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Constructible ∇ -modules on curves

Bernard Le Stum*

Version of December 15, 2010

Abstract

Let \mathcal{V} be a discrete valuation ring of mixed characteristic with perfect residue field. Let X be a geometrically connected smooth proper curve over \mathcal{V} . We introduce the notion of constructible convergent ∇ -module on the analytification X_K^{an} of the generic fibre of X . A constructible module is an $\mathcal{O}_{X_K^{\text{an}}}$ -module which is not necessarily coherent, but becomes coherent on a stratification by locally closed subsets of the special fiber X_k of X . The notions of connection, of (over-) convergence and of Frobenius structure carry over to this situation. We describe a specialization functor from the category of constructible convergent ∇ -modules to the category of $\mathcal{D}_{\hat{X}, \mathbf{Q}}^\dagger$ -modules. We show that if X is endowed with a lifting of the absolute Frobenius of \hat{X} , then specialization induces an equivalence between constructible F - ∇ -modules and perverse holonomic F - $\mathcal{D}_{\hat{X}, \mathbf{Q}}^\dagger$ -modules.

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Introduction

When we look for a category of coefficients for a cohomological theory, we usually get one which is too small inside another one which is too big and we aim at the perfect one that will sit in between.

For example, if we are interested in the singular cohomology of an algebraic variety over \mathbf{C} , we have on one side the category of local systems of finite dimensional vector spaces and on the other side the category of all sheaves of vector spaces. The perfect category will be the category of constructible sheaves. Similarly, for de Rham cohomology, we have on one side the category of coherent modules endowed with an integrable connection and on the other side the category of all \mathcal{D} -modules. The perfect one will be the category of regular holonomic \mathcal{D} -modules. Last, we may consider the category of finitely presented crystals and the category of all modules on the infinitesimal site. In between, we have the category of constructible pro-crystals (unpublished note of Deligne ([13])). We have equivalences of categories between local systems, modules with an integrable connection and finitely presented crystals (we need to add some regularity conditions when moving from analytic to algebraic side). At the derived category level, we also have an equivalences between constructible sheaves, regular holonomic \mathcal{D} -modules and constructible pro-crystals ([15] and [13]).

If we are interested in the p -adic cohomology of a variety of characteristic $p > 0$, there is no analog to the first theory. The closest would be the étale p -adic cohomology which is not satisfying. There is a very good analog to the second theory, which is the theory of arithmetic \mathcal{D} -modules (and overconvergent isocrystals) developed by Berthelot, Virrion, Huyghe, Trihan, Caro and others (see [7] for an overview). More recently, I started to develop a crystalline theory in [19]. I showed that the category of overconvergent isocrystals is equivalent to the category of finitely presented crystals on the overconvergent site. When I was visiting the university of Tokyo in october 2009, Shiho suggested that we should look for the perfect category in this new theory, and prove that it is equivalent to the one developed by Berthelot (an *overconvergent Deligne-Kashiwara correspondence*). I had a very naive idea of what this category of constructible overconvergent crystals should be and gave a lecture on this topic in March 2010 at Oxford University. The aim of this article is to show that the strategy works for curves.

More precisely, if X is a smooth proper curve over a complete discrete valuation ring \mathcal{V} of mixed characteristic p with fraction field K and perfect residue field k , we introduce the notion of *constructible convergent ∇ -module* E on X_K^{an} and show that its specialization $\text{R}\widetilde{\text{sp}}_* E$ is a *perverse* (complex of) $\mathcal{D}_{X_{\mathbf{Q}}}^{\dagger}$ -module. When there is a Frobenius on X , we show that specialization induces an equivalence between *constructible F - ∇ -modules* on X_K^{an} and perverse holonomic F - $\mathcal{D}_{X_{\mathbf{Q}}}^{\dagger}$ -modules. It should be remarked that the overconvergent isocrystals on an open subset U of X_k form a full subcategory of the category of all constructible convergent ∇ -modules on X_K^{an} and that our theorems extend Berthelot's equivalence theorems for specialization of (over-) convergent isocrystals.

Our definitions are very general and apply to any formal \mathcal{V} -scheme. A constructible overconvergent ∇ -module is an (not necessarily coherent) $\mathcal{O}_{X_K^{\text{an}}}$ -module E with a convergent (automatically integrable here) connection and we require that there exists a finite covering of the special fibre X_k by locally closed subset Y such that, if $i_Y : Y \hookrightarrow X_K^{\text{an}}$ denotes the inclusion map, then $i_Y^{-1} E$ is a coherent $i_Y^{-1} \mathcal{O}_{X_K^{\text{an}}}$ -module. In this definition, we view X_K^{an} as a Berkovich analytic space but we define $\text{R}\widetilde{\text{sp}}_* E := \text{R}\text{sp}_* E_G$ as the derived specialization

for the Tate topology.

We can give an explicit description of constructible convergent ∇ -modules and their specialization. If D is a smooth relative divisor on X with affine open complement $\text{Spec } A$, we can consider A_K^\dagger which is the generic fiber of the weak completion of A , and for each $a \in D_K$, the Robba ring \mathcal{R}_a on $K(a)$. Then, a constructible convergent ∇ -module is equivalent to the following data for some sufficiently big divisor D : an overconvergent ∇ -module M (of finite type) over A_K^\dagger and for each $a \in D_K$, a finite dimensional $K(a)$ -vector space H_a and a horizontal A_K^\dagger -linear map

$$M \rightarrow \mathcal{R}_a \otimes_{K(a)} H_a.$$

Now, we can describe $\text{R}\widetilde{\text{sp}}_* E$ as follows. Let $\mathcal{U}' = \text{Spf } \mathcal{A}'$ be an affine open subset of \hat{X} . First of all, if $\mathcal{D}' = \mathcal{U}' \cap \hat{D}$ is defined in \mathcal{U}' by an equation f , we may consider the weak completion $\mathcal{A}'[1/f]^\dagger$. Also, we can always write each \mathcal{R}_a as an extension of a $K(a)$ -vector space δ_a by the ring $\mathcal{O}_a^{\text{an}}$ of convergent functions on the open unit disc. We extend scalars on the left, project on the right and add all the above maps in order to get the complex

$$\mathcal{A}'[1/f]^\dagger \otimes_{A_K^\dagger} M \rightarrow \bigoplus_{a \in \mathcal{D}'_K} \delta_a \otimes_{K(a)} H_a.$$

This is $\text{R}\widetilde{\text{sp}}_* E$ on \mathcal{U}' . It is naturally a complex of $\mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger$ -modules on \mathcal{U}' . It is perverse in the sense that it has flat cohomology in degree 0, finite support in degree 1 and no other cohomology.

As already mentioned, we use Berkovich theory of ultrametric geometry (see [1] and [2] or [14]) and will try to recall the basic constructions when we meet them. We will generally avoid reference to basic results proved in the context of rigid analytic geometry and rather reprove them here in this new language when necessary. Anyway, the reader should keep in mind that a significant part of what follows is merely a reorganization of some of Berthelot's material using Berkovich theory instead of Tate's.

Note that I will use the terminology *finite module* for finitely presented module and *finite torsion module* if we ever have to consider a module which is finite as set. I will call ∇ -module a module endowed with a (integrable) connection. We denote with an upper $*$ the inverse image for a morphism of ringed spaces when the upper -1 is used for inverse image for a morphism of topological spaces. Also, we will only consider real numbers η inside $\sqrt{|K|}$ where K is our fixed ultrametric field. Finally, I will systematically identify a sheaf on a topological space reduced to one point with its global sections.

Many thanks to Daniel Caro and Pierre Berthelot who helped me a lot when I was struggling with arithmetic \mathcal{D} -modules problems. I am indebted too to Florian Ivorra who helped me clarify some questions related to adjunction of derived functors. I also recall that the problem which is solved here was pointed out by Atsushi Shiho.

1 Constructible modules

We fix a complete ultrametric field K with ring of integers \mathcal{V} , maximal ideal \mathfrak{m} and residue field k . We will assume that K is non trivial and fix some $\pi \in K$ with $0 < |\pi| < 1$. We will also assume that K has characteristic zero, that k has positive characteristic p , that the valuation is discrete and that k is perfect. When Frobenius enters the game, we also fix an isometry σ on K that lifts the absolute Frobenius of k .

We let X be a geometrically connected proper smooth \mathcal{V} -scheme of relative dimension one.

We may consider its special fiber X_k as well as its generic fiber X_K which are proper smooth geometrically connected curves and we will be particularly interested in the *Berkovich* analytification X_K^{an} of X_K . For the moment, we will simply use the fact that, since X is proper, we can identify X_K^{an} with the generic fiber \widehat{X}_K of the completion of X . In particular, X_K^{an} is covered (for Tate topology) by the affinoid domains $\mathcal{U}_K = \mathcal{M}(\mathcal{A}_K)$ (set of continuous multiplicative semi-norms) if \widehat{X} is covered by some affine subsets $\mathcal{U} = \text{Spf } \mathcal{A}$ (set of open primes).

We will consider the specialization map

$$\text{sp} : X_K^{\text{an}} = \widehat{X}_K \rightarrow \widehat{X} \simeq X_k$$

(where the last morphism is a homeomorphism). Locally, the specialization of a point $x \in \mathcal{U}_K$ is the open prime

$$\mathfrak{p} = \{f \in \mathcal{A}, \quad |f(x)| < 1\} \in \text{Spf } \mathcal{A} = \mathcal{U}$$

(see section 1 of [3] for the general construction). Note that if x is a semi-norm on a ring and f a function in this ring, we write $|f(x)|$ instead of $x(f)$.

When Y is a subset of X_k , we will consider its *tube*

$$]Y[:= \text{sp}^{-1}(Y) \subset X_K^{\text{an}}.$$

This is mostly used when Y is locally closed, in which case $]Y[$ is an analytic domain in X_K^{an} . We will give an explicit description below, but it should first be remarked that, since X_k is a connected curve, any locally closed subset Y of X_k is necessarily open or closed. Moreover, any open subset $U \subsetneq X_k$ is affine. Also, any closed subset $Z \subsetneq X_k$ is a finite set of closed points.

When U is open in X_k , then $]U[$ is a *closed* subset of X_K^{an} . More precisely, if \mathcal{U} is the formal lifting of U , then $]U[= \mathcal{U}_K$ (which is affinoid and therefore compact). Conversely, if Z is a closed subset of X_k , then $]Z[$ is an *open* subset of X_K^{an} : more precisely, if Z is defined inside some formal affine open subset \mathcal{U} by an equation $f = 0 \pmod{\mathfrak{m}}$, then $]Z[$ is defined in \mathcal{U}_K by $|f(x)| < 1$ (being open is local for Tate topology). It will also be convenient to consider the tubes $]Z]_\eta$ and $]Z[_\eta$ of radius $\eta < 1$ defined by $|f(x)| \leq \eta$ and $|f(x)| < \eta$ respectively (they are independent of the choice of f when η is close to 1).

The specialization map $\text{sp} : X_K^{\text{an}} \rightarrow X_k$ is surjective and may be used to give a description of X_K^{an} . First of all, there is only one point above the generic point ξ of X_k that we shall therefore denote by $]\xi[$ and call the *generic point* of X_K^{an} even if it depends on the model. Actually, if $U \subset X_k$ is a non empty affine subset, then $]\xi[$ is the *Shilov boundary* of $]U[$ (the point where any function reaches its maximum). It is also the usual boundary of $]U[$ as subset of the topological space X_K^{an} and the absolute boundary of $]U[$ in Berkovich sense. For example, in the case of $X = \mathbf{P}_\mathcal{V}^1$, then $]\xi[$ is the Gauss norm.

By definition, $X_K^{\text{an}} \setminus]\xi[$ is the disjoint union of all open subsets $]x[$ for x a closed point in X_k . These open subsets $]x[$ are called the *residual classes*. In other words, the generic point is used to connect the residual classes together (recall that X_K^{an} is simply connected). We can also mention that $\cap]U[=]\xi[$ when U runs through all the non empty (affine) open subsets of X_k .

Before giving an explicit description of the residual classes, recall that there exists also a canonical map $X_K^{\text{an}} \rightarrow X_K$. If $\text{Spec } A \subset X$ is an open subset, the analytification of

$\text{Spec } A_K$ is the set $\mathcal{M}^{\text{alg}}(A_K)$ of all multiplicative semi-norms x on A and x is sent to its kernel

$$\mathfrak{p} = \{f \in A, \quad |f(x)| = 0\} \in \text{Spec } A_K.$$

The map $X_K^{\text{an}} \rightarrow X_K$ induces a bijection between the rigid points of X_K^{an} (whose residue fields are finite over K) and the closed points of X_K (note that all the non rigid points are then sent to the generic point of X_K). We will implicitly identify the closed points of X_K with the rigid point of X_K^{an} .

We call a closed point $a \in X_K$ a lifting of a closed point $x \in X_k$ if $\text{sp}(a) = x$. Note that $K(a)$ is a finite extension of K and inherits a structure of complete ultrametric field whose ring of integers will be denoted $\mathcal{V}(a)$. The point a is said to be *unramified* if $K(a)/K$ is an unramified extension (i.e $\mathcal{V}(a)$ is étale over \mathcal{V}).

Here is a standard result on residual classes:

Proposition 1.1 *If $x \in X_k$ is a closed point and a is an unramified lifting of x , then $]x[_\eta \simeq \mathbf{D}_{K(a)}(0, 1^-)$. Actually, we have $]x]_\eta \simeq \mathbf{D}_{K(a)}(0, \eta^+)$ for $\eta < 1$ and close to 1.*

Proof: This follows from Berthelot's weak fibration theorem (see lemma 4.4 of [4]) as we can now recall. When x is a rational point, we can write x as the unique zero of an étale map

$$t : \mathcal{U} = \text{Spf } \mathcal{A} \rightarrow \widehat{\mathbf{A}}_{\mathcal{V}}^1.$$

The corresponding morphism $\mathcal{U}_K \rightarrow \mathbf{D}_K(0, 1^+)$ will induce an isomorphism $]x]_\eta \simeq \mathbf{D}(0, \eta^+)$. In general, we first have to extend the basis and consider the projection $X_{\mathcal{V}(a)} \rightarrow X$. We pick up some point y over x . Then, the corresponding morphism $X_{K(a)}^{\text{an}} \rightarrow X_K^{\text{an}}$ will induce an isomorphism $]y]_\eta \simeq]x]_\eta$. \square

We are finished with the study of finite closed subsets and we consider now the case of a non empty affine open subset $U \subset X_k$. We may always lift *non canonically* U to an affine open subset $\text{Spec } A \subset X$. Since k is perfect, we may even assume that $\text{Spec } A$ is the open complement of (the support of) a smooth (relative) divisor $D \subset X$. In other words, the complement of U in X_k is $D_k =: \{x_1, \dots, x_n\}$ and the complement of $\text{Spec } A_K$ in X_K is $D_K = \{a_1, \dots, a_n\}$ where each a_i is an unramified lifting of x_i . In this case, choosing A amounts to fixing the “center” a_i of each disc $]x_i[_$.

Actually, we will only use the weak completion A^\dagger of A and more precisely its generic fiber A_K^\dagger . In fact, if Z denotes the closed complement of U , the subsets $V_\lambda := X_K^{\text{an}} \setminus]Z[_\lambda$ form a cofinal family of affinoid neighborhoods of $]U[_$ and $A_K^\dagger = \varinjlim A_\lambda$ if we write $V_\lambda =: \mathcal{M}(A_\lambda)$. In particular, the algebra A_K^\dagger does not depend on the algebraic lifting but only on U . However, the geometric construction can be useful and we will stick to the original definition.

It is also easy to see that the subsets V_λ for various λ and U form a cofinal family of affinoid neighborhoods of the generic point $] \xi[_$. Actually, since X_K^{an} is compact, any open neighborhood V of $] \xi[_$ has a compact complement $C \subset \cup]x[_ = \cup]x]_\lambda$ and therefore, $C \subset \cup_{\text{finite}}]x]_\lambda =]T[_\lambda$. It follows that $V_\lambda := X_K^{\text{an}} \setminus]T[_\lambda \subset V$. In particular, we have

$$\mathcal{O}_{X_K^{\text{an}},] \xi[_} = \varinjlim A_K^\dagger$$

when $\text{Spec } A$ runs through the non empty affine open subsets of X . Note that $\mathcal{O}_{X_K^{\text{an}},] \xi[_}$ is a henselian field and that $|f(] \xi[_)| = \|f\|_{]U[_}$ if $f \in A_K^\dagger$ and $\text{Spec } A$ is a lifting of U .

If Y is a subset of X_k , we will denote by

$$i_Y :]Y[\hookrightarrow X_K^{\text{an}}$$

the corresponding inclusion. Recall that i_U is a *closed* embedding when U is an open subset in X_k and that i_Z is an *open* embedding when Z is a closed subset of X_k .

Definition 1.2 *An $\mathcal{O}_{X_K^{\text{an}}}$ -module E is constructible (resp. constructible-free) if there exists a finite covering of X_k by locally closed subsets Y such that $i_Y^{-1}E$ is a coherent (resp. free) $i_Y^{-1}\mathcal{O}_{X_K^{\text{an}}}$ -module.*

This definition makes sense in a very general situation but we really want to stick to curves here in order to give an explicit description of these constructible modules. Nevertheless, the following is purely formal:

Proposition 1.3 *Constructible (resp. constructible-free) modules form an abelian (resp. additive) subcategory stable under internal Hom, extensions and tensor product.*

Proof: Since the restriction maps $E \mapsto i_Y^{-1}E$ are exact and commute with internal hom and tensor product, all these assertions follow from the analogous results on coherent (resp. free) modules. \square

Note that if $Y' \subset Y$ and $i_Y^{-1}E$ is coherent, free or locally free, so is $i_{Y'}^{-1}E$ by restriction. It should also be remarked that if $Z \subset X_k$ is a closed subset, then $i_Z^{-1}E = E|_{]Z[}$ is the usual restriction to an open subset. However, when $U \subset X_k$ is open, then i_U is a closed embedding and $i_U^{-1}E$ is the sheaf of sections defined in the neighborhood of the compact subset $]U[$. More on this soon.

At some point, we will need to use some theorems that were formulated in the language of rigid analytic geometry. It will therefore be necessary to have a dictionary at our disposal. We want to explain this right now. We may consider the Grothendieck-Tate topology on X_K^{an} : admissible open subsets are analytic domains and coverings are *Tate* covering (see for example [2], page 25-26). If V is an analytic domain of X_K^{an} , we will denote by V_G the associated Grothendieck space. There is an obvious continuous map of Grothendieck spaces $\pi_V : V_G \rightarrow V$ which is simply the identity on underlying sets. Actually, π_V^* is exact and fully faithful and induces an equivalence on coherent sheaves. We will write $E_G := \pi_V^*E$. Finally, the set V^{rig} of rigid points of V has a natural structure of rigid analytic variety and the topos of V^{rig} is equivalent to the topos of V_G (affinoid domains and affinoid coverings coincide). We will denote by E_0 the rigid sheaf corresponding to E_G .

The next comparison theorem was proved in corollary 2.2.13 of [19] but it will not be long to recall how it works.

Proposition 1.4 *If U is an open subset of X_k , the functor $E \mapsto (i_{U*}E)_0$ induces an equivalence between the category of coherent $i_U^{-1}\mathcal{O}_{X_K^{\text{an}}}$ -modules on $]U[$ and the category of coherent $j^{\dagger}\mathcal{O}_{X_K^{\text{rig}}}$ -modules.*

Proof: If Z is the closed complement of U and $\lambda < 1$, we consider the affinoid domain $V_\lambda = X_K^{\text{an}} \setminus]Z[_\lambda$. Since X_K^{an} is locally compact, the category of coherent $i_U^{-1}\mathcal{O}_{X_K^{\text{an}}}$ -modules

is equivalent to the direct limit of the category of coherent \mathcal{O}_{V_λ} -modules (see corollary 2.2.5 of [19] for example). And we also know that the category of coherent $j^\dagger \mathcal{O}_{X_K^{\text{rig}}}$ -modules is equivalent to the direct limit of the category of coherent $\mathcal{O}_{V_\lambda^{\text{rig}}}$ -modules (theorem 5.4.4 of [18]). \square

In general, we will not use this comparison theorem because most results are a lot simpler to prove in the Berkovich topology and it sounds unnatural to use the rigid topology in order to derive them. For example, we have the following (so called theorem A and B) which is essentially proposition 5.4.8 of [18]:

Proposition 1.5 *If U is an affine open subset of X_k , then $i_U^{-1} \mathcal{O}_{X_K^{\text{an}}}$ is a coherent ring. Moreover, if $\text{Spec } A \subset X$ is an algebraic lifting of U , then*

$$\text{R}\Gamma(\text{]U[}, i_U^{-1} \mathcal{O}_{X_K^{\text{an}}}) = A_K^\dagger.$$

Finally, the functor $\text{R}\Gamma(\text{]U[}, -)$ induces an equivalence between coherent (resp. locally free, resp. free) $i_U^{-1} \mathcal{O}_{X_K^{\text{an}}}$ -modules and finite (resp. finite projective, resp. finite free) A_K^\dagger -modules.

Proof: Since]U[is a compact subset of X_K^{an} which is locally compact and $\mathcal{O}_{X_K^{\text{an}}}$ is coherent, the first assertion is purely formal (see for example proposition 2.2.3 of [19]). For the same reason, any coherent $i_U^{-1} \mathcal{O}_{X_K^{\text{an}}}$ -module E is the restriction of a coherent module on some neighborhood V of]U[in X_K^{an} and we get an equivalence of categories if we allow the shrinking of V (see corollary 2.2.5 of [19]). As mentioned above, there exists a cofinal family of affinoid neighborhoods $V_\lambda = \mathcal{M}(A_\lambda)$ with $A_K^\dagger = \varinjlim A_\lambda$. The rest of the proposition follows then immediately from the analogous results on affinoid varieties and the properties of filtered direct limits. \square

Lemma 1.6 *If E is a constructible (resp. constructible free) module and $Z \subset X_k$ is a finite closed subset, then $i_Z^{-1} E$ is coherent (resp. locally free).*

Proof: We may assume that Z is reduced to one point x . Then, there exists a locally closed subset $Y \subset X_k$ with $x \in Y$ and $i_Y^{-1} E$ coherent (resp. free) on]Y[. We may then restrict to]x[. \square

Proposition 1.7 *If E is a constructible module, there exists a non-empty open subset $U \subset X$ such that $i_U^{-1} E$ is a free $i_U^{-1} \mathcal{O}_{X_K^{\text{an}}}$ -module.*

Proof: By definition, if ξ is the generic point of X_k , there exists a locally closed subset $U \subset X_k$, which is necessarily open, such that $\xi \in U$ and $i_U^{-1} E$ is a coherent $i_U^{-1} \mathcal{O}_{X_K^{\text{an}}}$ -module. It follows that the stalk $E_{\text{]}\xi[}$ of E at $\text{]}\xi[$ is a finite dimensional vector space (recall that $\mathcal{O}_{X_K^{\text{an}}}$ is a field). At this point, it is convenient to write $i_U^{-1} E = i^{-1} \mathcal{F}$ where $i : \text{]U[} \hookrightarrow V$ if the inclusion of]U[in some neighborhood and \mathcal{F} is a coherent \mathcal{O}_V -module. Since $\mathcal{F}_{\text{]}\xi[} = E_{\text{]}\xi[}$ is a finite dimensional vector space and \mathcal{F} is coherent, we know that there exists a neighborhood of ξ in V on which \mathcal{F} becomes free. We saw above that the subsets $V_\lambda = X_K^{\text{an}} \setminus \text{]T[}_\lambda$ with T finite closed and $\lambda < 1$ form a cofinal family of neighborhoods of $\text{]}\xi[$. Therefore, there exists such a T and λ with $\mathcal{F}_{|V_\lambda}$ free. We may then remove a finite number of closed points in U and assume that $U \cap T = \emptyset$ in order to get $i_U^{-1} E$ free. \square

Corollary 1.8 *Let E be a constructible module. Then, the following are equivalent:*

1. E is constructible-free
2. $i_Z^{-1}E$ is locally free whenever $Z \subset X_k$ is a finite closed subset.
3. $i_x^{-1}E$ is free whenever $x \in X_k$ is a closed point

Proof: We know from proposition 1.1 that if $x \in X_k$ is any closed point, then $]x[$ is a disc. Moreover, since the valuation is discrete, a coherent locally free module on a disk is automatically free. It follows that assertion 2 and 3 are equivalent. And we showed in lemma 1.6 that they are automatically fulfilled when E is constructible-free. Conversely, since E is constructible, we know from the proposition that there exists a non-empty open subset $U \subset X_k$ such that $i_U^{-1}E$ is free. If we assume that assertion 3 holds, then, in particular, for all $x \notin U$, $i_x^{-1}E$ is free. Now, U and the points $x \notin U$ form a finite covering of X_k by locally closed subsets. \square

Corollary 1.9 *An $\mathcal{O}_{X_K^{\text{an}}}$ -module E is constructible-free if and only if there exists a finite covering of X_k by locally closed subsets Y such that $i_Y^{-1}E$ is a locally free $i_Y^{-1}\mathcal{O}_{X_K^{\text{an}}}$ -module.*

Proof: The condition is clearly necessary. Conversely, any such module is constructible and we may apply the previous corollary. \square

Corollary 1.10 *An $\mathcal{O}_{X_K^{\text{an}}}$ -module E is constructible (resp. constructible-free) if and only if there exists a non-empty affine open subset $U \subset X_k$ with closed complement Z such that $i_U^{-1}E$ is a coherent (resp. locally free) $i_U^{-1}\mathcal{O}_{X_K^{\text{an}}}$ -module and $i_Z^{-1}E$ is coherent (resp. locally free). Moreover, we may assume that $i_U^{-1}E$ is free.*

Proof: It follows from the previous corollary that the condition is necessary. Conversely, if E is constructible, we can find such a U thanks to the proposition and then apply lemma 1.6 to the complement Z of U in X_k . \square

Corollary 1.11 *Let E be a constructible-free module and $Y \subset X_k$ a locally closed subset. If $i_Y^{-1}E$ is coherent, it is necessarily locally free.*

Proof: If $a \in]Y[$ specializes to $x \in X_k$, then $a \in]x[$ and we know that $i_x^{-1}E$ is free. It follows that the stalk E_a of E at a is free. A coherent module whose stalks are free is necessarily locally free. \square

If $Z \subset X_k$ is a closed subset, then $i_Z :]Z[\hookrightarrow X_K^{\text{an}}$ is an open immersion and we may consider the extension by zero $i_{Z!}$ outside Z which is a left adjoint functor to i_Z^{-1} .

Proposition 1.12 *An $\mathcal{O}_{X_K^{\text{an}}}$ -module E is constructible (resp. constructible free) if and only if there exists an exact sequence*

$$0 \rightarrow i_{Z!}E_Z \rightarrow E \rightarrow i_{U*}E_U \rightarrow 0$$

where U is a non-empty affine open subset of X_k with closed complement Z , E_U is a coherent (resp. locally free) $i_U^{-1}\mathcal{O}_{X_K^{\text{an}}}$ -module and E_Z is coherent (resp. locally free). Moreover, we may assume that $i_U^{-1}E$ is free.

Proof: If we are given such an exact sequence, we may pull back along i_U and i_Z in order to obtain $E_U \simeq i_U^{-1}E$ and $E_Z \simeq i_Z^{-1}E$. And conversely, using corollary 1.10, we can find a non-empty affine open subset $U \subset X_k$ with closed complement Z such that $i_U^{-1}E$ is a coherent (resp. locally free) $i_U^{-1}\mathcal{O}_{X_k^{\text{an}}}$ -module and $i_Z^{-1}E$ is coherent (resp. locally free). There is an exact sequence

$$0 \rightarrow i_Z i_Z^{-1}E \rightarrow E \rightarrow i_{U*} i_U^{-1}E \rightarrow 0$$

and we can set $E_U := i_U^{-1}E$ and $E_Z := i_Z^{-1}E$. \square

The notion of constructibility is stable under pull back by the very definition but we will only use the following particular case:

Proposition 1.13 *Assume that $F : X \rightarrow X$ is a σ -linear lifting of the absolute Frobenius of X_k . If E is a constructible module on X_K^{an} , then $F_K^{\text{an}*}E$ is also constructible.*

Proof: Follows from the fact that coherence is stable under pull-back. \square

2 Classification

Recall that if Y is a locally closed subset of X_k , we denote by $i_Y :]Y[\hookrightarrow X_K^{\text{an}}$ the inclusion map. Recall also that ξ denotes the generic point of X_k .

Definition 2.1 *Let $T \subset X_k$ be a non empty finite closed subset. If \mathcal{F} is a coherent $\mathcal{O}_{]T[}$ -module, we will write $\mathcal{O}^{\text{an}}(\mathcal{F}) := \Gamma(]T[, \mathcal{F})$. The Robba module of \mathcal{F} is*

$$\mathcal{R}(\mathcal{F}) := (i_{T*}\mathcal{F})_{] \xi[}$$

and the Dirac space of \mathcal{F} is defined by the short exact sequence

$$0 \longrightarrow \mathcal{O}^{\text{an}}(\mathcal{F}) \longrightarrow \mathcal{R}(\mathcal{F}) \longrightarrow \delta(\mathcal{F}) \longrightarrow 0.$$

We will write $\mathcal{O}_T^{\text{an}} := \Gamma(]T[, \mathcal{O}_{]T[})$ and call

$$\mathcal{R}_T := (i_{T*}\mathcal{O}_{]T[})_{] \xi[} \quad \text{and} \quad \delta_T := \mathcal{R}_T / \mathcal{O}_T^{\text{an}}$$

the Robba ring and the Dirac space, respectively, of T .

Proposition 2.2 *If x is a closed point of X_k with unramified lifting a , then \mathcal{R}_x is (isomorphic to) the usual Robba ring \mathcal{R}_a over $K(a)$. In general, we have*

$$\mathcal{R}_T(\mathcal{F}) = \varinjlim \Gamma(]T[\setminus]T[_{\lambda}, \mathcal{F}).$$

Proof: As already mentioned, the subsets $X_K^{\text{an}} \setminus]Z[_{\lambda}$ form a cofinal family of affinoid neighborhoods of $] \xi[$ when Z runs through the finite subsets of X_k and $\lambda < 1$. It follows that

$$\mathcal{R}_T(\mathcal{F}) = (i_{T*}\mathcal{F}_{]T[})_{] \xi[} = \varinjlim \Gamma(X_K^{\text{an}} \setminus]Z[_{\lambda}, i_{T*}\mathcal{F}_{]T[}) = \varinjlim \Gamma(]T[\setminus]T[_{\lambda}, \mathcal{F}).$$

In particular, we obtain

$$\mathcal{R}_x = \varinjlim \Gamma(]x[\setminus]x[_{\lambda}, \mathcal{O}_{]x[}).$$

Since $]x[\simeq \mathbf{D}_{K(a)}(0, 1^-)$ and $]x[_\lambda \simeq \mathbf{D}_{K(a)}(0, \lambda^-)$, this is the (usual) Robba ring \mathcal{R}_a over $K(a)$. \square

If T is a non empty finite closed subset of X_k , the adjunction map

$$\mathcal{O}_{X_K^{\text{an}}} \rightarrow i_{T*} i_T^{-1} \mathcal{O}_{X_K^{\text{an}}}$$

will induce on the stalks a canonical morphism $\mathcal{O}_{X_K^{\text{an}},]\xi[} \rightarrow \mathcal{R}_T$. In particular, if U is a non empty affine open subset of X_k with algebraic lifting $\text{Spec } A$, there is a canonical morphism

$$A_K^\dagger \rightarrow \mathcal{O}_{X_K^{\text{an}},]\xi[} \rightarrow \mathcal{R}_T.$$

Definition 2.3 *Let T be a non empty finite closed subset of X_k . If E is a constructible module on X_K^{an} , the Robba fiber of E at T is*

$$R_T(E) := \mathcal{R}_T \otimes_{\mathcal{O}_{X_K^{\text{an}},]\xi[}} E_{] \xi[}.$$

Let U be a non empty affine open subset of X_k with algebraic lifting $\text{Spec } A$ and M a finite A_K^\dagger -module. Then, the Robba fiber of M at T is

$$R_T(M) := \mathcal{R}_T \otimes_{A_K^\dagger} M.$$

Note that the Robba fiber is a free module (of finite rank) because $\mathcal{O}_{X_K^{\text{an}},]\xi[}$ is a field. Note also that if $\text{Spec } B$ is some affine open subset of $\text{Spec } A$, then

$$\mathcal{R}_T(B_K^\dagger \otimes_{A_K^\dagger} M) = \mathcal{R}_T(M).$$

Finally, if $i_U^{-1}E$ is coherent and $M = \Gamma(]U[, i_U^{-1}E)$, then $\mathcal{R}_T(M) = \mathcal{R}_T(E)$. Be careful however that $\mathcal{R}_T(E) \neq \mathcal{R}(E_{]T[})$ in general (unless E is coherent) but we have the following:

Lemma 2.4 *Let U be a non empty affine open subset of X_k with algebraic lifting $\text{Spec } A$ and T be a non empty finite closed subset of U . Let M be a finite A_K^\dagger -module and M_T the restriction to $]T[$ of the corresponding $i_U^{-1}\mathcal{O}_{X_K^{\text{an}}}$ -module. Then, the Robba fiber of M at T is identical to the Robba module of M_T :*

$$\mathcal{R}_T(M) = \mathcal{R}(M_T).$$

Proof: Since we are working with right exact functors, we may assume that $M = A_K^\dagger$ in which case $M_T = \mathcal{O}_{]T[}$ and we fall back onto the definition. \square

Definition 2.5 *Let U be a non empty affine open subset of X_k with closed complement Z and $\text{Spec } A$ an algebraic lifting of U . Let M be a finite A_K^\dagger -module.*

An extension (at infinity) of M is a coherent module \mathcal{F} on $]Z[$ together with a linear map $\mathcal{R}_Z(M) \rightarrow \mathcal{R}(\mathcal{F})$. It is said to be free if M is projective and \mathcal{F} is locally free.

A morphism of extensions is pair made of a linear map $M' \rightarrow M$ and a linear map $\mathcal{F}' \rightarrow \mathcal{F}$ making commutative the obvious diagram.

In practice, we may just say that $\mathcal{R}_Z(M) \rightarrow \mathcal{R}(\mathcal{F})$ ‘is’ an extension, M and \mathcal{F} being understood as being part of the data.

Definition 2.6 1. Let U be a non empty affine open subset of X_k with closed complement Z and $\text{Spec } A$ an algebraic lifting of U . Let $\mathcal{R}_Z(M) \rightarrow \mathcal{R}(\mathcal{F})$ be an extension. Let $T \subset U$ be any finite closed subset and $W = \text{Spec } B$ some affine open subset of V which is a lifting of $U \setminus T$. Then, the extension of $N := B_K^\dagger \otimes_{A_K^\dagger} M$ defined by

$$\mathcal{R}_{Z \cup T}(N) = \mathcal{R}_{Z \cup T}(M) = \mathcal{R}_Z(M) \oplus \mathcal{R}_T(M) \rightarrow \mathcal{R}(\mathcal{F}) \oplus R(M_T).$$

is called the shrinking of $\mathcal{R}_Z(M) \rightarrow \mathcal{R}(\mathcal{F})$ to $U \setminus T$.

2. Two extensions $\mathcal{R}_Z(M) \rightarrow \mathcal{R}(\mathcal{F})$ on U and $\mathcal{R}_Z(M') \rightarrow \mathcal{R}(\mathcal{F}')$ on U' are said to be equivalent if their shrinkings to $U \cap U'$ are isomorphic.

Proposition 2.7 The category of constructible (resp. constructible-free) modules on X_K^{an} is equivalent to the category of extensions (resp. free extensions) modulo equivalence.

Proof: If we are given a closed subset Z of X with closed complement U , we know that $]U[$ is a closed subset with open complement $]Z[$ and it is a general fact that the category of $\mathcal{O}_{X_K^{\text{an}}}$ modules E on X_K^{an} is equivalent to the category of triples made of an $i_U^{-1} \mathcal{O}_{X_K^{\text{an}}}$ -module E_U , an $i_Z^{-1} \mathcal{O}_{X_K^{\text{an}}}$ -module E_Z and an $i_U^{-1} \mathcal{O}_{X_K^{\text{an}}}$ -linear map $E_U \rightarrow i_U^{-1} i_{Z*} E_Z$. This equivalence is described by the following morphism of short exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & i_{Z!} E_Z & \longrightarrow & i_{Z*} E_Z & \longrightarrow & i_{U*} i_U^{-1} i_{Z*} E_Z \longrightarrow 0 \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \longrightarrow & i_{Z!} E_Z & \longrightarrow & E & \longrightarrow & i_{U*} E_U \longrightarrow 0. \end{array}$$

We know from corollary 1.10 that E is constructible if and only if we can find such U and Z with both E_U and E_Z coherent. Then, we simply set $\mathcal{F} = E_Z$ and $M = \Gamma(]U[, E_U)$. Now, if $i'_\xi :]\xi[\hookrightarrow]U[$ denotes the inclusion map, we have

$$i_U^{-1} i_{Z*} \mathcal{F} = i'_{\xi*} i'^{-1}_\xi i_U^{-1} i_{Z*} \mathcal{F} = i'_{\xi*} \mathcal{R}(\mathcal{F})$$

and it follows that

$$\begin{aligned} \text{Hom}_{i_U^{-1} \mathcal{O}_{X_K^{\text{an}}}}(E_U, i_U^{-1} i_{Z*} E_Z) &= \text{Hom}_{i_U^{-1} \mathcal{O}_{X_K^{\text{an}}}}(E_U, i'_{\xi*} \mathcal{R}(\mathcal{F})) \\ &= \text{Hom}_{\mathcal{O}_{X_K^{\text{an}},]\xi[}}(E_U,]\xi[, \mathcal{R}(\mathcal{F})) = \text{Hom}_{\mathcal{R}_Z}(\mathcal{R}_Z(M), \mathcal{R}(\mathcal{F})) \end{aligned}$$

because $E_{]U[,]\xi[} = \mathcal{O}_{X_K^{\text{an}},]\xi[} \otimes_{A_K^\dagger} M$. Thus, we see that morphisms $E_U \rightarrow i_U^{-1} i_{Z*} E_Z$ correspond bijectively to morphisms $\mathcal{R}_Z(M) \rightarrow \mathcal{R}(\mathcal{F})$.

Of course, one easily checks that if we shrink U , the corresponding extension will be the shrinking of $\mathcal{R}_Z(M) \rightarrow \mathcal{R}(\mathcal{F})$. \square

In practice, a morphism $\mathcal{R}_Z(M) \rightarrow \mathcal{R}(\mathcal{F})$ corresponds to a morphism $i_{U*} E_U \rightarrow i_{\xi*} \mathcal{R}(\mathcal{F})$, and we can pull back the exact sequence

$$0 \rightarrow i_{Z!} \mathcal{F} \rightarrow i_{Z*} \mathcal{F} \rightarrow i_{\xi*} \mathcal{R}(\mathcal{F}) \rightarrow 0$$

in order to get

$$0 \rightarrow i_{Z!} E_Z \rightarrow E \rightarrow i_{U*} E_U \rightarrow 0.$$

3 Specialization

The specialization map $\text{sp} : X_K^{\text{an}} \rightarrow X_k$ is not continuous for the usual topology of X_K^{an} (actually, it is anticontinuous) and it is therefore necessary to use the Tate topology in order to define direct image.

Definition 3.1 *If E is a any module on X_K^{an} , the specialization of E is*

$$\widetilde{\text{sp}}_* E := \text{sp}_* E_G$$

where E_G denotes the module associated to E for the Tate topology.

Proposition 3.2 *The specialization functor $\widetilde{\text{sp}}_*$ is left exact. Moreover, if E is a any module on X_K^{an} , then*

$$\text{R}\widetilde{\text{sp}}_* E = \text{Rsp}_* E_G = \text{Rsp}_* E_0$$

where E_G (resp. E_0) denotes the module associated to E for the Tate topology (on the corresponding rigid space).

Proof: The functor is left exact as composition of two left exact functors. Moreover, since $E \mapsto E_G$ is even exact, we have $\text{R}\widetilde{\text{sp}}_* E := \text{Rsp}_* E_G$. Finally, we have $\Gamma(\text{]}U[, E_G) = \Gamma(\text{]}U[^{\text{rig}}, E_0)$ if U is any affine open subset of X_k . \square

Recall now from [5], 4.2.4, that if T is a finite closed subset of X_k , then the sheaf $\mathcal{O}_{\widehat{X}}(\dagger T)$ may be defined as follows. First of all, if $j : X_k \setminus Z \hookrightarrow X_k$ denotes the inclusion map, then $\mathcal{O}_{\widehat{X}}(\dagger T)$ is a subsheaf of $j_* j^{-1} \mathcal{O}_{\widehat{X}}$. Moreover, if T is defined in $\mathcal{U} = \text{Spf } \mathcal{A}$ by $f = 0 \pmod{\mathfrak{m}}$, then

$$\Gamma(U, \mathcal{O}_{\widehat{X}}(\dagger T)) := \varinjlim \mathcal{A}\{\pi/f^r\}$$

so that

$$\Gamma(U, \mathcal{O}_{\widehat{X}}(\dagger T)_{\mathbf{Q}}) := \mathcal{A}[1/f]_{\mathbf{Q}}^{\dagger}.$$

The next result is a Berkovich analog of a theorem of Berthelot (proposition 4.3.2 of [5], using proposition 1.4).

Proposition 3.3 *If U is a non empty open subset of X_k with closed complement Z , then*

$$\text{R}\widetilde{\text{sp}}_* i_{U*} i_U^{-1} \mathcal{O}_{X_K^{\text{an}}} = \mathcal{O}_{\widehat{X}}(\dagger Z)_{\mathbf{Q}}.$$

Moreover, the functor $\text{R}\widetilde{\text{sp}}_* i_{U*}$ induces an equivalence between coherent $i_U^{-1} \mathcal{O}_{X_K^{\text{an}}}$ -modules and coherent $\mathcal{O}_{\widehat{X}}(\dagger Z)_{\mathbf{Q}}$ -modules.

Proof: If E is a coherent $i_U^{-1} \mathcal{O}_{X_K^{\text{an}}}$ -module, we may always consider it as the restriction to $\text{]}U[$ of some coherent sheaf \mathcal{F} defined on some neighborhood $V_{\lambda} = X_K^{\text{an}} \setminus \text{]}Z[_{\lambda}$ of $\text{]}U[$. By definition, $\text{R}^q \widetilde{\text{sp}}_* i_{U*} E$ is the sheaf associated to

$$U' \mapsto \text{H}^q(\text{]}U'[, i_{U'}^* i_{U*} E)$$

If we set $V'_{\lambda} = \text{]}U'[\cap V_{\lambda}$ and denote by $i_{\lambda} : V'_{\lambda} \hookrightarrow V_{\lambda}$ the inclusion map, it is a general topological result (proposition 2.5 of [17] for example) that

$$\text{H}^q(\text{]}U'[, i_{U'}^* i_{U*} E) = \varinjlim \text{H}^q(V'_{\lambda}, i_{\lambda}^* \mathcal{F}).$$

Since V'_λ is affine, we have $H^q(\mathcal{O}_{U'}[i_U^* i_{U*} E])$ for $q > 0$ and therefore, $R^q \widetilde{\text{sp}}_* i_{U*} E_U = 0$ for $q > 0$. Moreover, we have $V'_\lambda =]U'[\setminus]Z'[_\lambda$ if $Z' = Z \cap U'$. If we call $\mathcal{U}' = \text{Spf } \mathcal{A}'$ the formal lifting of U' and if Z' is defined by $f = 0 \pmod{\mathfrak{m}}$ in \mathcal{U}' , we have $V'_\lambda = \text{Spf } \mathcal{A}'_K \{\pi/f^r\}$ for $\lambda = |\pi|^{1/r}$ and therefore,

$$\Gamma(\mathcal{O}_{U'}[i_U^{-1} \mathcal{O}_{X_K^{\text{an}}}) = \varinjlim \mathcal{A}'_K \{\pi/f^r\} = \Gamma(U, \mathcal{O}_{\widehat{X}}(\dagger Z)_{\mathbf{Q}}).$$

The last assertion then follows from theorem A and B for $\mathcal{O}_{\widehat{X}}(\dagger Z)_{\mathbf{Q}}$ (see proposition 4.3.2 of [5]). \square

In practice, if $\text{Spec } A$ is an algebraic lifting of U , a coherent $i_U^{-1} \mathcal{O}_{X_K^{\text{an}}}$ -module E corresponds to a finite A_K^\dagger -module M . Now, let U' be any affine open subset of X with formal lifting $\mathcal{U}' = \text{Spf } \mathcal{A}'$ and f an equation for Z in \mathcal{U}' . Then, there is a canonical map $A^\dagger \rightarrow \mathcal{A}'[1/f]^\dagger$ and

$$\Gamma(U', \widetilde{\text{sp}}_* i_{U*} E) = \mathcal{A}'[1/f]_{\mathbf{Q}}^\dagger \otimes_{A_K^\dagger} M.$$

We now study the case of a finite closed subset $Z \subset X_k$. We will write

$$\mathcal{H}_Z^\dagger := \mathcal{O}_{\widehat{X}}(\dagger Z) / \mathcal{O}_{\widehat{X}}.$$

For those who know the theory of arithmetic \mathcal{D} -modules, we recall that

$$R\underline{\Gamma}_Z^\dagger \mathcal{O}_{\widehat{X}\mathbf{Q}} \simeq \mathcal{H}_{Z\mathbf{Q}}^\dagger[-1].$$

This cohomology with support is closely related to the notion of Dirac space introduced in definition 2.1 as we shall see right now.

Lemma 3.4 *If Z is a finite closed subset of X_k , we have*

$$\mathcal{H}_{Z\mathbf{Q}}^\dagger = \bigoplus_{x \in Z} i_{x*} \delta_x$$

where $i_x : \{x\} \hookrightarrow \widehat{X}$ denotes the inclusion map.

Proof: Since $\mathcal{O}_{\widehat{X}}(\dagger Z)$ and $\mathcal{O}_{\widehat{X}}$ coincide outside Z , it is sufficient to show that there is an isomorphism on the stalks $\mathcal{H}_{Z\mathbf{Q},x}^\dagger \simeq \delta_x$ when $x \in Z$. We will now use the fact that if U is an affine neighborhood of x , then $]U[\setminus]x[_\lambda$ and $]x[_\lambda$ form an open covering of $]U[_\lambda$ with intersection $]x[\setminus]x[_\lambda$ and that this covering is acyclic for $\mathcal{O}_{]U[_\lambda}$. We call $\mathcal{U} = \text{Spf } \mathcal{A}$ the formal lifting of U and assume that x is defined in \mathcal{U} by an equation $f = 0 \pmod{\mathfrak{m}}$. We will also denote by $\mathcal{R}_{x,\lambda} := \Gamma(\mathcal{O}_{]x[\setminus]x[_\lambda}, \mathcal{O}_{]x[_\lambda})$ and $\delta_{x,\lambda} = \mathcal{R}_{x,\lambda} / \mathcal{O}_x^{\text{an}}$ with $\mathcal{O}_x^{\text{an}} = \Gamma(\mathcal{O}_{]x[_\lambda}, \mathcal{O}_{]x[_\lambda})$ as before. Then, the above remark implies that, in the following morphism of exact sequences, the last map is an isomorphism

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{A}_{\mathbf{Q}} & \longrightarrow & \mathcal{A}\{\pi/f^r\}_{\mathbf{Q}} & \longrightarrow & (\mathcal{A}\{\pi/f^r\}/\mathcal{A})_{\mathbf{Q}} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \simeq \\ 0 & \longrightarrow & \mathcal{O}_x^{\text{an}} & \longrightarrow & \mathcal{R}_{x,\lambda} & \longrightarrow & \delta_{x,\lambda} \longrightarrow 0 \end{array}$$

with $\lambda = |\pi|^{1/r}$. Taking limit on $\lambda < 1$ (and all $U \ni x$), we obtain the expected isomorphism $\mathcal{H}_{Z\mathbf{Q},x}^\dagger = \delta_x$. \square

Lemma 3.5 *If Z is a non empty finite closed subset of X_k , then*

$$\mathrm{R}\widetilde{\mathrm{sp}}_* i_{Z!} \mathcal{O}_{]Z[} \simeq \mathcal{H}_{Z\mathbf{Q}}^\dagger[-1] \quad (= \mathrm{R}\Gamma_Z^\dagger \mathcal{O}_{\hat{X}\mathbf{Q}}).$$

Proof: If U denotes the complement of Z in X_k , there is an exact sequence

$$0 \rightarrow i_{Z!} i_Z^{-1} \mathcal{O}_{X_K^{\mathrm{an}}} \rightarrow \mathcal{O}_{X_K^{\mathrm{an}}} \rightarrow i_{U*} i_U^{-1} \mathcal{O}_{X_K^{\mathrm{an}}} \rightarrow 0$$

from which we derive a triangle

$$\mathrm{R}\widetilde{\mathrm{sp}}_* i_{Z!} \mathcal{O}_{]Z[} \rightarrow \mathrm{R}\widetilde{\mathrm{sp}}_* \mathcal{O}_{X_K^{\mathrm{an}}} \rightarrow \mathrm{R}\widetilde{\mathrm{sp}}_* i_{U*} i_U^{-1} \mathcal{O}_{X_K^{\mathrm{an}}} \rightarrow$$

and then, using proposition 3.2, we obtain

$$\mathrm{R}\widetilde{\mathrm{sp}}_* i_{Z!} \mathcal{O}_{]Z[} \simeq [\mathcal{O}_{\hat{X}\mathbf{Q}} \rightarrow \mathcal{O}_{\hat{X}}(\dagger Z)\mathbf{Q}] \simeq \mathcal{H}_{Z\mathbf{Q}}^\dagger[-1]. \quad \square$$

Recall that we can always lift a finite closed subset Z to a smooth relative divisor $D \subset X$: if $Z = \{x_1, \dots, x_n\}$, we choose an unramified lifting $a_i \in X_K$ for each i and let $D = \mathrm{Spec} \prod \mathcal{V}(a_i)$. Note then that a (coherent) \mathcal{O}_{D_K} -module H is simply a finite collection of (finite dimensional) $K(a)$ -vector spaces H_a for $a \in D_K$. Note also that the canonical inclusion $D_K = D_K^{\mathrm{an}} = \hat{D}_K \hookrightarrow]Z[$ has a unique retraction contracting each disc onto its “center”. It also follows from lemma 3.4 that $\mathcal{H}_{Z\mathbf{Q}}^\dagger$ is an \mathcal{O}_{D_K} -module.

Proposition 3.6 *Let $D \subset X$ be a smooth divisor with reduction Z . If E is a locally free module on $]Z[$, there exists a coherent \mathcal{O}_{D_K} -module H such that $E \simeq \mathcal{O}_{]Z[} \otimes_{\mathcal{O}_{D_K}} H$ and we have*

$$\mathrm{R}\widetilde{\mathrm{sp}}_* i_{Z!} E \simeq (\mathcal{H}_{Z\mathbf{Q}}^\dagger \otimes_{\mathcal{O}_{D_K}} H)[-1]$$

Proof: The first assertion follows from the remark before and the second one then follows from lemma 3.5 since $\mathrm{R}\widetilde{\mathrm{sp}}_*$ is additive. \square

Definition 3.7 *A perverse sheaf on \hat{X} is a complex of $\mathcal{O}_{\hat{X}\mathbf{Q}}$ -modules \mathcal{E} such that $\mathcal{H}^0(\mathcal{E})$ is $\mathcal{O}_{\hat{X}\mathbf{Q}}$ -flat, $\mathcal{H}^1(\mathcal{E})$ has finite support and $\mathcal{H}^i(\mathcal{E}) = 0$ otherwise.*

It is sometimes convenient to split this definition in two (in order to see perverse sheaves as the heart of a t -structure). We can denote by $D_{\geq 0}(\hat{X})_{\mathbf{Q}}$ the category of bounded complexes of $\mathcal{O}_{\hat{X}\mathbf{Q}}$ -modules \mathcal{E} where $\mathcal{H}^0(\mathcal{E})$ is $\mathcal{O}_{\hat{X}\mathbf{Q}}$ -flat and $\mathcal{H}^i(\mathcal{E}) = 0$ for $i < 0$. Next, we denote by $D_{\leq 0}(\hat{X})_{\mathbf{Q}}$ the category of bounded complexes of $\mathcal{O}_{\hat{X}\mathbf{Q}}$ -modules \mathcal{E} where $\mathcal{H}^1(\mathcal{E})$ has finite support and $\mathcal{H}^i(\mathcal{E}) = 0$ for $i > 1$. Then the category of perverse sheaves is $D_{\geq 0}(\hat{X})_{\mathbf{Q}} \cap D_{\leq 0}(\hat{X})_{\mathbf{Q}}$.

Proposition 3.8 *If E is a constructible-free module on X_K^{an} , then $\mathrm{R}\widetilde{\mathrm{sp}}_* E$ is a perverse sheaf on \hat{X} . More precisely, we have*

$$\mathrm{R}\widetilde{\mathrm{sp}}_* E \simeq [\mathcal{E} \rightarrow \mathcal{H}_Z^\dagger \otimes_{\mathcal{O}_{D_K}} H]$$

where $Z \subset X_k$ is a finite closed subset, \mathcal{E} is a coherent locally free $\mathcal{O}_{\hat{X}}(\dagger Z)\mathbf{Q}$ -module, $D \subset X$ is a smooth divisor with reduction Z , and H is a coherent \mathcal{O}_{D_K} -module.

Proof: We showed in corollary 1.12 that there exists an exact sequence

$$0 \rightarrow i_{Z!}E_Z \rightarrow E \rightarrow i_{U*}E_U \rightarrow 0$$

where $Z \subset X_k$ is a finite closed subset with affine open complement U , E_U is a coherent locally free $i_U^{-1}\mathcal{O}_{X_K^{\text{an}}}$ -module and E_Z is a locally free $\mathcal{O}_{]Z[}$ -module.

There exists a smooth divisor $D \subset X$ with reduction Z and a coherent \mathcal{O}_{D_K} -module H such that $E \simeq \mathcal{O}_{]Z[} \otimes_{\mathcal{O}_{D_K}} H$. We saw in proposition 3.6 that

$$\text{R}\widetilde{\text{sp}}_* i_{Z!}E \simeq (\mathcal{H}_Z^\dagger \otimes_{\mathcal{O}_{D_K}} H)[-1].$$

On the other hand, we know from proposition 3.3 that

$$\text{R}\widetilde{\text{sp}}_* i_{U*}E_U = \mathcal{E}$$

where \mathcal{E} is a coherent locally free $\mathcal{O}_{\widehat{X}}(\dagger Z)_{\mathbf{Q}}$ -module. From the above exact sequence, we obtain the following exact triangle

$$(\mathcal{H}_Z^\dagger \otimes_{K_Z} H)[-1] \rightarrow \text{R}\widetilde{\text{sp}}_* E \rightarrow \mathcal{E} \rightarrow$$

and we are done. \square

In practice, a constructible module E is given by some extension $\mathcal{R}(M) \rightarrow \mathcal{R}(\mathcal{F})$ where M is a finite A_K^\dagger module (Spec A being an algebraic lifting of the open complement U of Z in X_k) and \mathcal{F} is a coherent module on $]Z[$. Now, let U' be any affine open subset of X with formal lifting $\mathcal{U}' = \text{Spf } \mathcal{A}'$ and f an equation for Z in \mathcal{U}' . Let $Z' := U' \cap Z$ and $\mathcal{F}' := i_{Z'}^{-1}E$. There are canonical maps $A^\dagger \rightarrow \mathcal{A}'[1/f]^\dagger$ as well as $\mathcal{A}'[1/f]^\dagger_{\mathbf{Q}} \rightarrow R_x$ when $x \in U'$. We can compose the map

$$\mathcal{A}'[1/f]^\dagger_{\mathbf{Q}} \otimes_{A_K^\dagger} M \rightarrow \mathcal{R}(\mathcal{F}')$$

obtained by scalars extensions from $M \rightarrow \mathcal{R}(M) \rightarrow \mathcal{R}(\mathcal{F}) \rightarrow \mathcal{R}(\mathcal{F}')$ with the projection $\mathcal{R}(\mathcal{F}') \rightarrow \delta(\mathcal{F}')$. When E is constructible-free (the other cases are not interesting for us), we get

$$\Gamma(U', \text{R}\widetilde{\text{sp}}_* E) = \left[\mathcal{A}'[1/f]^\dagger_{\mathbf{Q}} \otimes_{A_K^\dagger} M \rightarrow \delta(\mathcal{F}') \right].$$

Again, for latter use, we show that specialization commutes with Frobenius.

Proposition 3.9 *Assume that $F : X \rightarrow X$ is a lifting of the absolute Frobenius of X_k . If E is an $\mathcal{O}_{X_K^{\text{an}}}$ -module, there is a natural isomorphism $\widehat{F}^* \text{R}\widetilde{\text{sp}}_* E \simeq \text{R}\widetilde{\text{sp}}_* F_K^{\text{an}*} E$.*

Proof: We have to be a little careful because $\widetilde{\text{sp}}_*$ is not a direct image. The method is to start directly with an injective resolution I of E_G . It is then sufficient to check that $\widehat{F}^* \widetilde{\text{sp}}_* E \simeq \widetilde{\text{sp}}_* F_K^{\text{an}*} E$ and this easily follows from the fact that F is finite and flat. \square

4 Connections on constructible modules

We start with the very general following definition.

Definition 4.1 If $Y \subset X_k$ is any locally closed subset and $i_Y :]Y[\hookrightarrow X_K^{\text{an}}$ denotes the inclusion map, a connection on an $i_Y^{-1}\mathcal{O}_{X_K^{\text{an}}}$ -module E is a K -linear map

$$\nabla : E \rightarrow E \otimes_{i_{Y^{-1}}\mathcal{O}_{X_K^{\text{an}}}} i_Y^{-1}\Omega_{X_K^{\text{an}}}^1$$

satisfying the Leibnitz rule. A ∇ -module on $]Y[$ is an $i_Y^{-1}\mathcal{O}_{X_K^{\text{an}}}$ -module E endowed with a connection. A horizontal map between ∇ -modules on Y is a $i_Y^{-1}\mathcal{O}_{X_K^{\text{an}}}$ -linear map that commutes with the connection.

Of course, if $Y' \subset Y$, any ∇ -module E on $]Y[$ restricts to a ∇ -module on $]Y'[$. Note also that if $Z \subset X_k$ is a closed subset, meaning either a finite subset or X_k itself, then $]Z[$ is open in X_K^{an} and we fall back onto the usual notion of connection in analytic geometry. Finally, note - and this is important - that there is *no* finiteness condition at this point.

Proposition 4.2 A constructible ∇ -module E on X_K^{an} is constructible-free.

Proof: Since a coherent module with a connection on a disc is necessarily free when the valuation is discrete, we can apply corollary 1.8. \square

It is very convenient to be able to use the description of connections in terms of stratifications. We recall the definition of the first infinitesimal neighborhood P of the diagonal. If X is defined by some ideal \mathcal{I} into $X \times X$, then P is the closed subscheme of X defined by \mathcal{I}^2 . Recall that X and P have same underlying space and that there is a short exact sequence

$$0 \rightarrow \Omega_X^1 \rightarrow \mathcal{O}_P \rightarrow \mathcal{O}_X \rightarrow 0.$$

Actually, we will only need the analogous construction on X_K^{an} which can also be deduced from this one by functoriality. We will consider the maps $q_1, q_2 = P_K^{\text{an}} \rightarrow X_K^{\text{an}}$ and $\Delta : X_K^{\text{an}} \hookrightarrow P_K^{\text{an}}$ induced by the projections and the diagonal embedding.

Definition 4.3 If $Y \subset X_k$ is a locally closed subset, a 1-stratification on an $i_Y^{-1}\mathcal{O}_{X_K^{\text{an}}}$ -module E is an isomorphism

$$\epsilon : i_Y^{-1}q_2^*i_{Y*}E \simeq i_Y^{-1}q_1^*i_{Y*}E$$

such that $(i_Y^{-1}\Delta^*i_{Y*})(\epsilon)$ is the identity of E . A morphism of such is an $i_Y^{-1}\mathcal{O}_{X_K^{\text{an}}}$ -linear map that commutes with the 1-stratifications.

Alternatively, we can write this isomorphism as

$$i_Y^{-1}\mathcal{O}_{P_K^{\text{an}}} \otimes_{i_Y^{-1}\mathcal{O}_{X_K^{\text{an}}}} E \simeq E \otimes_{i_Y^{-1}\mathcal{O}_{X_K^{\text{an}}}} i_Y^{-1}\mathcal{O}_{P_K^{\text{an}}}.$$

The induced map $E \rightarrow E \otimes_{i_Y^{-1}\mathcal{O}_{X_K^{\text{an}}}} i_Y^{-1}\mathcal{O}_{P_K^{\text{an}}}$ coincides with the inclusion mod $i_Y^{-1}\mathcal{I}_K^{\text{an}}$ and the difference is a map

$$\nabla : E \rightarrow E \otimes_{i_{Y^{-1}}\mathcal{O}_{X_K^{\text{an}}}} i_Y^{-1}\Omega_{X_K^{\text{an}}}^1.$$

Proposition 4.4 This construction establishes an equivalence between the category of $i_Y^{-1}\mathcal{O}_{X_K^{\text{an}}}$ -modules endowed with a 1-stratification and the category of ∇ -modules on Y .

Proof: Standard. \square

Note that the category of ∇ -modules on Y is also equivalent to the category of left $i_Y^{-1}\mathcal{D}_{X_K^{\text{an}}}$ -modules if $\mathcal{D}_{X_K^{\text{an}}}$ denotes the sheaf of (algebraic) differential operators on X_K^{an} .

Let $U \subset X_k$ be an affine open subset and $\text{Spec } A \subset X$ an algebraic lifting. We will now recall the description of a connection on an A_K^\dagger -module in term of stratification as we just did above for $i_Y^{-1}\mathcal{O}_{X_K^{\text{an}}}$ -modules. If we set $P_A := (A \otimes A)/I^2$, where I is the ideal of multiplication $A \otimes A \rightarrow A$, then a 1-stratification on an A_K^\dagger -module M a P_{AK}^\dagger -linear isomorphism

$$P_{AK}^\dagger \otimes_{A_K^\dagger} M \simeq M \otimes_{A_K^\dagger} P_{AK}^\dagger$$

Again, this is equivalent to a connection on M .

Proposition 4.5 *Let $U \subset X_k$ be an affine open subset of X_k with algebraic lifting $\text{Spec } A$. Then, the functor $\Gamma(\text{]U[}, -)$ induces an equivalence between coherent (necessarily locally free) ∇ -modules on U and finite (necessarily projective) ∇ - A_K^\dagger -modules.*

Proof: We showed in proposition 1.5 that $\Gamma(\text{]U[}, -)$ induces an equivalence between coherent modules E on U and finite A_K^\dagger -modules M . It follows that there is an equivalence between 1-stratifications on E and 1-stratifications on M because, obviously, with the above notations, we have

$$\Gamma(\text{]U[}, i_U^{-1}\mathcal{O}_{P_K^{\text{an}}}) = P_{AK}^\dagger. \quad \square$$

Proposition 4.6 *If Y is a locally closed subset of X_k and E is a ∇ -module on]Y[, then $i_{Y*}E$ has a natural connection. If Z is a closed subset of X_k and E is a ∇ -module on]Z[, then $i_{Z!}E$ has a natural connection. These functors are fully faithful.*

Proof: We have for $j = 1, 2$,

$$q_j^* i_{Y*} E \simeq i_{Y*} i_Y^{-1} q_j^* i_{Y*} E.$$

Therefore, the 1-stratification extends canonically. The partial inverse is as usual induced by i_Y^{-1} . The proof follows the same lines for $i_{Z!}$ since we will have

$$q_j^* i_{Z!} E \simeq i_{Z!} i_Z^{-1} q_j^* i_{Z!} E. \quad \square$$

Proposition 4.7 *A ∇ -module E on X_K^{an} is constructible if and only if there exists an exact sequence*

$$0 \rightarrow i_{Z!} E_Z \rightarrow E \rightarrow i_{U*} E_U \rightarrow 0$$

where U is a non-empty affine open subset of X_k with closed complement Z , E_U is a coherent (necessarily locally free) ∇ -module on]U[and E_Z is a coherent (necessarily locally free) ∇ -module on]Z[. We can even assume that E_U is free.

Proof: Using 1-stratifications, it follows from proposition 1.12. \square

Recall that we defined above, when $T \subset X_k$ is a non empty finite closed subset and \mathcal{F} is a coherent $\mathcal{O}_{\text{]T[}}$ -module, the Robba module of \mathcal{F} as

$$\mathcal{R}(\mathcal{F}) := (i_{T*}\mathcal{F})_{\xi[}$$

where $i_T :]T[\hookrightarrow X_K^{\text{an}}$ is the inclusion map and $] \xi [$ is the “generic point” of X_K^{an} . In particular, we will consider the *Robba ring* $\mathcal{R}_T := \mathcal{R}(\mathcal{O}_T)$. If for each $x \in T$, we choose an unramified lifting a of x , then \mathcal{R}_T is the direct sum of the usual Robba rings over $K(a)$. We will denote by $\Omega_{\mathcal{R}_T}^1$ the module of finite differentials over \mathcal{R}_T .

Lemma 4.8 *If T is a non empty finite closed subset of X_k , then $\Omega_{\mathcal{R}_T}^1$ is canonically isomorphic to $\mathcal{R}(\Omega_{]T[}^1)$. Moreover, any connection on a coherent $\mathcal{O}_{]T[}$ -module \mathcal{F} induces an connection $\mathcal{R}(\mathcal{F})$.*

Proof: We may assume that T is reduced to one rational point x . Then, we know that $]x[$ is a disc with some parameter t , that \mathcal{R}_x is the usual Robba ring and we have

$$\mathcal{R}(\Omega_{]x[}^1) = \mathcal{R}(\mathcal{O}_{]x[} dt) = \mathcal{R}_x dt = \Omega_{\mathcal{R}_x}^1$$

Also, we have

$$\mathcal{R}(i_x^{-1} \mathcal{O}_{P_K^{\text{an}}}) \simeq \mathcal{R}_x[\tau]/\tau^2$$

where $\tau = p_2^{-1}(t) - p_1^{-1}(t)$. Therefore a 1-stratification on an $\mathcal{O}_{]x[}$ -module \mathcal{F} will induce an isomorphism

$$\mathcal{R}_x[\tau]/\tau^2 \otimes_{\mathcal{R}_x} \mathcal{R}(\mathcal{F}) \simeq \mathcal{R}(\mathcal{F}) \otimes_{\mathcal{R}_x} \mathcal{R}_x[\tau]/\tau^2.$$

This is a 1-stratification on $\mathcal{R}(\mathcal{F})$ that corresponds to a connection. \square

Recall that if $T \subset X_k$ is a non empty finite closed subset and $U \subset X_k$ is a non empty affine open subset with algebraic lifting $\text{Spec } A$, there is a canonical morphism $A_K^\dagger \rightarrow \mathcal{R}_T$ and that the *Robba fiber* of an A_K^\dagger -module M at T is

$$R_T(M) := \mathcal{R}_T \otimes_{A_K^\dagger} M.$$

Note that if M is endowed with a connection, then $R_T(M)$ inherits automatically a connection.

Lemma 4.9 *Let U be a non empty affine open subset of X_k with algebraic lifting $\text{Spec } A$ and T be a finite closed subset of U . Let M be a finite ∇ - A_K^\dagger -module and M_T the restriction to $]T[$ of the corresponding $i_U^{-1} \mathcal{O}_{X_K^{\text{an}}}$ -module. Then, there is a horizontal isomorphism*

$$\mathcal{R}_T(M) = \mathcal{R}(M_T).$$

Proof: Deduced from lemma 2.4 using 1-stratifications. \square

Definition 4.10 *Let U be a non empty affine open subset of X_k with closed complement Z and $\text{Spec } A$ an algebraic lifting of U . Let M be a finite ∇ - A_K^\dagger -module.*

A ∇ -extension (at infinity) of M is a coherent ∇ -module \mathcal{F} on $]Z[$ together with a horizontal map $\mathcal{R}_Z(M) \rightarrow \mathcal{R}(\mathcal{F})$.

A morphism of ∇ -extensions is a morphism of extensions given by horizontal maps.

Two ∇ -extensions $\mathcal{R}_Z(M) \rightarrow \mathcal{R}(\mathcal{F})$ on U and $\mathcal{R}_Z(M') \rightarrow \mathcal{R}(\mathcal{F})$ on U' are equivalent if their shrinkings to $U \cap U'$ are isomorphic as ∇ -extensions.

Proposition 4.11 *The category of constructible ∇ -modules on X_K^{an} is equivalent to the category of ∇ -extensions modulo equivalence.*

Proof: Using proposition 2.7, it is essentially sufficient to notice that if E is a constructible module, then q_2^*E too is constructible. Actually, if E is given by some extension $\mathcal{R}_Z(M) \rightarrow \mathcal{R}(\mathcal{F})$, then q_2^*E will be given by

$$\mathcal{R}_Z(P_{AK}^\dagger \otimes_{A_K^\dagger} M) \rightarrow \mathcal{R}(i_Z^{-1} q_2^* \mathcal{F})$$

And the analogous results holds for q_1^*E . Therefore, a 1-stratification on E is equivalent to a 1-stratification on M and a compatible 1-stratification on \mathcal{F} . \square

We denote by $\mathcal{D}_{\hat{X}}$ the sheaf of (algebraic) differential operators on \hat{X} . We call a $\mathcal{D}_{\hat{X}\mathbf{Q}}$ -module *perverse* if the underlying $\mathcal{O}_{\hat{X}\mathbf{Q}}$ is perverse (see definition 3.7).

Proposition 4.12 *$\mathrm{R}\widetilde{\mathrm{sp}}_*$ induces a functor from the category of constructible ∇ -modules on X_K^{an} to the category of perverse $\mathcal{D}_{\hat{X}\mathbf{Q}}$ -modules.*

Proof: It is essentially sufficient to recall the construction of $\mathrm{R}\widetilde{\mathrm{sp}}_*E$ from the corresponding extension $\mathcal{R}(M) \rightarrow \mathcal{R}(\mathcal{F})$. More precisely, let U' be an affine open subset of X with formal lifting $\mathcal{U}' = \mathrm{Spf} \mathcal{A}'$ and f an equation for Z in \mathcal{U}' . Let $Z' := U' \cap Z$ and $\mathcal{F}' := i_{Z'}^{-1} \mathcal{F}$. Then, we have

$$\Gamma(U', \mathrm{R}\widetilde{\mathrm{sp}}_*E) = \left[\mathcal{A}'[1/f]_{\mathbf{Q}}^\dagger \otimes_{A_K^\dagger} M \rightarrow \delta(\mathcal{F}') \right].$$

where the non-trivial map in this complex is obtained by scalar extension from the composite $M \rightarrow \mathcal{R}(M) \rightarrow \mathcal{R}(\mathcal{F})$, restriction of the image to the direct factor $\mathcal{R}(\mathcal{F}')$, and then, projection onto $\delta(\mathcal{F}')$. All those maps are horizontal. \square

We also want to state another result that we will need later on. If Z is a finite closed subset of \hat{X} , we set

$$\mathcal{D}_{\hat{X}}(\dagger Z)_{\mathbf{Q}} = \mathcal{O}_{\hat{X}}(\dagger Z)_{\mathbf{Q}} \otimes_{\mathcal{O}_{\hat{X}}} \mathcal{D}_{\hat{X}}.$$

Proposition 4.13 *If U is a non empty affine open subset of X_k with closed complement Z , then $\widetilde{\mathrm{sp}}_* i_{U*}$ establishes an equivalence between the category of coherent ∇ -modules on $U[$ and $\mathcal{D}_{\hat{X}}(\dagger Z)_{\mathbf{Q}}$ -modules that are coherent on $\mathcal{O}_{\hat{X}}(\dagger Z)_{\mathbf{Q}}$.*

Proof: Use 1-stratifications again and proposition 3.3. \square

Proposition 4.14 *Assume that $F : X \rightarrow X$ is a lifting of the absolute Frobenius of X_k .*

1. *If Y is a locally closed subset of X_k and E is a ∇ -module on $Y[$, then $i_Y^{-1} F_K^{\mathrm{an}*} i_{Y*} E$ is also a ∇ -module on $Y[$.*
2. *If E is a ∇ -module on X_K^{an} , then the isomorphism*

$$\hat{F}^* \mathrm{R}\widetilde{\mathrm{sp}}_* E \simeq \mathrm{R}\widetilde{\mathrm{sp}}_* F_K^{\mathrm{an}*} E$$

is horizontal.

Proof: If we denote by F_P the map induced on P by $F \times F$, then for $j = 1, 2$, we have

$$i_Y^{-1} q_j^* i_{Y*} i_Y^{-1} F_K^{\mathrm{an}*} i_{Y*} = i_Y^{-1} F_{PK}^{\mathrm{an}*} i_{Y*} i_Y^{-1} q_j^* i_{Y*}$$

and the first assertion follows. The second assertion is obtained by linearity by using 1-stratifications as usual. \square

Definition 4.15 Let $F : X \rightarrow X$ be an algebraic lifting of the absolute Frobenius of X_k and Y a locally closed subset of X_k . An F - ∇ -module on $]Y[$ is a ∇ -module E , endowed with an isomorphism $\phi : i_Y^{-1} F_K^{\text{an}*} i_{Y*} E \simeq E$.

We let the reader extend all the results of this section to F - ∇ -modules. This is straightforward since F is finite flat.

5 Overconvergent connections

Now, we embed X diagonally into $X \times X$. If $Y \subset X_k$ is any subset, we will denote by $]Y[[$ the tube of Y into $X_K^{\text{an}} \times X_K^{\text{an}}$. We call

$$p_1, p_2 :]X_k[[\rightarrow X_K^{\text{an}}$$

the maps induced by the projections. We will still denote by $i_Y :]Y[[\hookrightarrow]X_k[[$ the embedding, hoping that this will not create any confusion. Note that

$$]Y[[= p_j^{-1}(]Y[) \subset]X_k[[\quad \text{for } j = 1, 2.$$

Sometimes, even if we will try to avoid it, we may also denote by $p_1, p_2 :]Y[[\rightarrow]Y[[$ the maps induced by the projections (at least when Y is a closed subset).

Locally, the geometry of $]X_k[[$ is not too bad (this is the strong fibration theorem of Berthelot) as we can see right now. Since X is smooth of relative dimension 1, there exists locally an étale map t to the affine line \mathbf{A}_V^1 . We will then say that t is a *local parameter* on X . Assume that t is defined on some open subset $V \subset X$ with reduction $U \subset X_k$. Then $\tau := p_2^*(t) - p_1^*(t)$ defines an étale map $V \times V \rightarrow V \times \mathbf{A}_V^1$. More precisely, there is a commutative diagram

$$\begin{array}{ccc} V \times V & \xrightarrow{\quad} & V \times \mathbf{A}_V^1 \\ \swarrow p_1 & & \searrow p \\ \Delta & & 0 \end{array}$$

As in proposition 1.1, it follows from lemma 4.4 of [4] that this map induces an isomorphism $]U[[\simeq]U[\times \mathbf{D}(0, 1^-)$. Moreover, the morphism

$$V_K^{\text{an}} \times V_K^{\text{an}} \rightarrow V_K^{\text{an}} \times \mathbf{A}_K^{1, \text{an}}$$

is étale (formally étale and boundaryless). Thus, Proposition 4.3.4 of [2] implies that it induces an isomorphism between a neighborhood V' of $]U[[$ in $]X_k[[$ and a neighborhood V'' of $]U[\times \mathbf{D}(0, 1^-)$ into $X_K^{\text{an}} \times \mathbf{D}(0, 1^-)$. If we denote by Z the closed complement of U and let $V_\lambda = X_K^{\text{an}} \setminus]Z[_\lambda$ as usual, we may assume that $V'' = \cup_{\lambda, \eta} (V_\lambda \times \mathbf{D}(0, \eta^+))$ with $\lambda, \eta \rightarrow 1$. We can summarize the situation in the following diagram

$$\begin{array}{ccc} V_K^{\text{an}} \times V_K^{\text{an}} & \xrightarrow{\quad} & V_K^{\text{an}} \times \mathbf{A}_K^{1, \text{an}} \\ \uparrow \text{J} & & \uparrow \text{J} \\ V' & \xrightarrow{\simeq} & \cup_{\lambda, \eta} (V_\lambda \times \mathbf{D}(0, \eta^+)) \\ \uparrow \text{J} & & \uparrow \text{J} \\]U[[& \xrightarrow{\simeq} &]U[\times \mathbf{D}(0, 1^-) \end{array}$$

Of course, if $T \subset X_k$ is a finite closed subset, we may always assume that $T \subset U$, and we obtain $\mathbb{D}(0, 1^-)$. If x is a closed point with unramified lifting a , we even get

$$\mathbb{D}(0, 1^-) \simeq \mathbf{D}_{K(a)}(0, 1^-) \times \mathbf{D}(0, 1^-) \simeq \mathbf{B}_{K(a)}^2(0, 1^-)$$

which can also be proved directly as in proposition 1.1. Finally, we can describe the inverse image by specialization $\mathbb{D}(0, 1^-)$ of the generic point ξ of X_k : we have

$$\mathbb{D}(0, 1^-)$$

where the disc is over the completion $\mathcal{H}(\mathbb{D}(0, 1^-))$ of the field $\mathcal{O}_{X_k^{\text{an}}, \xi}$.

Following Berthelot, we will define now the overconvergence condition. Note before that the natural inclusion of the first infinitesimal neighborhood $P \hookrightarrow X \times X$, induces by functoriality a morphism $P_K^{\text{an}} \hookrightarrow X_k$ and the projections $q_1, q_2 : P_K^{\text{an}} \rightarrow X_K^{\text{an}}$ introduced in the previous section are induced by p_1 and p_2 .

Definition 5.1 *Let Y be a locally closed subset of X_k . A connection on an $i_Y^{-1}\mathcal{O}_{X_K^{\text{an}}}$ -module E is overconvergent if the corresponding 1-stratification is induced by a Taylor isomorphism*

$$\epsilon : i_Y^{-1}p_2^*i_{Y^*}E \simeq i_Y^{-1}p_1^*i_{Y^*}E$$

on Y which satisfies the cocycle condition (see below) on triple products. If Y is a closed subset, the connection is just said to be convergent. We will also say that the ∇ -module E is (over-) convergent.

Again, this definition applies in very general geometric situations. It is also important to insist on the fact that there are no finiteness conditions at that point. Note that the convergence condition is a lot simpler to describe: if Z is a closed subset in X_k (either finite or X_k itself), and if we still denote by $p_1, p_2 : Z \rightarrow Z$, the maps induced by the projections, we simply have $i_Y^{-1}p_j^*i_{Y^*}E = p_j^*E$ for $i = 1, 2$ and the Taylor isomorphism reads

$$\epsilon : p_2^*E \simeq p_1^*E$$

on Z .

It is quite easy to make the *cocycle condition* precise. We embed diagonally X into the triple product $X \times X \times X$. We denote by $\mathbb{D}(0, 1^-)$ the tube of $Y \subset X_k$ inside $X_K^{\text{an}} \times X_K^{\text{an}} \times X_K^{\text{an}}$ and by $p_{12}, p_{23}, p_{13} : \mathbb{D}(0, 1^-) \rightarrow X_k$ the maps induced by the projections. Then the cocycle condition reads

$$(i_Y^{-1}p_{12}^*i_{Y^*})(\epsilon) \circ (i_Y^{-1}p_{23}^*i_{Y^*})(\epsilon) = (i_Y^{-1}p_{13}^*i_{Y^*})(\epsilon).$$

Lemma 5.2 *Overconvergence is preserved under restriction to a locally closed subset $Y' \subset Y$.*

Proof: We denote by the same letter i both inclusion maps $Y' \hookrightarrow Y$ and $Y' \hookrightarrow Y$. Then we have for $j = 1, 2$,

$$\begin{aligned} i_{Y'}^{-1}p_j^*i_{Y'^*}i^{-1}E &= i_{Y'}^{-1}(\mathcal{O}_{X_k} \otimes_{p_j^{-1}\mathcal{O}_{X_K^{\text{an}}}} p_j^{-1}i_{Y'^*}i^{-1}E) \\ &= i_{Y'}^{-1}\mathcal{O}_{X_k} \otimes_{i_{Y'}^{-1}p_j^{-1}\mathcal{O}_{X_K^{\text{an}}}} i_{Y'}^{-1}p_j^{-1}i_{Y'^*}i^{-1}E = i_{Y'}^{-1}\mathcal{O}_{X_k} \otimes_{i_{Y'}^{-1}p_j^{-1}\mathcal{O}_{X_K^{\text{an}}}} i^{-1}p_j^{-1}i_{Y^*}E \end{aligned}$$

$$= i^{-1}(i_Y^{-1}\mathcal{O}_{X_k} \otimes_{i_Y^{-1}p_j^{-1}\mathcal{O}_{X_K^{\text{an}}}} p_j^{-1}i_{Y*}E) = i^{-1}(i_Y^{-1}p_j^*i_{Y*}E). \quad \square$$

We will use this explicit isomorphism below for the restriction from X_k to some Y in which case it says that the adjunction map is an isomorphism

$$i_Y^{-1}p_j^*E \simeq i_Y^{-1}p_j^*i_{Y*}i_Y^{-1}E.$$

Proposition 5.3 *If U is an open subset of X_k and E is an overconvergent ∇ -module on $]U[$, then $i_{U*}E$ is convergent. If Z is a closed subset of X_k and E is a convergent ∇ -module on $]Z[$, then $i_{Z!}E$ is convergent.*

Proof: We want to show that for $j = 1, 2$, we have

$$p_j^*i_{U*}E \simeq i_{U*}i_U^{-1}p_j^*i_{U*}E.$$

It is actually sufficient to show that

$$p_j^{-1}i_{U*}E \simeq i_{U*}i_U^{-1}p_j^{-1}i_{U*}E.$$

And this follows from the fact that there is a cartesian diagram

$$\begin{array}{ccc}]U[[\hookrightarrow]X_k[[& & \\ \downarrow & & \downarrow p_2 \\]U[\hookrightarrow X_K^{\text{an}} & & \end{array}$$

where the horizontal arrows are closed immersions. Therefore, the 1-stratification extends canonically.

The same type of argument shows that

$$p_j^*i_{Z!}E \simeq i_{Z!}i_Z^{-1}p_j^*i_{Z!}E$$

in the second case. \square

The next proposition will be the first illustration of the power of the overconvergence condition. Recall that a ∇ -module on an analytic variety is said to be (locally) *trivial* if it is (locally) generated by a finite set of horizontal sections. Note that a locally trivial ∇ -module on a disc is always trivial. For further use, we also state a lemma.

Lemma 5.4 *If $D \subset X$ is a smooth divisor with reduction Z , then the inclusion $D_K \hookrightarrow]Z[$ and its retraction induce an equivalence between coherent \mathcal{O}_{D_K} -module and locally trivial ∇ -modules on $]Z[$.*

Proof: We may assume that Z is reduced to one point x with unramified lifting a , in which case we are simply considering the inclusion of the origin $\{0\} \hookrightarrow D_{K(a)}(0, 1^-)$ into the disc. \square

Proposition 5.5 *If $Z \subset X_k$ is a finite closed subset, then any coherent convergent ∇ -module on $]Z[$ is locally trivial.*

Proof: We may assume that Z is reduced to a rational point x and lift it to a rational point $a \in X_K$. Think of a as the “center” of $]x[$. We set $V := a^*E$ which is a finite dimensional vector space. Next, we consider the composition $X \rightarrow \text{Spec } \mathcal{V} \hookrightarrow X$ of the projection p and the section a . Its graph $X \rightarrow X \times X$ induces a morphism $\gamma :]x[\hookrightarrow]x[[$ along which we can pull back the Taylor isomorphism. We obtain an isomorphism

$$\phi : \mathcal{O}_{]x[} \otimes_K V = p^*a^*E = \gamma^*p_2^*E \simeq \gamma^*p_1^*E = E.$$

Using the cocycle condition, one sees that the 1-stratification of E is compatible with the trivial one on the left hand side. More precisely, we use the map $X \times X \rightarrow X \times X \times X$ obtained by tensorizing the identity with the above graph. We may then pull the cocycle condition back along the induced map $]x[[\rightarrow]]x[[$ and get $\epsilon \circ p_2^*(\phi) = p_1^*(\phi)$. \square

Actually, the convergence condition on $\{x\}$ is also called the *Robba condition* and it is a classical result that a finite ∇ -module that satisfies the Robba condition on an open disc is automatically trivial.

Corollary 5.6 *If E is a constructible convergent ∇ -module on X_K^{an} and $x \in X_k$, then the restriction of E to $]x[$ is trivial. \square*

We now turn to the description of overconvergence on open subsets. Before doing anything else and although we will continue to prove most statements without referring to rigid cohomology, we should mention the following comparison result:

Proposition 5.7 *If U is an open subset of X_k , the functor $E \mapsto (i_{U*}E)_0$ induces an equivalence between the category of coherent overconvergent ∇ -modules on $]U[$ and overconvergent isocrystals on U .*

Proof: Using proposition 7.2.13 (and definition 7.2.10) of [18], this is simply a translation of the above overconvergence condition into the language of rigid geometry as in proposition 1.4. \square

Assume that t is a local parameter defined on some affine open subscheme $\text{Spec } A \subset X$ with reduction U . Write as usual $V_\lambda = X_K^{\text{an}} \setminus]Z[_\lambda = \mathcal{M}(A_\lambda)$ where Z is the closed complement of U . If M is a finite ∇ - A_K^\dagger -module, it extends as usual to some finite ∇ - A_λ module M_λ with $\lambda < 1$ and we may define the λ -radius of convergence of M as

$$R(M, \lambda) = \inf_{s \in M_\lambda} \inf \left\{ \lambda \quad , \quad \varliminf_k \left\| \frac{1}{k!} \frac{\partial^k}{\partial t^k}(s) \right\|_\lambda^{-\frac{1}{k}} \right\}$$

where $\| - \|_\lambda$ is a Banach norm on M_λ . We say that M is *overconvergent* if

$$\lim_{\lambda \rightarrow 1} R(M, \lambda) = 1.$$

Those who are interested in differential equations should notice that this condition is equivalent to requiring all the Robba fibers $\mathcal{R}_x(M)$ to be solvable whenever $x \in Z$ (use the “isometry” $A_K^\dagger \hookrightarrow \mathcal{R}_Z$). If one is willing to use rigid analytic geometry, the next proposition is a particular case of proposition 7.2.15 of [18].

Proposition 5.8 *Let $U \subset X_k$ be an affine open subset of X_k . Let t be a local parameter defined on an algebraic lifting $\text{Spec } A$ of U . Then, the functor $\Gamma(\text{]}U[, -)$ induces an equivalence between coherent overconvergent ∇ -modules on U and finite overconvergent $\nabla\text{-}A_K^\dagger$ -modules.*

Proof: Let E be a ∇ -module on $\text{]}U[$ and M the corresponding $\nabla\text{-}A_K^\dagger$ -module. One can check that the Taylor isomorphism is necessarily induced by the Taylor series

$$s \mapsto \sum \frac{1}{k!} \frac{\partial^k}{\partial t^k}(s) \tau^k$$

(see the remark after proposition 5.10 below). The overconvergence condition on $\text{]}U[$ means that this series converges on some neighborhood of $\text{]}U[[$ inside $\text{]}X_k[[$. And we may assume that this neighborhood has the form

$$\cup_{\lambda, \eta} (V_\lambda \times \mathbf{D}(0, \eta^+)) \quad \text{with} \quad \lambda, \eta \rightarrow 1$$

with V_λ as usual. The overconvergence condition for M is then the direct translation of the convergence of the series on this specific neighborhood. \square

We will need below general stratifications using all infinitesimal neighborhoods of X . We denote by $P^{(n)}$ the n -th infinitesimal neighborhood of X in $X \times X$ defined by \mathcal{I}^{n+1} if X is defined by \mathcal{I} into $X \times X$, and by $p_1^{(n)}, p_2^{(n)} = P_K^{(n), \text{an}} \rightarrow X_K^{\text{an}}$ the maps induced by the projections. With our former notations, we have $P = P^{(1)}$ and $q_j = p_j^{(1)}$.

Definition 5.9 *If $Y \subset X_k$ is a locally closed subset, a stratification on an $i_Y^{-1} \mathcal{O}_{X_K^{\text{an}}}$ -module E is a compatible family of isomorphisms*

$$\epsilon_n : i_Y^{-1} p_2^{(n)*} i_{Y*} E \simeq i_Y^{-1} p_1^{(n)*} i_{Y*} E$$

such that the cocycle condition holds on triple products and ϵ_0 is the identity. A morphism of such is an $i_Y^{-1} \mathcal{O}_{X_K^{\text{an}}}$ -linear map that commutes with the stratifications.

Proposition 5.10 *The category of stratified $i_Y^{-1} \mathcal{O}_{X_K^{\text{an}}}$ -modules is equivalent the category of ∇ -modules on Y .*

Proof: Standard. \square

It should also be mentioned that the stratification is automatically induced by the Taylor isomorphism when E is overconvergent.

Proposition 5.11 *Let E be a ∇ -module on X_K^{an} , U an affine open subset of X and Z its closed complement. Assume that $i_U^{-1} E$ is a coherent overconvergent ∇ -module and that $i_Z^{-1} E$ is a locally trivial ∇ -module. Then, E is convergent.*

Proof: The point is to show that the stratification is induced by a Taylor isomorphism $\epsilon : p_2^* E \simeq p_1^* E$ on $\text{]}X_k[[$. We already have isomorphisms

$$\epsilon_U : i_U^{-1} p_2^* E \simeq i_U^{-1} p_1^* E \quad \text{and} \quad \epsilon_Z : i_Z^{-1} p_2^* E \simeq i_Z^{-1} p_1^* E$$

coming from the overconvergence of E on U and Z (see the remark following lemma 5.2). And we can use again the fact that a sheaf on a topological space is uniquely determined by its restriction to an open subset, its restriction to the closed complement and the adjunction map. It is therefore sufficient to show that the diagram

$$\begin{array}{ccc} i_U^{-1}p_2^*E & \longrightarrow & i_U^{-1}i_{Z*}i_Z^{-1}p_2^*E \\ \downarrow \epsilon_U & & \downarrow \epsilon_Z \\ i_U^{-1}p_1^*E & \longrightarrow & i_U^{-1}i_{Z*}i_Z^{-1}p_1^*E \end{array}$$

is commutative. Since the analogous diagram with stratifications is commutative by hypothesis (we have a stratification on E), it is sufficient to prove that the canonical map

$$\mathrm{Hom}_{i_U^{-1}\mathcal{O}_{\llbracket X_k \rrbracket}}(i_U^{-1}p_2^*E, i_U^{-1}i_{Z*}i_Z^{-1}p_1^*E) \rightarrow \varprojlim \mathrm{Hom}_{i_U^{-1}\mathcal{O}_{P_K^{(n),\mathrm{an}}}}(i_U^{-1}p_2^{(n)*}E, i_U^{-1}i_{Z*}i_Z^{-1}p_1^{(n)*}E)$$

is injective. Since $i_U^{-1}E$ is finitely presented and $i_Z^{-1}E$ is trivial, it is sufficient to prove that the canonical map

$$\Gamma(\llbracket U \rrbracket, i_U^{-1}i_{Z*}i_Z^{-1}p_1^*\mathcal{O}_{\llbracket X_k \rrbracket}) \rightarrow \varprojlim \Gamma(\llbracket U \rrbracket, i_U^{-1}i_{Z*}i_Z^{-1}p_1^{(n)*}\mathcal{O}_{P_K^{(n),\mathrm{an}}})$$

is injective. By considering the stalks and since $\llbracket \xi \rrbracket$ is closed in $\llbracket X_k \rrbracket$, one sees that

$$i_U^{-1}i_{Z*}i_Z^{-1}p_1^*\mathcal{O}_{\llbracket X_k \rrbracket} = i_U^{-1}i_{Z*}\mathcal{O}_{\llbracket Z \rrbracket} = i_{\xi*}i_{\xi}^{-1}i_{Z*}\mathcal{O}_{\llbracket Z \rrbracket}.$$

Also, if we fix a local parameter t on X , we can easily identify the right hand side with $\mathcal{R}_Z[[\tau]]$. We are led to check that the canonical map

$$\Gamma(\llbracket \xi \rrbracket, i_{\xi}^{-1}i_{Z*}\mathcal{O}_{\llbracket Z \rrbracket}) \rightarrow \mathcal{R}_Z[[\tau]]$$

is injective.

We may assume that the local parameter t is defined on an affine open subscheme $\mathrm{Spec} A$ whose reduction U contains Z . As usual, if T denotes the closed complement of U , we let $V_{\lambda} := X_K^{\mathrm{an}} \setminus T[\lambda]$. It is sufficient to show that if V' is a neighborhood of $\llbracket \xi \rrbracket$ inside $\llbracket X_k \rrbracket$, the map

$$\Gamma(V' \cap Z, \mathcal{O}_{V'}) \rightarrow \mathcal{R}_Z[[\tau]]$$

is injective. After removing some points in U if necessary, we may assume that

$$V' = \cup_{\lambda, \eta} (V_{\lambda} \times \mathbf{D}(0, \eta^+))$$

with $\lambda, \eta \rightarrow 1$. We are then reduced to showing that the map

$$\Gamma(Z \setminus Z[\lambda \times \mathbf{D}(0, \eta^+), \mathcal{O}] \rightarrow \mathcal{R}_Z[[\tau]]$$

is injective. An it is sufficient to consider the obvious injective map

$$\Gamma([Z]_{\mu} \setminus Z[\lambda \times \mathbf{D}(0, \eta^+), \mathcal{O}] \hookrightarrow \Gamma([Z]_{\mu} \setminus Z[\lambda, \mathcal{O}])[[\tau]]$$

and take inverse limit when $\mu \rightarrow 1$. \square

Corollary 5.12 *Let E be a constructible ∇ -module on X_K^{an} . The connection on E is convergent if and only if there exists a finite covering of X_k by locally closed subsets Y such that the connection is overconvergent on each Y .*

Proof: Follows also from proposition 5.11. \square

Corollary 5.13 *A ∇ -module E on X_K^{an} is constructible convergent if and only if there exists an exact sequence*

$$0 \rightarrow i_{Z!}E_Z \rightarrow E \rightarrow i_{U*}E_U \rightarrow 0$$

where U is a non-empty affine open subset of X_k with closed complement Z , E_U is a coherent overconvergent ∇ -module on $]U[$ and E_Z is a locally trivial ∇ -module on $]Z[$. We can even assume that E_U is free.

Proof: Follows from propositions 4.7 and 5.11. \square

Recall that the notion of ∇ -extension was introduced in definition 4.10.

Definition 5.14 *A ∇ -extension $\mathcal{R}(M) \rightarrow \mathcal{R}(\mathcal{F})$ is said to be convergent if M is overconvergent and \mathcal{F} is locally trivial.*

In other words, a convergent ∇ -extension is given by a an overconvergent ∇ - A_K^\dagger -module M where $\text{Spec } A \subset X$ is non empty affine open subset with reduction U , a collection of finite dimensional $K(a)$ -vector spaces H_a for each point x not in U , where a is an unramified lifting of x , and horizontal A_K^\dagger -linear maps $M \rightarrow \mathcal{R}_a \otimes_{K(a)} H_a$.

Theorem 5.15 *The category of constructible convergent ∇ -modules on X_K^{an} is equivalent to the category of convergent ∇ -extensions modulo equivalence.*

Proof: Follows from propositions 4.11 and 5.11. \square

Finally, we study the relation between overconvergence and Frobenius:

Proposition 5.16 *Assume that $F : X \rightarrow X$ is a lifting of the absolute Frobenius of X_k . If Y is a locally closed subset of X_k and E is an overconvergent ∇ -module on $]Y[$, then $i_Y^{-1}F_K^{\text{an}*}i_{Y*}E$ is also overconvergent.*

Proof: If we denote by $F_{]]X_k[[}$ the map induced on $]]X_k[[$ by $F \times F$, then for $j = 1, 2$, we have

$$i_Y^{-1}p_j^*i_{Y*}i_Y^{-1}F_K^{\text{an}*}i_{Y*} = i_Y^{-1}F_{]]X_k[[}^*i_{Y*}i_Y^{-1}p_j^*i_{Y*}$$

and the result follows. \square

Proposition 5.17 *If $F : X \rightarrow X$ is a lifting of the absolute Frobenius of X_k , then any constructible F - ∇ -module on X_K^{an} is convergent.*

Proof: This is Dwork's trick. Using proposition 5.11, we may assume either that $E = i_{U*}E_U$ with $U \subset X_k$ affine open and E_U a coherent $i_U^{-1}\mathcal{O}_{X_K^{\text{an}}}$ -module or $E = i_{x!}E_x$ with $x \in X_k$ closed and E_x coherent on $]x[$. The first case follows from proposition 5.8 using for example corollary 8.3.9 of [18]. The second case is the classical fact that an (finite) F - ∇ -module on an open disc is always trivial. \square

6 Constructibility and \mathcal{D} -modules

We introduce now the ring of arithmetic differential operators on \hat{X} . It can be described as follows. First of all, if $\mathcal{D}_{\hat{X}}$ denotes the sheaf of algebraic differential operators on \hat{X} as before, and $\widehat{\mathcal{D}}_{\hat{X}}$ is its p -adic completion, we have

$$\mathcal{D}_{\hat{X}} \subset \mathcal{D}_{\hat{X}}^{\dagger} \subset \widehat{\mathcal{D}}_{\hat{X}}.$$

Moreover, as shown in proposition 2.4.4 of [5], if t is a local parameter on X defined on some subset $\mathcal{U} = \mathrm{Spf} \mathcal{A} \subset \hat{X}$, then

$$\Gamma(\mathcal{U}, \mathcal{D}_{\hat{X}}^{\dagger}) = \left\{ \sum_{k=0}^{\infty} \frac{1}{k!} f_k \frac{\partial^k}{\partial t^k}, \quad \exists c > 0, \eta < 1, \quad \|f_k\| \leq c\eta^k \right\}.$$

We are interested in $\mathcal{D}_{\hat{X}\mathbf{Q}}^{\dagger}$ -modules. The case of coherent $\mathcal{D}_{\hat{X}\mathbf{Q}}^{\dagger}$ -modules with finite support is easy to settle. If $D \subset X$ is a smooth divisor with reduction Z , then the direct image functor

$$\begin{array}{ccc} D_{\mathrm{coh}}^b(\mathcal{O}_{\hat{D}\mathbf{Q}}) & \xrightarrow{i_{Z+}} & D_{\mathrm{coh}}^b(\mathcal{D}_{\hat{X}\mathbf{Q}}^{\dagger}) \\ H & \longmapsto & i_{Z*}(i_Z^* \mathrm{Hom}_{\mathcal{O}_{\hat{X}}}(\Omega_{\hat{X}}^1, \mathcal{D}_{\hat{X}}^{\dagger})_{\mathbf{Q}} \otimes_{\mathcal{O}_{\hat{D}\mathbf{Q}}} H) \end{array}$$

induces an equivalence between coherent $\mathcal{O}_{\hat{D}\mathbf{Q}}$ -modules and coherent $\mathcal{D}_{\hat{X}\mathbf{Q}}^{\dagger}$ -modules with support in Z (see section 5.3.3 of [7] for the general statement). Its inverse is induced by the exceptional inverse image $i_Z^!$.

Recall that if Z is a finite closed subset of X , there is a short exact sequence

$$0 \rightarrow \mathcal{O}_{\hat{X}} \rightarrow \mathcal{O}_{\hat{X}}(\dagger Z) \rightarrow \mathcal{H}_Z^{\dagger} \rightarrow 0$$

of $\mathcal{D}_{\hat{X}}^{\dagger}$ -module. Moreover, $\mathcal{O}_{\hat{X}\mathbf{Q}}$, $\mathcal{O}_{\hat{X}}(\dagger Z)_{\mathbf{Q}}$ and $\mathcal{H}_{Z\mathbf{Q}}^{\dagger}$ are all $\mathcal{D}_{\hat{X}\mathbf{Q}}^{\dagger}$ -coherent.

The next result is also well-known (formula 4.4.5.2 of [7]) but rather easy to check (and instructive) in our situation.

Proposition 6.1 *If $D \subset X$ is a smooth divisor with reduction Z , and H is a coherent $\mathcal{O}_{\hat{D}\mathbf{Q}}$ -module, there is a canonical isomorphism of $\mathcal{D}_{\hat{X}\mathbf{Q}}^{\dagger}$ -modules*

$$i_{Z+}H \simeq \mathcal{H}_{Z\mathbf{Q}}^{\dagger} \otimes_{\mathcal{O}_{\hat{D}\mathbf{Q}}} H.$$

Proof: We may assume that $Z = \{x\}$ where x is the only zero of a local parameter t defined on some formal affine open subset $\mathcal{U} = \mathrm{Spf} \mathcal{A}$, and that $H = K(a)$ with a an unramified lifting of x . We have a commutative diagram with exact rows (for multiplication on the left)

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma(\mathcal{U}, \mathcal{D}_{\hat{X}}^{\dagger}) & \xrightarrow{\frac{\partial}{\partial t}} & \Gamma(\mathcal{U}, \mathcal{D}_{\hat{X}}^{\dagger}) & \longrightarrow & \mathcal{A} \longrightarrow 0 \\ & & \parallel & & \downarrow t & & \downarrow \\ 0 & \longrightarrow & \Gamma(\mathcal{U}, \mathcal{D}_{\hat{X}}^{\dagger}) & \xrightarrow{t \frac{\partial}{\partial t} + 1} & \Gamma(\mathcal{U}, \mathcal{D}_{\hat{X}}^{\dagger}) & \xrightarrow{1 \mapsto \frac{1}{t}} & \mathcal{A}[1/t]^{\dagger} \longrightarrow 0 \end{array}$$

from which we deduce an isomorphism

$$\Gamma(\mathcal{U}, \mathcal{D}_{\hat{X}}^\dagger) / \Gamma(\mathcal{U}, \mathcal{D}_{\hat{X}}^\dagger) t \simeq \mathcal{A}[1/t]^\dagger / \mathcal{A}.$$

The right hand side is exactly $\Gamma(\mathcal{U}, \mathcal{H}_Z^\dagger)$. On the other hand, we have

$$\Gamma(\mathcal{U}, i_{x+} K(a)) = K(a) \otimes_{\mathcal{A}_K} \text{Hom}_{\mathcal{A}_K}(\mathcal{A}_K dt, \Gamma(\mathcal{U}, \mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger) \simeq \Gamma(\mathcal{U}, \mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger) / \Gamma(\mathcal{U}, \mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger) t.$$

We get an isomorphism as expected which is easily seen to be independent of the choices (just multiply t by an invertible element in \mathcal{A}_K). \square

The next proposition is a very fancy way of stating an almost trivial result. However, it may be seen as an analog of theorem 6.5 of [16] in a very simple situation. This may also be seen as a special case of theorem 2.5.10 of [10].

Proposition 6.2 *If Z is a finite closed subset of X_k , then $\text{R}\widetilde{\text{sp}}_* i_{Z!}$ induces an equivalence between the category of constructible convergent ∇ -modules on $]Z[$ and the category complexes of $\mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger$ -modules with support in Z and coherent cohomology concentrated in degree 1.*

Proof: If $D \subset X$ is a smooth divisor with reduction Z , and H is a coherent $\mathcal{O}_{\hat{D}\mathbf{Q}}$ -module, it follows from proposition 6.1 and proposition 3.6 that

$$\text{R}\widetilde{\text{sp}}_* i_{Z!}(\mathcal{O}_{]Z[} \otimes_{\mathcal{O}_{\hat{D}\mathbf{Q}}} H) = i_{Z+} H[1].$$

Moreover, we know that $\mathcal{O}_{]Z[} \otimes_{\mathcal{O}_{\hat{D}}} -$ is an equivalence of categories between coherent $\mathcal{O}_{\hat{D}_K}$ -modules and locally trivial ∇ -modules on $]Z[$, and that i_{Z+} is an equivalence of categories between coherent $\mathcal{O}_{\hat{D}\mathbf{Q}}$ -modules and coherent $\mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger$ -modules with support in Z . Since specialization $\hat{D}_K \rightarrow Z$ is bijective, we can identify $\mathcal{O}_{\hat{D}\mathbf{Q}}$ with $\mathcal{O}_{\hat{D}_K}$ and consequently $\mathcal{O}_{\hat{D}\mathbf{Q}}$ -modules with $\mathcal{O}_{\hat{D}_K}$ -modules. Finally, we know from proposition 5.5 that constructible convergent ∇ -modules on $]Z[$ are locally trivial (and conversely). \square

We now turn to the case of an affine open subset of X_k . It is necessary to introduce the ring of arithmetic differential operators $\mathcal{D}_{\hat{X}}^\dagger(\dagger Z)$ with overconvergent poles along a non empty finite closed subset Z . Since we are only interested in this ring modulo torsion, it is more convenient to use the modified definition given in 2.6.2 of [20].

Just before proposition 4.13, we introduced the ring $\mathcal{D}_{\hat{X}}(\dagger Z)$ of algebraic differential operators with overconvergent poles. If $j : X_k \setminus Z \hookrightarrow X_k$ denotes the inclusion map, we will have

$$\text{both } \mathcal{D}_{\hat{X}}^\dagger \text{ and } \mathcal{D}_{\hat{X}}(\dagger Z) \subset \mathcal{D}_{\hat{X}}^\dagger(\dagger Z) \subset j_* j^{-1} \mathcal{D}_{\hat{X}}^\dagger.$$

If there exists a local parameter t defined on some open subset $\mathcal{U} = \text{Spf } \mathcal{A}$ such that $Z \cap \mathcal{U} = \{x\}$ where x is the only zero of t , then

$$\Gamma(\mathcal{U}, \mathcal{D}_{\hat{X}}^\dagger(\dagger Z)) = \left\{ \sum_{k,l=0}^{\infty} \frac{1}{k!} \frac{f_{k,l}}{t^l} \frac{\partial^k}{\partial t^k}, \quad \exists c > 0, \eta < 1, \quad \|f_{k,l}\| \leq c \eta^{k+l} \right\}.$$

Note that $\mathcal{D}_{\hat{X}}^\dagger(\dagger Z)_{\mathbf{Q}}$ -modules that are $\mathcal{O}_{\hat{X}\mathbf{Q}}(\dagger Z)$ -coherent form a full subcategory of the category of $\mathcal{D}_{\hat{X}}(\dagger Z)_{\mathbf{Q}}$ modules. In other words, the forgetful functor is fully faithful (use Spencer complexes to show this).

We will have to consider scalar extension and we will write when \mathcal{E} is a coherent $\mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger$ -module,

$$\mathcal{E}(\dagger Z) := \mathcal{D}_{\hat{X}}^\dagger(\dagger Z)_{\mathbf{Q}} \otimes_{\mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger} \mathcal{E}.$$

Then, there exists an exact triangle

$$i_{Z+} i_Z^\dagger \mathcal{E} \rightarrow \mathcal{E} \rightarrow \mathcal{E}(\dagger Z) \rightarrow .$$

We want to explain now, as in proposition 4.4.3 of [5], that if $U = X \setminus Z$, and E is a coherent overconvergent ∇ -module on $]U[$, then the action of $\mathcal{D}_{\hat{X}}^\dagger(\dagger Z)$ on $\widehat{\mathrm{sp}}_* i_{U*} E$ extends to an action of $\mathcal{D}_{\hat{X}}^\dagger(\dagger Z)$.

Assume that t is a local parameter defined on some open subset $\mathcal{U}' = \mathrm{Spf} \mathcal{A}'$ with $Z \cap \mathcal{U}' = \{x\}$ where x is the only zero of t and that t is defined on a lifting $\mathrm{Spec} A$ of U . The coherent ∇ -module E is given by a finite ∇ - A_K^\dagger -module M and

$$\Gamma(\mathcal{U}', \widehat{\mathrm{sp}}_* i_{U*} E) = \mathcal{A}'[1/t]_{\mathbf{Q}}^\dagger \otimes_{A_K^\dagger} M =: M'.$$

Also, we can write $M = \varinjlim_{\lambda < 1} M_\lambda$ where M_λ is a finite ∇ - A_λ -module so that

$$M' = \varinjlim_{\lambda < 1} M'_\lambda \quad \text{with} \quad M'_\lambda := \mathcal{A}'_K \{\lambda/t\} \otimes_{A_\lambda} M.$$

If $P \in \Gamma(\mathcal{U}', \mathcal{D}_{\hat{X}}^\dagger(\dagger Z))$, then there exists $c > 0$ and $\eta < 1$ such that

$$P = \sum_{k,l=0}^{\infty} \frac{1}{k!} \frac{f_{k,l}}{t^l} \frac{\partial^k}{\partial t^k} \quad \text{with} \quad \|f_{k,l}\| \leq c\eta^{k+l}.$$

If M is overconvergent and λ close enough to 1, we have $\|\frac{1}{k!} \frac{\partial^k(s)}{\partial t^k}\| \eta^k \rightarrow 0$ for $s \in M_\lambda$. Of course, we will also have $\lambda > \eta$ for λ close enough to one. Then, we see that if s' denotes the restriction of s to $\mathcal{U}'_K \cap V_\lambda$, we have

$$\left\| \frac{1}{k!} \frac{f_{k,l}}{t^l} \frac{\partial^k(s')}{\partial t^k} \right\| \leq c \frac{\eta^{k+l}}{\lambda^l} \left\| \frac{1}{k!} \frac{\partial^k(s')}{\partial t^k} \right\| \leq c \left\| \frac{1}{k!} \frac{\partial^k(s')}{\partial t^k} \right\| \eta^k \rightarrow 0$$

and we can set

$$P(s') = \sum_{k,l=0}^{\infty} \frac{1}{k!} \frac{f_{k,l}}{t^l} \frac{\partial^k(s')}{\partial t^k}.$$

In other words, we can extend the action by continuity on s' . Since the question is local on \hat{X} (and also, that it is sufficient to define the action on a finite set of generators), we see that the action of $\mathcal{D}_{\hat{X}}^\dagger(\dagger Z)$ on $\widehat{\mathrm{sp}}_* i_{U*} E$ does extend to an action of $\mathcal{D}_{\hat{X}}^\dagger(\dagger Z)$.

Of course, by continuity, any horizontal map will give a $\mathcal{D}_{\hat{X}}^\dagger(\dagger Z)$ -linear map and we have proven the following:

Proposition 6.3 *If U is a non empty affine open subset of X_k and Z denotes its closed complement, then the functor $\widehat{\mathrm{sp}}_* i_{U*}$ induces a fully faithful functor from the category of coherent overconvergent ∇ -modules E on $]U[$ to the category of $\mathcal{D}_{\hat{X}}^\dagger(\dagger Z)_{\mathbf{Q}}$ -modules. \square*

Actually, it is a lot better:

Theorem 6.4 (Berthelot) *The essential image of this functor is the category of coherent $\mathcal{D}_{\hat{X}}^\dagger(\dagger Z)_{\mathbf{Q}}$ -modules that are $\mathcal{O}_{\hat{X}}(\dagger Z)_{\mathbf{Q}}$ -coherent.*

Proof: This is exactly the content of a letter from Berthelot to Caro ([8]). Actually, the condition is even weaker since we can simply assume that the restriction to the formal lifting \mathcal{U} of U is $\mathcal{O}_{\mathcal{U}}$ -coherent. \square

We want to extend these results to constructible convergent ∇ -modules on X_K^{an} . For the moment, we can prove the following:

Proposition 6.5 *$\text{R}\widetilde{\text{sp}}_*$ induces a functor from the category of constructible convergent ∇ -modules on X_K^{an} to the category of perverse $\mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger$ -modules.*

Proof: We know from proposition 4.12 that $\text{R}\widetilde{\text{sp}}_*E$ is a perverse $\mathcal{D}_{\hat{X}\mathbf{Q}}$ -module. Moreover, we gave in proposition 3.8 a description of the terms of this complex. Since \mathcal{H}_Z^\dagger is a $\mathcal{D}_{\hat{X}}^\dagger$ -module, it follows from proposition 6.4 that both terms of $\text{R}\widetilde{\text{sp}}_*E$ are $\mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger$ -modules. It only remains to check that the non-trivial map in this complex is $\mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger$ -linear.

We showed in theorem 5.15 that a constructible convergent ∇ -module E on X_K^{an} is given by an overconvergent ∇ - A_K^\dagger -module M where $\text{Spec } A \subset X$ is non empty affine open subset with reduction U , a collection of finite dimensional $K(a)$ -vector spaces H_a for each point $x \notin U$, where a is an unramified lifting of x , and horizontal A_K^\dagger -linear maps $M \rightarrow \mathcal{R}_a \otimes_{K(a)} H_a$. We explained at the end of section 3 how to construct $\text{R}\widetilde{\text{sp}}_*E$ from these data. If t is a local parameter defined on some open subset $\mathcal{U}' = \text{Spf } \mathcal{A}'$ with $Z \cap \mathcal{U}' = \{x\}$ where x is the only zero of t , and t is actually defined on a lifting $\text{Spec } A$ of U , we have

$$\Gamma(\mathcal{U}', \text{R}\widetilde{\text{sp}}_*E) = \left[\mathcal{A}'[1/t]_{\mathbf{Q}}^\dagger \otimes_{A_K^\dagger} M \rightarrow \delta_a \otimes_{K(a)} H_a \right].$$

Thus, we must show that the morphisms

$$\mathcal{A}'[1/t]_{\mathbf{Q}}^\dagger \otimes_{A_K^\dagger} M \rightarrow \delta_a \otimes_{K(a)} H_a$$

are $\Gamma(\mathcal{U}', \mathcal{D}_{\hat{X}}^\dagger)$ -linear. This follows from the fact that they are horizontal and continuous (recall that the action is defined on both sides by continuity). \square

Lemma 6.6 *Assume that $F : X \rightarrow X$ is a lifting of the absolute Frobenius of X_k . If E is a constructible convergent ∇ -module on X_K^{an} , then the isomorphism*

$$\hat{F}^* \text{R}\widetilde{\text{sp}}_*E \simeq \text{R}\widetilde{\text{sp}}_*F_K^{\text{an}*}E$$

is $\mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger$ -linear.

Proof: Again, this follows from the fact that it is horizontal and a continuity argument. \square

The following is a consequence of the local monodromy theorem:

Proposition 6.7 *Let $F : X \rightarrow X$ be a lifting of the absolute Frobenius of X_k . If E is a constructible F - ∇ -module on X_K^{an} , then $\text{R}\widetilde{\text{sp}}_*E$ is a holonomic F - $\mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger$ -module.*

Proof: Thanks to propositions 5.13 and 6.2, we may assume that $E = i_{U*}E_U$ with E_U coherent. We showed in proposition 3.2 that $\widetilde{\text{sp}}_* i_{U*}E = \text{sp}_*(i_{U*}E_U)_0$ and in proposition 5.7 that $(i_{U*}E_U)_0$ is an overconvergent F -isocrystal. Then, it is shown in theorem 4.3.4 of [9] (see also [21]) that $\text{sp}_*(i_{U*}E_U)_0$ is holonomic. \square

However, it is not clear that $\text{R}\widetilde{\text{sp}}_*$ is fully faithful on constructible F - ∇ -modules or that its image is exactly the category of perverse holonomic F - $\mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger$ -modules. Some more work is necessary.

7 Formal fibers

If $x \in X_k$ is a closed point, we can consider the completion \hat{x} of X along x (usually written \hat{X}^x). Note that if a is an unramified point over x , then $\hat{x} \simeq \text{Spf } \mathcal{V}(a)[[t]]$. As explained in [11] and [12], although \hat{x} is not a p -adic formal scheme, there exists a beautiful theory of arithmetic \mathcal{D} -modules on \hat{x} which is completely analogous to the theory for \hat{X} . The ring $\mathcal{O}_{\hat{x}\mathbf{Q}}$ is isomorphic to the set of bounded function on the open unit disc over $K(a)$. The ring $\mathcal{R}_x^{\text{bd}} := \mathcal{O}_{\hat{x}}(\dagger x)_{\mathbf{Q}}$ is isomorphic to the bounded Robba ring $\mathcal{R}_a^{\text{bd}}$ over $K(a)$. And we have

$$\delta_x = \mathcal{R}_x / \mathcal{O}_x^{\text{an}} \simeq \mathcal{R}_x^{\text{bd}} / \mathcal{O}_{\hat{x}\mathbf{Q}}.$$

The rings $\mathcal{D}_{\hat{x}}^\dagger$ and $\mathcal{D}_{\hat{x}}^\dagger(\dagger x)$ have exactly the same description as above (using the Gauss norm).

There is a canonical morphism of formal schemes $i_{\hat{x}} : \hat{x} \rightarrow \hat{X}$ which is formally étale. One can define for a coherent $\mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger$ -module \mathcal{E} , its *exceptional inverse image*

$$i_{\hat{x}}^! \mathcal{E} := \mathcal{D}_{\hat{x}\mathbf{Q}}^\dagger \otimes_{\mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger} \mathcal{E}$$

which is a coherent $\mathcal{D}_{\hat{x}\mathbf{Q}}^\dagger$ -module, and we have the following adjunction formula:

Lemma 7.1 *If \mathcal{E} is a coherent $\mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger$ -module (or perfect complex) and $x \in X_k$ is a closed point, then*

$$\text{RHom}_{\mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger}(\mathcal{E}, \mathcal{H}_{x\mathbf{Q}}^\dagger) \simeq i_{x*} \text{RHom}_{\mathcal{D}_{\hat{x}\mathbf{Q}}^\dagger}(i_{\hat{x}}^! \mathcal{E}, \delta_x).$$

Proof: We may clearly assume that $\mathcal{E} = \mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger$ and we fall back on lemma 3.4. \square

Definition 7.2 *Let \mathcal{E} be a coherent $\mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger$ -module (or perfect complex) and $x \in X_k$, a closed point. Then, the bounded Robba fiber of \mathcal{E} at x is*

$$\mathcal{R}_x^{\text{bd}}(\mathcal{E}) := i_{\hat{x}}^! \mathcal{E}(\dagger x).$$

If $Z \subset X_k$ is a finite closed subset and $x \in Z$, we can also define for a coherent $\mathcal{D}_{\hat{X}}^\dagger(\dagger Z)_{\mathbf{Q}}$ -module \mathcal{E} , its *exceptional inverse image*

$$i_{\hat{x}}^! \mathcal{E} := \mathcal{D}_{\hat{x}}^\dagger(\dagger x)_{\mathbf{Q}} \otimes_{\mathcal{D}_{\hat{X}}^\dagger(\dagger Z)_{\mathbf{Q}}} \mathcal{E}.$$

Note that if \mathcal{E} is a coherent $\mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger$ -module, we have

$$\mathcal{R}_x^{\text{bd}}(\mathcal{E}) \simeq i_x^! \mathcal{E}(\dagger Z).$$

In particular, the bounded Robba fiber is a generic invariant in the sense that

$$\mathcal{R}_x^{\text{bd}}(\mathcal{E}) \simeq \mathcal{R}_x^{\text{bd}}(\mathcal{E}(\dagger Z))$$

whenever Z is a finite closed subset of X_k . Finally, if \mathcal{E} is a $\mathcal{D}_{\hat{X}}^\dagger(\dagger Z)_{\mathbf{Q}}$ -module which is coherent both as $\mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger$ -module and as $\mathcal{O}_{\hat{X}}(\dagger Z)_{\mathbf{Q}}$ -module, then

$$\mathcal{R}_x^{\text{bd}}(\mathcal{E}) \simeq \mathcal{R}_x^{\text{bd}} \otimes_{\mathcal{O}_{\hat{X}}(\dagger Z)_{\mathbf{Q}}} \mathcal{E}$$

is a finite ∇ -module on $\mathcal{R}_x^{\text{bd}}$.

Definition 7.3 *Let \mathcal{E} be a coherent $\mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger$ -module. Then, \mathcal{E} has Frobenius type at a closed point x (resp. has Frobenius type) if $\mathcal{R}_x^{\text{bd}}(\mathcal{E})$ has a Frobenius structure (resp. a Frobenius structure at all closed points $x \in X_k$).*

Of course, a coherent $F\text{-}\mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger$ -module has Frobenius type.

Proposition 7.4 *Let $Z \subset X_k$ be a finite closed subset and $x \in Z$. Let \mathcal{E} be a $\mathcal{D}_{\hat{X}}^\dagger(\dagger Z)_{\mathbf{Q}}$ -module which is coherent both as $\mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger$ -module and as $\mathcal{O}_{\hat{X}}(\dagger Z)_{\mathbf{Q}}$ -module. Assume that \mathcal{E} has Frobenius type at x . Then, we have*

$$\text{RHom}_{\mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger}(\mathcal{E}, \mathcal{H}_x^\dagger) \simeq i_{x*} \text{RHom}_{\nabla}(\mathcal{R}_x(\mathcal{E}), \mathcal{R}_x).$$

Proof: Note first that, in our situation, we have $i_x^! \mathcal{E} = \mathcal{R}_x^{\text{bd}}(\mathcal{E})$. Thus, it follows from lemma 7.1 that

$$\text{RHom}_{\mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger}(\mathcal{E}, \mathcal{H}_x^\dagger) \simeq i_{x*} \text{RHom}_{\mathcal{D}_{\hat{x}\mathbf{Q}}^\dagger}(\mathcal{R}_x^{\text{bd}}(\mathcal{E}), \delta_x).$$

Since $\mathcal{R}_x^{\text{bd}}(\mathcal{E})$ is a finite $F\text{-}\nabla$ -module on $\mathcal{R}_x^{\text{bd}}$, it follows from theorem 3.3 of [11] and proposition 5.6 of [12] that

$$\text{RHom}_{\mathcal{D}_{\hat{x}\mathbf{Q}}^\dagger}(\mathcal{R}_x^{\text{bd}}(\mathcal{E}), \mathcal{O}_x^{\text{an}}) = 0.$$

But there is an exact sequence

$$0 \rightarrow \mathcal{O}_x^{\text{an}} \rightarrow \mathcal{R}_x \rightarrow \delta_x \rightarrow 0,$$

and we obtain

$$\begin{aligned} \text{RHom}_{\mathcal{D}_{\hat{x}\mathbf{Q}}^\dagger}(\mathcal{R}_x^{\text{bd}}(\mathcal{E}), \delta_x) &= \text{RHom}_{\mathcal{D}_{\hat{x}\mathbf{Q}}^\dagger}(\mathcal{R}_x^{\text{bd}}(\mathcal{E}), \mathcal{R}_x) \\ &= \text{RHom}_{\mathcal{D}_{\hat{x}}^\dagger(\dagger x)_{\mathbf{Q}}}(\mathcal{R}_x^{\text{bd}}(\mathcal{E}), \mathcal{R}_x) = \text{RHom}_{\nabla}(\mathcal{R}_x^{\text{bd}}(\mathcal{E}), \mathcal{R}_x) \\ &= \text{RHom}_{\nabla}(\mathcal{R}_x(\mathcal{E}), \mathcal{R}_x) \quad \square \end{aligned}$$

We consider now the analog definitions on the constructible side. Recall that if $x \in X_k$ is a closed point, we introduced in definition 2.1, the Robba ring at x as

$$\mathcal{R}_x := (i_{x*} \mathcal{O}_{]x[})_{] \xi[}.$$

Since $\mathcal{O}_{X_K^{\text{an}}] \xi[}$ is a field, the adjunction map $\mathcal{O}_{X_K^{\text{an}}] \xi[} \rightarrow \mathcal{R}_x$ takes values into the maximum subfield of \mathcal{R}_x which is exactly $\mathcal{R}_x^{\text{bd}}$.

Definition 7.5 Let $x \in X_k$ be a closed point. If E be a constructible module, the bounded Robba fiber of E at x is

$$\mathcal{R}_x^{\text{bd}}(E) := \mathcal{R}_x^{\text{bd}} \otimes_{\mathcal{O}_{X_K^{\text{an}}}]_{\xi]} E]_{\xi]}.$$

A constructible ∇ -module E has Frobenius type at x if $\mathcal{R}_x^{\text{bd}}(E)$ has a Frobenius structure.

Note that this definition is generic in the sense that if $U \subset X_k$ is any open subset, then $\mathcal{R}_x^{\text{bd}}(i_{U*}i_U^{-1}E) = \mathcal{R}_x^{\text{bd}}(E)$. We globalize now the definition:

Definition 7.6 A constructible convergent ∇ -module E has Frobenius type if

1. E has Frobenius type at all closed points $x \in X_k$.
2. $\text{R}\widetilde{\text{sp}}_*E$ is a coherent $\mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger$ -module.

It is very likely that the second condition is not necessary and it is also possible that the convergence condition is automatic. Anyway, we have the following:

Proposition 7.7 If $F : X \rightarrow X$ is a lifting of the absolute Frobenius of X_k , then any constructible F - ∇ -module on X_K^{an} has Frobenius type.

Proof: The Frobenius structure on E will induce a Frobenius structure on all the bounded Robba fibers. Moreover, we saw in proposition 6.7 that $\text{R}\widetilde{\text{sp}}_*E$ is holonomic and in particular coherent. \square

For computations, it is convenient to have also at our disposal a more algebraic approach. If $\text{Spec } A \subset X$ is an algebraic lifting of the complement U of Z , taking global sections on the map $\mathcal{O}_{\hat{X}}(\dagger Z)_{\mathbf{Q}} \rightarrow i_{\hat{x}*}\mathcal{O}_{\hat{x}}(\dagger x)_{\mathbf{Q}}$ will give a morphism $A_K^\dagger \rightarrow \mathcal{R}_x^{\text{bd}}$. Alternatively, this is the composite map

$$A_K^\dagger \rightarrow \mathcal{O}_{X_K^{\text{an}}}]_{\xi]} \rightarrow \mathcal{R}_x^{\text{bd}}.$$

Definition 7.8 Let $x \in X_k$ be a closed point. Let U be a non empty affine open subset of X_k with algebraic lifting $\text{Spec } A$ and M a finite A_K^\dagger -module. Then, the bounded Robba fiber of M at x is

$$\mathcal{R}_x^{\text{bd}}(M) := \mathcal{R}_x^{\text{bd}} \otimes_{A_K^\dagger} M.$$

A finite ∇ - A_K^\dagger -module M has Frobenius type at a closed point x (resp. has Frobenius type) if the bounded Robba fiber at x (resp. at all closed points $x \in X_k$) has a Frobenius structure.

Recall that we also introduced in definition 2.3 the Robba fiber of M and that, clearly, $\mathcal{R}_x(M) = \mathcal{R}_x \otimes_{\mathcal{R}_x^{\text{bd}}} \mathcal{R}_x^{\text{bd}}(M)$.

Lemma 7.9 Let $x \in X_k$ be a closed point.

1. Let $Z \subset X_k$ be a finite closed subset and \mathcal{E} be a $\mathcal{D}_{\hat{X}}^\dagger(\dagger Z)_{\mathbf{Q}}$ -module which is coherent both as $\mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger$ -module and as $\mathcal{O}_{\hat{X}}(\dagger Z)_{\mathbf{Q}}$ -module. If $M := \Gamma(\hat{X}, \mathcal{E})$, we have

$$\mathcal{R}_x^{\text{bd}}(\mathcal{E}) = \mathcal{R}_x^{\text{bd}}(M).$$

2. Let U be an open subset of X_k and E a coherent $i_U^{-1}\mathcal{O}_{X_K^{\text{an}}}$ -module. If $M := \Gamma(\text{]U[}, E)$, we have

$$\mathcal{R}_x^{\text{bd}}(E) = \mathcal{R}_x^{\text{bd}}(M).$$

Proof: Since all functors are right exact, the assertions follow from the cases $\mathcal{E} = \mathcal{O}_{\hat{X}}(\dagger Z)_{\mathbf{Q}}$ and $E = i_U^{-1}\mathcal{O}_{X_K^{\text{an}}}$. \square

Proposition 7.10 *Let E be a constructible convergent ∇ -module and $\mathcal{E} := \text{R}\widetilde{\text{sp}}_*E$. Assume that \mathcal{E} is a coherent $\mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger$ -module. Then, if $x \in X_k$ is any closed point, we have*

$$\mathcal{R}_x^{\text{bd}}(\mathcal{E}) = \mathcal{R}_x^{\text{bd}}(E).$$

*In particular E has Frobenius type (at x) if and only if $\text{R}\widetilde{\text{sp}}_*E$ has Frobenius type (at x).*

Proof: If U is an open subset of X_k with closed complement Z , we know that

$$\mathcal{R}_x^{\text{bd}}(\mathcal{E}) = \mathcal{R}_x^{\text{bd}}(\mathcal{E}(\dagger Z)) \quad \text{and} \quad R_x^{\text{bd}}(E) = R_x^{\text{bd}}(i_{U*}i_U^{-1}E).$$

We may therefore assume that $E = i_{U*}E_U$ with E_U coherent as $i_U^{-1}\mathcal{O}_{X_K^{\text{an}}}$ -module. And then use lemma 7.9. \square

8 The Deligne-Kashiwara correspondence

On the analytic side, we will try to stick to the ‘‘connection’’ vocabulary and write $\text{R}\mathcal{H}om_{\nabla}(E', E'')$ when E' and E'' are two ∇ -modules on X_K^{an} for example, but we will systematically identify this space with $\text{R}\mathcal{H}om_{\mathcal{D}_{X_K^{\text{an}}}}(E', E'')$.

Proposition 8.1 *If E is a ∇ -module on X_K^{an} , then*

$$\text{R}\widetilde{\text{sp}}_*\text{R}\mathcal{H}om_{\nabla}(\mathcal{O}_{X_K^{\text{an}}}, E) \simeq \text{R}\mathcal{H}om_{\mathcal{D}_{\hat{X}\mathbf{Q}}}(\mathcal{O}_{\hat{X}\mathbf{Q}}, \text{R}\widetilde{\text{sp}}_*E).$$

Actually, if E is a constructible convergent ∇ -module, we have

$$\text{R}\widetilde{\text{sp}}_*\text{R}\mathcal{H}om_{\nabla}(\mathcal{O}_{X_K^{\text{an}}}, E) \simeq \text{R}\mathcal{H}om_{\mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger}(\mathcal{O}_{\hat{X}\mathbf{Q}}, \text{R}\widetilde{\text{sp}}_*E).$$

Proof: The *analytic Spencer complex*

$$[\mathcal{H}om_{\mathcal{O}_{X_K^{\text{an}}}}(\Omega_{X_K^{\text{an}}}^1, \mathcal{D}_{X_K^{\text{an}}}) \rightarrow \mathcal{D}_{X_K^{\text{an}}}],$$

which is locally given by $u \mapsto u(dt)\frac{\partial}{\partial t}$, is a locally free resolution of $\mathcal{O}_{X_K^{\text{an}}}$. It follows that

$$\text{R}\mathcal{H}om_{\mathcal{D}_{X_K^{\text{an}}}}(\mathcal{O}_{X_K^{\text{an}}}, E) = E \otimes_{\mathcal{O}_{X_K^{\text{an}}}} \Omega_{X_K^{\text{an}}}^\bullet.$$

But it is also true that the *algebraic Spencer complex*

$$[\mathcal{H}om_{\mathcal{O}_{\hat{X}\mathbf{Q}}}(\Omega_{\hat{X}\mathbf{Q}}^1, \mathcal{D}_{\hat{X}\mathbf{Q}}) \rightarrow \mathcal{D}_{\hat{X}\mathbf{Q}}]$$

is a resolution of $\mathcal{O}_{\hat{X}\mathbf{Q}}$ so that

$$\text{R}\mathcal{H}om_{\mathcal{D}_{\hat{X}\mathbf{Q}}}(\mathcal{O}_{\hat{X}\mathbf{Q}}, \text{R}\widetilde{\text{sp}}_*E) = [\text{R}\widetilde{\text{sp}}_*E \otimes_{\mathcal{O}_{\hat{X}\mathbf{Q}}} \Omega_{\hat{X}\mathbf{Q}}^\bullet].$$

The first result follows since

$$\mathrm{R}\widetilde{\mathrm{sp}}_*(E \otimes_{\mathcal{O}_{X_K^{\mathrm{an}}}} \Omega_{X_K^{\mathrm{an}}}^\bullet) = [\mathrm{R}\widetilde{\mathrm{sp}}_* E \otimes_{\mathcal{O}_{\hat{X}\mathbf{Q}}} \Omega_{\hat{X}\mathbf{Q}}^\bullet].$$

Now, if E is a constructible convergent ∇ -module, we can consider the *arithmetic Spencer complex*

$$[\mathcal{H}om_{\mathcal{O}_{\hat{X}\mathbf{Q}}}(\Omega_{\hat{X}\mathbf{Q}}^1, \mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger) \rightarrow \mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger],$$

which is also a resolution of $\mathcal{O}_{\hat{X}\mathbf{Q}}$ (proposition 4.3.3 of [6]) and get

$$\mathrm{R}\mathcal{H}om_{\mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger}(\mathcal{O}_{\hat{X}\mathbf{Q}}, \mathrm{R}\widetilde{\mathrm{sp}}_* E) = [\mathrm{R}\widetilde{\mathrm{sp}}_* E \otimes_{\mathcal{O}_{\hat{X}\mathbf{Q}}} \Omega_{\hat{X}\mathbf{Q}}^\bullet]. \quad \square$$

Corollary 8.2 *If U is an open subset of X_k and E' and E'' are two coherent ∇ -modules on $]U[$, then*

$$\mathrm{R}\widetilde{\mathrm{sp}}_* \mathrm{R}\mathcal{H}om_{\nabla}(i_{U*} E', i_{U*} E'') \simeq \mathrm{R}\mathcal{H}om_{\mathcal{D}_{\hat{X}\mathbf{Q}}}(\widetilde{\mathrm{sp}}_* i_{U*} E', \widetilde{\mathrm{sp}}_* i_{U*} E'').$$

Actually, if E' and E'' are overconvergent, we have

$$\mathrm{R}\widetilde{\mathrm{sp}}_* \mathrm{R}\mathcal{H}om_{\nabla}(i_{U*} E', i_{U*} E'') \simeq \mathrm{R}\mathcal{H}om_{\mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger}(\widetilde{\mathrm{sp}}_* i_{U*} E', \widetilde{\mathrm{sp}}_* i_{U*} E'')$$

Proof: It follows from proposition 3.3 that if

$$E := \mathcal{H}om_{i_U^{-1} \mathcal{O}_{X_K^{\mathrm{an}}}}(E', E'')$$

then, we have

$$E \simeq \mathcal{H}om_{\mathcal{O}_{\hat{X}}(\dagger Z)\mathbf{Q}}(\widetilde{\mathrm{sp}}_* i_{U*} E', \widetilde{\mathrm{sp}}_* i_{U*} E'').$$

One can use the 1-stratifications in order to show that this isomorphism is horizontal. Moreover, since E' is coherent, E will be overconvergent when E' and E'' are. And we can then apply the proposition to $i_{U*} E$. \square

Lemma 8.3 *Let U be an open subset of X_k with closed complement Z . If E' is a locally trivial ∇ -module on $]Z[$ and E'' is a coherent overconvergent ∇ -module on $]U[$, we have*

$$\mathrm{R}\mathcal{H}om_{\nabla}(i_{Z!} E', i_{U*} E'') = 0 \quad \text{and} \quad \mathrm{R}\mathcal{H}om_{\mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger}(\mathrm{R}\widetilde{\mathrm{sp}}_* i_{Z!} E', \widetilde{\mathrm{sp}}_* i_{U*} E'') = 0.$$

Proof: First of all, we have $i_U^{-1} i_{Z!} E' = 0$ and therefore, by adjunction,

$$\mathrm{R}\mathcal{H}om_{\mathcal{O}_{X_K^{\mathrm{an}}}}(i_{Z!} E', i_{U*} E'') = \mathrm{R}\mathcal{H}om_{i_U^{-1} \mathcal{O}_{X_K^{\mathrm{an}}}}(i_U^{-1} i_{Z!} E', E'') = 0.$$

The first assertion follows. For the second one, we can write

$$\mathrm{R}\widetilde{\mathrm{sp}}_* i_{Z!} E' = i_{Z+} H[-1] \quad \text{and} \quad \widetilde{\mathrm{sp}}_* i_{U*} E'' = \mathcal{E}$$

where H is a coherent \mathcal{O}_{D_K} -module, D being a smooth lifting of Z , and \mathcal{E} is a $\mathcal{D}_{\hat{X}}^\dagger(\dagger Z)\mathbf{Q}$ -module. Note that

$$(i_{Z+} H)(\dagger Z) := \mathcal{D}_{\hat{X}}^\dagger(\dagger Z) \otimes_{\mathcal{D}_{\hat{X}}^\dagger} i_{Z+} H = 0.$$

But then, by adjunction, we have

$$\mathrm{R}\mathcal{H}om_{\mathcal{D}_{\hat{X}}^\dagger} (i_{Z+}H, \mathcal{E}) = \mathrm{R}\mathcal{H}om_{\mathcal{D}_{\hat{X}}^\dagger(i_Z)\mathcal{Q}} ((i_{Z+}H)(\dagger Z), \mathcal{E}) = 0. \quad \square$$

Recall that we defined for a constructible module E , its Robba fiber (resp. bounded Robba fiber) at a closed point $x \in X_k$, as

$$\mathcal{R}_x(E) = \mathcal{R}_x \otimes_{\mathcal{O}_{X_K^{\mathrm{an}}}]_{\xi[\]} E]_{\xi[\]} \quad (\text{resp.} \quad \mathcal{R}_x^{\mathrm{bd}}(E) = \mathcal{R}_x^{\mathrm{bd}} \otimes_{\mathcal{O}_{X_K^{\mathrm{an}}}]_{\xi[\]} E]_{\xi[\]}).$$

Lemma 8.4 *Let U be an open subset of X_k and $x \notin U$. If E is a coherent $i_U^{-1}\mathcal{O}_{X_K^{\mathrm{an}}}$ -module, then*

$$\mathrm{R}\mathcal{H}om_{\mathcal{O}_{X_K^{\mathrm{an}}}]_{\xi[\]} (i_{U*}E, i_{x!}\mathcal{O}_{]x[}) = i_{\xi*}\mathrm{Hom}_{\mathcal{R}_x}(\mathcal{R}_x(E), \mathcal{R}_x)$$

If E is a coherent ∇ -module on $]U[$, we have

$$\mathrm{R}\mathcal{H}om_{\nabla}(i_{U*}E, i_{x!}\mathcal{O}_{]x[}) = i_{\xi*}\mathrm{Hom}_{\nabla}(\mathcal{R}_x(E), \mathcal{R}_x).$$

Proof: By adjunction, we have

$$\mathrm{R}\mathcal{H}om_{\mathcal{O}_{X_K^{\mathrm{an}}}]_{\xi[\]} (i_{U*}E, i_{x!}\mathcal{O}_{]x[}) = i_{\xi*}\mathrm{Hom}_{\mathcal{O}_{X_K^{\mathrm{an}}}]_{\xi[\]} (E]_{\xi[\]}, \mathcal{R}_x)$$

and the first assertion follows by extending scalars. Then, we can write

$$\begin{aligned} \mathrm{R}\mathcal{H}om_{\nabla}(i_{U*}E, i_{x!}\mathcal{O}_{]x[}) &= \mathrm{R}\mathcal{H}om_{\mathcal{D}_{X_K^{\mathrm{an}}}}(\mathcal{O}_{X_K^{\mathrm{an}}}, \mathrm{R}\mathcal{H}om_{\mathcal{O}_{X_K^{\mathrm{an}}}]_{\xi[\]} (i_{U*}E, i_{x!}\mathcal{O}_{]x[})) \\ &= \mathrm{R}\mathcal{H}om_{\mathcal{D}_{X_K^{\mathrm{an}}}}(\mathcal{O}_{X_K^{\mathrm{an}}}, i_{\xi*}\mathrm{Hom}_{\mathcal{O}_{X_K^{\mathrm{an}}}]_{\xi[\]} (E]_{\xi[\]}, \mathcal{R}_x)) \\ &= i_{\xi*}\mathrm{R}\mathcal{H}om_{\mathcal{D}_{X_K^{\mathrm{an}}}]_{\xi[\]}(\mathcal{O}_{X_K^{\mathrm{an}}}]_{\xi[\]}, \mathrm{Hom}_{\mathcal{O}_{X_K^{\mathrm{an}}}]_{\xi[\]} (E]_{\xi[\]}, \mathcal{R}_x)) \\ &= i_{\xi*}\mathrm{R}\mathcal{H}om_{\nabla}(E]_{\xi[\]}, \mathcal{R}_x) = i_{\xi*}\mathrm{R}\mathcal{H}om_{\nabla}(\mathcal{R}_x(E), \mathcal{R}_x) \quad \square \end{aligned}$$

Recall that a constructible ∇ -module E is said to have Frobenius type at a closed point $x \in X_k$ if its bounded Robba Fiber at this point has a Frobenius structure.

Proposition 8.5 *Let E be a coherent overconvergent ∇ -module on $]U[$ and $x \notin U$. Assume E has Frobenius type at x and that $\widetilde{\mathrm{sp}}_*i_{U*}E$ is $\mathcal{D}_{\hat{X}\mathcal{Q}}^\dagger$ -coherent. Then, we have*

$$\mathrm{R}\widetilde{\mathrm{sp}}_*\mathrm{R}\mathcal{H}om_{\nabla}(i_{U*}E, i_{x!}\mathcal{O}_{]x[}) \simeq \mathrm{R}\mathcal{H}om_{\mathcal{D}_{\hat{X}\mathcal{Q}}^\dagger} (\widetilde{\mathrm{sp}}_*i_{U*}E, \mathcal{H}_x^\dagger)[-1].$$

Proof: Follows from lemma 8.4 and proposition 7.1. \square

Recall from definition 7.6, that a constructible convergent ∇ -module E has Frobenius type if it has Frobenius type at all closed points and $\mathrm{R}\widetilde{\mathrm{sp}}_*E$ is a coherent $\mathcal{D}_{\hat{X}\mathcal{Q}}^\dagger$ -module.

Theorem 8.6 *Let E' and E'' be two constructible convergent ∇ -module on X_K^{an} . Assume that E' has Frobenius type. Then, we have*

$$\mathrm{R}\widetilde{\mathrm{sp}}_*\mathrm{R}\mathcal{H}om_{\nabla}(E', E'') \simeq \mathrm{R}\mathcal{H}om_{\mathcal{D}_{\hat{X}\mathcal{Q}}^\dagger} (\mathrm{R}\widetilde{\mathrm{sp}}_*E', \mathrm{R}\widetilde{\mathrm{sp}}_*E'').$$

Proof: Since E' and E'' are constructible, there exists an open subset U of X_k such that both $i_U^{-1}E'$ and $i_U^{-1}E''$ are coherent. If we use corollary 5.13 for E'' , it follows from corollary 8.2 and proposition 8.5 that

$$\mathrm{R}\widetilde{\mathrm{sp}} \mathrm{R}\mathcal{H}om_{\nabla}(i_{U*}i_U^{-1}E', E'') \simeq \mathrm{R}\mathcal{H}om_{\mathcal{D}_{\hat{X}\mathbf{Q}}^{\dagger}}(\widetilde{\mathrm{sp}}_*i_{U*}i_U^{-1}E', \mathrm{R}\widetilde{\mathrm{sp}}_*E'').$$

If we apply this result to the case $E' = \mathcal{O}_{X_K^{\mathrm{an}}}$ and use corollary 5.13 for $\mathcal{O}_{X_K^{\mathrm{an}}}$, then proposition 8.1 gives

$$\mathrm{R}\widetilde{\mathrm{sp}} \mathrm{R}\mathcal{H}om_{\nabla}(i_{x!}\mathcal{O}_{]x[}, E'') \simeq \mathrm{R}\mathcal{H}om_{\mathcal{D}_{\hat{X}\mathbf{Q}}^{\dagger}}(\mathcal{H}_x^{\dagger}[-1], \mathrm{R}\widetilde{\mathrm{sp}}_*E'').$$

And we can apply again corollary 5.13, but to E' this time. \square

Corollary 8.7 *The functor $\mathrm{R}\widetilde{\mathrm{sp}}_*$ induces a fully faithful functor from the category of constructible convergent ∇ -modules of Frobenius type to the category of $\mathcal{D}_{\hat{X}\mathbf{Q}}^{\dagger}$ -modules.*
 \square

We want now to understand its essential image. Since we are only interested in curves, we can make the following ad hoc definition:

Definition 8.8 *A coherent $\mathcal{D}_{\hat{X}\mathbf{Q}}^{\dagger}$ -module \mathcal{E} is holonomic if there exists a finite closed subset Z of X_k such that $\mathcal{E}(\dagger Z)$ is $\mathcal{O}_{\hat{X}}(\dagger Z)_{\mathbf{Q}}$ -coherent. A bounded complex of $\mathcal{D}_{\hat{X}\mathbf{Q}}^{\dagger}$ -modules is holonomic if it has holonomic cohomology.*

Note that an F - $\mathcal{D}_{\hat{X}\mathbf{Q}}^{\dagger}$ -module is holonomic in the usual sense if and only if the underlying module is holonomic in this sense.

Proposition 8.9 *If \mathcal{E} is a perverse holonomic $\mathcal{D}_{\hat{X}\mathbf{Q}}^{\dagger}$ -module of Frobenius type, there exists a constructible convergent ∇ -module E on X_K^{an} such that $\mathrm{R}\widetilde{\mathrm{sp}}_*E = \mathcal{E}$.*

Proof: Let \mathcal{E} be a perverse holonomic $\mathcal{D}_{\hat{X}\mathbf{Q}}^{\dagger}$ -module. By definition, it is a complex with holonomic cohomology concentrated in degree 0 and 1, flat in degree 0 and finitely supported in degree 1. Let Z be a finite closed subset of X_k such that $\mathcal{H}^1(\mathcal{E})$ is supported in Z and $\mathcal{H}^q(\mathcal{E})(\dagger Z)$ is $\mathcal{O}_{\hat{X}}(\dagger Z)_{\mathbf{Q}}$ -coherent (for $q = 0, 1$). Since \mathcal{E} is coherent, we have an exact triangle

$$i_{Z+}i_Z^!\mathcal{E} \rightarrow \mathcal{E} \rightarrow \mathcal{E}(\dagger Z) \rightarrow \dots$$

First of all, we see that $\mathcal{H}^q(\mathcal{E}(\dagger Z)) = \mathcal{H}^q(\mathcal{E})(\dagger Z) = 0$ when $q \neq 0$. In other words, $\mathcal{E}(\dagger Z)$ is a coherent $\mathcal{D}_{\hat{X}\mathbf{Q}}^{\dagger}$ -module which is also $\mathcal{O}_{\hat{X}}(\dagger Z)_{\mathbf{Q}}$ -coherent. Thanks to theorem 6.4, we can write $\mathcal{E}(\dagger Z) = \widetilde{\mathrm{sp}}_*i_{U*}E_U$ with E_U a coherent overconvergent ∇ -module on $]U[$.

By definition, if x is a point in Z and a a non ramified lifting of x , we have

$$i_x^!\mathcal{E} = \mathrm{Li}_x^*\mathcal{E}[-1] = K(a) \otimes_{\mathcal{O}_{\hat{X}\mathbf{Q},x}}^{\mathrm{L}} \mathcal{E}_x[-1]$$

If x is defined as the zero of a local parameter t we can use the free left resolution

$$\mathcal{O}_{\hat{X}\mathbf{Q},x} \xrightarrow{t} \mathcal{O}_{\hat{X}\mathbf{Q},x}$$

of $K(a)$ to compute $i_x^! \mathcal{E}$. Then, there is a spectral sequence (or workout the bicomplex if you prefer)

$$E_2^{p,q} := \mathcal{H}^p \left(\mathcal{H}^q(\mathcal{E}_x) \xrightarrow{t} \mathcal{H}^q(\mathcal{E}_x) \right) \Rightarrow i_x^! \mathcal{E}$$

with $H^q(\mathcal{E}_x)$ flat in degree 0, t -torsion in degree 1 and 0 otherwise. It follows that $i_x^! \mathcal{E}$ is just a finite dimensional vector space placed in degree 1. Thanks to proposition 6.2, we can therefore write $i_{Z+} i_Z^! \mathcal{E} = \mathrm{R}\widetilde{\mathrm{sp}}_* i_Z^! E_Z$ with E_Z a locally trivial ∇ -module on $]Z[$. And the assertion then follows from theorem 8.6. \square

Corollary 8.10 *The functor $\mathrm{R}\widetilde{\mathrm{sp}}_*$ induces an equivalence between the category of constructible convergent ∇ -modules of Frobenius type on X_K^{an} and perverse holonomic $\mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger$ -modules of Frobenius type.* \square

We can at last prove the overconvergent Deligne-Kashiwara correspondence for curves (when there exists a global lifting of Frobenius):

Theorem 8.11 *Assume that $F : X \rightarrow X$ is a lifting of the absolute Frobenius of X_k . Then, the functor $\mathrm{R}\widetilde{\mathrm{sp}}_*$ induces an equivalence between constructible F - ∇ -modules on X_K^{an} and perverse holonomic F - $\mathcal{D}_{\hat{X}\mathbf{Q}}^\dagger$ -modules on \hat{X} .*

Proof: Follows from corollary 8.7, proposition 6.7 and proposition 8.9. \square

Note that this theorem extends to the case where there is no global lifting of Frobenius. Actually, one expects an analog result in higher dimension. However, we believe that our methods reach their limit here and that a crystalline approach will be more appropriate to go further.

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