a PSWF approximation based method outperforms in terms of spatial resolution and stability of timestep, the classical approximation methods based on Legendre or Tchebyshev polynomials. The authors of [8] were among the first to study the quality of approximation by the PSWFs in the Sobolev space $H^s(I)$, $s > 0$, $I = [-1, 1]$. In particular, they have given an estimate of the decay of the PSWFs expansion coefficients of a function $f \in H^s(I)$, see also [6]. Recently, in [30], the author studied the speed of convergence of the expansion of such a function in a basis of PSWFs. We should mention that the methods used in the previous three references are heavily based on the use of the properties of the PSWFs as eigenfunctions of the differential operator L_c , given by (2). They pose the problem of the best choice of the value of the band-width $c > 0$, for approximating well a given $f \in H^s(I)$, but their answer is mainly experimental. It has been numerically checked in [6, 30] that the smaller the value of s , the larger the value of c should be.

Our study tries to give a satisfactory answer to this important problem of the choice of the parameter c . More precisely, we show that if $f \in H^s(I)$, for some positive real number $s > 0$, then for any integer $N \geq 1$, we have

$$\|f - S_N f\|_{L^2(I)} \leq K(1 + c^2)^{-s/2} \|f\|_{H^s(I)} + K\sqrt{\lambda_N(c)} \|f\|_{L^2(I)}. \quad (10)$$

Here, $S_N f = \sum_{k=0}^N \langle f, \psi_{n,c} \rangle \psi_{n,c}$ and K is a constant depending only on s . Moreover, we study an $L^2(I)$ -convergence rate of the projection $S_N f$ to f . This is done by using the decay of the eigenvalues $(\lambda_n(c))_n$ as well as the use of some estimates of Legendre expansion coefficients of PSWFs, combined with the following exponential decay rate of the PSWFs expansion coefficients for the exponential trigonometric functions

$$|\langle e^{ik\pi x}, \psi_{n,c}(x) \rangle| \leq M' e^{-an}, \quad |k| \leq n/M, \quad n \geq \max(cM, 3). \quad (11)$$

Here, $c \geq 1$, $M \geq 1.40$ and $M', a > 0$ are two positive constants. Under these hypotheses and notations, our rate of convergence of $S_N f$ to $f \in H^s(I)$, $s > 0$, $s \notin \frac{1}{2} + \mathbb{N}$, is given as follows

$$\|f - S_N(f)\|_{L^2(I)} \leq M'(1 + (\pi N)^2)^{-s/2} \|f\|_{H^s} + M' e^{-aN} \|f\|_{L^2}. \quad (12)$$

This work is organized as follows. In Section 2, we list some estimates of the PSWFs and their associated eigenvalues $\chi_n(c)$. In Section 3, we prove a sharp exponential decay rate of the eigenvalues $\lambda_n(c)$ associated with the integral operator \mathcal{Q}_c . In section 4, we first give some useful bounds of the moments of the PSWFs, then we give some practical and useful estimates of the decay of the Legendre expansion coefficients of the PSWFs. In Section 5 we first give the quality of approximation by the PSWFs in the set of almost time and band-limited functions. Then, we combine these results with those of Section 4 and give a first $L^2(I)$ -error bound of approximating a function $f \in H^s(I)$ by its N th terms truncated PSWFs series expansion. The proof of this bound is based on the use of the quality of approximation of almost bandlimited functions by the PSWFs. Then, we study a more elaborated error analysis of the spectral approximation by the PSWFs in the periodic Sobolev space. This quality of approximation is then extended to the usual Sobolev space $H^s(I)$. These new estimates provide us with a way for the choice of the appropriate bandwidth $c > 0$ to be used by a PSWFs based method for the approximation in a given Sobolev space $H^s(I)$. In Section 6, we provide the reader with some numerical examples that illustrate the different results of this work.

We will frequently skip the parameter c in $\chi_n(c)$ and $\psi_{n,c}$, when there is no doubt on the value of the bandwidth. We then note $q = c^2/\chi_n$ and skip both parameters n and c when their values are obvious from the context.

2 Estimates of PSWFs and eigenvalues $\chi_n(c)$.

Here we first list some classical as well as some recent results on PSWFs and their eigenvalues $\chi_n(c)$, then we push forward the methods and adapt them to our study. We systematically use the same notations as in [5]. It is well known that the eigenvalues χ_n satisfy the classical inequalities

$$n(n+1) \leq \chi_n \leq n(n+1) + c^2. \quad (13)$$

In case where $q = c^2/\chi_n \leq 1$, the following better lower bound of χ_n has been recently given in [4],

$$n(n+1) + (3 - 2\sqrt{2})c^2 \leq \chi_n. \quad (14)$$

Next, we recall the elliptic Legendre integral of the first and second kind, that are given respectively, by

$$\mathbf{K}(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad \mathbf{E}(k) = \int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}} dt, \quad 0 \leq k \leq 1. \quad (15)$$

Osipov has proved in [22] that the condition $q = \frac{c^2}{\chi_n} < 1$ is fulfilled when $c < \frac{\pi n}{2}$, while it is not when $c > \frac{\pi(n+1)}{2}$. This is part of the following statement, which gives precise lower and upper bounds of the quantity $\sqrt{q} = \frac{c}{\sqrt{\chi_n}}$, see [5].

Lemma 1. *For all $c > 0$ and $n \geq 2$ we have*

$$\Phi\left(\frac{2c}{\pi(n+1)}\right) < \frac{c}{\sqrt{\chi_n}} < \Phi\left(\frac{2c}{\pi n}\right), \quad (16)$$

where Φ is the inverse of the function $k \mapsto \frac{k}{\mathbf{E}(k)} = \Psi(k)$, $0 \leq k \leq 1$.

This is equivalent to the fact that

$$\frac{\pi n}{2\mathbf{E}(\sqrt{q})} < \sqrt{\chi_n} < \frac{\pi(n+1)}{2\mathbf{E}(\sqrt{q})}. \quad (17)$$

The left hand side is due to Osipov [22]. Note that $\Phi(0) = 0$ and $\Phi(1) = 1$. Also, we should mention that since

$$\Psi'(x) = \frac{\mathbf{E}(x) - x\mathbf{E}'(x)}{(\mathbf{E}(x))^2} = \frac{\mathbf{K}(x)}{(\mathbf{E}(x))^2}, \quad 0 \leq x < 1, \quad (18)$$

then

$$0 \leq \Phi'(x) = \frac{(\mathbf{E}(\Phi(x)))^2}{\mathbf{K}(\Phi(x))} \leq \frac{(\mathbf{E}(0))^2}{\mathbf{K}(0)} = \frac{\pi}{2}, \quad 0 \leq x < 1. \quad (19)$$

Hence, Φ is an increasing function on $[0, 1]$. Moreover, since $\frac{2}{\pi} \leq \frac{1}{\mathbf{E}(x)} \leq 1$, we have

$$\frac{2x}{\pi} \leq \Psi(x) \leq x.$$

One gets the following useful bounds of Φ ,

$$x \leq \Phi(x) \leq \frac{\pi x}{2}, \quad 0 \leq x \leq 1. \quad (20)$$

We will use bounds for $\psi_{n,c}$ given in [5], which have been established under the condition that $(1-q)\sqrt{\chi_n} > 3.5\mathbf{E}(\sqrt{q})$. Compared to [5], where the condition $(1-q)\sqrt{\chi_n(c)} > 3.5\mathbf{E}(\sqrt{q})$ is systematically used to develop the uniform estimates over $[-1, 1]$ of the ψ_n , we leave some flexibility for the choice of the constant κ . We will only need estimates at 1, which we give here in a slightly different form compared to [5].

Let us first recall some notations.

$$\varepsilon_n = ((1-q)\sqrt{\chi_n})^{-1}, \quad \alpha = 1.5, \quad \beta = 0.37. \quad (21)$$

At this moment we do not systematically replace α and β by numerical values to simplify further improvements.

Lemma 2. *Let $n \geq 3$. We assume that the condition*

$$(1-q)\sqrt{\chi_n(c)} > \kappa \quad (22)$$

is satisfied for some $\kappa \geq 4$. Then, there exists a constant $\delta(\kappa)$ (independent of c and n) such that one has the following bounds for $A = \psi_{n,c}(1)\chi_n(c)^{-1/4}$.

$$\frac{\pi}{2\mathbf{K}(\sqrt{q})} (1 - \delta(\kappa) \varepsilon_n) \leq A^2 \leq \frac{\pi}{2\mathbf{K}(\sqrt{q})} (1 + \delta(\kappa) \varepsilon_n). \quad (23)$$

We refer to [4], Theorem 3, for the proof. Explicit values for the constant $\delta(\kappa)$ can also be deduced from [5]. We can choose

$$\delta(\kappa) = \eta \left(2 + \frac{\eta}{\kappa} \right), \quad \eta = C(\kappa) \left(\frac{\beta}{1 + (1 - \kappa^{-1}\beta)^{1/2}} + \frac{\sqrt{2}\alpha\kappa}{\kappa - \alpha} \right) \quad (24)$$

with $C(\kappa)^{-1} = (1 - \kappa^{-1}\beta)^{1/2} - \frac{\sqrt{2}\alpha}{\kappa - \alpha}$.

In any case, we see that the theoretical values of $\delta(\kappa)$ are larger than 4.6. This corresponds to a systematic error in the approximation of the PSWFs. We find approximatively $\delta(4) \approx 90$,

$\delta(12) \approx 7.7$. Numerical tests (see Example 1 in Section 6) tend to prove that the quantities $\delta(\kappa)$ may be taken far much smaller.

We have proved in [4] that one has the inequality

$$|A| = |\psi_{n,c}(1)|\chi_n(c)^{-1/4} \leq 2 \quad \text{for } c \leq \frac{\pi(n+1)}{2}. \quad (25)$$

So in particular the right hand side bound of (23) is not accurate when κ is small.

We need to translate Condition (22) in terms of the parameters n, c , which can be done by using [Proposition 4, [5]]. The inequality given there is the following. For $n \geq 2$ and $q < 1$,

$$(1-q)\sqrt{\chi_n} \geq \frac{(n - \frac{2c}{\pi}) - e^{-1}}{\log n + 5}, \quad (26)$$

A further improvement of the previous inequality is given by the following lemma:

Lemma 3. *Let $n \geq 3$, $q < 1$ and $\kappa \geq 4$. Then one of the following conditions,*

$$c \leq n - \kappa, \quad (27)$$

$$\frac{\pi n}{2} - c > \frac{\kappa}{4}(\ln(n) + 9), \quad (28)$$

implies the inequality (22), that is,

$$(1-q)\sqrt{\chi_n(c)} > \kappa.$$

Moreover, if we assume already that $c > \frac{n+1}{2}$, then the condition $\frac{\pi n}{2} - c > \frac{\kappa}{4}(\ln(n) + 6)$ is sufficient.

Proof. Let $\gamma = \frac{2c}{\pi n}$. It follows from (17) that

$$1 - \gamma < 1 - \sqrt{q} + \frac{\mathbf{E}(\sqrt{q}) - 1}{\mathbf{E}(\sqrt{q})}. \quad (29)$$

We claim that

$$\mathbf{E}(x) - 1 \leq (1 - x^2) \left(\frac{1}{4} \ln \left(\frac{1}{1 - x^2} \right) + \ln 2 \right). \quad (30)$$

Let us assume this and go on with the proof. It follows that

$$1 - \gamma < \frac{1 - q}{\mathbf{E}(\sqrt{q})} \left(\frac{1}{4} \ln \left(\frac{1}{1 - q} \right) + \frac{\mathbf{E}(\sqrt{q})}{1 + \sqrt{q}} + \ln 2 \right). \quad (31)$$

We then use the elementary inequality, valid for $0 < s < 1$,

$$s \ln(1/s) \leq 1/n + s \ln(n/e).$$

It implies that

$$1 - \gamma - \frac{1}{4n\mathbf{E}(\sqrt{q})} < \frac{1 - q}{\mathbf{E}(\sqrt{q})} \left(\frac{1}{4} \ln(n/e) + \frac{\mathbf{E}(\sqrt{q})}{1 + \sqrt{q}} + \ln 2 \right).$$

We use also (17) to conclude that

$$(1 - q)\sqrt{\chi_n} \geq \frac{\pi n}{2\mathbf{E}(\sqrt{q})}(1 - q) > \kappa \quad (32)$$

whenever

$$\frac{\pi n}{2} - c > \kappa \left(\frac{1}{4} \ln(n/e) + \frac{\mathbf{E}(\sqrt{q})}{1 + \sqrt{q}} + \ln 2 \right) + \frac{1}{4n}.$$

This is the case, in particular, when $\frac{\pi n}{2} - c > \frac{\kappa}{4} (\ln(n) + 9)$, using the fact that $\frac{\mathbf{E}(\sqrt{q})}{1+\sqrt{q}} \leq \frac{\pi}{2}$.

The condition $c \geq \frac{n+1}{2}$ implies that $q > \frac{1}{\pi}$. Then, by using the value of $\mathbf{E}(\sqrt{\pi^{-1}})$, the constant 9 in (28) can be replaced by 6.

It remains to prove (30). We write

$$\mathbf{E}(x) - 1 \leq (1-x^2) \int_0^1 \frac{1}{(\sqrt{1-x^2t^2} + \sqrt{1-t^2})} \frac{t dt}{\sqrt{1-t^2}} \quad (33)$$

$$= \int_0^1 \frac{ds}{(1-x^2+s^2x^2)^{\frac{1}{2}} + s}. \quad (34)$$

We cut the last integral into two parts. For the first one, from $\sqrt{1-x^2}$ to 1, we replace the denominator by $2s$ and find the logarithm term. For the second one we replace the denominator by $\sqrt{1-x^2} + s$ and find $\ln 2$. \square

We will need another inequality of the same type:

$$1 - \frac{2c}{\pi n} \leq 2(1-q)\mathbf{K}(\sqrt{q}). \quad (35)$$

This is a consequence of (29), using the fact that $\mathbf{E}(x) - 1 \leq (1-x^2)\mathbf{K}(x)$, which comes directly from (33).

We end this section by giving bounds for the values of the successive derivatives of ψ_n at 0. We have proved in [4] that

$$|\psi_n(0)|^2 + \chi_n^{-1} |\psi_n'(0)|^2 \leq \begin{cases} 1 & \text{if } 0 \leq q \leq 2 \\ \frac{q+1}{\sqrt{q}} & \text{if } q > 2. \end{cases} \quad (36)$$

Let us prove that, furthermore, successive derivatives at 0 may be also bounded.

Proposition 1. *Assume that $q = \frac{c^2}{\chi_n} < 1$. Then for any integer $k \geq 0$ satisfying $k(k+1) \leq \chi_n$, we have*

$$\left| \psi_n^{(k)}(0) \right| \leq (\sqrt{\chi_n})^k. \quad (37)$$

Proof. Because of (36), it is sufficient to prove that $m_k = (\sqrt{\chi_n})^{-k} \left| \psi_n^{(k)}(0) \right|$ is bounded by $(|\psi_n(0)|^2 + \chi_n^{-1} |\psi_n'(0)|^2)^{1/2}$. Moreover, since $\psi_{n,c}$ has same parity as n , then it is sufficient to consider even derivatives or odd derivatives depending on the parity of n . Assume first that n is even and consider $k = 2l$. We show that for a fixed n , $\psi_{n,c}^{(2l)}(0)$ has alternating signs, that is $\psi_{n,c}^{(k)}(0)\psi_{n,c}^{(k-2)}(0) < 0$. Indeed, by an iterative use of the identity

$$(1-x^2)\psi_n''(x) = 2x\psi_n'(x) + (c^2x^2 - \chi_n)\psi_n(x),$$

one can easily check that the $\psi_{n,c}^{(k)}(0) = \psi^{(k)}(0)$ are given by the following recurrence relation,

$$\psi^{(k+2)}(0) = (k(k+1) - \chi_n)\psi^{(k)}(0) + k(k-1)c^2\psi^{(k-2)}(0), \quad k \geq 0, \quad (38)$$

with $\psi(0) > 0$, $\psi^{(2)}(0) = -\chi_n\psi(0)$. Note that $\psi^{(2)}(0)\psi(0) < 0$. Assume that $\psi^{(k)}(0)\psi^{(k-2)}(0) < 0$. Multiplying both sides of (38) by $\psi^{(k)}(0)$, using the assumption that $k(k+1) \leq \chi_n$ as well as the induction hypothesis, one concludes that the induction assumption holds for the order k . Consequently, we have,

$$\left| \psi^{(k+2)}(0) \right| = (\chi_n - k(k+1)) \left| \psi^{(k)}(0) \right| + k(k-1)c^2 \left| \psi^{(k-2)}(0) \right|, \quad k \geq 0. \quad (39)$$

This may be rewritten as

$$m_{k+2} = \left(1 - \frac{k(k+1)}{\chi_n}\right) m_k + k(k-1) \frac{q}{\chi_n} m_{k-2}. \quad (40)$$

The fact that all m_{2l} are bounded by $m_0 = |\psi(0)| = m_2$ follows at once by induction. For n odd the proof follows the same lines. \square

As a consequence of the previous proposition, we have the following corollary concerning the sign and the bounds of the different moments of the ψ_n .

Corollary 1. *Let $c > 0$, be a positive real number. We assume that $q = c^2/\chi_n < 1$. Then, for $j(j+1) \leq \chi_n$, all moments $\int_{-1}^1 y^j \psi_n(y) dy$ of the same parity as n have the same sign and*

$$\left| \int_{-1}^1 y^j \psi_n(y) dy \right| \leq \left(\frac{1}{q}\right)^{j/2} |\mu_n(c)|. \quad (41)$$

Proof. By taking the j -th derivative at zero on both sides of $\int_{-1}^1 e^{icxy} \psi_n(y) dy = \mu_n(c) \psi_n(x)$, one gets

$$\int_{-1}^1 y^j \psi_n(y) dy = (-i)^j c^{-j} \mu_n(c) \psi_n^{(j)}(0), \quad \text{with } i^2 = -1. \quad (42)$$

Since $\psi_n^{(j)}(0)$ and $\psi_n^{(j+2)}(0)$ have opposite signs, then the previous equation implies that moments have the same sign for any positive integer j with $j(j+1) \leq \chi_n$. The second inequality of (41) follows from the previous proposition. \square

3 Sharp decay estimates of eigenvalues $\lambda_n(c)$.

In this section, we use some of the estimates we have given in the previous section and we prove a sharp over-exponential decay rate of the eigenvalues $(\lambda_n(c))_n$. We first recall that these $\lambda_n(c)$ are governed by the following differential equation, see for example [33],

$$\partial_c \ln \lambda_n(c) = \frac{2|\psi_{n,c}(1)|^2}{c}. \quad (43)$$

As a consequence, for fixed n there exists a unique value of c for which $\lambda_n(c) = 1/2$, which we call c_n^* . We know from [14] that it can be bounded below and above, namely

$$\frac{\pi}{2}(n-1) \leq c_n^* \leq \frac{\pi}{2}(n+1) \quad \text{with} \quad \lambda_n(c_n^*) = \frac{1}{2}. \quad (44)$$

By combining (43) and (44), one gets

$$\lambda_n(c) = \frac{1}{2} \exp \left(-2 \int_c^{c_n^*} \frac{(\psi_{n,\tau}(1))^2}{\tau} d\tau \right). \quad (45)$$

Our main result is the following theorem.

Theorem 2. *There exist three constants $\delta_1 \geq 1, \delta_2, \delta_3, \geq 0$ such that, for $n \geq 3$ and $c \leq \frac{\pi n}{2}$,*

$$\delta_1^{-1} n^{-\delta_2} \left(\frac{c}{c+1} \right)^{\delta_3} \leq \frac{\widetilde{\lambda_n(c)}}{\lambda_n(c)} \leq \delta_1 n^{\delta_2} \left(\frac{c}{c+1} \right)^{-\delta_3}, \quad (46)$$

where

$$\widetilde{\lambda}_n(c) = \frac{1}{2} \exp \left(-\frac{\pi^2(n + \frac{1}{2})}{2} \int_{\Phi(\frac{2c}{\pi(n+\frac{1}{2})})}^1 \frac{1}{t(\mathbf{E}(t))^2} dt \right). \quad (47)$$

The factor $\frac{c}{c+1}$ can be replaced by 1 when $c > 1$ and replaced by c when $c < 1$. We have written the formula this way to avoid to have to distinguish between the two cases, $c \geq 1$ and $0 < c < 1$.

It is simpler to write equivalent inequalities for logarithms, which is done in the following proposition. We keep the same notations for constants, which are of course not the same. We note $\ln^+(x)$ the positive part of the Logarithm, that is, $\max(0, \ln(x))$. The following theorem is a fundamental theorem that is required in the proof of the main Theorem 2.

Theorem 3. *There exist three non negative constants $\delta_1, \delta_2, \delta_3$ such that, for $n \geq 3$ and $c \leq \frac{\pi n}{2}$, we have*

$$\int_c^{c_n^*} \frac{(\psi_{n,\tau}(1))^2}{\tau} d\tau = \frac{\pi^2(n + \frac{1}{2})}{4} \int_{\Phi(\frac{2c}{\pi(n+\frac{1}{2})})}^1 \frac{1}{t(\mathbf{E}(t))^2} dt + \mathcal{E}, \quad (48)$$

with

$$|\mathcal{E}| \leq \delta_1 + \delta_2 \ln(n) + \delta_3 \ln^+(1/c). \quad (49)$$

Let us make some comments before starting the proof. At this moment the three constants are not sufficiently small and cannot be used reasonably to obtain numerical values. But they can be computed and are not that enormous. There is no hope, of course, to have found an exact formula for $\lambda_n(c)$ and (47) gives only an approximation. But these theoretical approximation errors may be seen as a kind of theoretical validation of the quality of approximation, which we test numerically in Section 6.

It has been observed by many authors, and predicted by the work of Landau and Widom [15], that for fixed c the eigenvalues $\lambda_n(c)$ decrease first exponentially in some interval starting at $[\frac{2c}{\pi}] + 1$ with length a multiple of $\ln(c)$, then super-exponentially as in the asymptotic behavior given by Widom. This is what one observes in Formula (47), but the error terms do not allow to observe the decay rate at the small decay starting region. In fact the tools that we use, that is, the lower and upper bounds for $\psi_{n,\tau}(1)^2$, are only valid for $c_n^* - \tau$ sufficiently large in terms of $\ln(n)$.

We try to have small constants at each step but are certainly far from the best possible. We give an explicit bound for \mathcal{E} in (70).

The following notations will be used frequently in the sequel. We define

$$I(a, b) = \int_a^b \frac{(\psi_{n,\tau}(1))^2}{\tau} d\tau. \quad (50)$$

$$\mathcal{J}(y) = \frac{\pi^2}{4} \int_{\Phi(\frac{2y}{\pi})}^1 \frac{1}{t(\mathbf{E}(t))^2} dt \quad (51)$$

We should mention that the proofs of Theorems 2 and Theorem 3, require many steps, so we start by giving a sketch of these proofs.

Sketch of the proofs.

We want to prove that

$$I(c, c_n^*) \approx (n + \frac{1}{2}) \mathcal{J} \left(\frac{c}{n + \frac{1}{2}} \right).$$

Lemma 2 expresses the fact that, under some condition depending on a parameter κ , we have

$$\psi_{n,\tau}(1)^2 \approx \frac{\pi \sqrt{\chi_n(\tau)}}{2\mathbf{K}(\sqrt{q(\tau)})} = \frac{\pi \tau}{2\sqrt{q(\tau)}\mathbf{K}(\sqrt{q(\tau)})}.$$

The parameter κ is related with the quality of approximation and Lemma 3 proves that the condition for this may be written $c < c_n^\kappa$ for some c_n^κ . From the last equivalence, it follows that

$$I(c, c_n^\kappa) \approx \int_c^{c_n^\kappa} \frac{\pi d\tau}{2\sqrt{q(\tau)}\mathbf{K}(\sqrt{q(\tau)})}.$$

Then Lemma 1 will be interpreted as the fact that

$$\sqrt{q(\tau)}\mathbf{K}(\sqrt{q(\tau)}) \approx \Phi\left(\frac{2\tau}{\pi(n+\frac{1}{2})}\right)\mathbf{K}\circ\Phi\left(\frac{2\tau}{\pi(n+\frac{1}{2})}\right).$$

It is then elementary to rely the new integral with the function \mathcal{J} and finally find that

$$I(c, c_n^\kappa) \approx \left(n + \frac{1}{2}\right)\mathcal{J}\left(\frac{c}{n + \frac{1}{2}}\right).$$

It remains to bound the tails of the integrals $I(c_n^\kappa, c_n^*)$, which we can do because the two values are sufficiently close.

Let us start the proof itself. We need a set of intermediate results that can be classified into three main steps. The first step will concern the properties of the function \mathcal{J} . In the second step, we give bounds of the tails of the integrals. Finally, in the third step, we use the results of the previous two steps and complete the proofs of Theorems 2 and 3.

First step: Properties of \mathcal{J} .

We define

$$\mathcal{J}_l(c) = \frac{\pi}{2} \int_c^{\frac{\pi l}{2}} \frac{d\tau}{\Phi\left(\frac{2\tau}{\pi l}\right)\mathbf{K}\circ\Phi\left(\frac{2\tau}{\pi l}\right)}. \quad (52)$$

Such integrals are clearly involved in the proof as seen in the sketch. We first see that they are related with \mathcal{J} .

Lemma 4. *We have the identity*

$$\mathcal{J}_l(c) = l\mathcal{J}(c/l). \quad (53)$$

Proof. We consider the substitution

$$s = \Phi\left(\frac{2\tau}{\pi l}\right), \quad \tau = \frac{\pi l}{2}\Psi(s). \quad (54)$$

We have already seen in (18) that $\Psi'(x) = \frac{\mathbf{K}(x)}{(\mathbf{E}(x))^2}$. Hence, we have

$$\mathcal{J}_l(c) = l \int_{\Phi\left(\frac{2c}{\pi l}\right)}^1 \frac{ds}{s(\mathbf{E}(s))^2} = l\mathcal{J}(c/l).$$

□

The following proposition gives us upper and lower bounds, as well as the asymptotic behavior of \mathcal{J} .

Proposition 2. *For $x \in (0, \pi/2)$, one has the upper and lower bounds*

$$\ln^+\left(\frac{1}{x}\right) \leq \mathcal{J}(x) \leq \frac{\pi^2}{4} \ln\left(\frac{\pi}{2x}\right). \quad (55)$$

Moreover, one can write

$$\mathcal{J}(x) = \frac{\pi^2}{4} \int_{\Phi(2x/\pi)}^1 \frac{dt}{t(\mathbf{E}(t))^2} = \ln\left(\frac{4}{ex}\right) + \mathcal{E}', \quad (56)$$

with $|\mathcal{E}'| \leq \frac{\pi^2 x^2}{8}$.

Proof. The first inequalities are an easy consequence of bounds below and above of Φ , given by (20). Let us prove (56). We first write, for $0 < y < 1$,

$$\frac{\pi^2}{4} \int_y^1 \frac{dt}{t(\mathbf{E}(t))^2} + \ln(y) = \Delta - \int_0^y \frac{\frac{\pi^2}{4} - \mathbf{E}(t)^2}{t(\mathbf{E}(t))^2} dt = \Delta - I_1(y). \quad (57)$$

Here

$$\Delta = \int_0^1 \frac{\frac{\pi^2}{4} - \mathbf{E}(t)^2}{t(\mathbf{E}(t))^2} dt.$$

It is probably well-known that

$$\Delta = \ln\left(\frac{4}{e}\right) \quad (58)$$

but we did not find any reference. We will see it as a corollary of Widom's Theorem. The integral $I_1(y)$ is bounded by $\frac{\pi^2 y^2}{8}$. This is a consequence of the elementary inequalities

$$1 \leq \mathbf{E}(s) \leq \frac{\pi}{2}, \quad \frac{\pi}{2} - \mathbf{E}(s) \leq x^2 \int_0^1 \frac{t^2 dt}{\sqrt{1-t^2}} = \frac{\pi s^2}{4}.$$

Let us now fix $y = \Phi(2x/\pi)$. At this point we have proved that

$$0 \leq \ln\left(\frac{x}{y}\right) - \mathcal{E}' = I_1(y) \leq \frac{\pi^2 y^2}{8}.$$

From the inequalities

$$\frac{2y}{\pi} \leq \frac{2x}{\pi} = \frac{y}{\mathbf{E}(y)} \leq \frac{2y}{\pi} \left(1 - \frac{y^2}{2}\right)^{-1} \leq \frac{2y}{\pi} (1 + y^2),$$

it follows that $0 \leq \ln\left(\frac{x}{y}\right) + y^2$. We have proved the proposition. \square

This proposition leads to the following corollary, where we recognize the equivalent given by Widom.

Corollary 2. *We have the double inequality*

$$\frac{1}{2} \left(\frac{ec}{4(n + \frac{1}{2})} \right)^{2n+1} e^{-\frac{\pi^2}{4} \frac{c^2}{n + \frac{1}{2}}} \leq \widetilde{\lambda}_n(c) \leq \frac{1}{2} \left(\frac{ec}{4(n + \frac{1}{2})} \right)^{2n+1} e^{+\frac{\pi^2}{4} \frac{c^2}{n + \frac{1}{2}}}. \quad (59)$$

Proof. Just note that $\widetilde{\lambda}_n(c) = \frac{1}{2} \exp(-(2n+1)\mathcal{J}(c/(n+1/2)))$ and use (56) with $x = \frac{c}{n+1/2}$. \square

Let us go back to quantities \mathcal{J}_l . It is a straightforward consequence of (53) that the quantity $\mathcal{J}_l(c)$ increases with l . The next lemma gives reverse inequalities.

Lemma 5. *We have the inequalities*

$$\mathcal{J}_{n+1}(c) - \frac{\pi^2}{8} \ln\left(\frac{\pi(n+1)}{2c}\right) - \frac{\pi^3}{16} \leq \mathcal{J}_{n+\frac{1}{2}}(c) \leq \mathcal{J}_n(c) - \frac{\pi^2}{8} \ln\left(\frac{\pi(n+\frac{1}{2})}{2c}\right) + \frac{\pi^3}{16}. \quad (60)$$

Proof. We will prove only one of the inequalities, the other one being identical. Elementary computations give

$$\mathcal{J}_{n+1}(c) - \mathcal{J}_{n+\frac{1}{2}}(c) \leq \frac{1}{2} \mathcal{J}\left(\frac{c}{n+1}\right) + \frac{\pi^2}{4} \left(n + \frac{1}{2}\right) \ln\left(\frac{\Phi\left(\frac{2c}{\pi(n+\frac{1}{2})}\right)}{\Phi\left(\frac{2c}{\pi(n+1)}\right)}\right).$$

We use (55) for the first term. The second one is bounded by

$$\frac{\pi^2}{4} \left(n + \frac{1}{2}\right) \frac{\Phi\left(\frac{2c}{\pi(n+\frac{1}{2})}\right) - \Phi\left(\frac{2c}{\pi(n+1)}\right)}{\Phi\left(\frac{2c}{\pi(n+1)}\right)} \leq \frac{\pi^3}{16}.$$

Indeed, this is a consequence of the fact that $\Phi'(x) \leq \pi/2$ and $\frac{x}{\Phi(x)} \leq 1$, for $0 < x \leq 1$. \square

Second step: tails of the integrals.

We fix some constant $\kappa \geq 4$ (for instance $\kappa = 12$) and we assume that $n \geq 2\kappa + 1$. Then, we know from Lemma 3, that the condition (22), that is,

$$(1 - q)\sqrt{\chi_n} > \kappa,$$

is satisfied for $c < \frac{n+1}{2}$. Next, if we define

$$c_n^\kappa = \max\left(\frac{\pi n}{2} - \frac{\kappa}{4}(\ln(n) + 6), \frac{n+1}{2}\right) \quad (61)$$

then, we have the following.

Lemma 6. *For $n \geq 2\kappa + 1$ we have the inequality*

$$I(c_n^\kappa, c_n^*) \leq \pi\kappa \ln(n) + 6\pi\kappa + 2\pi^2. \quad (62)$$

Proof. Recall that $|\psi_{n,c}(1)| \leq 2\chi_n^{1/4}$ and $\sqrt{\chi_n(c)} \leq \frac{\pi}{2}(n+1)$, [22], so that

$$|\psi_{n,\tau}(1)|^2 \leq 4\sqrt{\chi_n(\tau)} \leq 2\pi(n+1).$$

Hence, we have

$$\int_{c_n^\kappa}^{c_n^*} \frac{(\psi_{n,\tau}(1))^2}{\tau} d\tau \leq 2\pi(n+1) \ln\left(1 + \frac{\frac{\pi}{2} + \frac{\kappa}{4}(\ln(n) + 6)}{c_n^\kappa}\right).$$

We conclude by using the fact that $c_n^\kappa \geq \frac{n+1}{2}$. \square

We claim that we can conclude the proof of Theorem 3 when $n \geq 2\kappa + 1$ and $c < c_n^\kappa$. More precisely, we get, under these conditions, the inequalities

$$-\frac{\pi^2}{16}(\kappa \ln(n) + 6\pi\kappa + \pi) \leq I(c, c_n^*) - \left(n + \frac{1}{2}\right) \mathcal{J}\left(\frac{c}{n + \frac{1}{2}}\right) \leq \pi\kappa \ln(n) + 6\pi\kappa + 2\pi^2 \quad (63)$$

The right hand side comes from the previous lemma, the left hand side from (55).

We will conclude this paragraph by showing that we have also the conclusions of Theorem 3 and Theorem 2 for the finite number of missing values of n , that is, $n \leq 2\kappa + 1$. There is no problem to have upper bounds and lower bounds that do not depend on c for $c < 1$. From Corollary 2, we have a precise estimate in terms of c^{2n+1} for $\widetilde{\lambda_n(c)}$. The same is given for $\lambda_n(c)$ by the following lemma.

Lemma 7. *Assume that $n \geq 1$ is fixed and let $0 < c < 1$. Then there exist two constants $\delta(n), \delta'(n)$ such that*

$$\delta(n)c^{2n+1} \leq \lambda_n(c) \leq \delta'(n)c^{2n+1}. \quad (64)$$

Proof. We first note that $I(1, c_n^*) \leq I(1, \frac{\pi(n+1)}{2})$. We recall that on this interval we have the inequality $|\psi_{n,\tau}(1)|^2 \leq 4\frac{\pi(n+1)}{2}$. So $I(1, c_n^*) \leq 2\pi(n+1)\ln(\frac{\pi(n+1)}{2})$. Inside the integral defining $I(c, 1)$ we use the following inequality, that may be found in [5],

$$\left| |\psi_{n,\tau}(1)| - \sqrt{n + \frac{1}{2}} \right| \leq \frac{\tau^2}{2}. \quad (65)$$

So $|I(c, 1) - (n + \frac{1}{2})\ln(\frac{1}{c})| \leq 1$, from which we conclude. \square

Third step: Proofs of Theorems 2 and 3.

We fix $\kappa > 4$. Because of the previous steps, which allowed to conclude in the other cases, we can assume that

$$n \geq 2\kappa + 1 \quad c < c_n^\kappa = \max\left(\frac{\pi n}{2} - \frac{\kappa}{4}(\ln(n) + 6), \frac{n+1}{2}\right).$$

In view of (48), we want to give a bound to

$$\mathcal{E} = I(c, c_n^*) - \left(n + \frac{1}{2}\right) \mathcal{J}\left(\frac{c}{n + \frac{1}{2}}\right).$$

We have already given a bound to a first error term

$$\mathcal{E}_1 = I(c, c_n^*) - I(c, c_n^\kappa).$$

Because of (62) we know that

$$0 \leq \mathcal{E}_1 \leq \pi\kappa \ln(n) + 6\pi\kappa + 2\pi^2. \quad (66)$$

Next, the conditions on κ allow to use the double inequality (23). Namely,

$$(\psi_{n,\tau}(1))^2 = \frac{\pi}{2\mathbf{K}(\sqrt{q})} \sqrt{\chi_n(\tau)} + \mathcal{R}(\tau), \quad |\mathcal{R}(\tau)| \leq \frac{\delta(\kappa)}{(1-q(\tau))\mathbf{K}(\sqrt{q(\tau)})}, \quad 0 \leq \tau \leq c_n^\kappa. \quad (67)$$

This leads to a second error,

$$\mathcal{E}_2 = I(c, c_n^\kappa) - \frac{\pi}{2} \int_c^{c_n^\kappa} \frac{d\tau}{\sqrt{q(\tau)}\mathbf{K}(\sqrt{q(\tau)})},$$

which is bounded by

$$|\mathcal{E}_2| \leq \delta(\kappa) \int_c^{c_n^\kappa} \frac{1}{(1-q(\tau))\mathbf{K}(\sqrt{q(\tau)})} \frac{d\tau}{\tau}.$$

Lemma 8. *We have the inequality*

$$|\mathcal{E}_2| \leq 2\delta(\kappa) \left(\left(1 + \frac{\pi\kappa}{4}\right) \ln(n) + \ln^+(c) + \frac{3\pi\kappa}{2} \right). \quad (68)$$

Proof. By (35), we know that

$$2(1-q(\tau))\mathbf{K}(\sqrt{q(\tau)}) \geq 1 - \frac{2\tau}{\pi n}.$$

So we have the inequality

$$|\mathcal{E}_2| \leq 2\delta(\kappa) \int_{\frac{2c}{\pi n}}^{\frac{2c_n^\kappa}{\pi n}} \frac{ds}{(1-s)s} \leq 2\delta(\kappa) \left(\ln\left(\frac{n}{c}\right) + \ln\left(\frac{1}{1 - \frac{2c_n^\kappa}{\pi n}}\right) \right).$$

We conclude at once. \square

It remains to consider the main term, that is,

$$I_{\text{main}}(c, c_n^\kappa) = \frac{\pi}{2} \int_c^{c_n^\kappa} \frac{\sqrt{\chi_n(\tau)} d\tau}{\mathbf{K}(\sqrt{q(\tau)} \tau)} = \frac{\pi}{2} \int_c^{c_n^\kappa} \frac{d\tau}{\sqrt{q(\tau)} \mathbf{K}(\sqrt{q(\tau)})}. \quad (69)$$

We will use the monotonicity properties of $\sqrt{q(\tau)} \mathbf{K}(\sqrt{q(\tau)})$, namely

$$\Phi\left(\frac{2\tau}{\pi(n+1)}\right) \mathbf{K} \circ \Phi\left(\frac{2\tau}{\pi(n+1)}\right) \leq \sqrt{q(\tau)} \mathbf{K}(\sqrt{q(\tau)}) \leq \Phi\left(\frac{2\tau}{\pi n}\right) \mathbf{K} \circ \Phi\left(\frac{2\tau}{\pi n}\right).$$

It follows that

$$\mathcal{J}_n(c) - \mathcal{J}_n(c_n^\kappa) \leq I_{\text{main}}(c, c_n^\kappa) \leq \mathcal{J}_{n+1}(c).$$

So the last error,

$$\mathcal{E}_3 = I_{\text{main}}(c, c_n^\kappa) - \mathcal{J}_{n+\frac{1}{2}}(c) = I_{\text{main}}(c, c_n^\kappa) - \left(n + \frac{1}{2}\right) \mathcal{J}\left(\frac{c}{n + \frac{1}{2}}\right),$$

satisfies the inequalities

$$\mathcal{J}_n(c) - \mathcal{J}_{n+\frac{1}{2}}(c) - \mathcal{J}_n(c_n^\kappa) \leq \mathcal{E}_3 \leq \mathcal{J}_{n+1}(c) - \mathcal{J}_{n+\frac{1}{2}}(c).$$

It remains to use (60) and (55) to conclude. We finally find that

$$|\mathcal{E}| \leq \pi\kappa \ln(n) + 6\pi\kappa + 2\pi^2 + 2\delta(\kappa) \left(\left(1 + \frac{\pi\kappa}{4}\right) \ln(n) + \ln^+(c) + \frac{3\pi\kappa}{2} \right) + \frac{\pi^2}{8} \ln\left(\frac{\pi(n + \frac{1}{2})}{2c}\right) + \frac{\pi^3}{16}. \quad (70)$$

So we can take the following values for $\delta_1, \delta_2, \delta_3$.

$$\begin{aligned} \delta_1 &= 22 + 3\pi\kappa(2 + \delta(\kappa)) \\ \delta_2 &= \frac{\pi^2}{8} + \pi\kappa + 2\delta(\kappa)\left(1 + \frac{\pi\kappa}{4}\right) \\ \delta_3 &= \frac{\pi^2}{8} + 2\delta(\kappa)\left(1 + \frac{\pi\kappa}{4}\right). \end{aligned}$$

It is easy to see from the proof above that this bound is also valid for $c_n^\kappa < c < \frac{n\pi}{2}$, that is, under the assumptions of Theorem 3, except for the values $n \leq 2\kappa + 1$. These estimates are not sharp enough to justify a further study to minimize the sum by a specific choice of κ . When $\kappa = 12$ we find $\delta_2 \approx 200$. We could have improved bounds at each step, but not significantly. Numerical experiments tend to prove that they are much smaller.

This concludes the proofs of Theorem 3 and Theorem 2.

From Theorem 3 and Corollary 2 we get the following corollary:

Corollary 3. *There exist three constants $\delta_1 \geq 1, \delta_2, \delta_3, \geq 0$ such that, for $n \geq 3$ and $c \leq \frac{\pi n}{2}$,*

$$A(n, c)^{-1} \left(\frac{ec}{2(2n+1)} \right)^{2n+1} \leq \lambda_n(c) \leq A(n, c) \left(\frac{ec}{2(2n+1)} \right)^{2n+1}. \quad (71)$$

with

$$A(n, c) = \delta_1 n^{\delta_2} \left(\frac{c}{c+1} \right)^{-\delta_3} e^{+\frac{\pi^2}{4} \frac{c^2}{n}}.$$

Widom's Theorem says that $A(n, c)$ can be replaced by a quantity that tends to 1 for n tending to ∞ . We cannot give such an asymptotic behavior at this moment, but we can estimate errors for fixed c and n , which he does not. Remark that we used the fact that $\Delta = \ln(4/e)$, see (58), without proving it or giving a reference. This is a consequence of the asymptotic behavior found by Widom, which cannot be valid at the same time as (71) if $e/4$ is replaced by another constant. This implies in particular Theorem 1.

It may be useful to give also the following corollary.

Corollary 4. *There exist constants $a > 0$ and $\delta \geq 1$ such that, for $c \geq 1$ and $n > 1.35c$, we have*

$$\lambda_n(c) \leq \delta e^{-an}. \quad (72)$$

Proof. The constant 1.35 has been chosen so that $2 \ln(\frac{4n}{cc}) > \frac{\pi^2 c^2}{4n^2}$ for $n > 1.35c$. \square

Also, by using (71), one gets the following corollary.

Corollary 5. *Let $c \geq 1$, then for any $0 \leq a < \frac{4}{e}$, there exists $N_a \in \mathbb{N}$ such that*

$$\lambda_n(c) \leq e^{-2n \log(\frac{an}{c})}, \quad \forall n \geq N_a.$$

Moreover, for any $b > \frac{4}{e}$, there exists $N_b \in \mathbb{N}$ such that

$$\lambda_n(c) > e^{-2n \log(\frac{bn}{c})}, \quad \forall n \geq N_b.$$

4 Decay estimates of the Legendre expansion coefficients

Recently, there is an extensive amount of work devoted to new highly accurate computational methods of the PSWFs, see [3, 12, 33]. In particular, the methods given in [3, 33] are based on an efficient quadrature method on the unit circle that provides highly accurate values of the PSWFs inside $[-1, 1]$, as well as accurate approximations of the different eigenvalues $\mu_n(c)$, $n \geq 0$. The methods developed in [12] for computing the values of the $\psi_{n,c}(x)$ inside $[-1, 1]$ and the eigenvalues $\mu_n(c)$ are based on an appropriate matrix representation of the finite Fourier transform operator \mathcal{F}_c , given by (1). Also, we should mention a classical method known as Flammer's method, [9] that uses the differential operator \mathcal{L}_c , is extensively used to compute the PSWFs and their eigenvalues. This method is based on the following Legendre expansion of the PSWFs,

$$\psi_n(x) = \sum_{k \geq 0} \beta_k^n \overline{P_k}(x). \quad (73)$$

Recall that ψ_n has the same parity as n . Hence, the previous Legendre expansion coefficients of the ψ_n satisfy $\beta_k^n = 0$ if n and k have different parities. The expansion (73) is in particular used to compute the eigenvalues in terms of the coefficients β_k^n . Indeed, using the fact that the Fourier transform of the Legendre polynomials can be expressed in terms of Bessel functions, as well as the property of ψ_n of being an eigenfunction of \mathcal{F}_c , we have

$$\psi_n(x) = \frac{\sqrt{2\pi}}{|\mu_n(c)|} \sum_{k \geq 0} (-1)^k \beta_k^n \sqrt{k+1/2} \frac{J_{k+1/2}(cx)}{\sqrt{cx}}, \quad (74)$$

which extends analytically outside I . Here J_α denotes the Bessel function of the first kind and order $\alpha > -1$. As a consequence, Slepian has proved in [27] that

$$\mu_n(c) = \frac{2\pi}{c} \left[\frac{\sum_{k \geq 0} i^k \sqrt{k+1/2} \beta_k^n J_{k+1/2}(c)}{\sum_{k \geq 0} \beta_k^n \sqrt{k+1/2}} \right], \quad (75)$$

is the exact value of the n -th eigenvalue of the finite Fourier transform operator \mathcal{F}_c .

It is well known that the different expansion coefficients $(\beta_k^n)_k$ as well as the corresponding eigenvalues χ_n are obtained by solving the following eigensystem

$$\begin{aligned} & \frac{(k+1)(k+2)}{(2k+3)\sqrt{(2k+5)(2k+1)}}c^2\beta_{k+2}^n + \left(k(k+1) + \frac{2k(k+1)-1}{(2k+3)(2k-1)}c^2\right)\beta_k^n \\ & + \frac{k(k-1)}{(2k-1)\sqrt{(2k+1)(2k-3)}}c^2\beta_{k-2}^n = \chi_n(c)\beta_k^n, \quad k \geq 0. \end{aligned} \quad (76)$$

A useful decay estimate of the β_k^n is based on the following positivity result of the β_k^n .

Lemma 9. *Let $c > 0$, be a fixed positive real number. Then, for all positive integers k, n such that $k(k-1) + 1.13c^2 \leq \chi_n(c)$, we have $\beta_k^n \geq 0$.*

Proof. We recall that the β_k^n are given by the eigensystem (76). Let us first consider $k = 2$ (when n is even) and $k = 3$ (when n is odd) satisfying the condition. We compute

$$\beta_2^n = \frac{3\sqrt{5}}{2c^2} \left(\chi_n - \frac{c^2}{3} \right) \beta_0^n \geq \beta_0^n \geq 0, \quad \beta_3^n = \frac{5\sqrt{21}}{6c^2} \left(\chi_n - 2 - \frac{3c^2}{5} \right) \beta_1^n \geq \beta_1^n \geq 0.$$

For $k \geq 2$, taking upper bounds for the fractions as in [8], Equation (76) implies that

$$\frac{2c^2}{3\sqrt{5}}(\beta_{k+2}^n + \beta_{k-2}^n) \geq (\chi_n(c) - k(k+1) - \frac{11c^2}{21})\beta_k^n.$$

The constant 1.13 has been chosen so that $1.13 > \frac{4}{3\sqrt{5}} + \frac{11}{21}$. By the assumption on k it is not possible for β_{k+2} and β_{k-2} to be bounded by β_k , with $\beta_k > 0$. The same is valid at each step $k-2, k-4, \dots$. Since $\beta_2 \geq \beta_0 > 0$ (resp. $\beta_3 \geq \beta_1 > 0$) depending on the parity of n , the sequence β_{2j} (resp. β_{2j+1}) is non decreasing for $2j \leq k+2$ (resp. $2j+1 \leq k+2$). This implies the positivity. \square

The following proposition provides us with a useful decay rate of the expansion coefficients β_k^n .

Proposition 3. *Let $c > 0$, be a fixed positive real number. Then, for all positive integers n, k such that $k(k-1) + 1.13c^2 \leq \chi_n(c)$, we have*

$$|\beta_0^n| \leq \frac{1}{\sqrt{2}}|\mu_n(c)| \quad \text{and} \quad |\beta_k^n| \leq \sqrt{\frac{5}{4\pi}} \left(\frac{2}{\sqrt{q}} \right)^k |\mu_n(c)|. \quad (77)$$

Proof. The first inequality follows from Corollary 1 and the fact that $\beta_0^n = \frac{1}{\sqrt{2}} \int_{-1}^1 \psi_n(y) dy$. To prove the second inequality, we first note that the moments of the normalized Legendre polynomials are non-negative and they are given in [2], by

$$a_{jk} = \int_{-1}^1 x^j \overline{P_k}(x) dx = \begin{cases} 0 & \text{if } j < k \text{ or } j - k \text{ is odd} \\ \frac{\sqrt{\pi} j!}{2^j \binom{j-k}{2}! \Gamma(\frac{k+j+3}{2})} & \text{if } j - k \geq 0 \text{ and } j - k \text{ is even.} \end{cases} \quad (78)$$

Since $x^j = \sum_{k=0}^j a_{jk} \overline{P_k}(x)$, then the moments of the ψ_n are related to the PSWFs Legendre expansion coefficients by the following rule,

$$\int_{-1}^1 x^j \psi_n(x) dx = \sum_{k=0}^j a_{jk} \beta_k^n.$$

Since by the previous lemma, we have $\beta_k^n \geq 0$, for any $0 \leq k \leq j$ and since the a_{jk} are positive, then the previous equality implies that

$$\beta_j^n \leq \frac{1}{a_{jj}} \int_{-1}^1 x^j \psi_n(x) dx \leq \frac{1}{a_{jj}} \left(\frac{1}{q}\right)^{j/2} |\mu_n(c)|. \quad (79)$$

The last inequality follows from the previous corollary. On the other hand, we have

$$a_{jj} = \frac{\sqrt{\pi} \sqrt{j+1/2} j!}{2^j \Gamma(j+3/2)} = \frac{\sqrt{\pi} j!}{2^j \sqrt{j+1/2} \Gamma(j+1/2)}.$$

Moreover, it is well known that $j^{1-s} \leq \frac{\Gamma(j+1)}{\Gamma(j+s)} \leq (j+1)^{1-s}$. Hence, we have

$$\frac{1}{a_{jj}} \leq \frac{2^j}{\sqrt{\pi}} \sqrt{1 + \frac{1}{2j}} \leq 2^j \sqrt{\frac{5}{4\pi}}, \quad \forall j \geq 1. \quad (80)$$

By combining (79) and (80), one gets the second inequality of (77). \square

Remark 1. The condition $k(k-1) + 1.13c^2 \leq \chi_n(c)$ of the previous proposition can be replaced with the following more explicit condition. Consider a real number $A > 1$, then by using (14), one concludes that if $n \geq c\alpha \frac{A}{\sqrt{A^2-1}}$ with $\alpha = \sqrt{1.13 + 2\sqrt{2} - 3} \approx 0.979$ and $k \leq n/A$, then the conditions for (77) are satisfied. Moreover, the previous constant 0.979 is certainly not optimal, it is a consequence of the non optimal lower bound of $\chi_n(c)$, given by (14).

Remark 2. In [8], by using the eigensystem (76), the authors have obtained under a stronger condition, a decay of the Legendre coefficients $(\beta_k^n)_k$ which is similar but less precise to the one we have given by (77). More precisely, they have shown that if $0 < k \leq 2m$ with $m = O(n^{2/3})$ and $2m(2m+1) < \frac{\log 2}{2} \chi_n$, then there exists a constant D such that

$$|\beta_k^n| \leq D \left(\frac{2}{\sqrt{q}}\right)^k |\beta_0^n|, \quad \text{for even } k, \quad |\beta_k^n| \leq D \left(\frac{2}{\sqrt{q}}\right)^k |\beta_1^n|, \quad \text{for odd } k.$$

5 Quality of the spectral approximation by the PSWFs

In this section, we first study the quality of approximation of almost band-limited functions by the classical PSWFs, ψ_n that are concentrated on $[-b, b]$, for some $b > 0$. Then, we extend this study to the case of periodic and non periodic Sobolev space $H^s([-1, 1])$, $s > 0$.

5.1 Approximation of almost time and band-limited functions

In this paragraph, $\|\cdot\|_2$ denotes the norm in $L^2(\mathbb{R})$. We show that the set $\{\psi_n(x), n \geq 0\}$ is well adapted for the representation of almost time-limited and almost band-limited functions, which are defined as follows.

Definition 1. Let $T = [-a, +a]$ and $\Omega = [-b, +b]$ be two intervals. A function f , which we assume to be normalized in such a way that $\|f\|_2 = 1$, is said to be ϵ_T -concentrated in T and ϵ_Ω -band concentrated in Ω if

$$\int_{T^c} |f(t)|^2 dt \leq \epsilon_T^2, \quad \frac{1}{2\pi} \int_{\Omega^c} |\widehat{f}(\omega)|^2 d\omega \leq \epsilon_\Omega^2.$$

Up to a re-scaling of the function f , we can always assume that $T = [-1, 1]$ and $\Omega = [-c, +c]$, with $c := ab$. Indeed, for f that is ϵ_T -concentrated in $T = [-a, +a]$ and ϵ_Ω -band concentrated in $\Omega = [-b, +b]$, the normalized function $g(t) = \sqrt{a}f(at)$ is ϵ_T -concentrated in $[-1, +1]$ and ϵ_Ω -band concentrated in $[-ab, +ab]$.

Before stating the theorem, let us give some notations. For $f \in L^2(\mathbf{R})$, we consider its expansion $f = \sum_{n \geq 0} a_n \psi_{n,c}$ in $L^2([-1, +1])$. Due to the normalization of the functions $\psi_{n,c}$ given by (3), the following equality holds,

$$\int_{-1}^{+1} |f(t)|^2 dt = \sum_{n \geq 0} |a_n|^2. \quad (81)$$

We call $S_{N,c}f$, the N -th partial sum, defined by

$$S_{N,c}f(t) = \sum_{n < N} a_n \psi_{n,c}(t). \quad (82)$$

We write more simply $S_N f$ when there is no ambiguity. In the next lemma, we prove that $S_N f$ tends to f rapidly when f belongs to the space of band-limited functions. This statement is both very simple and classical, see for instance [26, 27] or Theorem 3.1 in [30].

Lemma 10. *Let $f \in B_c$ be an L^2 normalized function. Then*

$$\int_{-1}^{+1} |f - S_N f|^2 dt \leq \lambda_N(c). \quad (83)$$

Proof. Since the set of functions $\psi_{n,c}$ is also an orthogonal basis of B_c , the function f may be written on \mathbb{R} as $f = \sum_{n \geq 0} a_n \psi_{n,c}$, with

$$\int_{\mathbb{R}} |f(t)|^2 dt = \sum_{n \geq 0} |\lambda_n(c)|^{-1} |a_n|^2. \quad (84)$$

The two expansions coincide on $[-1, +1]$, and, from (84) applied to $f - S_N f$, it follows that

$$\int_{-1}^{+1} |f - S_N f|^2 dt \leq \sup_{n \geq N} |\lambda_n(c)| \sum_{n \geq N} |\lambda_n(c)|^{-1} |a_n|^2.$$

We use the fact that the sequence $|\lambda_n(c)|$ decreases and (84) to conclude. \square

Next we define the time-limiting operator P_T and the band-limiting operator Π_Ω by:

$$P_T(f)(x) = \chi_T(x) f(x), \quad \Pi_\Omega(f)(x) = \frac{1}{2\pi} \int_{\Omega} e^{ix\omega} \widehat{f}(\omega) d\omega.$$

The following proposition provides us with the quality of approximation of almost time- and band-limited functions by the PSWFs.

Proposition 4. *If f is an L^2 normalized function that is ϵ_T -concentrated in $T = [-1, +1]$ and ϵ_Ω -band concentrated in $\Omega = [-c, +c]$, then for any positive integer N , we have*

$$\left(\int_{-1}^{+1} |f - S_N f|^2 dt \right)^{1/2} \leq \epsilon_\Omega + \sqrt{\lambda_N(c)} \quad (85)$$

and, as a consequence,

$$\|f - P_T S_N f\|_2 \leq \epsilon_T + \epsilon_\Omega + \sqrt{\lambda_N(c)}. \quad (86)$$

More generally, if f is an L^2 normalized function that is ϵ_T -concentrated in $T = [-a, +a]$ and ϵ_Ω -band concentrated in $\Omega = [-b, +b]$ then, for $c = ab$ and for any positive integer N , we have

$$\|f - P_T S_{N,c,a} f\|_2 \leq \epsilon_T + \epsilon_\Omega + \sqrt{\lambda_N(c)} \quad (87)$$

where $S_{N,c,a}$ gives the N -th partial sum for the orthonormal basis $\frac{1}{\sqrt{a}}\psi_{n,c}(t/a)$ on $[-a, +a]$.

Proof: We first prove (85) by writing f as the sum of $\Pi_\Omega f$ and g . Remark first that $\int_{-1}^{+1} |g - S_N g|^2 dt \leq \|g\|_2 \leq \epsilon_\Omega$. We then use Lemma 10 for the band limited function $\Pi_\Omega f$ to conclude. The rest of the proof follows at once. \square

Remark 3. Let f be a normalized L^2 function that vanishes outside I and we assume that $f \in H^s(\mathbb{R})$. Then f gives an example of 0-concentrated in I and ϵ_c -band concentrated in $[-c, +c]$, with $\epsilon_c \leq M_f/c^s$ and $M_f^2 = \frac{1}{2\pi} \int |\widehat{f}(\xi)|^2 |\xi|^{2s} d\xi$.

5.2 Approximation by the PSWFs in Sobolev spaces

In this paragraph, we study the quality of approximation by the PSWFs in the Sobolev space $H^s([-1, 1])$. We provide an $L^2([-1, 1])$ -error bound of the approximation of a function $f \in H^s([-1, 1])$ by the N -th partial sum of its expansion in the basis of PSWFs.

To simplify notation we will write $I = [-1, 1]$. We should mention that different spectral approximation results by the PSWFs in $H^s(I)$ have been already given in [6, 8, 30]. More precisely, the following result has been proved in [8]. Here $a_k(f) = \int_{-1}^1 f(x)\psi_k(x) dx$.

Theorem 4. (Theorem 3.1 in [8]). Let $f \in H^s(I)$, $s \geq 0$. Then

$$|a_N(f)| \leq C \left(N^{-2/3s} \|f\|_{H^s(I)} + \left(\sqrt{\frac{c^2}{\chi_N(c)}} \right)^{\delta N} \|f\|_{L^2(I)} \right),$$

where C, δ are independent of f, N and c .

In [30], the author has used a different approach for the study of the spectral approximation by the PSWFs. More precisely, by considering the weighted Sobolev space $\tilde{H}^r(I)$, associated with the differential operator \mathcal{L}_c defined by

$$\tilde{H}^r(I) = \left\{ f \in L^2(I), \|f\|_{\tilde{H}^r(I)}^2 = \|\mathcal{L}_c^{r/2} f\|^2 = \sum_{k \geq 0} (\chi_k)^r |f_k|^2 < +\infty \right\},$$

where $f = \sum f_k$ is the expansion in the basis of PSWFs. The following result has been given in [30].

Theorem 5. (Theorem 3.3 in [30]). For any $f \in \tilde{H}^r(I)$, with $r \geq 0$, we have

$$\|f - S_N f\|_{L^2(I)} \leq (\chi_N(c))^{-r/2} \|f\|_{\tilde{H}^r(I)} \leq N^{-r} \|f\|_{\tilde{H}^r(I)}.$$

It is important to mention that the error bounds of the spectral approximations given by the previous two theorems, do not indicate how to choose a “good” value of the bandwidth c to approximate a given $f \in H^s(I)$. By a simultaneous use of the properties of the PSWFs as eigenfunctions of the differential operator \mathcal{L}_c and the integral operator \mathcal{F}_c , we give a first answer to this question. This is the subject of the following theorem.

Theorem 6. *Let $c > 0$ be a positive real number. Assume that $f \in H^s(I)$, for some positive real number $s > 0$. Then for any integer $N \geq 1$, we have*

$$\|f - S_N f\|_{L^2(I)} \leq K(1 + c^2)^{-s/2} \|f\|_{H^s(I)} + K\sqrt{\lambda_N(c)} \|f\|_{L^2(I)}. \quad (88)$$

Here, the constant K depends only on s . Moreover it can be taken equal to 1 when f belongs to the space $H_0^s(I)$.

Proof. To prove (88), we first use the fact that for any real number $s \geq 0$, there exists a linear and continuous extension operator $E : H^s(I) \rightarrow H^s(\mathbb{R})$. Moreover, if $f \in H^s(I)$ and $F = E(f) \in H^s(\mathbb{R})$, then there exists a constant $K > 0$ such that

$$\|F\|_{L^2(\mathbb{R})} \leq K\|f\|_{L^2(I)}, \quad \|F\|_{H^s(\mathbb{R})} \leq K\|f\|_{H^s(I)}. \quad (89)$$

We recall that the Sobolev norm of a function F on \mathbb{R} is given by

$$\|F\|_{H^s(\mathbb{R})}^2 = \frac{1}{2\pi} \int_{\mathbb{R}} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi.$$

In particular, for F c -bandlimited, one has

$$\|F\|_{L^2(\mathbb{R})}^2 \leq (1 + c^2)^{-s} \|F\|_{H^s(\mathbb{R})}^2.$$

Next, if \mathcal{F} denotes the Fourier transform operator and if

$$\mathcal{G} = \mathcal{F}^{-1}(\widehat{F} \cdot 1_{[-c,c]}), \quad \mathcal{H} = \mathcal{F}^{-1}(\widehat{F} \cdot (1 - 1_{[-c,c]})),$$

then \mathcal{G} is c -bandlimited and $F = \mathcal{G} + \mathcal{H}$. Moreover, since $\|\widehat{\mathcal{G}}\|_{L^2(\mathbb{R})} \leq \|\widehat{F}\|_{L^2(\mathbb{R})}$ and $\|\mathcal{H}\|_{L^2(\mathbb{R})} \leq c^{-s} \|F\|_{H^s(\mathbb{R})}$, then by using (89), one gets

$$\|\mathcal{G}\|_{L^2(\mathbb{R})} \leq K\|f\|_{L^2(I)}, \quad \|\mathcal{H}\|_{L^2(I)} \leq K(1 + c^2)^{-s/2} \|f\|_{H^s(I)}. \quad (90)$$

Finally, by using the previous inequalities and the fact that \mathcal{G} is c -bandlimited, one concludes that

$$\begin{aligned} \|f - S_N f\|_{L^2(I)} &\leq \|\mathcal{G} - S_N \mathcal{G}\|_{L^2(I)} + \|\mathcal{H} - S_N \mathcal{H}\|_{L^2(I)} \\ &\leq \sqrt{\lambda_N(c)} \|\mathcal{G}\|_{L^2(\mathbb{R})} + \|\mathcal{H}\|_{L^2(I)} \\ &\leq \sqrt{\lambda_N(c)} K \|f\|_{L^2(I)} + K(1 + c^2)^{-s} \|f\|_{H^s(I)}. \end{aligned}$$

This concludes the proof for general f . When f is in the subspace $H_0^s(I)$, one can take as extension operator the extension by 0 outside I , so that the constant K can be replaced by 1. \square

Remark 4. *This should be compared with the results of [30], given by Theorem 5. This has the advantage to give an error term for all values of c , while the first term in (88) is only small for c large enough. On the other hand, Wang compares his specific Sobolev space with the classical one and finds that*

$$\|f\|_{\widetilde{H}^s(I)} \leq C(1 + c^2)^{s/2} \|f\|_{H^s(I)}.$$

For large values of N we clearly have $\frac{(1 + c^2)}{\chi_N} \ll (1 + c^2)^{-1}$, but it goes the other way around when χ_N and $1 + c^2$ are comparable. So it may be useful to have both kinds of estimates in mind for numerical purpose and for the choice of the value of c .

Remark 5. *The error bound given by the previous theorem has the advantage to be explicitly given in terms of c and $\lambda_n(c)$. Nonetheless, it has a drawback that it does not imply a rate of convergence, nor the convergence of $S_N(f)$ to f in the usual $L^2(I)$ -norm. To overcome this problem, we devote the remaining of this section to a more elaborated convergence analysis in the 2-periodic Sobolev space H_{per}^s , then we extend this analysis to the usual $H^s(I)$ -space.*

Next, we consider the subspace H_{per}^s of functions in $H^s(I)$ that extend into 2-periodic functions of the same regularity. For such functions, one can also use the norm

$$\|f\|_{H_{per}^s} = \sum_{k \in \mathbb{Z}} (1 + (k\pi)^2)^s |b_k(f)|^2.$$

Here,

$$b_k(f) = \frac{1}{\sqrt{2}} \int_{-1}^{+1} f(x) e^{-i\pi kx} dx = \frac{1}{\sqrt{2}} \widehat{f}(k\pi)$$

is the coefficient of the Fourier series expansion of f . We then have the following theorem.

Theorem 7. *Let $c \geq 1$, then there exist constants $M > 1.40$ and M' , $a > 0$ such that, when $N \geq \max(cM, 3)$ and $f \in H_{per}^s$, $s > 0$, we have the inequality*

$$\|f - S_N(f)\|_{L^2(I)} \leq M'(1 + (\pi N)^2)^{-s/2} \|f\|_{H_{per}^s} + M'e^{-aN} \|f\|_{L^2}. \quad (91)$$

Proof. We start with reductions of the problem, which are analogous to the ones that we have detailed above. It is sufficient to prove this separately with the constant $M'/2$ for periodic functions g and $h = f - g$, where g is the projection of f onto the subspace of H_{per}^s whose Fourier coefficients $b_k(f)$ are zero for $|k| > N/M$. Moreover, we have directly the inequality without a second term, since the L^2 norm of h may be bounded by the first term multiplied by some constant. So, let us prove the inequality for g . This time we will prove that the inequality holds without the first term, that is,

$$\|g - S_N(g)\|_{L^2(I)} \leq \frac{M'}{2} e^{-aN} \|g\|_{L^2(I)}.$$

The next reduction consists of restricting to exponentials $e^{ik\pi x}$, with $|k| \leq N/M$. Indeed, assume that we prove the previous inequality for all of them, with a uniform bound by $M''e^{-a'N}$. Then, by linearity we will have

$$\|g - S_N(g)\|_{L^2(I)} \leq M''e^{-a'N} \sum |b_k(g)| \leq M''e^{-a'N} \sqrt{2[N/M] + 1} e^{-aN} \|g\|_{L^2(I)}.$$

This in turn gives constants the required form by choosing $a < a'$.

So we content ourselves to consider $f(x) = e^{ik\pi x}$, with $|k| \leq N/M$. Finally, since $\|f - S_N f\|_{L^2(I)}^2 = \sum \langle f, \psi_n \rangle^2$, it is sufficient to have such an estimate for each $n > N$, and conclude by taking the sum $\sum_{n > N} e^{-an}$. So the proof is a consequence of the following lemma. \square

Remark 6. *The previous theorem gives the rate of convergence of the truncated PSWFs series expansion of a function f from H_{per}^s . This rate of convergence will be generalized in the sequel to the usual $H^s(I)$ -space. Note that this rate of convergence drastically improves the one given by [8]. Moreover, unlike the error bound given in [30], the decay of the error bound given by the previous theorem is still even when N is comparable to c . Nonetheless, in practice, Theorem 6 is useful in the sense that provides us with a criteria for the choice of the bandwidth $c > 0$, that depends on magnitude of the Sobolev exponent $s > 0$. The smaller s , the larger c should be and vice versa.*

Lemma 11. *Let $c \geq 1$, then there exist constants $M > 1.40$ and M' , $a > 0$ such that, when $n \geq \max(cM, 3)$ and $f(x) = e^{ik\pi x}$ with $|k| \leq n/M$, we have*

$$|\langle f, \psi_n \rangle| \leq M'e^{-an}. \quad (92)$$

Proof. This scalar product can be written by using (74)

$$\begin{aligned}
\langle e^{ik\pi x}, \psi_n \rangle &= \int_{-1}^1 e^{ik\pi x} \psi_n(x) dx = \sum_{m \geq 0} \beta_m^n \langle e^{ik\pi x}, \bar{P}_m \rangle = \sum_{m \geq 0} \beta_m^n \sqrt{\frac{2}{k}} \sqrt{m+1/2} J_{m+1/2}(k\pi) \\
&= \sum_{m=0}^{[n/M]} \beta_m^n \sqrt{\frac{2}{k}} \sqrt{m+1/2} J_{m+1/2}(k\pi) + \sum_{m \geq [n/M]+1} \beta_m^n \sqrt{\frac{2}{k}} \sqrt{m+1/2} J_{m+1/2}(k\pi) \\
&= I_1^n + I_2^n.
\end{aligned}$$

To bound I_1^n , we first remark that the Fourier transform of $\bar{P}_n \chi_{[-1,1]}$ is bounded by 1 and then we use remark 1 to check that (77) is satisfied whenever $n \geq cM$ with $M \geq 1.40$. Hence, we have

$$\begin{aligned}
|I_1^n| &\leq \sum_{m=0}^{[n/M]} |\beta_m^n| \leq \sqrt{\frac{5}{4\pi}} |\mu_n(c)| \sum_{m=0}^{[n/M]} \left(\frac{2\sqrt{\chi_n}}{c} \right)^m \\
&\leq K \left(\frac{2\sqrt{\chi_n}}{c} \right)^{[n/M]+1} |\mu_n(c)|.
\end{aligned}$$

Moreover, taking into account the decay of the $\mu_n(c)$ given by (72) and using the upper bound of χ_n , we conclude that

$$|I_1^n| \leq K' \left(\frac{\pi(n+1)}{c} \right)^{\frac{n}{M}+1} e^{-\delta n} \leq K'' e^{-an} \quad (93)$$

for some sufficiently small positive real number a , as soon as $M > 1.40$. To bound I_2^n , it suffices to use the fact that $|\beta_k^n| \leq 1$ and the bound of the Bessel function given by [2],

$$|J_\alpha(x)| \leq \frac{|x|^\alpha}{2^\alpha \Gamma(\alpha+1)}, \quad \forall \alpha > -1/2, \quad \forall x \in \mathbb{R}, \quad (94)$$

one concludes that

$$\begin{aligned}
|I_2^n| &\leq \sum_{m \geq n/M} \sqrt{2/k} \sqrt{m+1/2} |J_{m+1/2}(k\pi)| \leq \sum_{m \geq [n/2]+1} \sqrt{2/k} \sqrt{m+1/2} \frac{(k\pi)^{m+1/2}}{2^{m+1/2} \Gamma(m+3/2)} \\
&\leq \sum_{m \geq [n/M]+1} \frac{(k\pi)^m}{2^m \sqrt{m+1/2} \Gamma(m+1/2)}.
\end{aligned}$$

Moreover, since $\Gamma(m+1/2) \geq m!/\sqrt{m+1}$ and $m! \geq (m/e)^m \sqrt{2\pi m}$, each term is bounded by an exponential e^{-an} and we find the required estimate for $|I_2^n|$. \square

Remark 7. We also have a bound of the error for ordinary polynomials. Indeed, if we consider the polynomial $f(x) := x^j$, then

$$a_n(f) = \int_{-1}^1 y^j \psi_{n,c}(y) dy = (-i)^j c^{-j} \mu_n(c) \psi_{n,c}^{(j)}(0), \quad \text{with } i^2 = -1.$$

We can then use Proposition 5 to conclude that if $c^2/\chi_N < 1$, then

$$\|f - S_N f\|_2^2 \leq C^2 \sum_{k \geq N} \left(\frac{\chi_k(c)}{c^2} \right)^j |\mu_k(c)|^2. \quad (95)$$

As a corollary of the previous theorem and remark, we obtain the following corollary that extends the result of the previous theorem to the case of the usual Sobolev space $H^s([-1,1])$.

Corollary 6. *Let $c \geq 1$, and let $s > 0$ with $[s] = m \in \mathbb{N}$, and $s \notin \frac{1}{2} + \mathbb{N}$. Let $f \in H^s(I)$, then there exist constants $M \geq 1.40$ and $M', M'_s > 0$ such that, when $N \geq \max(cM, 3)$, we have the inequality*

$$\|f - S_N(f)\|_{L^2(I)} \leq M'_s(1 + N^2)^{-s/2} \|f\|_{H^s([-1,1])} + M' e^{-aN} \|f\|_{L^2([-1,1])}. \quad (96)$$

Proof. Since $f \in H^s([-1,1])$ with $[s] = m$, and $s \notin \frac{1}{2} + \mathbb{N}$, then there exists a polynomial P , of degree at most m , such that $f + P \in H_{per}^s$. Consequently, by using the previous theorem and the inequality (95), one concludes for (96). \square

6 Numerical results

In this section, we illustrate the results of the previous sections by various numerical examples.

Example 1: In this first example, we illustrate the fact that the actual values of the constants κ and $\delta(\kappa)$, given by (22) and (23), respectively, are far much smaller than the theoretical values given in the proof of Lemma 2. We are interested in these values for $n \geq 2c/\pi$. For this purpose, we have considered the values of $c = m\pi, m = 10, 20, 30, 40$. Then, we have used Flammer's method and computed high accurate values of $\chi_n(c)$ and $\psi_{n,c}(1)$. Then, we have computed the smallest value of κ , denoted by κ_c and ensuring the bounds (23). Also, we have computed the corresponding values $\delta(\kappa_c)$ so that A^2 is equal to its upper bound given in (23). It turns out that κ_c , the critical value of κ , is obtained for n -th eigenvalues $\chi_n(c)$ with $n = n_c = [2c/\pi]$. Also, by considering various consecutive values of $n_c \leq n \leq n_c + 40$ and by computing the corresponding values of κ and $\delta(\kappa)$, we found that the $\max \delta(\kappa)$ is of the same size as κ_c . Table 1 shows the values of the critical values κ_c and $\delta(\kappa_c)$ for the different values of the bandwidth c . Also, we give the values of $\max \delta(\kappa)$.

c	n_c	κ_c	$\delta(\kappa_c)$	$\max \delta(\kappa)$.
10π	20	0.447	0.058	0.091
20π	40	0.413	0.051	0.084
30π	60	0.394	0.047	0.080
40π	80	0.335	0.025	0.048

Table 1: Critical values of κ , $\delta(\kappa)$ and $\max \delta(\kappa)$ for different values of c .

Example 2: In this example, we compare the explicit formula given by Theorem 2 to compute highly accurate values of $\lambda_n(c)$. For this purpose, we have considered the values of $c = 10\pi, 20\pi, 30\pi$ and computed $\lambda_n(c)$ by using the method given in [12]. Then, we have implemented our formula (47) in a Maple computing software code. Figure 1 (a), (b), (c) show the graph of $\ln(\lambda_n(c))$ versus the graph of $\ln(\tilde{\lambda}_n(c))$, for the different values of c and n . Also, we have plotted in Figure 2, the graphs of the corresponding values of $\ln\left(\frac{\lambda_n(c)}{\tilde{\lambda}_n(c)}\right)$. These figures illustrate the surprising precision of the explicit formula of Theorem 3 for computing the $\lambda_n(c)$ which is numerically valid whenever $q < 1$.

Next, to illustrate the quality of approximation by the ψ_n in the Sobolev space $H^s(I)$, we first describe a numerical method for the computation of the PSWFs series expansion coefficients of a function from the Sobolev space $H^s(I)$. Note that if $f \in H_{per}^s, s > 0$, then its different PSWFs

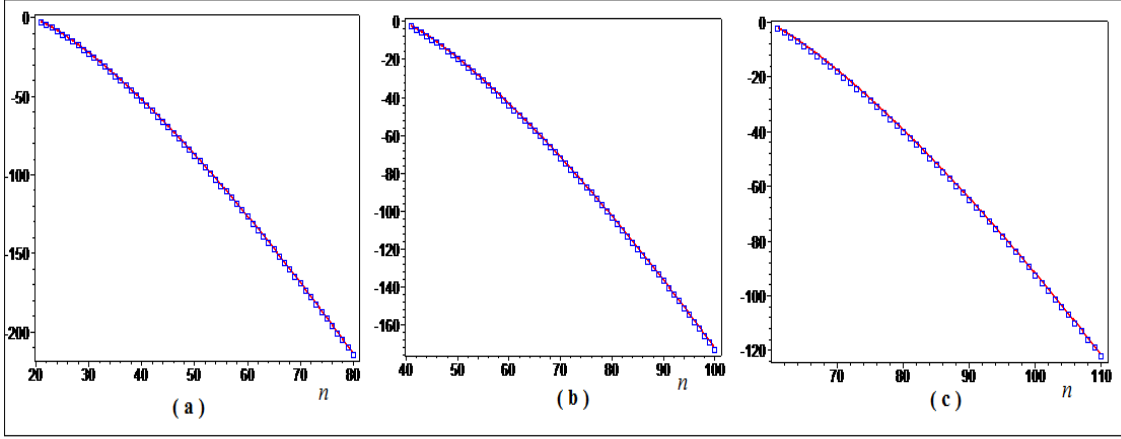


Figure 1: Graphs of $\ln(\widetilde{\lambda}_n(c))$ (boxes) and $\ln(\lambda_n(c))$ (red) with $c = 10\pi$ for (a), $c = 20\pi$ for (b) and $c = 30\pi$ for (c).

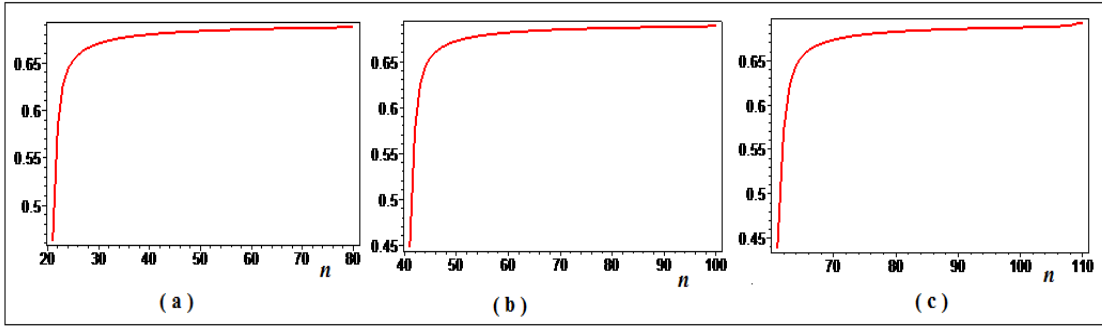


Figure 2: Graphs of $\ln\left(\frac{\widetilde{\lambda}_n(c)}{\lambda_n(c)}\right)$ with $c = 10\pi$ for (a), $c = 20\pi$ for (b) and $c = 30\pi$ for (c).

series expansion coefficients can be easily approximated as follows. For a positive integer K , an approximation $a_n^K(f)$ to $a_n(f)$ is given by the following formula

$$a_n^K(f) = \frac{\mu_n(c)}{\sqrt{2}} \sum_{k=-K}^K b_k(f) \psi_{n,c}\left(\frac{k\pi}{c}\right) = a_n(f) + \epsilon_K, \quad (97)$$

where the $b_k(f)$ are the Fourier coefficients of f and where $\epsilon_K = \frac{1}{\sqrt{2}} \sum_{|k| \geq K+1} \mu_n(c) b_k(f) \psi_{n,c}\left(\frac{k\pi}{c}\right)$.

Moreover, from the well known asymptotic behavior of the $\psi_{n,c}(x)$, for large values of x , see for example [12], one can easily check that $\epsilon_K = o\left(\frac{1}{((K+1)\pi)^{1+s}}\right)$. This computational method of the $a_n(f)$ has the advantage to work for small as well as large values of the smoothness coefficient $s > 0$.

Also, note that if $f \in H^s([-1, 1])$, where $s > 1/2 + 2m, m \geq 1$, is an integer, then $f \in C^{2m}([-1, 1])$. Moreover since $\psi_{n,c} \in C^\infty(\mathbb{R})$, then the classical Gaussian quadrature method, see

Table 2: Values of $E_N(s)$ for various values of N and s .

	$s = 0.75$	$s = 1$	$s = 1.25$	$s = 1.5$	$s = 1.75$	$s = 2.0$
N	$E_N(s)$	$E_N(s)$	$E_N(s)$	$E_N(s)$	$E_N(s)$	$E_N(s)$
20	4.57329E-01	4.66173E-01	4.85990E-01	5.05973E-01	5.23232E-01	5.37227E-01
30	3.15869E-01	3.11677E-01	3.28241E-01	3.48562E-01	3.67260E-01	3.82963E-01
40	1.06843E-01	1.52009E-01	1.91237E-01	2.20969E-01	2.43432E-01	2.60523E-01
50	4.09844E-02	6.88472E-02	1.01827E-01	1.26518E-01	1.44809E-01	1.58520E-01
60	3.30178E-02	2.09084E-02	3.25551E-02	4.28999E-02	5.06959E-02	5.65531E-02
70	3.15097E-02	8.82446E-03	2.51157E-03	7.35725E-04	2.33066E-04	1.04137E-04
80	3.01566E-02	8.55598E-03	2.40312E-03	6.87458E-04	1.98993E-04	5.80481E-05
90	2.67972E-02	7.64167E-03	2.14661E-03	6.15062E-04	1.78461E-04	5.22848E-05
100	2.39141E-02	6.72825E-03	1.82818E-03	5.10057E-04	1.45036E-04	4.19238E-05

for example [2] gives us the following approximate value $\tilde{a}_n(f)$ of the $(n + 1)$ -th expansion coefficient $a_n(f) = \langle f, \psi_{n,c} \rangle$,

$$\tilde{a}_n(f) = \sum_{l=1}^m \omega_l f(x_l) \psi_{n,c}(x_l) = a_n(f) + \epsilon_n, \quad (98)$$

with $|\epsilon_n| \leq \sup_{\eta \in [-1,1]} \frac{1}{b_m^2} \frac{(f \cdot \psi_{n,c})^{(2m)}(\eta)}{(2m)!}$. Here, b_m is the highest coefficient of $\overline{P_m}$, and the different weights ω_l and nodes x_l , are easily computed by the special method given in [2].

The following examples illustrate the quality of approximation in $H^s(I)$ by the PSWFs.

Example 3: In this example, we consider the Weierstrass function

$$W_s(x) = \sum_{k \geq 0} \frac{\cos(2^k x)}{2^{ks}}, \quad -1 \leq x \leq 1. \quad (99)$$

Note that $W_s \in H^{s-\epsilon}([-1,1])$, $\forall \epsilon < s$, $s > 0$. We have considered the value of $c = 100$, and computed $W_{s,N}$, the N -th terms truncated PSWFs series expansion of W_s with different values of $\frac{3}{4} \leq s \leq 2$ and different values of $20 \leq N \leq 100$. Also, for each pair (s, N) , we have computed the corresponding

approximate L^2 -error bound $E_N(s) = \left[\frac{1}{50} \sum_{k=-50}^{50} (W_{s,N}(k/50) - W_s(k/50))^2 \right]^{1/2}$. Table 2 lists the obtained values of $E_N(s)$. Note that the numerical results given by Table 2, follow what has been predicted by the theoretical results of the previous section. In fact, the L^2 -errors $\|W_s - \Pi_N W_s\|_2$ is of order $O(N^{-s})$, whenever $N \geq N_c \sim \left\lceil \frac{2c}{\pi} \right\rceil + 4$. In the case, where $c = 100$, $N_c = 67$. The graphs of $W_{3/4}(x)$ and $W_{3/4,N}(x)$, $N = 90$ are given by Figure 3.

Example 4: In this example, we let $s > 0$ be any positive real number and we consider the Brownian motion function $B_s(x)$ given by as follows.

$$B_s(x) = \sum_{k \geq 1} \frac{X_k}{k^s} \cos(k\pi x), \quad -1 \leq x \leq 1. \quad (100)$$

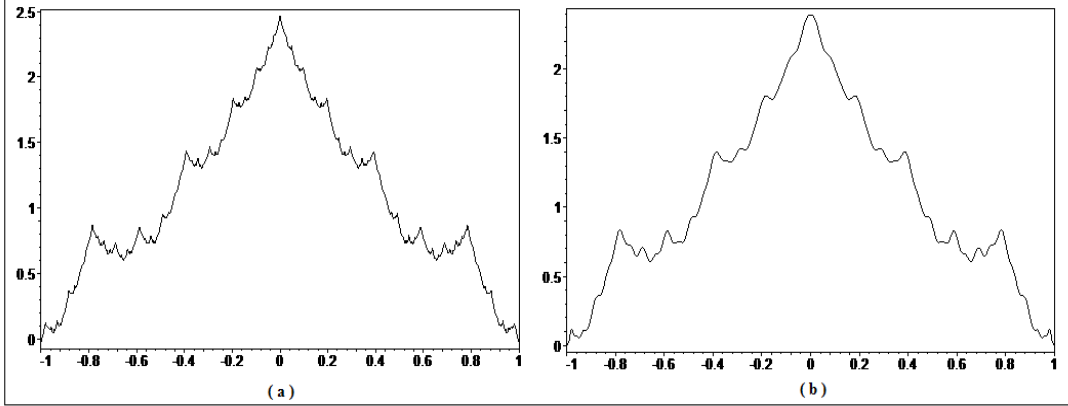


Figure 3: (a) graph of $W_{3/4}(x)$, (b) graph of $W_{3/4,N}(x)$, $N = 90$.

Here, X_k is a Gaussian random variable. It is well known that $B_s \in H^s([-1, 1])$. For the special case $s = 1$, we consider the band-width $c = 100$, a truncation order $N = 80$ and compute $B_{1,N}$ the approximation of B_1 by its N -th terms truncated PSWFs series expansion. The graphs of B_1 and $B_{1,N}$ are given by Figure 4.

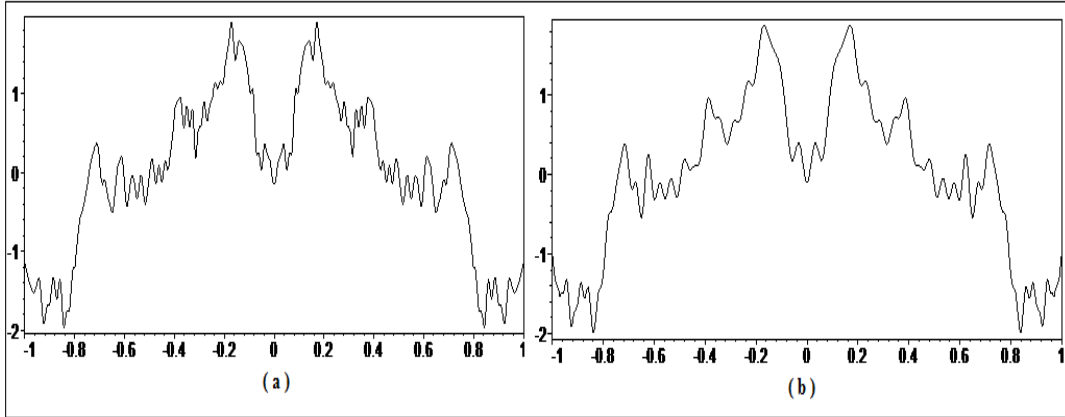


Figure 4: (a) graph of $B_1(x)$, (b) graph of $B_{1,N}(x)$, $N = 80$.

Remark 8. From the quality of approximation in the Sobolev spaces $H^s([-1, 1])$ given in this paper and in [6, 8, 30], one concludes that for any value of the bandwidth $c \geq 0$, the approximation error $\|f - S_N f\|_2$ has the asymptotic order $O(N^{-s})$. Nonetheless, for a given $f \in H^s([-1, 1])$, $s > 0$ which we may assume to have a unit L^2 -norm and for a given error tolerance ϵ , the appropriate value of the bandwidth $c \geq 0$, corresponding to the minimum truncation order N , ensuring that $\|f - S_N f\|_2 \leq \epsilon$, depends on whether or not, f has some significant Fourier expansion coefficients, corresponding to large frequency components. In other words, the faster decay to zero of the Fourier coefficients of f , the smaller the value of the bandwidth should be and vice versa.

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