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Sharp rate for the dual quantization problem

Gilles Pagès*

Benedikt Wilbertz†

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Abstract

In this paper we establish the sharp rate of the optimal dual quantization problem. The notion of dual quantization was recently introduced in the paper [8], where it was shown that, at least in an Euclidean setting, dual quantizers are based on a Delaunay triangulation, the dual counterpart of the Voronoi tessellation on which “regular” quantization relies. Moreover, this new approach shares an intrinsic stationarity property, which makes it very valuable for numerical applications.

We establish in this paper the counterpart for dual quantization of the celebrated Zador theorem, which describes the sharp asymptotics for the quantization error when the quantizer size tends to infinity. The proof of this theorem relies among others on an extension of the so-called Pierce Lemma by means of a random quantization argument.

Keywords: Quantization, Rate, Zador’s Theorem, Pierce’s Lemma, Dual Quantization, Delaunay triangulation, Random Quantization.

MSC: 60F25

1 Introduction

In [8], we introduced a new notion of vector quantization called *dual quantization* (or *Delaunay quantization* in an Euclidean framework). The principle of dual quantization is to map an \mathbb{R}^d -valued random vector (r.v.) onto a finite grid $\Gamma \subset \mathbb{R}^d$ using an appropriate random splitting operator $\mathcal{J}_\Gamma : \Omega_0 \times \mathbb{R}^d \rightarrow \Gamma$ (defined on an exogenous probability space $(\Omega_0, \mathcal{S}_0, \mathbb{P}_0)$) satisfying an intrinsic *stationary property*

$$\forall \xi \in \text{conv}(\Gamma), \quad \mathbb{E}_{\mathbb{P}_0}(\mathcal{J}_\Gamma(\xi)) = \int_{\Omega_0} \mathcal{J}_\Gamma(\omega_0, \xi) \mathbb{P}_0(d\omega_0) = \xi. \quad (1)$$

Then, for every random vector (r.v.) X taking values in $\text{conv}(\Gamma)$ defined on a probability space $(\Omega, \mathcal{S}, \mathbb{P})$ (once canonically extended on $(\Omega_0 \times \Omega, \mathcal{S}_0 \otimes \mathcal{S}, \mathbb{P}_0 \otimes \mathbb{P})$),

$$\mathbb{E}_{\mathbb{P}_0 \otimes \mathbb{P}}(\mathcal{J}_\Gamma(X) | X) = X.$$

This means that the resulting approximation $\mathcal{J}_\Gamma(X)$ of X always satisfies a *reverse stationarity* property which can be compared to the one satisfied by the nearest neighbour projection $\text{Proj}_\Gamma(X)$ of X onto Γ , namely $\mathbb{E}(X | \text{Proj}_\Gamma(X)) = \text{Proj}_\Gamma(X)$ which is mainly satisfied by optimal grids for

*Laboratoire de Probabilités et Modèles aléatoires, UMR 7599, Université Paris 6, case 188, 4, pl. Jussieu, F-75252 Paris Cedex 05. E-mail: gilles.pages@upmc.fr

†Laboratoire de Probabilités et Modèles aléatoires, UMR 7599, Université Paris 6, case 188, 4, pl. Jussieu, F-75252 Paris Cedex 05. E-mail: benedikt.wilbertz@upmc.fr

the mean (regular) quadratic quantization error (see below) in an Euclidean framework. It has been emphasized in [8, 6, 7] how to take advantage of this intrinsic stationary property to produce more accurate cubature formulae for (conditional) expectation approximation regardless of any optimality property of the grid(s). This new quantization modulus leads to an optimal dual quantization problem

$$d_{n,p}(X) = \inf \left\{ \|F_p(X; \Gamma)\|_p, \Gamma \subset \mathbb{R}^d, |\Gamma| \leq n \right\},$$

where F_p denotes the local dual quantization error function

$$F_p(\xi; \Gamma) = \inf \left\{ \left(\sum_{x \in \Gamma} \lambda_x \|\xi - x\|^p \right)^{\frac{1}{p}}, \lambda_x \in [0, 1], \sum_{x \in \Gamma} \lambda_x x = \xi, \sum_{x \in \Gamma} \lambda_x = 1 \right\}.$$

Since this notion only makes sense for compactly supported r.v. X , we also consider the extension to unbounded r.v. X (see [8]) defined by

$$\bar{F}_p(\xi; \Gamma) := F_p(\xi; \Gamma) \mathbf{1}_{\text{conv}(\Gamma)}(\xi) + \text{dist}(X, \Gamma) \mathbf{1}_{\text{conv}(\Gamma)^c}(\xi).$$

and the extended dual quantization error given by

$$\bar{d}_{n,p}(X) = \inf \left\{ \|\bar{F}_p(X; \Gamma)\|_p, \Gamma \subset \mathbb{R}^d, |\Gamma| \leq n \right\}.$$

Recall that the ‘‘regular’’ Voronoi optimal quantization problem reads

$$e_{n,p}(X) = \inf \left\{ \left(\mathbb{E} \min_{x \in \Gamma} \|X - x\|^p \right)^{\frac{1}{p}}, \Gamma \subset \mathbb{R}^d, |\Gamma| \leq n \right\}$$

It is well-known that $e_{n,p}(X) \downarrow 0$ as soon as $n \rightarrow \infty$ and $X \in L^p(\mathbb{P})$. Moreover, this rate of convergence to 0 of $e_{n,p}(X)$ is ruled by the celebrated Zador Theorem (see [3])

Theorem 1. *Let $X \in L^p_{\mathbb{R}^d}(\mathbb{P})$, $p' > p$. Assume the distribution \mathbb{P}_X of X is decomposed as $\mathbb{P}_X = h.\lambda_d + \nu$, $\nu \perp \lambda_d$. Then*

$$\lim_{n \rightarrow \infty} n^{\frac{1}{d}} e_{n,p}(X) = Q_{\|\cdot\|,p,d}^{vq} \|h\|_{\frac{d}{p+d}}^{\frac{1}{p}}$$

where

$$Q_{\|\cdot\|,p,d}^{vq} = \inf_{n \rightarrow \infty} n^{\frac{1}{d}} e_{n,p}(U([0, 1]^d)) \in (0, \infty).$$

This rate depending on d is known as the *curse of dimensionality*. Its statement and proof go back to Zador in 1954 for uniform distribution, with an extension to possibly unbounded absolutely continuous distributions by Bucklew and Wise (see [1]). It has been finally established rigorously (as far as mathematical standard are concerned) in [3] in 2000. A comprehensive survey of the history of quantization can be found in [4].

The aim of this paper is to prove for any $p > 0$ and any norm on \mathbb{R}^d a counterpart of Zador’s Theorem in the framework of dual quantization for both $d_{n,p}$ and $\bar{d}_{n,p}$ error moduli.

Theorem 2. (a) *Let $X \in L^{\infty}_{\mathbb{R}^d}(\mathbb{P})$. Assume the distribution \mathbb{P}_X of X reads $\mathbb{P}_X = h.\lambda_d + \nu$, $\nu \perp \lambda_d$. Then*

$$\lim_{n \rightarrow \infty} n^{\frac{1}{d}} d_{n,p}(X) = \lim_{n \rightarrow \infty} n^{\frac{1}{d}} \bar{d}_{n,p}(X) = Q_{\|\cdot\|,p,d}^{dq} \|h\|_{\frac{d}{p+d}}^{\frac{1}{p}}$$

where

$$Q_{\|\cdot\|,p,d}^{dq} = \inf_{n \rightarrow \infty} n^{\frac{1}{d}} d_{n,p}(U([0, 1]^d)) \in (0, \infty).$$

(b) Let $X \in L_{\mathbb{R}^d}^{p'}(\mathbb{P})$, $p' > p$. Assume the distribution \mathbb{P}_X of X reads $\mathbb{P}_X = h \cdot \lambda_d + \nu$, $\nu \perp \lambda_d$. Then

$$\lim_{n \rightarrow \infty} n^{\frac{1}{d}} \bar{d}_{n,p}(X) = Q_{\|\cdot\|,p,d}^{dq} \|h\|_{\frac{p}{p+d}}^{\frac{1}{p}}.$$

(c) If $d = 1$, then

$$d_{n,p}(U([0, 1])) = \left(\frac{2}{(p+1)(p+2)} \right)^{\frac{1}{p}} \frac{1}{n-1},$$

which implies $Q_{|\cdot|,p,1}^{dq} = \left(\frac{2^{p+1}}{p+2} \right)^{\frac{1}{p}} Q_{|\cdot|,p,1}^{vq}$.

Moreover we will also establish in Section 4 a upper bound for the dual quantization coefficient $Q_{\|\cdot\|,p,d}^{dq}$ when $\|\cdot\| = |\cdot|_{\ell^r}$.

Proposition 1. *Let $r, p \in [1, \infty)$ and $r \leq p$. Then it holds for every $d \in \mathbb{N}$*

$$Q_{|\cdot|_{\ell^r},p,d}^{dq} \leq d^{\frac{1}{r}} \cdot Q_{|\cdot|,p,1}^{dq}.$$

Since this upper bound realizes the same asymptotic rate as in the case of regular quantization (cf. Cor. 9.4 in [3]), we believe the rate of $d^{\frac{1}{r}}$ to be also the true one for $Q_{\|\cdot\|,p,d}^{dq}$ as $d \rightarrow \infty$.

The paper is entirely devoted to the proof of Theorem 2. Our global strategy of proof is close to that adopted in [3] for the original Zador's Theorem. However, it differs at some points when dealing with the extended modulus $\bar{d}_{n,p}(X)$. In one dimension the exact rate $O(n^{-1})$ for $d_{n,p}(X)$ and $\bar{d}_{n,p}(X)$ follows from a random quantization argument detailed in Section 3 which is an extension of the so-called Pierce Lemma $d_{n,p}(X)$ (in fact, we even state a slightly more general result than requested for or purpose). This rate can be transferred to $O(n^{-\frac{1}{d}})$ in a d -dimensional framework using a product (dual) quantization argument (see Section 2.2). Finally the sharp upper bound is obtained in Section 4 by successive approximation procedures of the density of X which follow the approach developed in [3] to prove the corresponding part of Zador's Lemma, whereas the lower bound relies on a new "firewall" Lemma.

NOTATIONS: $\text{conv}(A)$ stands for the convex hull of A , $|A|$ for its cardinality and $[x]$ will denote the (lower) integral part of the real x .

2 Dual quantization: definition and basic properties

2.1 Definitions

Assume \mathbb{R}^d equipped with a norm $\|\cdot\|$. First we recall the definition of the regular quantization problem for a random vector (r.v.) $X : (\Omega, \mathcal{S}, \mathbb{P}) \rightarrow (\mathbb{R}^d, \mathcal{B}^d)$ and

Definition 1. *Let $X \in L_{\mathbb{R}^d}^p(\mathbb{P})$ and $\Gamma \subset \mathbb{R}^d$.*

1. *We define the L^p -mean regular quantization error for a grid Γ as*

$$e_p(X; \Gamma) = (\mathbb{E} \min_{x \in \Gamma} \|X - x\|^p)^{1/p} = \|d(X, \Gamma)\|_{L^p}.$$

2. *The optimal regular quantization error, which can be achieved by a grid Γ of size not exceeding n is given by*

$$e_{n,p}(X) = \inf \{ e_p(X; \Gamma) : \Gamma \subset \mathbb{R}^d, |\Gamma| \leq n \}.$$

Following [8], the dual quantization error can be introduced as follows.

Definition 2. Let $X \in L^p(\mathbb{P})$ and $\Gamma \subset \mathbb{R}^d$.

1. We define the local p -dual quantization error for a grid Γ as

$$F_p(\xi; \Gamma) = \inf \left\{ \left(\sum_{x \in \Gamma} \lambda_x \|\xi - x\|^p \right)^{\frac{1}{p}} : \lambda_x \in [0, 1], \sum_{x \in \Gamma} \lambda_x x = \xi, \sum_{x \in \Gamma} \lambda_x = 1 \right\}$$

2. The L^p -mean dual quantization error for X induced by a grid Γ is then given by

$$\begin{aligned} d_p(X; \Gamma) &= \|F_p(X; \Gamma)\|_{L^p} \\ &= \left(\mathbb{E} \inf \left\{ \sum_{x \in \Gamma} \lambda_x \|\xi - x\|^p : \lambda_x \in [0, 1] \text{ and } \sum_{x \in \Gamma} \lambda_x x = \xi, \sum_{x \in \Gamma} \lambda_x = 1 \right\} \right)^{1/p}. \end{aligned}$$

3. The optimal dual quantization error, which can be achieved by a grid Γ of size not exceeding n will be denoted by

$$d_{n,p}(X) = \inf \{ d_p(X; \Gamma) : \Gamma \subset \mathbb{R}^d, |\Gamma| \leq n \}.$$

4. The extended L^p -mean dual quantization error induces by a grid Γ is defined by

$$\bar{d}_p(X, \Gamma) = \left\| F_p(X; \Gamma) \mathbf{1}_{\text{conv}(\Gamma)}(X) + \text{dist}(X, \Gamma) \mathbf{1}_{\text{conv}(\Gamma)^c}(X) \right\|_{L^p}.$$

5. The optimal extended dual quantization error, which can be achieved by a grid Γ of size not exceeding n will be denoted by

$$\bar{d}_{n,p}(X) = \inf \{ \bar{d}_p(X; \Gamma) : \Gamma \subset \mathbb{R}^d, |\Gamma| \leq n \}.$$

Remarks. 1. Since the above quantities only depend on the distribution of the r.v. X we will also write $d_p(\mathbf{P}, \Gamma)$ for $d_p(X, \Gamma)$ and $d_{n,p}(\mathbf{P})$ for $d_{n,p}(X)$ where $\mathbf{P} = \mathbb{P}_X$.

2. To alleviate notations, we will use throughout the paper F^p , d^p and \bar{d}^p , ... instead of F_p^p , d_p^p and \bar{d}_p^p , ...

In fact the terminology *dual quantization* refers to a canonical example of intrinsic stationary splitting operator: the dual quantization operator.

To be more precise, assume \mathbb{R}^d is equipped with a norm $\|\cdot\|$ and let $p \in [1, +\infty)$. Let $\Gamma = \{x_1, \dots, x_n\} \subset \mathbb{R}^d$ be a grid of size $n \geq d + 1$ such that $\text{aff.dim}(\Gamma) \geq d + 1$.

The idea is to “split” $\xi \in \text{conv}(\Gamma)$ among at most $d + 1$ affinely independent points in Γ (which convex hull contains ξ) proportionally to its barycentric coordinates. There are usually many possible choices so we introduced a minimal inertia based criterion to select the most appropriate “neighbours” of ξ , namely the function $F_p(\xi; \Gamma)$ defined for every ξ as the value of the minimization problem

$$F_p(\xi; \Gamma) = \inf_{(\lambda_1, \dots, \lambda_n)} \left\{ \left(\sum_{i=1}^n \lambda_i \|\xi - x_i\|^p \right)^{\frac{1}{p}}, \lambda_i \geq 0, \sum_i \lambda_i \begin{bmatrix} x_i \\ 1 \end{bmatrix} = \begin{bmatrix} \xi \\ 1 \end{bmatrix} \right\}$$

Owing to the compactness of constraint set, there exist at least one solution $\lambda^*(\xi)$ and for any such solution, one shows using convex extremality arguments that the set $I^*(\xi) := \{i \in I \text{ s.t. } \lambda_i^*(\xi) > 0\}$ defines an affinely independent subset $\{x_i, i \in I^*(\xi)\}$.

When this solution is always unique, the dual quantization operator is simply defined on $\text{conv}(\Gamma)$ by

$$\forall \xi \in \text{conv}(\Gamma), \forall \omega_0 \in \Omega_0, \quad \mathcal{J}_\Gamma^*(\omega_0, \xi) = \sum_{i \in I(\xi)^*} x_i \mathbf{1}_{\{\sum_{j=1}^{i-1} \lambda_j^*(\xi) \leq U(\omega_0) < \sum_{j=1}^i \lambda_j^*(\xi)\}}.$$

Thus in the quadratic ($p = 2$) Euclidean case and when Γ is in the so-called “general position”, then $\{\{\xi \text{ s.t. } I^*(\xi) = I\}, |I| \leq d+1\}$ makes up a Borel partition of $\text{conv}(\Gamma)$ (with possibly empty elements), known in 2-dimension as the *Delaunay triangulation* of Γ (see [10] for the connection to Delaunay triangulations). In a general framework, we refer to [8] for a construction of dual quantization operators.

These operators plays the role of the nearest neighbour projections for “regular” Voronoi quantization and one checks that

$$\begin{aligned} \|\mathcal{J}_\Gamma^*(X) - X\|_{L^p(\mathbb{P}_0 \otimes \mathbb{P})} &= \|F_p(X; \Gamma)\|_{L^p(\mathbb{P})} \\ &= \mathbb{E} \inf \left\{ \left(\sum_{i=1}^n \lambda_i \|X - x_i\|^p \right)^{\frac{1}{p}}, \lambda_i \geq 0, \sum_i \lambda_i \begin{bmatrix} x_i \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ 1 \end{bmatrix} \right\}. \end{aligned}$$

The second step of the optimization process is to find grids which are optimally “fitting” (the distribution of) X *i.e.* that are solution to the second level optimization problem

$$d_{n,p}(X) = \inf \left\{ \|\mathcal{J}_\Gamma^*(X) - X\|_{L^p(\mathbb{P}_0 \otimes \mathbb{P})}, \mathcal{J}_\Gamma^* : \Omega_0 \times \text{conv}(\Gamma) \rightarrow \Gamma, \text{conv}(\Gamma) \supset \text{supp} \mathbb{P}_X, |\Gamma| \leq n \right\}.$$

Note that if $X \in L^\infty(\mathbb{P})$, $d_{n,p}(X) < +\infty$ iff $n \geq d+1$ (and is identically infinite if X is not essentially bounded). The existence of an optimal grid (or dual quantizer) has been established in [8] as well as the following characterization $d_{n,p}(X)$ as the lowest L^p -mean approximation error by r.v. taking at most n values and satisfying the intrinsic stationary property *i.e.*

$$d_{n,p}(X) = \inf \left\{ \|X - \widehat{X}\|_{L^p(\mathbb{P}_0 \otimes \mathbb{P})}, |\widehat{X}(\Omega_0 \times \Omega)| \leq n, \mathbb{E}_{\mathbb{P}_0 \otimes \mathbb{P}}(\widehat{X} | X) = X \right\}.$$

A stochastic optimization procedure is devised in [8] based on a stochastic gradient approach to compute optimal grids w.r.t. various distributions.

When a random vector X is not essentially bounded, the above approach cannot be developed since no finite grid can contain its support. In that case, we need to extend the definition of our splitting operator \mathcal{J}_Γ outside the convex hull of Γ . One way to proceed (see [8]) is to consider again a (deterministic) nearest neighbour projection Proj_Γ

$$\forall \xi \in \mathbb{R}^d \setminus \text{conv}(\Gamma), \quad \mathcal{J}_\Gamma(\omega_0, \xi) = \text{Proj}_\Gamma(\xi).$$

We loose the intrinsic stationary property, however we were able to show the existence of an optimal grid solution to the resulting minimization problem

$$\bar{d}_{n,p}(X) = \inf \left\{ \|F_p(X; \Gamma) \mathbf{1}_{\{X \in \text{conv}(\Gamma)\}} + \text{dist}(X, \Gamma) \mathbf{1}_{\{X \notin \text{conv}(\Gamma)\}}\|_{L^p(\mathbb{P})} \right\}.$$

It is clear that $d_{n,p}(X)$ and $\bar{d}_{n,p}(X)$ do not coincide even for bounded r.v. but one can show that

$$\min(d_{n,p}(X), \bar{d}_{n,p}(X)) \geq e_{n,p}(X)$$

where $e_{n,p}(X)$ is the “regular” Voronoi L^p -mean quantization error at level n defined by

$$e_{n,p}(X) = \inf \left\{ \|X - \text{Proj}_\Gamma(X)\|_{L^p(\mathbb{P})}, |\Gamma| \leq n \right\}.$$

The above dual quantization problem is characterized in terms of best approximation in L^p by the following theorem established in [8].

Theorem 3. Let $X \in L^0(\Omega, \mathcal{S}, \mathbb{P})$ and $n \in \mathbb{N}$. Then

$$\begin{aligned} d_{n,p}(X) &= \inf \left\{ \mathbb{E} \|X - \mathcal{J}_\Gamma(X)\|_{L^p} : \mathcal{J}_\Gamma : \Omega_0 \times \mathbb{R}^d \rightarrow \Gamma, \text{ intrinsic stationary,} \right. \\ &\quad \left. \text{supp}(\mathbb{P}_X) \subset \text{conv}(\Gamma), |\Gamma| \leq n \right\} \\ &= \inf \left\{ \mathbb{E} \|X - \widehat{X}\|_{L^p} : \widehat{X} : (\Omega_0 \times \Omega, \mathcal{S}_0 \otimes \mathcal{S}, \mathbb{P}_0 \otimes \mathbb{P}) \rightarrow \mathbb{R}^d, \right. \\ &\quad \left. |\widehat{X}(\Omega_0 \times \Omega)| \leq n, \mathbb{E}(\widehat{X}|X) = X \right\} \leq +\infty. \end{aligned}$$

These quantities are finite iff $X \in L^\infty(\Omega, \mathcal{S}, \mathbb{P})$.

As already mentioned, we established in [8] the existence of dual quantizers at level $n \in \mathbb{N}$ for the L^p -norm when $p \in (1, \infty)$. We reproduce this result only for the reader's convenience:

Theorem 4 (Existence of optimal quantizers). Let $X \in L^p(\mathbb{P})$ for some $p \in (1, \infty)$.

- (a) If $\text{supp}(\mathbb{P}_X)$ is compact, then there exists for every $n \in \mathbb{N}$ a grid $\Gamma_n^* \subset \mathbb{R}^d$, $|\Gamma_n^*| \leq n$ such that $d_p(X; \Gamma_n^*) = d_{n,p}(X)$.
- (b) If \mathbb{P}_X is strongly continuous in the sense that it assigns mass zero to all hyperplanes in \mathbb{R}^d , then there exists for every $n \in \mathbb{N}$ a grid $\Gamma_n^* \subset \mathbb{R}^d$, $|\Gamma_n^*| \leq n$ such that $\bar{d}_p(X; \Gamma_n^*) = \bar{d}_{n,p}(X)$.

If furthermore $|\text{supp}(\mathbb{P}_X)| \geq n$, then the above statements hold with $|\Gamma_n^*| = n$.

2.2 Local properties of the dual quantization functional

We establish in this paragraph some general properties for the local dual quantization functional F^p , which will be needed for the final proof of Theorem 2.

Proposition 2. Let $\Gamma_1, \Gamma_2 \subset \mathbb{R}^d$ be finite grids and let $\xi \in \mathbb{R}^d$. Then

$$\Gamma_1 \subset \Gamma_2 \implies F_p(\xi; \Gamma_2) \leq F_p(\xi; \Gamma_1).$$

Proof. Assume $\Gamma_1 = \{x_1, \dots, x_m\}$ and $\Gamma_2 = \{x_1, \dots, x_m, x_{m+1}, \dots, x_n\}$. Then

$$\begin{aligned} F^p(\xi; \Gamma_2) &= \min_{\lambda \in \mathbb{R}^n} \sum_{i=1}^n \lambda_i \|\xi - x_i\|^p \leq \min_{\lambda = (\lambda_1, 0)} \sum_{i=1}^n \lambda_i \|\xi - x_i\|^p \\ &\quad \text{s.t. } \begin{bmatrix} x_1 & \dots & x_n \\ 1 & \dots & 1 \end{bmatrix} \lambda = \begin{bmatrix} \xi \\ 1 \end{bmatrix}, \lambda \geq 0 \quad \text{s.t. } \begin{bmatrix} x_1 & \dots & x_m \\ 1 & \dots & 1 \end{bmatrix} \lambda_1 = \begin{bmatrix} \xi \\ 1 \end{bmatrix}, \lambda_1 \geq 0 \\ &= \min_{\lambda \in \mathbb{R}^m} \sum_{i=1}^m \lambda_i \|\xi - x_i\|^p \\ &\quad \text{s.t. } \begin{bmatrix} x_1 & \dots & x_m \\ 1 & \dots & 1 \end{bmatrix} \lambda = \begin{bmatrix} \xi \\ 1 \end{bmatrix}, \lambda \geq 0 \\ &= F^p(\xi; \Gamma_1). \end{aligned}$$

□

Moreover, we will make use of the following three properties established in [8].

Proposition 3 (Scalar bound). Let $\Gamma = \{x_1, \dots, x_n\} \subset \mathbb{R}$ with $x_1 \leq \dots \leq x_n$. Then

$$\forall \xi \in [x_1, x_n], \quad F^p(\xi; \Gamma) \leq \max_{1 \leq i \leq n-1} \left(\frac{x_{i+1} - x_i}{2} \right)^p.$$

Proposition 4 (Local product Quantization). *Let $\|\cdot\| = |\cdot|_{\ell^p}$ defined for every $\xi = (\xi^1, \dots, \xi^d) \in \mathbb{R}^d$ by $|\xi|_{\ell^p} = \left(\sum_{1 \leq i \leq d} |\xi^i|^p\right)^{1/p}$ and $\Gamma = \prod_{j=1}^d \Gamma_j$ for some $\Gamma_j \subset \mathbb{R}$. Then*

$$F_p(\xi; \Gamma) = \left(\sum_{j=1}^d F^p(\xi^j, \Gamma_j) \right)^{\frac{1}{p}}.$$

One then may derive in the next proposition a first upper bound for the asymptotics of the optimal dual quantization error of distributions with bounded support when the size of the grid tends to infinity.

Proposition 5 (Product Quantization). *Let $C = a + \ell[0, 1]^d$, $a = (a_1, \dots, a_d) \in \mathbb{R}^d$, $\ell > 0$, be a hypercube, parallel to the coordinate axis with common edge length ℓ . Let Γ be the product quantizer of size $(m+1)^d$ defined by*

$$\Gamma = \prod_{k=1}^d \left\{ a_j + \frac{i\ell}{m}, i = 0, \dots, m \right\}.$$

There exists a positive real constant $C_{d, \|\cdot\|}$ such that

$$\forall \xi \in C, \quad F^p(\xi; \Gamma) \leq C_{d, \|\cdot\|} \cdot \left(\frac{\ell}{2}\right)^p \cdot m^{-p}. \quad (2)$$

3 Extended Pierce lemma and applications

The aim of this section is to provide a non-asymptotic upper-bound for the optimal dual quantization error in the spirit of [9], which achieves nevertheless the optimal rate of convergence when the size n goes to infinity. Like for “regular” Voronoi quantization this upper-bound deeply relies on a random quantization argument. In fact, it can be established for a (slightly) more general family of error functionals than the ones considered so far for dual and regular quantization.

3.1 One dimensional extended Pierce Lemma

Let

$$\mathcal{I}_n := \{(x_1, \dots, x_n) \in \mathbb{R}^n, -\infty < x_1 \leq x_2 \leq \dots \leq x_n < +\infty\}$$

be the set of “non-decreasing” n -tuples of \mathbb{R}^n .

Definition 3. *Let (Ω_0, \mathcal{A}) be a measurable space and let $n \geq 1$ be an integer. A measurable functional $\Phi_n : (\Omega_0 \times \mathcal{I}_n \times \mathbb{R}, \mathcal{A} \otimes \text{Bor} \mathcal{I}_n \otimes \text{Bor}(\mathbb{R})) \rightarrow (\mathbb{R}, \text{Bor}(\mathbb{R}))$ is called a splitting functional at level n if it satisfies:*

$\forall (x_1, \dots, x_n) \in \mathcal{I}_n, \forall \xi \in \mathbb{R}$,

$$\begin{cases} (i) & \xi \in [x_i, x_{i+1}], i = 1, \dots, n-1 & \implies & \forall \omega \in \Omega_0, \Phi_n(\omega, x_1, \dots, x_n, \xi) \in [x_i, x_{i+1}], \\ (ii) & \xi \in (-\infty, x_1] & \implies & \forall \omega \in \Omega_0, \Phi_n(\omega, x_1, \dots, x_n, \xi) = x_1, \\ (iii) & \xi \in [x_n, \infty) & \implies & \forall \omega \in \Omega_0, \Phi_n(\omega, x_1, \dots, x_n, \xi) = x_n \end{cases}$$

EXAMPLES: (a) *Nearest neighbour/Voronoi quantization.* For every $i = 1, \dots, n-1$, let $x_{i+\frac{1}{2}} := \frac{x_i + x_{i+1}}{2}$. Set

$$\Phi_n(\omega, x_1, \dots, x_n, \xi) = \begin{cases} x_i & \text{if } \xi \in [x_i, x_{i+\frac{1}{2}}), \\ \in \{x_i, x_{i+1}\} & \text{if } \xi = x_{i+\frac{1}{2}}, \\ x_{i+1} & \text{if } \xi \in (x_{i+\frac{1}{2}}, x_{i+1}]. \end{cases}$$

(b) *Dual quantization.* $\Omega_0 = [0, 1]$ and

$$\begin{aligned} \Phi_n(\omega, x_1, \dots, x_n, \xi) &= \sum_{i=1}^{n-1} \left(x_i \mathbf{1}_{\{\omega \leq \frac{x_{i+1} - \xi}{x_{i+1} - x_i}\}} \right. \\ &\quad \left. + x_{i+1} \mathbf{1}_{\{\frac{\xi - x_i}{x_{i+1} - x_i} \geq x_n\}} \right) \mathbf{1}_{[x_i, x_{i+1})}(\xi) + x_1 \mathbf{1}_{\{\xi < x_1\}}(\xi) + x_n \mathbf{1}_{\{\xi \geq x_n\}}(\xi). \end{aligned}$$

It follows from (i) that a splitting functional at level n satisfies for every $p > 0$, $\omega \in \Omega_0$, $(x_1, \dots, x_n) \in \mathcal{I}_n$, $\xi \in \mathbb{R}$,

$$d(\xi, \{x_1, \dots, x_n\})^p \leq |\xi - \Phi_n(\omega, x_1, \dots, x_n, \xi)|^p \leq A_{p,n}(x_1, \dots, x_n, \xi)^p \quad (3)$$

where

$$A_{p,n}(x_1, \dots, x_n, \xi) = \left(\sum_{i=1}^{n-1} (x_{i+1} - x_i)^p \mathbf{1}_{\{x_i \leq \xi < x_{i+1}\}} + (\xi - x_n)^p \mathbf{1}_{\{\xi \geq x_n\}} + (x_1 - \xi)^p \mathbf{1}_{\{\xi < x_1\}} \right)^{\frac{1}{p}}.$$

Let X be random variable defined on $(\Omega_0, \mathcal{A}, \mathbb{P})$.

$$X \in L^p(\mathbb{P}) \implies X - \Phi_n(\cdot, x_1, \dots, x_n, X), A(x_1, \dots, x_n, X) \in L^p(\mathbb{P}).$$

Furthermore, it follows from (3) that

$$\inf_{(x_1, \dots, x_n) \in \mathcal{I}_n} \|A_{p,n}(x_1, \dots, x_n, X)\|_{L^p} \geq \inf_{(x_1, \dots, x_n) \in \mathcal{I}_n} \|X - \Phi_n(\cdot, x_1, \dots, x_n, X)\|_{L^p} \geq e_{n,p}(X).$$

The functionals $A_{p,n}$ share two important properties extensively used in what follows:

- *Consistency*: if, for every $(x_1, \dots, x_n) \in \mathcal{I}_n$ and for every $\xi \in \mathbb{R}$,

$$\forall i \in \{1, \dots, n-1\}, A_{p,n}(x_1, \dots, x_n, \xi) = A_{p,n+1}(x_1, \dots, x_{i-1}, x_i, x_i, x_{i+1}, \dots, x_n, \xi).$$

As a straightforward consequence, it follows that

$$n \mapsto \inf_{(x_1, \dots, x_n) \in \mathcal{I}_n} \|A_{p,n}(x_1, \dots, x_n, X)\|_{L^p} \text{ is non-increasing.} \quad (4)$$

- *Scaling*: $\forall \omega \in \Omega_0, \forall (x_1, \dots, x_n) \in \mathcal{I}_n, \forall \xi \in \mathbb{R}, \forall \alpha \in \mathbb{R}_+, \forall \beta \in \mathbb{R}$

$$\begin{aligned} A_{p,n}(\alpha x_1 + \mu, \dots, \alpha x_n + \beta, \xi) &= \alpha A_{p,n}(x_1, \dots, x_n, \xi), \\ A_{p,n}(x_1, \dots, x_n, -\xi) &= A_{p,n}(-x_n, \dots, -x_1, \xi). \end{aligned}$$

The main result of this section shows the existence of a universal non-asymptotic upper bounds for the error induced by splitting functionals which appears as an extension of the so-called Pierce Lemma established in [3] (see also [5]) as crucial step towards Zador's Theorem for regular Voronoi quantization.

Theorem 5. *Let $p, \eta > 0$. There exists a positive real constant $C_{p,\eta} > 0$ and an integer $n_{p,\eta} \geq 1$ such that for any random variables $X : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow \mathbb{R}$ and any sequence of splitting functionals $(\Phi_n)_{n \geq 1}$ defined on a probability space $(\Omega_0, \mathcal{A}_0, \mathbb{P}_0)$*

$$\forall n \geq n_{p,\eta}, \inf_{(x_1, \dots, x_n) \in \mathcal{I}_n} \|X - \Phi_n(\cdot, x_1, \dots, x_n, X)\|_{L^p} \leq C_{p,\eta} \|X\|_{L^{p+\eta}(\mathbb{P}_0 \otimes \mathbb{P})} n^{-1}$$

(where X and Φ_n have been canonically extended to $\Omega_0 \times \Omega$).

Proof. STEP 1. We first assume that X is $[1, +\infty)$ -valued. Let $(Y_n)_{n \geq 1}$ be a sequence of i.i.d. Pareto(δ)-distributed random variables (with probability density $f_Y(y) = \delta y^{\delta-1} \mathbf{1}_{\{y \geq 1\}}$) defined on a probability space $(\Omega', \mathcal{A}', \mathbb{P}')$. By considering $\tilde{\Omega} = \Omega \times \Omega'$, $\tilde{\mathcal{A}} = \mathcal{A} \otimes \mathcal{A}'$, $\tilde{\mathbb{P}} = \mathbb{P} \otimes \mathbb{P}'$, one may assume without loss of generality that X and the sequence $(Y_n)_{n \geq 1}$ are independent (and defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$). For convenience we will denote throughout the proof by $\|\cdot\|_{L^p}$ the L^p -norm on $(\Omega_0 \times \Omega, \mathcal{A}_0 \otimes \mathcal{A}, \mathbb{P}_0 \otimes \mathbb{P})$.

Let $\delta = \delta(p, \eta) \in (0, \frac{p}{\eta})$, let $\ell = \ell(p, \eta) = \frac{p}{\delta(p, \eta)}$. For every $n \geq \ell(p, \eta)$, let $\tilde{n} = n - \ell + 2$.

$$\begin{aligned} \inf_{(x_1, \dots, x_n) \in \mathcal{I}_n} \|X - \Phi_n(\cdot, x_1, \dots, x_n, X)\|_{L^p} &\leq \inf_{(x_1, \dots, x_n) \in \mathcal{I}_n} \|A_{p,n}(x_1, \dots, x_n, X)\|_{L^p} \\ &\leq \inf_{(1, x_2, \dots, x_{\tilde{n}}) \in \mathcal{I}_{\tilde{n}}} \|A_{p,\tilde{n}}(1, x_2, \dots, x_{\tilde{n}}, X)\|_{L^p} \\ &\leq \|A_{p,\tilde{n}}(1, Y_1^{(n)}, \dots, Y_{\tilde{n}}^{(n)}, X)\|_{L^p} \end{aligned}$$

where, for every $n \geq 1$, $Y^{(n)} = (Y_1^{(n)}, \dots, Y_n^{(n)})$ denotes the standard order statistics of the first n terms of the sequence $(Y_n)_{n \geq 1}$. For notational convenience we set $Y_0^{(n)} = 1$. Then, using that X and $(Y_k)_{k \geq 1}$ are independent, we get

$$\begin{aligned} \mathbb{E} A_{p,\tilde{n}}(1, Y_1^{(n)}, \dots, Y_{\tilde{n}}^{(n)}, X)^p &\leq \sum_{i=0}^{n-\ell} \mathbb{E} \left((Y_{i+1}^{(n)} - Y_i^{(n)})^p \mathbf{1}_{\{X \in [Y_i^{(n)}, Y_{i+1}^{(n)}]\}} \right) \\ &\quad + \mathbb{E} \left((X - Y_{n-\ell+1}^{(n)})^p \mathbf{1}_{\{X \geq Y_{n-\ell+1}^{(n)}\}} \right). \end{aligned}$$

STEP 3. Now we will compute the successive terms of the above sum. Set $\kappa = p + \eta$. Let $i \in \{1, \dots, n - \ell\}$.

$$\begin{aligned} \mathbb{E} \left((Y_{i+1}^{(n)} - Y_i^{(n)})^p \mathbf{1}_{\{X \in [Y_i^{(n)}, Y_{i+1}^{(n)}]\}} \right) &\leq \mathbb{E} \left((Y_{i+1}^{(n)} - Y_i^{(n)})^p \mathbf{1}_{\{X \geq Y_i^{(n)}\}} \right) \\ &\leq \mathbb{E} \left((Y_{i+1}^{(n)} - Y_i^{(n)})^p (Y_i^{(n)})^{-\kappa} \right) \mathbb{E} X^\kappa. \end{aligned}$$

where we used that X and $(Y_k)_{k \geq 1}$ are independent. Now, denoting by $F_Y(u) = (1 - u^{-\delta}) \mathbf{1}_{\{u \geq 1\}}$ the distribution function of the Pareto(δ)-distribution, elementary computations show that

$$\begin{aligned} &\mathbb{E} \left((Y_{i+1}^{(n)} - Y_i^{(n)})^p (Y_i^{(n)})^{-\kappa} \right) \\ &= \int_{1 \leq u \leq v} du dv u^{-\kappa} (v - u)^p F_Y(u)^{i-1} (1 - F_Y(u))^{n-i-1} f_Y(u) f_Y(v) \frac{n!}{(i-1)!(n-i-1)!} \\ &= \delta B(n - i + \frac{\eta}{\delta}, i) B((n - i)\delta - r, r + 1) \frac{\Gamma(n + 1)}{\Gamma(i)\Gamma(n - i)} \\ &= \delta \frac{\Gamma(n - i + \frac{\eta}{\delta})\Gamma(i)}{\Gamma(n + \frac{\eta}{\delta})} \frac{\Gamma((n - i)\delta - r)\Gamma(p + 1)}{\Gamma((n - i)\delta + 1)} \frac{\Gamma(n + 1)}{\Gamma(i)\Gamma(n - i)} \end{aligned}$$

where the functions

$$\Gamma(t) = \int_0^\infty u^{t-1} e^{-u} du, \quad t > 0 \quad \text{and} \quad B(a, b) = \int_0^1 u^{a-1} (1 - u)^{b-1} du, \quad a, b > 0$$

are known to satisfy $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

One checks likewise that the above equality still holds for $i = 0$.

Then one derives

$$\begin{aligned} \sum_{i=0}^{n-\ell} \mathbb{E} \left((Y_{i+1}^{(n)} - Y_i^{(n)})^p (Y_i^{(n)})^{-\kappa} \right) &= \delta \Gamma(p+1) \frac{\Gamma(n+1)}{\Gamma(n+\frac{\eta}{\delta})} \sum_{i=0}^{n-\ell} \frac{\Gamma(n-i+\frac{\eta}{\delta}) \Gamma((n-i)\delta-r)}{\Gamma((n-i)\delta+1) \Gamma(n-i)} \\ &= \delta \Gamma(p+1) \frac{\Gamma(n+1)}{\Gamma(n+\frac{\eta}{\delta})} \sum_{i=\ell}^n \frac{\Gamma(i+\frac{\eta}{\delta}) \Gamma(i\delta-r)}{\Gamma(i\delta+1) \Gamma(i)}. \end{aligned}$$

Using that, for every $a > 0$, $\frac{\Gamma(x+a)}{\Gamma(x)} \sim x^a$ as $x \rightarrow \infty$, we derive that

$$\frac{\Gamma(i+\frac{\eta}{\delta})}{\Gamma(i)} \sim i^{\frac{\eta}{\delta}}, \quad \frac{\Gamma(i\delta-r)}{\Gamma(i\delta+1)} \sim (i\delta-r)^{-(p+1)} \sim (i\delta)^{-(p+1)} \quad \text{as } i \rightarrow \infty$$

and

$$\frac{\Gamma(n+1)}{\Gamma(n+\frac{\eta}{\delta})} \sim n^{1-\frac{\eta}{\delta}} \quad \text{as } n \rightarrow \infty.$$

Consequently,

$$\begin{aligned} \sum_{i=0}^{n-\ell} \mathbb{E} \left((Y_{i+1}^{(n)} - Y_i^{(n)})^p (Y_i^{(n)})^{-\kappa} \right) &\sim n^{1-\frac{\eta}{\delta}} \sum_{i=1}^n \delta^{-(p+1)} \frac{1}{i^{p+1-\frac{\eta}{\delta}}} \\ &= \delta^{-(p+1)} n^{-p} \frac{1}{n} \sum_{i=\ell}^n \left(\frac{i}{n} \right)^{\frac{\eta}{\delta}-r-1} \\ &\sim \delta^{-(p+1)} n^{-p} \underbrace{\int_0^1 u^{\frac{\eta}{\delta}-r-1} du}_{<+\infty \text{ since } \frac{\eta}{\delta} > p} \quad \text{as } n \rightarrow \infty \end{aligned}$$

so that

$$\sum_{i=0}^{n-\ell} \mathbb{E} \left((Y_{i+1}^{(n)} - Y_i^{(n)})^p (Y_i^{(n)})^{-\kappa} \right) \sim C_{p,\eta} n^{-p} \quad \text{as } i \rightarrow \infty.$$

The remaining term can be treated as follows.

$$\begin{aligned} \mathbb{E} \left((X - Y_{n-\ell+1}^{(n)})^p \mathbf{1}_{\{X \geq Y_{n-\ell+1}^{(n)}\}} \right) &\leq \mathbb{E} X^p \mathbf{1}_{\{X \geq Y_{n-\ell+1}^{(n)}\}} \\ &\leq \mathbb{E} X^p \frac{X^\eta}{(Y_{n-\ell+1}^{(n)})^\eta} \\ &= \mathbb{E} X^\kappa \mathbb{E} (Y_{n-\ell+1}^{(n)})^{-\eta}. \end{aligned}$$

Note that

$$\begin{aligned} \mathbb{E} (Y_{n-\ell+1}^{(n)})^{-\eta} &= \frac{\Gamma(n+1)}{\Gamma(n-\ell+1) \Gamma(\ell)} \int_0^1 (1-v)^{n-\ell} v^{\ell+\frac{\eta}{\delta}-1} dv \\ &= \frac{\Gamma(n+1)}{\Gamma(n-\ell+1) \Gamma(\ell)} \frac{\Gamma(n-\ell+1) \Gamma(\ell+\frac{\eta}{\delta})}{\Gamma(n+\frac{\eta}{\delta})} \\ &\sim \frac{\Gamma(\ell+\frac{\eta}{\delta})}{\Gamma(\ell)} n^{-\frac{\eta}{\delta}} = o(n^{-p}) \end{aligned}$$

since $\frac{\eta}{\delta} > r$. Finally

$$\limsup_n n^p \left(\sum_{i=0}^{n-\ell} \mathbb{E} \left((Y_{i+1}^{(n)} - Y_i^{(n)})^p (Y_i^{(n)})^{-\kappa} \right) + \mathbb{E} (Y_{n-\ell+1}^{(n)})^{-\eta} \right) < +\infty$$

so that

$$\sup_{n \geq \ell(p, \eta)} n^p \left(\sum_{i=0}^{n-\ell} \mathbb{E} \left((Y_{i+1}^{(n)} - Y_i^{(n)})^p (Y_i^{(n)})^{-\kappa} \right) + \mathbb{E} (Y_{n-\ell+1}^{(n)})^{-\eta} \right) < +\infty.$$

This shows that for every $n \geq n_{p, \eta} := \ell(p, \eta)$,

$$\inf_{(x_1, \dots, x_n) \in \mathcal{I}_n} \|A_{p, n}(x_1, \dots, x_n, X)\|_{L^p} \leq C_{p, \eta} \frac{\|X\|_{L^{p+\eta}}^{1+\frac{p}{\eta}}}{n}.$$

STEP 4. If X is a non-negative random variable, applying the second step to $X + 1$ and using the scaling property satisfied by $A_{p, n}$ yields for $n \geq n_{p, \eta}$ (as defined in Step 3),

$$\begin{aligned} \inf_{(1, x_2, \dots, x_n) \in \mathcal{I}_n} \|A_{p, n}(1, x_2, \dots, x_n, X)\|_{L^p} &= \inf_{(0, x_2, \dots, x_n) \in \mathcal{I}_n} \|A_{p, n}(x_1, \dots, x_n, X + 1)\|_{L^p} \\ &\leq C_{p, \eta} \frac{\|1 + X\|_{L^{p+\eta}}^{1+\frac{p}{\eta}}}{n} \\ &\leq C'_{p, \eta} \frac{(1 + \|X\|_{L^{p+\eta}})^{1+\frac{p}{\eta}}}{n}. \end{aligned}$$

We may assume that $\|X\|_{L^{p+\eta}} \in (0, \infty)$. Then, applying the above bound to the non-negative random variable $\tilde{X} = \frac{X}{\|X\|_{L^{p+\eta}}}$ yields using positive homogeneity

$$\inf_{(0, x_2, \dots, x_n) \in \mathcal{I}_n} \|A_{p, n}(0, x_2, \dots, x_n, X)\|_{L^p} \leq \|X\|_{L^{p+\eta}} C'_{p, \eta} \frac{1+1}{n}.$$

STEP 5. Let X be a real-valued random variable and let for every integer $n \geq 1$, $x_1, \dots, x_n \in (-\infty, 0)$, $x_{n+1} = 0$ and $x_{n+2}, \dots, x_{2n+1} \in (0, +\infty)$. It follows that

$$\begin{aligned} A_{p, 2n+1}(x_1, \dots, x_{2n+1}, X)^p &= A_{p, n+1}(x_1, \dots, x_{n+1}, X_+)^p \mathbf{1}_{\{X \geq 0\}} \\ &\quad + A_{p, n+1}(x_1, \dots, x_{n+1}, -X_-)^p \mathbf{1}_{\{X < 0\}} \\ &= A_{p, n+1}(x_1, \dots, x_{n+1}, X_+)^p + A_{p, n+1}(-x_{n+1}, \dots, -x_1, X_-)^p. \end{aligned}$$

Consequently, if $p \geq 1$, we get using that $u^{\frac{1}{p}} + v^{\frac{1}{p}} \leq (u + v)^{\frac{1}{p}}$, $u, v \geq 0$,

$$\begin{aligned} \inf_{(x_1, \dots, x_{2n+1}) \in \mathcal{I}_{2n+1}} \|A_{p, 2n+1}(x_1, \dots, x_{2n+1}, X)\|_{L^p} &\leq \inf_{(0, x_2, \dots, x_{n+1}) \in \mathcal{I}_{n+1}} \|A_{p, n+1}(0, x_2, \dots, x_{n+1}, X_+)\|_{L^p} \\ &\quad + \inf_{(0, x_2, \dots, x_{n+1}) \in \mathcal{I}_{n+1}} \|A_{p, n+1}(0, x_2, \dots, x_{n+1}, X_-)\|_{L^p}. \end{aligned}$$

Hence, it follows from Step 3 that, for every $n \geq n_{p, \eta} - 1$,

$$\begin{aligned} \inf_{(x_1, \dots, x_{2n+1}) \in \mathcal{I}_{2n+1}} \|A_{p, 2n+1}(x_1, \dots, x_{2n+1}, X)\|_{L^p} &\leq C'_{p, \eta} \left(\|X_-\|_{L^{p+\eta}} + \|X_+\|_{L^{p+\eta}} \right) \frac{1}{n+1}, \\ &= C'_{p, \eta} \|X\|_{L^{p+\eta}} \frac{1}{n+1} \end{aligned}$$

where we used that $\|X\|_{L^{p+\eta}} \leq \|X_-\|_{L^{p+\eta}} + \|X_+\|_{L^{p+\eta}}$. Finally, the monotonicity property (4) implies that, for every $n \geq 2n_{p, \eta}$,

$$\inf_{(x_1, \dots, x_n) \in \mathcal{I}_n} \|A_{p, n}(x_1, \dots, x_n, X)\|_{L^p} \leq 2C'_{p, \eta} \frac{\|X\|_{L^{p+\eta}}}{n}.$$

If $p \in (0, 1)$, one obtains using directly (5) that

$$\inf_{(x_1, \dots, x_{2n+1}) \in \mathcal{I}_{2n+1}} \|A_{p, 2n+1}(x_1, \dots, x_{2n+1}, X)\|_{L^p}^p \leq (C'_{p, \eta})^p \left(\|X_-\|_{L^{p+\eta}}^p + \|X_+\|_{L^{p+\eta}}^p \right) \frac{1}{(n+1)^p}.$$

Now

$$\|X_-\|_{p+\eta}^p + \|X_+\|_{L^{p+\eta}}^p \leq (\|X_-\|_{L^{p+\eta}}^{p+\eta} + \|X_+\|_{L^{p+\eta}}^{p+\eta})^{\frac{p}{p+\eta}} = \|X\|_{L^{p+\eta}}^p$$

so that the conclusion remains the same. \square

3.2 A d -dimensional non-asymptotic upper-bound for the dual quantization error

Using Proposition 4 one easily shows the following d -dimensional version of the extended Pierce Lemma.

Proposition 6 (d -dimensional extended Pierce Lemma). *Let $p, \eta > 0$. There exists an integer $n_{d,p,\eta} \geq 1$ and a real constant $C_{d,p,\eta}$ such that, for every $n \geq n_{d,p,\eta}$ and every random variable $X \in L_{\mathbb{R}^d}^{p+\eta}(\Omega_0, \mathcal{A}, \mathbb{P})$,*

$$\bar{d}_{n,p}(X) \leq C_{d,p,\eta} \|X\|_{L^{p+\eta}} n^{-1/d}.$$

If $\text{supp}(\mathbb{P}_X)$ is compact then the same inequality holds true for $d_{n,p}(X)$.

Proof. $d = 1$. In this one dimensional setting one may consider only ordered n -tuples $\gamma = (x_1, \dots, x_n)$. One derives from Theorem 5 and the example (b) that follows that, for every $n \geq n_{p,\eta}$,

$$\bar{d}_{n,p}(X) = \inf_{\gamma \in \mathcal{I}_n} \|X - \Phi_n^{dq}(U, \gamma)\|_{-p} \leq C_{p,\eta} \|X\|_{L^{p+\eta}} n^{-1}$$

where $U \sim U([0, 1])$ is independent of X .

$d \geq 2$. Let γ be an optimal quantizer of size $n_1 \cdots n_d \leq n$. Then if $X = (X^1, \dots, X^d)$ denote the components of X , one has if $\min_\ell n_\ell \geq n_{p,\eta}$ (from the one dimensional case) using Proposition 4

$$\begin{aligned} \bar{d}_{n,p}^p(X, \Gamma_\gamma) &= \mathbb{E} \bar{F}_{n,p}(X; \gamma) \\ &\leq C_{d,\|\cdot\|,p} \sum_{\ell=1}^d \mathbb{E} \bar{F}_{n,p}(X^\ell; n_\ell) \\ &\leq C_{d,\|\cdot\|,p} C_{p,\eta} \sum_{\ell=1}^d \|X^\ell\|_{L^{p+\eta}} n_\ell^{-1} \\ &\leq C_{d,\|\cdot\|,p,\eta} \max_{1 \leq \ell \leq d} \|X^\ell\|_{L^{p+\eta}} \times d \lfloor n^{\frac{1}{d}} \rfloor^{-1} \\ &\leq C'_{d,\|\cdot\|,p,\eta} \|X\|_{L^{p+\eta}} n^{-\frac{1}{d}}. \end{aligned}$$

\square

4 Proof of the sharp rate theorem

On the way to proof the sharp rate theorem, we have to establish few further propositions.

Proposition 7 (Sub-linearity). *Let $\mathbf{P} = \sum_{i=1}^m s_i \mathbf{P}_i$, $\sum_{i=1}^m s_i = 1$ and $\sum_{i=1}^m n_i \leq n$. Then*

$$d_{n,p}^p(\mathbf{P}) \leq \sum_{i=1}^m s_i d_{n_i,p}^p(\mathbf{P}_i).$$

Proof. For $\varepsilon > 0$ and every $i = 1, \dots, m$, let $\Gamma_i \subset \mathbb{R}^d$, $|\Gamma_i| \leq n_i$ such that

$$d^p(\mathbf{P}_i; \Gamma_i) \leq (1 + \varepsilon) d_{n_i}^p(\mathbf{P}_i).$$

Then by Proposition 2 and with $\Gamma = \bigcup_{i=1}^m \Gamma_i$

$$\begin{aligned}
d_{n,p}^p(\mathbf{P}) &\leq d_{n,p}^p(\mathbf{P}; \Gamma) \\
&= \sum_{i=1}^m s_i \int F_p^p(\xi; \Gamma) d\mathbf{P}_i(\xi) \\
&\leq \sum_{i=1}^m s_i \int F_p^p(\xi; \Gamma_i) d\mathbf{P}_i(\xi) \\
&\leq (1 + \varepsilon) \sum_{i=1}^m s_i d_{n_i,p}^p(\mathbf{P}_i),
\end{aligned}$$

so that sending $\varepsilon \rightarrow 0$ yields the assertion. \square

Remark. Proposition 7 does not hold for \bar{d}_n^p , which causes substantial difficulties in the proof of the sharp rate compared to the regular quantization setting.

Proposition 8 (Scaling property). *Let $C = a + \rho[0, 1]^d$ be a d -dimensional hypercube, parallel to the coordinate axis, with edge length $\rho > 0$. Then*

$$d_{n,p}(\mathcal{U}(C)) = \rho \cdot d_{n,p}(\mathcal{U}([0, 1]^d)).$$

Proof. We have

$$\begin{aligned}
d^p(\mathcal{U}(C); \{a + \rho x_1, \dots, a + \rho x_n\}) &= \int_{[0, \rho]^d} \min_{\lambda \in \mathbb{R}^n} \sum_{i=1}^n \lambda_i \|\xi - \rho x_i\|^p \frac{d\lambda^d(\xi)}{\lambda^d([0, \rho]^d)} \\
&\quad \text{s.t. } \begin{bmatrix} \rho x_1 & \dots & \rho x_n \\ 1 & \dots & 1 \end{bmatrix} \lambda = \begin{bmatrix} \xi \\ 1 \end{bmatrix}, \lambda \geq 0 \\
&= \int_{[0, 1]^d} \min_{\lambda \in \mathbb{R}^n} \sum_{i=1}^n \lambda_i \|\rho \xi - \rho x_i\|^p d\lambda^d(\xi) \\
&\quad \text{s.t. } \begin{bmatrix} \rho x_1 & \dots & \rho x_n \\ 1 & \dots & 1 \end{bmatrix} \lambda = \begin{bmatrix} \rho \xi \\ 1 \end{bmatrix}, \lambda \geq 0 \\
&= \rho^p \int_{[0, 1]^d} \min_{\lambda \in \mathbb{R}^n} \sum_{i=1}^n \lambda_i \|\xi - x_i\|^p d\lambda^d(\xi) \\
&\quad \text{s.t. } \begin{bmatrix} x_1 & \dots & x_n \\ 1 & \dots & 1 \end{bmatrix} \lambda = \begin{bmatrix} \xi \\ 1 \end{bmatrix}, \lambda \geq 0 \\
&= \rho^p \cdot d^p(\mathcal{U}([0, 1]^d); \{x_1, \dots, x_n\}),
\end{aligned}$$

which yields the assertion. \square

The following Lemma shows that also for $\bar{d}_{n,p}$ the convex hull of spanned by a sequence of “semi-optimal” quantizers asymptotically covers the interior of $\text{supp}(\mathbb{P}_X)$, a fact which is trivial for $d_{n,p}$ and compact support.

Lemma 1. *Let $K = \text{conv}\{a_1, \dots, a_k\} \subset \overbrace{\text{supp}(\mathbf{P})}^{\circ}$ be a set with $\overset{\circ}{K} \neq \emptyset$ and let Γ_n be a sequence of quantizers such that $\bar{d}_{n,p}(\mathbf{P}, \Gamma_n) \rightarrow 0$ as $n \rightarrow \infty$. Then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$*

$$K \subset \text{conv}(\Gamma_n).$$

Proof. Set $a_0 = \frac{1}{k} \sum_{i=1}^k a_i$ and define for $\rho > 0$

$$\tilde{K}(\rho) = \text{conv}\{\tilde{a}_1(\rho), \dots, \tilde{a}_k(\rho)\} \quad \text{with} \quad \tilde{a}_i = a_0 + (1 + \rho)(a_i - a_0).$$

Since $K \subset \overbrace{\text{supp}(\mathbf{P})}^{\circ}$ there exists a $\rho_0 > 0$ such that $\tilde{K} = \tilde{K}(\rho_0) \subset \text{supp}(\mathbb{P}_X)$. We then also denote $\tilde{a}_i(\rho_0)$ by \tilde{a}_i . Since moreover $\tilde{a}_i \in \text{supp}(\mathbb{P}_X)$, there exists a sequence $(a_i^n)_{n \geq 1}$ having values in $\text{conv}(\Gamma_n)$ and converging to \tilde{a}_i . Otherwise there would be $\varepsilon_0 > 0$ and a subsequence (n') such that $B(\tilde{a}_i, \varepsilon_0) \subset \text{conv}(\Gamma_{n'})^c$. Then $\bar{d}_{n',p}(X, \Gamma_{n'}) \geq (\frac{\varepsilon_0}{2})^p \mathbf{P}(B(\tilde{a}_i, \varepsilon_0/2))$ since $\tilde{a}_i \in \text{supp}(P)$ which contradicts the assumption on the sequence $(\Gamma_n)_{n \geq 1}$.

Since K has a nonempty interior, it follows that $\text{aff. dim}\{a_1, \dots, a_k\} = \text{aff. dim}\{\tilde{a}_1, \dots, \tilde{a}_k\} = d$. Consequently, we may choose a set $I^* \subset \{1, \dots, k\}$, $|I^*| = d+1$ so that $\{\tilde{a}_j : j \in I^*\}$ is an affinely independent system in \mathbb{R}^d and furthermore there exists a $n_0 \in \mathbb{N}$ such that the same holds for $\{a_j^n : j \in I^*\}$ and every $n \geq n_0$. Hence, we may write for $n \geq n_0$

$$\tilde{a}_i = \sum_{j \in I^*} \mu_j^{n,i} a_j^n, \quad \sum_{j \in I^*} \mu_j^{n,i} = 1, \quad i = 1, \dots, k. \quad (5)$$

This linear system has the unique asymptotic solution $\mu_j^{\infty,i} = \delta_{ij}$ (Kronecker symbol), which implies $\mu_j^{n,i} \rightarrow \delta_{ij}$ for $n \rightarrow \infty$.

Now let $\xi \in K \subset \tilde{K}$ and write

$$\xi = \sum_{i=1}^k \lambda_i a_i \text{ for some } \lambda_i \geq 0, \sum_{i=1}^k \lambda_i = 1.$$

One easily verifies that it also holds

$$\xi = \sum_{i=1}^k \tilde{\lambda}_i \tilde{a}_i \text{ for } \tilde{\lambda}_i = \frac{\rho_0}{k(1+\rho_0)} + \frac{\lambda_i}{1+\rho_0} \geq \frac{\rho_0}{k(1+\rho_0)} > 0 \quad \text{and} \quad \sum_{i=1}^k \tilde{\lambda}_i = 1,$$

and we furthermore may choose a $n_1 \geq n_0$ such that for every $n \geq n_1$

$$\mu_i^{n,i} > \frac{1}{2} \quad \text{and} \quad |\mu_j^{n,i}| \leq \frac{\rho_0}{4k(1+\rho_0)} \quad \forall j \neq i.$$

Using (5) this leads to

$$\xi = \sum_{j \in I^*} \left(\sum_{i=1}^k \tilde{\lambda}_i \mu_j^{n,i} \right) a_j^n$$

and

$$\sum_{i=1}^k \tilde{\lambda}_i \mu_j^{n,i} > \tilde{\lambda}_j \mu_j^{n,j} - \sum_{i=1, i \neq j}^k \tilde{\lambda}_i |\mu_j^{n,i}| > \frac{\rho_0}{2k(1+\rho_0)} - \frac{\rho_0}{4k(1+\rho_0)} = \frac{\rho_0}{2k(1+\rho_0)} > 0, \quad j \in I^*.$$

Thus, noting that

$$\sum_{j \in I^*} \sum_{i=1}^k \tilde{\lambda}_i \mu_j^{n,i} = \sum_{i=1}^k \tilde{\lambda}_i \sum_{j \in I^*} \mu_j^{n,i} = 1$$

finally completes the proof. \square

As already said, Proposition 7 does not hold anymore for $\bar{d}_{n,p}$. As a consequence we have to establish an asymptotic firewall Lemma, which will help us in the sequel to overcome this problem also in the non-compact setting.

Lemma 2 (Firewall). *Let $K \subset \mathbb{R}^d$ be compact and convex with $\overset{\circ}{K} \neq \emptyset$. Moreover, let $\varepsilon > 0$ such that*

$$K_\varepsilon = \{x \in K : \text{dist}(x, K^c) \geq \varepsilon\} \neq \emptyset,$$

and denote by $\Gamma_{\alpha, \varepsilon}$ a subset of the lattice $\alpha\mathbb{Z}^d$ with edge length $\alpha > 0$ satisfying

$$K \setminus K_\varepsilon \subset \text{conv}(\Gamma_{\alpha, \varepsilon}).$$

and for every $x \in K \setminus K_\varepsilon$, $\text{dist}(x, \Gamma_{\alpha, \varepsilon}) \leq C_{\|\cdot\|} \alpha$ where $C_{\|\cdot\|} > 0$ is real constant which only depends on the norm $\|\cdot\|$.

Then, for every grid $\Gamma \subset \mathbb{R}^d$, $\eta \in (0, 1)$ and $\xi \in K_\varepsilon$, it holds

$$F^p(\xi; \Gamma) \geq \frac{1}{(1 + \eta)^p} F^p(\xi; (\Gamma \cap \overset{\circ}{K}) \cup \Gamma_{\alpha, \varepsilon}) - \left(1 + \frac{1}{\eta}\right)^p \tilde{C}_{\|\cdot\|} \alpha^p.$$

Remark. An almost minimal choice for $\Gamma_{\alpha, \varepsilon}$ is

$$\Gamma_{\alpha, \varepsilon} = \alpha\mathbb{Z}^d \cap (K \setminus K_\varepsilon) + \{0, \pm \alpha e^i : i = 1, \dots, d\}$$

where (e^1, \dots, e^d) denotes the canonical basis of \mathbb{R}^d .

Proof. Let $\Gamma = \{x_1, \dots, x_n\}$ and let $\xi \in K_\varepsilon$. Then we may choose $I = I(\xi) \subset \{1, \dots, n\}$, $|I| \leq d + 1$ such that

$$F^p(\xi; \Gamma) = \sum_{i \in I} \lambda_j \|\xi - x_i\|^p, \quad \sum_{i \in I} \lambda_i x_i = \xi, \quad \lambda_i \geq 0, \quad \sum_{i \in I} \lambda_i = 1.$$

Assume now that there is a $i_0 \in I$ such that $x_{i_0} \in \Gamma \setminus \overset{\circ}{K}$ and $\lambda_{i_0} > 0$ (otherwise the assertion is trivial). Note that there are at most d such components in $I(\xi)$ and choose $\theta = \theta(i_0) \in (0, 1)$ such that

$$\tilde{x}_{i_0} = \xi + \theta(x_{i_0} - \xi) \in K \setminus K_\varepsilon.$$

Setting

$$\tilde{\lambda}_i^0 = \frac{\lambda_i \theta}{\theta + \lambda_{i_0}(1 - \theta)}, \quad i \in I \setminus \{i_0\}, \quad \tilde{\lambda}_{i_0}^0 = \frac{\lambda_{i_0}}{\theta + \lambda_{i_0}(1 - \theta)}$$

we arrive at

$$\tilde{\lambda}_{i_0}^0 \tilde{x}_{i_0} + \sum_{i \in I \setminus \{i_0\}} \tilde{\lambda}_i^0 x_i = \xi, \quad \tilde{\lambda}_i^0 \geq 0, \quad \sum_{i \in I} \tilde{\lambda}_i^0 = 1$$

so that

$$\begin{aligned} \tilde{\lambda}_{i_0}^0 \|\xi - \tilde{x}_{i_0}\|^p + \sum_{j \in I \setminus \{i_0\}} \tilde{\lambda}_j^0 \|\xi - x_j\|^p &= \frac{\lambda_{i_0}^0 \theta^p}{\theta + \lambda_{i_0}^0 (1 - \theta)} \|\xi - x_{i_0}\|^p \\ &\quad + \sum_{i \in I \setminus \{i_0\}} \frac{\lambda_i^0 \theta}{\theta + \lambda_{i_0}^0 (1 - \theta)} \|\xi - x_i\|^p \\ &\leq \frac{\theta}{\theta + \lambda_{i_0}^0 (1 - \theta)} \sum_{i \in I} \lambda_i^0 \|\xi - x_i\|^p \\ &< F^p(\xi; \Gamma) \end{aligned} \tag{6}$$

where we used that $\theta^p \leq \theta$ since $p \geq 1$. Repeating the procedure (at most d times) for every $x_i \in \Gamma \setminus \overset{\circ}{K}$ finally yields by induction the existence of $\tilde{x}_i \in K \setminus K_\varepsilon$ and $\tilde{\lambda}_i$, $i \in I$ such that

$$\sum_{i \in I: x_i \notin \overset{\circ}{K}} \tilde{\lambda}_i \tilde{x}_i + \sum_{i \in I: x_i \in \overset{\circ}{K}} \tilde{\lambda}_i x_i = \xi, \quad \tilde{\lambda}_i \geq 0, \quad \sum_{i \in I} \lambda_i = 1$$

and

$$F^p(\xi; \Gamma) > \sum_{i \in I: x_i \notin \hat{K}} \tilde{\lambda}_i \|\xi - \tilde{x}_i\|^p + \sum_{i \in I: x_i \in \hat{K}} \tilde{\lambda}_i \|\xi - x_i\|^p.$$

Let us denote $\Gamma_{\alpha, \varepsilon} = \{a_1, \dots, a_m\}$ and let \tilde{x}_{i_0} be a ‘‘modified’’ $x_{i_0} (\in \Gamma \setminus \hat{K})$. By construction $\tilde{x}_{i_0} \in K \setminus K_\varepsilon \subset \text{conv}(\Gamma_{\alpha, \varepsilon})$ and there is $J_{i_0} \subset \{1, \dots, m\}$ such that

$$F^p(\tilde{x}_{i_0}, \Gamma_{\alpha, \varepsilon}) = \sum_{j \in J_{i_0}} \mu_j^{i_0} \|\tilde{x}_{i_0} - a_j\|^p, \quad \sum_{j \in J_{i_0}} \mu_j^{i_0} x_j = \tilde{x}_{i_0}, \quad \mu_j^{i_0} \geq 0, \quad \sum_{j \in J_{i_0}} \mu_j^{i_0} = 1$$

and

$$\forall j \in J_{i_0}, \quad \|\tilde{x}_{i_0} - a_j\| \leq C_{\|\cdot\|} \alpha$$

for a real constant $C_{\|\cdot\|} > 0$, which only depends on the norm $\|\cdot\|$.

A possible explicit construction (when $\Gamma_{\alpha, \varepsilon}$ is as in the above remark) is the following: one may select $\underline{k} \in \mathbb{Z}^d$ such that $\alpha \underline{k}$ is the nearest neighbour of \tilde{x}_{i_0} in $\Gamma_{\alpha, \varepsilon} \cap K \setminus K_\varepsilon$. Then there exists $\varepsilon_1^{j_0}, \dots, \varepsilon_d^{j_0} \in \{\pm 1\}$ such that

$$\tilde{x}_{i_0} \in \text{conv}(\alpha \underline{k}, \alpha \underline{k} + \varepsilon_j^{i_0} e^j).$$

The resulting index set J_{i_0} clearly satisfies the above claim.

Using the elementary inequality

$$\forall \eta > 0, \quad \forall u, v \geq 0, \quad (u + v)^p \leq (1 + \eta)^p u^p + \left(1 + \frac{1}{\eta}\right)^p v^p,$$

we conclude for every $j \in J_{i_0}$

$$\begin{aligned} \|\xi - a_j\|^p &\leq (\|\xi - \tilde{x}_{i_0}\| + \|\tilde{x}_{i_0} - a_j\|)^p \\ &\leq (1 + \eta)^p \|\xi - \tilde{x}_{i_0}\|^p + \left(1 + \frac{1}{\eta}\right)^p C_{\|\cdot\|}^p \alpha^p. \end{aligned}$$

As a consequence,

$$\sum_{j \in J_{i_0}} \mu_j^{i_0} \|\xi - a_j\|^p \leq (1 + \eta)^p \|\xi - \tilde{x}_{i_0}\|^p + \left(1 + \frac{1}{\eta}\right)^p C_{\|\cdot\|}^p \alpha^p$$

which in turn implies

$$\|\xi - \tilde{x}_{i_0}\|^p \geq \frac{1}{(1 + \eta)^p} \sum_{j \in J_{i_0}} \mu_j^{i_0} \|\xi - a_j\|^p - \left(1 + \frac{1}{\eta}\right)^p C_{\|\cdot\|}^p \alpha^p.$$

Plugging this inequality in (6) yields

$$\begin{aligned} F^p(\xi; \Gamma) &> \sum_{i \in I: x_i \in \hat{K}} \tilde{\lambda}_i \|\xi - x_i\|^p \\ &+ \frac{1}{(1 + \eta)^p} \sum_{i \in I: x_i \notin \hat{K}} \tilde{\lambda}_i \sum_{j \in J_i} \mu_j^i \|\xi - a_j\|^p \\ &- d \left(1 + \frac{1}{\eta}\right)^p C_{\|\cdot\|}^p \alpha^p \\ &\geq \frac{1}{(1 + \eta)^p} F^p(\xi; (\Gamma \setminus \{x_{j_0}\}) \cup \Gamma_{\alpha, \varepsilon}) - \left(1 + \frac{1}{\eta}\right)^p \tilde{C}_{\|\cdot\|}^p \alpha^p \end{aligned}$$

where $\tilde{C}_{\|\cdot\|}^p = d C_{\|\cdot\|}^p > 0$. □

Now we can establish the sharp rate for the uniform distribution $U([0, 1]^d)$.

Proposition 9 (Uniform distribution). *For every $p \geq 1$,*

$$Q_{\|\cdot\|, p, d}^{dq} := \inf_{n \geq 0} n^{1/d} d_{n,p}(\mathcal{U}([0, 1]^d)) = \lim_{n \rightarrow \infty} n^{1/d} d_{n,p}(\mathcal{U}([0, 1]^d)).$$

Proof. Let $n, m \in \mathbb{N}$, $m < n$ and set $k = k(n, m) = \left\lfloor \left(\frac{n}{m}\right)^{1/d} \right\rfloor$.

Covering the unit hypercube $[0, 1]^d$ by k^d translates C_1, \dots, C_{k^d} of the hypercube $\left[0, \frac{1}{k}\right]^d$, we arrive at $\mathcal{U}([0, 1]^d) = k^{-d} \sum_{i=1}^{k^d} \mathcal{U}(C_i)$. Hence, Proposition 7 yields

$$d_{n,p}^p(\mathcal{U}([0, 1]^d)) \leq k^{-d} \sum_{i=1}^{k^d} d_m^p(\mathcal{U}(C_i)).$$

Furthermore, Proposition 8 states

$$d_{m,p}(\mathcal{U}(C_i)) = k^{-1} d_{m,p}(\mathcal{U}([0, 1]^d)),$$

so that we may conclude for all $n, m \in \mathbb{N}$, $m < n$

$$d_{n,p}(\mathcal{U}([0, 1]^d)) \leq k^{-1} d_{m,p}(\mathcal{U}([0, 1]^d)).$$

Thus, we arrive at

$$\begin{aligned} n^{1/d} d_{n,p}(\mathcal{U}([0, 1]^d)) &\leq k^{-1} n^{1/d} d_{m,p}(\mathcal{U}([0, 1]^d)) \\ &\leq \frac{k+1}{k} m^{1/d} d_{m,p}(\mathcal{U}([0, 1]^d)), \end{aligned}$$

which yields for every integer $m \geq 1$

$$\limsup_{n \rightarrow \infty} n^{1/d} d_{n,p}(\mathcal{U}([0, 1]^d)) \leq m^{1/d} d_{m,p}(\mathcal{U}([0, 1]^d)),$$

since $\lim_{n \rightarrow \infty} k(n, m) = +\infty$.

This finally implies

$$\lim_{n \rightarrow \infty} n^{1/d} d_{n,p}(\mathcal{U}([0, 1]^d)) = \inf_{m \geq 0} m^{1/d} d_{m,p}(\mathcal{U}([0, 1]^d)).$$

□

Proposition 10. *For every $p \geq 1$,*

$$Q_{\|\cdot\|, p, d}^{dq} = \lim_{n \rightarrow \infty} n^{1/d} d_{n,p}(\mathcal{U}([0, 1]^d)) = \lim_{n \rightarrow \infty} n^{1/d} \bar{d}_{n,p}(\mathcal{U}([0, 1]^d))$$

Proof. Since we have $\bar{d}_{n,p}(X) \leq d_{n,p}(X)$ it remains to show

$$Q_{\|\cdot\|, p, d}^{dq} \leq \liminf_{n \rightarrow \infty} n^{1/d} \bar{d}_{n,p}(\mathcal{U}([0, 1]^d)).$$

Let (Γ_n) be a sequence of optimal quantizers for $\bar{d}_{n,p}(\mathcal{U}([0, 1]^d))$ and

For $0 < \varepsilon < 1/2$ let $C_\varepsilon = (1/2, \dots, 1/2) + \frac{1-\varepsilon}{2}[-1, 1]^d$ be the centered hypercube in $[0, 1]^d$ with edge length $1 - \varepsilon$ and midpoint $(1/2, \dots, 1/2)$. Moreover let (Γ_n) be a sequence of quantizers such that

$$\bar{d}_p(\mathcal{U}([0, 1]^d); \Gamma_n) \leq (1 + \varepsilon) \bar{d}_{n,p}(\mathcal{U}([0, 1]^d)).$$

Owing to Lemma 1, there is an integer $n_\varepsilon \in \mathbb{N}$ such that

$$\forall n \geq n_\varepsilon, \quad C_\varepsilon \subset \text{conv}(\Gamma_n).$$

We therefore get for any $n \geq n_\varepsilon$

$$(1 + \varepsilon)d_{n,p}(\mathcal{U}([0, 1]^d)) \geq d_p(\mathcal{U}([0, 1]^d); \Gamma_n) \geq d_p(\mathcal{U}([0, 1]^d)|_{\Gamma_n}; \Gamma_n) \geq d_p(\mathcal{U}([0, 1]^d)|_{C_\varepsilon}; \Gamma_n).$$

Normalizing $\mathcal{U}([0, 1]^d)|_{C_\varepsilon}$ into a probability distribution and applying Proposition 8, we get

$$(1 + \varepsilon)d_{n,p}(\mathcal{U}([0, 1]^d)) \geq d_{n,p}(\mathcal{U}([0, 1]^d)|_{C_\varepsilon}) = (\lambda^d(C_\varepsilon))^{\frac{1}{p}} d_{n,p}(\mathcal{U}(C_\varepsilon)) = (1 - \varepsilon)^{1+d/p} d_{n,p}(\mathcal{U}([0, 1]^d)).$$

Hence, we obtain for all $0 < \varepsilon < 1/2$

$$\liminf_{n \rightarrow \infty} n^{1/d} d_{n,p}^p(\mathcal{U}([0, 1]^d)) \geq \frac{(1 - \varepsilon)^{1+d/p}}{1 + \varepsilon} Q_{\|\cdot\|, p, d}^{\text{dq}},$$

so that letting $\varepsilon \rightarrow 0$ yields the assertion. \square

Proposition 11. *Let $\mathbf{P} = \sum_{i=1}^m s_i \mathcal{U}(C_i)$, $\sum_{i=1}^m s_i = 1$, $s_i > 0$, $i = 1, \dots, m$, where $C_i = a_i + [0, l]^d$, $i = 1, \dots, m$, are pairwise disjoint hypercubes in \mathbb{R}^d with common edge-length l . Set*

$$h := \frac{d\mathbf{P}}{d\lambda^d} = \sum_{i=1}^m s_i l^{-d} \mathbb{1}_{C_i}.$$

Then

$$(a) \quad \limsup_{n \rightarrow \infty} n^{1/d} d_{n,p}(\mathbf{P}) \leq Q_{\|\cdot\|, p, d}^{\text{dq}} \cdot \|h\|_{d/(d+p)}^{\frac{1}{p}}.$$

$$(b) \quad \liminf_{n \rightarrow \infty} n^{1/d} \bar{d}_{n,p}(\mathbf{P}) \geq Q_{\|\cdot\|, p, d}^{\text{dq}} \cdot \|h\|_{d/(d+p)}^{\frac{1}{p}}.$$

Proof. (a) For $n \in \mathbb{N}$, set

$$t_i = \frac{s_i^{d/(d+p)}}{\sum_{j=1}^m s_j^{d/(d+p)}} \quad \text{and} \quad n_i = \lfloor t_i n \rfloor, \quad 1 \leq i \leq m.$$

Then, by Proposition 7 and Proposition 8, we get for every $n \geq \max_{1 \leq i \leq m} (1/t_i)$

$$d_{n,p}^p(\mathbf{P}) \leq \sum_{i=1}^m s_i d_{n,p}^p(\mathcal{U}(C_i)) = l^p \sum_{i=1}^m s_i d_{n_i}^p(\mathcal{U}([0, 1]^d)).$$

Proposition 9 then yields

$$n^{\frac{p}{d}} d_{n_i}^p(\mathcal{U}([0, 1]^d)) = \left(\frac{n}{n_i}\right)^{\frac{p}{d}} n_i^{\frac{p}{d}} d_{n_i}^p(\mathcal{U}([0, 1]^d)) \rightarrow t_i^{-\frac{p}{d}} Q_{\|\cdot\|, p, d}^{\text{dq}} \quad \text{as } n \rightarrow \infty.$$

Noting that $\|h\|_{d/(d+p)} = l^p \left(\sum s_i^{d/(d+p)}\right)^{(d+p)/d}$, we get

$$\limsup_{n \rightarrow \infty} n^{\frac{p}{d}} d_{n,p}^p(\mathbf{P}) \leq Q_{\|\cdot\|, p, d}^{\text{dq}} l^p \sum_{i=1}^m s_i t_i^{-\frac{p}{d}} \leq Q_{\|\cdot\|, p, d}^{\text{dq}} \cdot \|h\|_{d/(d+p)}.$$

(b) Let $\varepsilon \in (0, l/2)$ and denote by $C_{i,\varepsilon}$ the closed hypercube with the same center as C_i and edge-length $l - \varepsilon$. For $\alpha \in (0, \varepsilon/2)$, we denote by $\gamma_{\alpha_n, \varepsilon, i}$ the lattice with edge-length $\tilde{\alpha} = \frac{l}{\lceil l/\alpha \rceil}$ covering $C_i \setminus C_{i,\varepsilon}$.

We then get for every $i \in \{1, \dots, m\}$

$$|\gamma_{\alpha_n, \varepsilon, i}| = \left(\frac{l}{\tilde{\alpha}} + 1\right)^d - \left(\frac{l}{\tilde{\alpha}} - 2 \left\lceil \frac{\varepsilon/2}{\tilde{\alpha}} \right\rceil - 1\right)^d$$

and define for $\varepsilon \in (0, l/2), \alpha \in (0, \varepsilon/2)$

$$g_{l,\varepsilon}(\alpha) = \alpha^d |\gamma_{\alpha_n, \varepsilon, i}| = \left(\alpha \frac{l}{\tilde{\alpha}} + \alpha\right)^d - \left(\alpha \frac{l}{\tilde{\alpha}} - 2\alpha \left\lceil \frac{\varepsilon/2}{\tilde{\alpha}} \right\rceil - \alpha\right)^d.$$

Since $\frac{\alpha}{\tilde{\alpha}} \rightarrow 1$ and $2\alpha \left\lceil \frac{\varepsilon/2}{\tilde{\alpha}} \right\rceil \rightarrow \varepsilon$ as $\alpha \rightarrow 0$, we conclude

$$\forall \varepsilon \in (0, l/2), \quad \lim_{\alpha \rightarrow 0} g_{l,\varepsilon}(\alpha) = l^d - (l - \varepsilon)^d. \quad (7)$$

Let $\eta \in (0, 1)$ and denote by Γ_n a sequence of n -quantizers such that $\bar{d}^p(\mathbf{P}; \Gamma_n) \leq (1 + \eta)d_n^p(\mathbf{P})$. It follows from Proposition 6 that $\bar{d}^p(\mathbf{P}; \Gamma_n) \rightarrow 0$ for $n \rightarrow \infty$ so that Lemma 1 yields the existence of $n_\varepsilon \in \mathbb{N}$ such that for any $n \geq n_\varepsilon$

$$\bigcup_{1 \leq i \leq m} C_{i,\varepsilon} \subset \text{conv}(\Gamma_n).$$

We then derive from Lemma 2

$$\begin{aligned} \bar{d}^p(\mathcal{U}(C_i); \Gamma_n) &= l^{-d} \int_{C_i} \bar{F}^p(\xi; \Gamma_n) \lambda^d(d\xi) \\ &\geq l^{-d} \int_{C_{i,\varepsilon}} \bar{F}^p(\xi; \Gamma_n) \lambda^d(d\xi) = l^{-d} \int_{C_{i,\varepsilon}} F^p(\xi; \Gamma_n) \lambda^d(d\xi) \\ &\geq \frac{l^{-d} (l - \varepsilon)^d}{(1 + \eta)^p} d^p(\mathcal{U}(C_{i,\varepsilon}); (\Gamma_n \cap \dot{C}_i) \cup \gamma_{\alpha_n, \varepsilon, i}) - l^{-d} (l - \varepsilon)^d \left(1 + \frac{1}{\eta}\right)^p \cdot C_{\|\cdot\|} \cdot \alpha^p. \end{aligned}$$

At this stage, we set for every $\rho > 0$

$$\alpha = \alpha_n = \left(\frac{m}{\rho n}\right)^{1/d} \quad (8)$$

and denote

$$n_i = |(\Gamma_n \cap \dot{C}_i) \cup \gamma_{\alpha_n, \varepsilon, i}|.$$

Since $d_{n_i, p}(\mathcal{U}(C_{i,\varepsilon})) = (l - \varepsilon)d_{n_i, p}(\mathcal{U}([0, 1]^d))$ owing to Proposition 8, we get

$$\begin{aligned} n^{\frac{p}{d}} \bar{d}_{n,p}^p(\mathbf{P}) &\geq \frac{1}{1 + \eta} \sum_{i=1}^m s_i n^{\frac{p}{d}} \bar{d}^p(\mathcal{U}(C_i); \Gamma_n) \\ &\geq \frac{l^{-d} (l - \varepsilon)^d}{(1 + \eta)^{p+1}} \sum_{i=1}^m s_i n^{\frac{p}{d}} d^p(\mathcal{U}(C_{i,\varepsilon}); (\Gamma_n \cap \dot{C}_i) \cup \gamma_{\alpha_n, \varepsilon, i}) \\ &\quad - l^{-d} (l - \varepsilon)^d \frac{(1 + \eta)^{p+1}}{\eta^p} \sum_{i=1}^m s_i \cdot C_{\|\cdot\|} \cdot \alpha^p \cdot n^{\frac{p}{d}} \\ &\geq \frac{l^{-d} (l - \varepsilon)^{d+p}}{(1 + \eta)^{p+1}} \sum_{i=1}^m s_i n^{\frac{p}{d}} d_{n_i}^p(\mathcal{U}([0, 1]^d)) - l^{-d} (l - \varepsilon)^d \frac{(1 + \eta)^{p+1}}{\eta^p} \cdot C_{\|\cdot\|} \cdot \left(\frac{m}{\rho}\right)^{\frac{p}{d}}. \end{aligned} \quad (9)$$

Since

$$\frac{n_i}{n} \leq \frac{|\Gamma_n \cap \hat{C}_i|}{n} + \frac{g_{l,\varepsilon}(\alpha_n)}{n\alpha^d} = \frac{|\Gamma_n \cap \hat{C}_i|}{n} + \frac{\rho}{m} g_{l,\varepsilon}(\alpha_n),$$

we conclude from (7) and (8)

$$\limsup_{n \rightarrow \infty} \sum_{i=1}^m \frac{n_i}{n} \leq 1 + \rho(l^d - (l - \varepsilon)^d).$$

We may choose a subsequence (still denoted by (n)), such that

$$n^{1/d} \bar{d}_{n,p}(\mathbf{P}) \rightarrow \liminf_{n \rightarrow \infty} n^{1/d} d_{n,p}(\mathbf{P}) \quad \text{and} \quad \frac{n_i}{n} \rightarrow v_i \in [0, 1 + \rho(l^d - (l - \varepsilon)^d)].$$

As a matter of fact, it holds $v_i > 0$, $1 \leq i \leq m$: otherwise Proposition 9 would yield

$$\begin{aligned} n^{\frac{p}{d}} \bar{d}_{n,p}^p(\mathbf{P}) &\geq \frac{l^{-d} (l - \varepsilon)^{d+p}}{(1 + \eta)^{p+1}} \sum_{i=1}^m s_i \left(\frac{n_i}{n}\right)^{-\frac{p}{d}} n_i^{\frac{p}{d}} d_{n_i}^p(\mathcal{U}([0, 1]^d)) \\ &\quad - l^{-d} (l - \varepsilon)^d \frac{(1 + \eta)^{p+1}}{\eta^p} \cdot C_{\|\cdot\|} \cdot \left(\frac{m}{\rho}\right)^{\frac{p}{d}} \\ &\rightarrow +\infty, \end{aligned}$$

which contradicts (a).

Consequently, we may normalize the v_i 's by setting

$$\tilde{v}_i = \frac{v_i}{1 + \rho(l^d - (l - \varepsilon)^d)}$$

so that $\sum_{i=1}^m \tilde{v}_i \leq 1$. We conclude from Proposition 9

$$\begin{aligned} \liminf_{n \rightarrow \infty} \sum_{i=1}^m s_i n^{\frac{p}{d}} d_{n_i}^p(\mathcal{U}([0, 1]^d)) &= \sum_{i=1}^m s_i \lim_{n \rightarrow \infty} \left(\frac{n(1 + \rho(l^d - (l - \varepsilon)^d))}{n_i} \right)^{\frac{p}{d}} n_i^{\frac{p}{d}} d_{n_i}^p(\mathcal{U}([0, 1]^d)) \\ &= Q_{\|\cdot\|, p, d}^{\text{dq}} \sum_{i=1}^m s_i \tilde{v}_i^{-\frac{p}{d}} \\ &\geq Q_{\|\cdot\|, p, d}^{\text{dq}} \inf_{\sum_i y_i \leq 1, y_i \geq 0} \sum_{i=1}^m s_i y_i^{-\frac{p}{d}} \\ &= Q_{\|\cdot\|, p, d}^{\text{dq}} \left(\sum_{i=1}^m s_i^{d/(d+p)} \right)^{(d+p)/d}. \end{aligned}$$

Hence, we derive from (9)

$$\begin{aligned} \liminf_{n \rightarrow \infty} n^{\frac{p}{d}} \bar{d}_{n,p}^p(\mathbf{P}) &\geq \frac{l^{-d} (l - \varepsilon)^{d+p}}{(1 + \eta)^{p+1} (1 + \rho(l^d - (l - \varepsilon)^d))^{\frac{p}{d}}} Q_{\|\cdot\|, p, d}^{\text{dq}} \left(\sum_{i=1}^m s_i^{d/(d+p)} \right)^{(d+p)/d} \\ &\quad - l^{-d} (l - \varepsilon)^d \frac{(1 + \eta)^{p+1}}{\eta^p} \cdot C_{\|\cdot\|} \cdot \left(\frac{m}{\rho}\right)^{\frac{p}{d}} \end{aligned}$$

so that sending $\varepsilon \rightarrow 0$ implies

$$\begin{aligned} \liminf_{n \rightarrow \infty} n^{\frac{p}{d}} \bar{d}_{n,p}^p(\mathbf{P}) &\geq \frac{l^p}{(1 + \eta)^{p+1}} Q_{\|\cdot\|, p, d}^{\text{dq}} \left(\sum_{i=1}^m s_i^{d/(d+p)} \right)^{(d+p)/d} - \frac{(1 + \eta)^{p+1}}{\eta^p} C_{\|\cdot\|} \left(\frac{m}{\rho}\right)^{\frac{p}{d}} \\ &= \frac{1}{(1 + \eta)^{p+1}} Q_{\|\cdot\|, p, d}^{\text{dq}} \cdot \|h\|_{d/(d+p)} - \frac{(1 + \eta)^{p+1}}{\eta^p} C_{\|\cdot\|} \left(\frac{m}{\rho}\right)^{\frac{p}{d}} \end{aligned}$$

and, finally, letting successively ρ go to ∞ and η go to 0 yields the assertion. \square

Proposition 12. *Assume that \mathbf{P} is absolutely continuous w.r.t. λ^d with compact support. Then*

$$(a) \quad \limsup_{n \rightarrow \infty} n^{\frac{p}{d}} d_{n,p}(\mathbf{P}) \leq Q_{\|\cdot\|,p,d}^{dq} \cdot \|h\|_{d/(d+p)}^{\frac{1}{p}}.$$

$$(b) \quad \liminf_{n \rightarrow \infty} n^{\frac{p}{d}} \bar{d}_{n,p}(\mathbf{P}) \geq Q_{\|\cdot\|,p,d}^{dq} \cdot \|h\|_{d/(d+p)}^{\frac{1}{p}}.$$

Proof. Let $C = [-l/2, l/2]^d$ be a closed hypercube centered at the origin, parallel to the coordinate axis with edge-length l , such that $\text{supp}(\mathbf{P}) \subset C$. For $k \in \mathbb{N}$ consider the tessellation of C into k^d closed hypercubes with common edge-length l/k . To be precise, for every $\underline{i} = (i_1, \dots, i_d) \in \mathbb{Z}^d$, we set

$$C_{\underline{i}} = \prod_{r=1}^d \left[-\frac{l}{2} + \frac{i_r l}{k}, -\frac{l}{2} + \frac{(i_r + 1)l}{k} \right].$$

Set $h = \frac{d\mathbf{P}}{d\lambda^d}$ and

$$\mathbf{P}_k = \sum_{\substack{\underline{i} \in \mathbb{Z}^d \\ 0 \leq i_r < k}} \mathbf{P}(C_{\underline{i}}) \mathcal{U}(C_{\underline{i}}), \quad h_k = \frac{d\mathbf{P}_k}{d\lambda^d} = \sum_{\substack{\underline{i} \in \mathbb{Z}^d \\ 0 \leq i_r < k}} \frac{\mathbf{P}(C_{\underline{i}})}{\lambda^d(C_{\underline{i}})} \mathbb{1}_{C_{\underline{i}}}. \quad (10)$$

By differentiation, of measures we obtain $h_k \rightarrow h$, λ^d -a.s. as $k \rightarrow \infty$. which in turn implies owing to Scheffé's Lemma

$$\lim_{k \rightarrow \infty} \|h_k - h\|_1 = 0,$$

and

$$\lim_{k \rightarrow \infty} \|h_k\|_{d/(d+p)} = \|h\|_{d/(d+p)},$$

since $\|h_k - h\|_{d/(d+p)} \leq \left(\lambda^d(C) \right)^{\frac{p}{d}} \|h_k - h\|_1$, owing to Jensen's Inequality applied to the probability measure $\frac{\lambda^d|_C}{\lambda^d(C)}$. Moreover, by Proposition 11 we have

$$\lim_{n \rightarrow \infty} n^{1/d} d_{n,p}(\mathbf{P}_k) = Q_{\|\cdot\|,p,d}^{dq} \|h_k\|_{d/(d+p)}^{\frac{1}{p}}. \quad (11)$$

Likewise, we define an inner approximation to \mathbf{P} : Denote by

$$C^k = \bigcup_{\underline{i} \in \text{supp}(\mathbf{P})} C_{\underline{i}}$$

the union of the hypercubes $C_{\underline{i}}$ lying in the interior of $\text{supp}(\mathbf{P})$. Setting

$$\mathring{\mathbf{P}}_k = \sum_{\underline{i} \in \text{supp}(\mathbf{P})} \mathbf{P}(C_{\underline{i}}) \mathcal{U}(C_{\underline{i}}),$$

$$\mathring{h}_k = \frac{d\mathring{\mathbf{P}}_k}{d\lambda^d} = h_k \mathbb{1}_{C^k},$$

we have as above that

$$\mathring{h}_k \rightarrow h, \quad \lambda^d - \text{a.s.} \quad \text{for } k \rightarrow \infty.$$

Consequently

$$\lim_{k \rightarrow \infty} \|\mathring{h}_k - h\|_1 = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \|\mathring{h}_k\|_{d/(d+p)} = \|h\|_{d/(d+p)}.$$

We get likewise by Proposition 11 that, for every $k \in \mathbb{N}$,

$$\lim_{n \rightarrow \infty} n^{1/d} d_{n,p}(\mathring{\mathbf{P}}_k) = Q_{\|\cdot\|,p,d}^{\text{dq}} \cdot \|h_k\|_{d/(d+p)}^{\frac{1}{p}}. \quad (12)$$

(a) Let $0 < \varepsilon < 1$ and $n \geq 2^d/\varepsilon$. If we divide each edge of the hypercube C into

$$m = \lfloor (\varepsilon n)^{1/d} \rfloor - 1$$

intervals of equal length l/m , the interval endpoints define $m + 1$ grid points on each edge. Denoting by $\Gamma_1 = \Gamma_1(\varepsilon, n)$ the product quantizer made up by this procedure, we clearly have

$$|\Gamma_1| = (m + 1)^d = \lfloor (\varepsilon n)^{1/d} \rfloor^d =: n_1.$$

For this product quantizer it holds for all $\xi \in C$ by Propositions 4 and 3

$$\begin{aligned} F^p(\xi; \Gamma_1) &\leq C_{\|\cdot\|} \sum_{i=1}^d \left(\frac{l}{2m} \right)^p \\ &\leq C_{\|\cdot\|,p,d} \frac{l^p}{(\varepsilon n)^{\frac{p}{d}}}. \end{aligned}$$

For $n_2 = \lfloor (1 - \varepsilon)n \rfloor$ let Γ_2 be a n_2 -quantizer such that $d^p(\mathbf{P}_k; \Gamma_2) \leq (1 + \varepsilon)d_{n_2}^p(\mathbf{P}_k)$. We clearly have $|\Gamma_1 \cup \Gamma_2| \leq n$ and

$$\begin{aligned} n^{\frac{p}{d}} \left| \int F^p(\xi; \Gamma_1 \cup \Gamma_2) d\mathbf{P}_k(\xi) - \int F^p(\xi; \Gamma_1 \cup \Gamma_2) d\mathbf{P}(\xi) \right| \\ \leq n^{\frac{p}{d}} \int F^p(\xi; \Gamma_1 \cup \Gamma_2) |h_k(\xi) - h(\xi)| d\lambda^d \xi \\ \leq C_{\|\cdot\|,p,d} \frac{l^p}{\varepsilon^{\frac{p}{d}}} \|h_k - h\|_1 \\ = c_{1,\varepsilon} \|h_k - h\|_1 \end{aligned}$$

for $k \in \mathbb{N}$ and $n \geq \max\left\{\frac{2^d}{\varepsilon}, \frac{1}{1-\varepsilon}\right\}$. This implies

$$\begin{aligned} n^{\frac{p}{d}} d_{n,p}^p(\mathbf{P}) &\leq n^{\frac{p}{d}} \int F^p(\xi; \Gamma_1 \cup \Gamma_2) d\mathbf{P}(\xi) \\ &\leq n^{\frac{p}{d}} \int F^p(\xi; \Gamma_1 \cup \Gamma_2) d\mathbf{P}_k(\xi) + c_1 \|h_k - h\|_1 \\ &\leq n^{\frac{p}{d}} \int F^p(\xi; \Gamma_2) d\mathbf{P}_k(\xi) + c_1 \|h_k - h\|_1 \\ &\leq (1 + \varepsilon) n^{\frac{p}{d}} d_{n_2}^p(\mathbf{P}_k) + c_{1,\varepsilon} \|h_k - h\|_1, \end{aligned}$$

so that we conclude from (11)

$$\limsup_{n \rightarrow \infty} n^{\frac{p}{d}} d_{n,p}^p(\mathbf{P}) \leq \frac{1 + \varepsilon}{(1 - \varepsilon)^{\frac{p}{d}}} (Q_{\|\cdot\|,p,d}^{\text{dq}})^p \|h_k\|_{d/(d+p)} + c_{1,\varepsilon} \|h_k - h\|_1.$$

Letting first k go to infinity and then letting ε go to zero yields

$$\limsup_{n \rightarrow \infty} n^{1/d} d_{n,p}(\mathbf{P}) \leq Q_{\|\cdot\|,p,d}^{\text{dq}} \|h_k\|_{d/(d+p)}^{\frac{1}{p}}.$$

(b) Assume that Γ_3 is a n_2 -quantizer such that $\bar{d}^p(\mathbf{P}; \Gamma_3) \leq (1 + \varepsilon) \bar{d}_{n_2}^p(\mathbf{P})$. Again it holds $|\Gamma_1 \cup \Gamma_3| \leq n$ and we derive as above

$$n^{\frac{p}{d}} \left| \int F^p(\xi; \Gamma_1 \cup \Gamma_3) d\dot{\mathbf{P}}_k(\xi) - \int F^p(\xi; \Gamma_1 \cup \Gamma_3) d\mathbf{P}(\xi) \right| \leq c_{2,\varepsilon} \|\dot{h}_k - h\|_1. \quad (13)$$

Moreover, Lemma 1 yields for every $k \in \mathbb{N}$ the existence of $n_{k,\varepsilon} \in \mathbb{N}$ such that, for all $n \geq n_{k,\varepsilon}$,

$$\begin{aligned} (1 + \varepsilon) \bar{d}_{n_2,p}^p(\mathbf{P}) &\geq \bar{d}_p^p(\mathbf{P}; \Gamma_3) \geq \int_{\text{conv}(\Gamma_3)} F^p(\xi; \Gamma_3) d\mathbf{P}(\xi) \\ &\geq \int_{C^k} F^p(\xi; \Gamma_3) d\mathbf{P}(\xi) \geq \int_{C^k} F^p(\xi; \Gamma_1 \cup \Gamma_3) d\mathbf{P}(\xi). \end{aligned}$$

Thus, we derive from (13) that, for every $n \geq \max\left(n_{k,\varepsilon}, \frac{2^d}{\varepsilon}, \frac{1}{1-\varepsilon}\right)$,

$$\begin{aligned} (1 + \varepsilon) n^{\frac{p}{d}} \bar{d}_{n_2,p}^p(\mathbf{P}) &\geq n^{\frac{p}{d}} \int_{C^k} F^p(\xi; \Gamma_1 \cup \Gamma_3) d\mathbf{P}(\xi) \\ &\geq n^{\frac{p}{d}} \int_{C^k} F^p(\xi; \Gamma_1 \cup \Gamma_3) d\dot{\mathbf{P}}_k(\xi) - c_{2,\varepsilon} \|\dot{h}_k - h\|_1 \\ &\geq n^{\frac{p}{d}} d_{n,p}^p(\dot{\mathbf{P}}_k) - c_{2,\varepsilon} \|\dot{h}_k - h\|_1, \end{aligned}$$

which yields, once combined with (12),

$$\frac{1 + \varepsilon}{(1 - \varepsilon)^{\frac{p}{d}}} \liminf_{n \rightarrow \infty} n^{\frac{p}{d}} \bar{d}_{n_2,p}^p(\mathbf{P}) \geq Q_{\|\cdot\|,p,d}^{\text{dq}} \|\dot{h}_k\|_{d/(d+p)} - c_{2,\varepsilon} \|\dot{h}_k - h\|_1.$$

Letting first k go to ∞ and then letting ε go to 0, we get

$$\liminf_{n \rightarrow \infty} n^{\frac{p}{d}} \bar{d}_{n,p}^p(\mathbf{P}) \geq Q_{\|\cdot\|,p,d}^{\text{dq}} \|h\|_{\frac{p}{d/(d+p)}}.$$

□

Proposition 13 (Singular distribution). *Assume that \mathbf{P} is singular with respect to λ^d and has compact support. Then*

$$\limsup_{n \rightarrow \infty} n^{\frac{p}{d}} \bar{d}_{n,p}^p(\mathbf{P}) = 0.$$

Proof. (closely follows Step 4 in Graf and Luschgy's proof of Zador's Theorem). Let A be a Borel set such that $\mathbf{P}(A) = 1$ and $\lambda^d(A) = 0$. Let $\varepsilon > 0$; by the outside regularity of λ^d there exists an open set $O = O(\varepsilon)$ set such that $\lambda^d(O) \leq \varepsilon$ (and $\mathbf{P}(O) = 1$). Let C be an open hypercube with edges of length ℓ parallel to the coordinate axis containing the closure of A .

Let $C_k = \prod_{i=1}^d [c_{k,i}, c_{k,i} + \ell_i)$, $k \in \mathbb{N}$ be a countable partition of A consisting of nonempty half-open hypercubes, still with edges parallel to the coordinate axis (see, e.g. Lemma 1.4.2 in [2]).

Let $m = m(\varepsilon) \in \mathbb{N}$ such that $\sum_{k \geq m+1} \mathbf{P}(C_k) \leq \varepsilon^{\frac{p}{d}} \ell^{-p}$.

Let $n \in \mathbb{N}$, $n \geq 2^{d+1}$ and let $n_1, \dots, n_d \geq 2$ be integers such that the product $n_1^d + \dots + n_m^d \leq n/2$. One designs a grid Γ as follows.

For every $k \in \{1, \dots, m\}$, we consider the lattice of C_k of size n_i^d defined by

$$\prod_{i=1}^d \left\{ c_{k,i} + \frac{r_i}{n_k - 1} \ell_i, r_i = 0, \dots, n_k - 1, i = 1, \dots, d \right\}.$$

Then one defines likewise the lattice of C of size $n_{m+1}^d \leq n/2$

$$\prod_{i=1}^d \left\{ C_{k,i} + \frac{r_i}{n_{m+1} - 1} \ell_i, r_i = 0, \dots, n_{m+1} - 1, i = 1, \dots, d \right\}.$$

The grid Γ is made up with all the points of the $m + 1$ above finite lattices. Now let $\xi \in A$. It is clear from the definition of the function F_p that

$$F_p(\xi; \Gamma) \leq \begin{cases} C_{\|\cdot\|} (\ell_k/n_k)^p & \text{if } \xi \in \bigcup_{k=1}^m C_k \\ C_{\|\cdot\|} (\ell/n_{m+1})^p & \text{if } \xi \in C \setminus \bigcup_{k=1}^m C_k \end{cases}$$

where $C_{\|\cdot\|} > 0$ is a real constant only depending on the norm. As a consequence

$$\begin{aligned} d_{n,p}^p(\mathbf{P}) &= \sum_{k=1}^m \int_{C_k} F^p(\xi; \Gamma) dP(\xi) + \int_{C \setminus \bigcup_{k=1}^m C_k} F^p(\xi; \Gamma) dP(\xi) \\ &\leq C_{\|\cdot\|} \left(\sum_{k=1}^m (\ell_k/n_k)^p P(C_k) + (\ell/n_{m+1})^p P(C \setminus \bigcup_{k=1}^m C_k) \right). \end{aligned}$$

Set for every $k \in \{1, \dots, m\}$,

$$n_k = \left\lfloor \frac{\ell_k (n/2)^{\frac{1}{d}}}{(\sum_{k'=1}^d \ell_{k'}^d)^{\frac{1}{d}}} \right\rfloor$$

and $n_{m+1} = \lfloor (n/2)^{\frac{1}{d}} \rfloor$. Note that

$$\sum_{k'=1}^d \ell_{k'}^d = \sum_{k=1}^m \lambda^d(C_k) \leq \lambda^d(A) \leq \varepsilon.$$

Elementary computations show that for large enough n all the n_k are greater than 1 and that

$$\sum_{k=1}^m (\ell_k/n_k)^p P(C_k) + (\ell/n_{m+1})^p P(C \setminus \bigcup_{k=1}^m C_k) \leq \left(\sum_{k'=1}^d \ell_{k'}^d \right)^{\frac{p}{d}} (n/2)^{-\frac{p}{d}} \mathbf{P}(\cup_{1 \leq k \leq m} C_k) + (n/2)^{-\frac{p}{d}} \ell^p \mathbf{P}(C \setminus \bigcup_{k=1}^m C_k)$$

so that

$$\limsup_n n^{\frac{p}{d}} d_{n,p}^p(\mathbf{P}) \leq C_{\|\cdot\|} (\varepsilon/2)^{\frac{p}{d}}$$

which in turn implies by letting ε go to 0 that

$$\limsup_n n^{\frac{p}{d}} d_{n,p}^p(\mathbf{P}) = 0.$$

□

PROOF OF THEOREM 2: The assertion (a) follows directly from Propositions 12 and 13 and the fact that it holds $\bar{d}_{n,p}(X) \leq d_{n,p}(X)$ for every $n \in \mathbb{N}$. Furthermore, part (c) was derived in [8], Section 5.1. Hence, it remains to prove (b).

Proof. **STEP 1.** (Lower bound) If X is compactly supported, the assertion follows from Proposition 12. Otherwise, set for every $R \in (0, \infty)$,

$$C_R = [-R, R]^d$$

and for $k \in \mathbb{N}$

$$\mathbf{P}(\cdot|C_k) = \frac{h\mathbb{1}_{C_k}}{\mathbf{P}(C_k)}\lambda^d.$$

Proposition 12 yields again

$$\lim_{n \rightarrow \infty} n^{\frac{1}{d}} \bar{d}_{n,p}(\mathbf{P}(\cdot|C_k)) = Q_{\|\cdot\|,p,d}^{\text{dq}} \cdot \|h\mathbb{1}_{C_k}/\mathbf{P}(C_k)\|_{d/(d+p)}^{\frac{1}{p}}, \quad (14)$$

so that $\bar{d}_{n,p}^p(\mathbf{P}) \geq \mathbf{P}(\cdot|C_k)\bar{d}_{n,p}^p(\mathbf{P}(\cdot|C_k))$ implies for all $k \in \mathbb{N}$

$$\liminf_{n \rightarrow \infty} n^{\frac{1}{d}} \bar{d}_{n,p}(\mathbf{P}) \geq Q_{\|\cdot\|,p,d}^{\text{dq}} \cdot \|h\mathbb{1}_{C_k}\|_{d/(d+p)}^{\frac{1}{p}}.$$

Sending k to infinity, we arrive at

$$\liminf_{n \rightarrow \infty} n^{\frac{1}{d}} \bar{d}_{n,p}(\mathbf{P}) \geq Q_{\|\cdot\|,p,d}^{\text{dq}} \cdot \|h\|_{d/(d+p)}^{\frac{1}{p}}.$$

STEP 2 (*Upper bound*, $\text{supp}(\mathbf{P}) = \mathbb{R}^d$). Let $\rho \in (0, 1)$. Set $K = C_{k+\rho}$ and $K_\rho = C_k$. Let $\Gamma_{k,\alpha,\rho}$ be the lattice grid associated to $K \setminus K_\rho$ with edge $\alpha > 0$ as defined in the remark that follows the firewall lemma. It is straightforward that there exists a real constant $C > 0$ such that

$$\forall k \in \mathbb{N}, \forall \rho \in (0, 1), \forall \alpha \in (0, \rho) : |\Gamma_{\alpha,\rho}| \leq Cd\rho k^{d-1}\alpha^{-d}.$$

Let $\varepsilon \in (0, 1)$. For every $n \geq 1$, set $\alpha_n = \tilde{\alpha}_0 n^{-\frac{1}{d}}$ where $\tilde{\alpha}_0 \in (0, 1)$ is a real constant and

$$n_0 = |\Gamma_{k,\alpha_n,\rho}|, \quad n_1 = \lfloor (1 - \varepsilon)(n - n_0) \rfloor, \quad n_2 = \lfloor \varepsilon(n - n_0) \rfloor,$$

so that $\alpha_n \in (0, \rho)$, $n_0 + n_1 + n_2 \leq n$ and $n_i \geq 1$ for large enough n . For every $\xi \in K_\rho = C_k$, for every grid $\Gamma \subset \mathbb{R}^d$, we know by the firewall lemma that

$$F^p(\xi; (\Gamma \cap \overset{\circ}{K}) \cup \Gamma_{\alpha,\rho}) \leq (1 + \eta)^p F^p(\xi; \Gamma) + (1 + \eta)^p (1 + 1/\eta)^p C_{\|\cdot\|} \alpha^p.$$

Let $\Gamma_1 = \Gamma_1(n_1, k)$ be a n_1 quantizer such that $d_{n_1}^p(\mathbf{P}(\cdot|C_k); \Gamma_1) \leq (1 + \eta)d_{n_1}^p(\mathbf{P}(\cdot|C_k))$. Set $\Gamma'_1 = ((\Gamma_1 \cap \overset{\circ}{C}_{k+\rho}) \cup \Gamma_{k,\alpha_n,\rho})$. One has $\Gamma'_1 \subset C_{k+2\rho}$ for large enough n (so that $\alpha_n < \rho$). Let moreover $\Gamma_2 = \Gamma_2(n_2, k)$ be a n_2 quantizer such that $\bar{d}_{n_2}^p(\mathbf{P}(\cdot|C_k^c); \Gamma_2) \leq (1 + \eta)\bar{d}_{n_2}^p(\mathbf{P}(\cdot|C_k^c))$. For $n \geq n_\rho$, we may assume that $C_{k+2\rho} \subset \text{conv} \Gamma_2$ owing to Lemma 1 since $C_{k+2\rho} = \text{conv}(C_{k+2\rho} \setminus C_{k+\frac{3}{2}\rho})$ and $C_{k+2\rho} \setminus C_{k+\frac{3}{2}\rho} \subset \overbrace{\text{supp} \mathbf{P}(\cdot|C_k^c)}^{\text{supp} \mathbf{P}(\cdot|C_k^c)}$. As a consequence $\Gamma'_1 \subset \text{conv}(\Gamma_2)$ so that $\text{conv}(\Gamma'_1) \subset \text{conv}(\Gamma_2) = \text{conv}(\Gamma)$ where $\Gamma = \Gamma'_1 \cup \Gamma_2$ and

$$C_{k+\rho} \subset \text{conv}(\Gamma) = \text{conv}(\Gamma_2).$$

Now

$$\begin{aligned} \bar{d}_{n,p}^p(\mathbf{P}) &\leq \int_{C_k} \left(F^p(\xi; \Gamma) \mathbf{1}_{\{\xi \in \text{conv}(\Gamma_2)\}} + \underbrace{d(\xi, \Gamma)^p \mathbf{1}_{\{\xi \notin \text{conv}(\Gamma_2)\}}}_{=0} \right) d\mathbf{P}(\xi) \\ &\quad + \int_{C_k^c} (F^p(\xi; \Gamma) \mathbf{1}_{\{\xi \in \text{conv}(\Gamma_2)\}} + d(\xi, \Gamma)^p \mathbf{1}_{\{\xi \notin \text{conv}(\Gamma_2)\}}) d\mathbf{P}(\xi). \end{aligned}$$

Using that, for every $\xi \in C_k$,

$$\begin{aligned} F^p(\xi; \Gamma) &\leq F^p(\xi; \Gamma'_1) \\ &\leq (1 + \eta)^p \left(F^p(\xi; \Gamma_1) + (1 + 1/\eta)^p C_{\|\cdot\|} \alpha_n^p \right) \end{aligned}$$

implies

$$\begin{aligned} \bar{d}_{n,p}^p(\mathbf{P}) &\leq \mathbf{P}(C_k)(1+\eta)^p \left((1+\eta) d_{n_1,p}^p(\mathbf{P}(\cdot|C_k)) + (1+1/\eta)^p C_{\|\cdot\|} \tilde{\alpha}_0 n^{-\frac{1}{d}} \right) \\ &\quad + \mathbf{P}(C_k^c) (1+\eta) \bar{d}_{n_2,p}(\mathbf{P}(\cdot|C_k^c)). \end{aligned}$$

Consequently

$$\begin{aligned} n^{\frac{p}{d}} \bar{d}_{n,p}^p(\mathbf{P}) &\leq \mathbf{P}(C_k)(1+\eta)^p \left[(1+\eta) \left(\frac{n}{n_1} \right)^{\frac{p}{d}} n_1^{\frac{p}{d}} d_{n_1,p}^p(\mathbf{P}(\cdot|C_k)) + (1+1/\eta)^p C_{\|\cdot\|} \tilde{\alpha}_0 \right] \\ &\quad + (1+\eta) \left(\frac{n}{n_2} \right)^{\frac{p}{d}} \mathbf{P}(C_k^c) n_2^{\frac{p}{d}} \bar{d}_{n_2,p}(\mathbf{P}(\cdot|C_k^c)) \end{aligned}$$

which in turn implies, using the d -dimensional version of the extended Pierce Lemma (Proposition 6),

$$\begin{aligned} \limsup_n n^{\frac{p}{d}} \bar{d}_{n,p}^p(\mathbf{P}) &\leq \mathbf{P}(C_k)(1+\eta)^p \left(\left(\frac{(1+\eta)^{-p/d}}{(1-\varepsilon)(1-Cd\rho k^{d-1}\tilde{\alpha}_0^{-d})} \right)^{\frac{p}{d}} Q_{\|\cdot\|}^{dq} \|h\mathbf{1}_{C_k}\|_{L^{\frac{d}{d+p}}} \right. \\ &\quad \left. + (1+1/\eta)^p C_{\|\cdot\|} \tilde{\alpha}_0 \right) \\ &\quad + \mathbf{P}(C_k^c) (1+\eta) C_{p,d} \|X\mathbf{1}_{\{X \in C_k^c\}}\|_{L^{p+\delta}}^p \left(\frac{1}{\varepsilon(1-Cd\rho k^{d-1}\tilde{\alpha}_0^{-d})} \right)^{\frac{p}{d}}. \end{aligned}$$

One concludes by letting successively ρ , $\tilde{\alpha}_0$ and η go to 0, sending $k \rightarrow \infty$ and finally ε to 0. **STEP 3.** (Upper bound: general case). Let $\rho \in (0, 1)$. Set $\mathbf{P}_\rho = \rho\mathbf{P} + (1-\rho)\mathbf{P}_0$ where $\mathbf{P}_0 = \mathcal{N}(0; I_d)$ (d -dimensional normal distribution). It is clear from the very definition of $\bar{d}_{n,p}$ that $\bar{d}_{n,p}(\mathbf{P}) \leq \frac{1}{\rho} \bar{d}_{n,p}(\mathbf{P}_\rho)$ since $\mathbf{P} \leq \frac{1}{\rho} \mathbf{P}_\rho$. The distribution \mathbf{P}_ρ has $h_\rho = \rho h + (1-\rho)h_0$ (with obvious notations) and one concludes by noting that

$$\lim_{\rho \rightarrow 0} \|h_\rho\|_{d/(d+p)} = \|h\|_{d/(d+p)}$$

owing to the Lebesgue dominated convergence Theorem. \square

PROOF OF PROPOSITION 1:

Proof. Using Hoelder's inequality one easily checks that for $0 \leq r \leq p$ and $x \in \mathbb{R}^d$ it holds

$$|x|_{\ell^r} \leq d^{\frac{1}{r}-\frac{1}{p}} |x|_{\ell^p}.$$

Moreover, for $m \in \mathbb{N}$ set $n = m^d$ and let Γ' be an optimal quantizer for $d_{m,p}(\mathcal{U}([0, 1]))$ (or at least $(1+\varepsilon)$ -optimal for $\varepsilon > 0$). Denoting $\Gamma = \prod_{i=1}^d \Gamma'$, it then follows from Proposition 4 that

$$n^{\frac{p}{d}} d_n^p(\mathcal{U}([0, 1]^d)) \leq n^{\frac{p}{d}} d^p(\mathcal{U}([0, 1]^d); \Gamma) = m^p \sum_{i=1}^d d^p(\mathcal{U}([0, 1]); \Gamma') = d m^p d_m^p(\mathcal{U}([0, 1])).$$

Combining both results and reminding that $Q_{\|\cdot\|, p, d}^{\text{dq}}$ holds as an infimum, we arrive for $0 \leq r \leq p$ at

$$(Q_{|\cdot|_{\ell^r}, p, d}^{\text{dq}})^p \leq d^{\frac{p}{r}-1} n^{\frac{p}{d}} d_{n, |\cdot|_{\ell^r}}^p(\mathcal{U}([0, 1]^d)) \leq d^{\frac{p}{r}} m^p d_m^p(\mathcal{U}([0, 1])),$$

which finally proves the assertion by sending $m \rightarrow \infty$. \square

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