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REMARKS ON SYMMETRIES OF 2
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V.ARTAMONOV, S.SÀNCHEZ

ABSTRACT. The paper presents a strict mathematical proof of the classifications of finite groups of symmetries of 2-dimensional quasicrystals given in [4].

INTRODUCTION

There are various approaches to a construction of mathematical models of quasicrystals and a definition of their symmetries, see for example [1], [2, Chapter 6], [4], [5], [9]. We shall adopt the following model which is usually call *cut and project scheme* [9]. Nowadays the common definition of this method is the following one. Let V be an additive locally compact topological Abelian group, U an additive group of a real *physical* or *internal* vector space of dimension d and M a discrete subgroup in $E = U \oplus V$ such that E/M is compact and $M \cap V$ consists only of zero vector. The group E is often called a *hyperspace* or *embedding space*. Consider the diagram of projections of groups

$$\begin{array}{ccc}
 U & \xleftarrow{\pi} & E = U \oplus V \xrightarrow{\rho} V \\
 & & \cup \\
 & & M
 \end{array}$$

Since the intersection of V and M consists only of zero element the projection π is injective on M . It is assumed that $\rho(M)$ is dense in V .

Let us take a convex compact subset W in V called a *window*. Let us put $Q = \rho^{-1}(W) \cap M$. Then the set $\pi(Q)$ is called a *model set* or a *cut and project set*. Note that the set Q under the projection π is mapping injectively into U .

It is necessary to mention [8, Chapter 2, §3.2] that any locally compact Abelian group V is an inverse limit of direct product of additive groups $\mathbb{R}^m \times \mathbb{T}^s \times D$ where \mathbb{T} is a one-dimensional torus and D is a

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discrete group. So the case when V is a real vector space is in fact one of the most important.

Throughout the paper we shall consider the special case of the model when V is a normed real vector space of a finite dimension $n - d > 0$. Then E is a vector space of dimension $n > d$ and M is a lattice in E . It means that M is discrete and $E = \mathbb{R} \otimes_{\mathbb{Z}} M$. Equivalently the rank of the free Abelian group M is equal to n . We shall assume that Q contains a basis of M as a free Abelian group.

Since all norms in E are equivalent without loss of generality we can assume that E is an Euclidean space.

According to [1] and [2, Chapter 6] a symmetry of the model Q in this case is the group of all affine transformations of E which map Q bijectively onto itself. Since Q contains a basis of M each symmetry leaves the lattice M invariant. We shall now call all these symmetries *proper*. **It is necessary to mention that there exist non-proper symmetries. For example for Fibonacci tilings powers of the matrix**

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

show examples of non-proper symmetries.

Suppose that M has also zero intersection with V . If f is a symmetry of Q and 0 is the origin then $f(0) \in M \cap U = 0$. Hence each symmetry f is in fact a linear operator. As in [1] we can show that U is invariant under an action of f .

Generalizing this idea we shall denote by $\text{Sym } Q$ the subgroup in the group $\text{GL}(E)$ of all invertible linear operators in E such that both U and M are invariant under an action of each element from $\text{Sym } Q$. The group of proper symmetries is a subgroup of $\text{Sym } Q$. Each element of the group $\text{Sym } Q$ in some special basis of E has the lower left zero block from (1). Moreover there exists an invertible matrix $Z \in \text{GL}(n, \mathbb{R})$ such that $Z^{-1} \text{Sym } Q Z \subset \text{GL}(n, \mathbb{Z})$. These two conditions are equivalent to the definition of the group $\text{Sym } Q$.

In the paper we shall consider the case when $\dim U = \dim V = 2$ and G is a finite subgroup of $\text{Sym } Q$. The main results of the paper are in Theorems 2.5, 2.11, 2.12, 2.13, 2.14, 2.15. The list of these finite groups is given in [4] without proof. Here we justify this classification and expose strict mathematical proofs based on group theory approach. The main results of the present paper are complementary to those in [4], [3] where realizations of some symmetries are presented. We give a list of group and in also in §3 their representation in 4-dimensional hyperspace.

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1. Matrix representations of symmetries

Suppose that e_1, \dots, e_n is a basis of the lattice M . Then these elements form a basis of E . If $\mathcal{A} \in \text{Sym } Q$ and A is a matrix of \mathcal{A} in the basis e_1, \dots, e_n . Since M is invariant under \mathcal{A} we can conclude that A is an integer matrix, $A \in \text{GL}(n, \mathbb{Z})$.

Let u_1, \dots, u_d be a basis of U and u_{d+1}, \dots, u_n a basis of V . Suppose that

$$(u_1, \dots, u_n) = (e_1, \dots, e_n)C, \quad C \in \text{GL}(n, \mathbb{R}).$$

Since U is invariant under \mathcal{A} we have

$$C^{-1}AC = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, \quad (1)$$

$$A_1 \in \text{GL}(d, \mathbb{R}), \quad A_3 \in \text{GL}(n-d, \mathbb{R}), \quad A_2 \in \text{Mat}(d \times (n-d), \mathbb{R}).$$

We shall use throughout the paper these properties of matrices representing elements from $\text{Sym } Q$.

2. 2D symmetries

In this section we shall assume that $n = 4$, $d = 2$ and G is a finite subgroup in $\text{Sym } Q$. Recall that a dihedral group D_k is a group of orthogonal symmetries of a regular k -gon [2]. D_k is a subgroup of the group of orthogonal matrices $\text{O}(2, \mathbb{R})$ generated by two matrices

$$a = \begin{pmatrix} \cos \frac{2\pi}{k} & -\sin \frac{2\pi}{k} \\ \sin \frac{2\pi}{k} & \cos \frac{2\pi}{k} \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (2)$$

It is known [2] that D_k consists of $2k$ elements

$$1, a, \dots, a^{k-1}, b, ba, \dots, ba^{k-1},$$

where $a^k = (ba^l)^2 = 1$ for any l . In particular $ba^l = a^{-l}b$. Note that

$$a^l = \begin{pmatrix} \cos \frac{2\pi l}{k} & -\sin \frac{2\pi l}{k} \\ \sin \frac{2\pi l}{k} & \cos \frac{2\pi l}{k} \end{pmatrix}, \quad ba^l = \begin{pmatrix} \cos \frac{2\pi l}{k} & \sin \frac{2\pi l}{k} \\ \sin \frac{2\pi l}{k} & -\cos \frac{2\pi l}{k} \end{pmatrix} \quad (3)$$

In order to classify finite subgroups in $\text{Sym } Q$ we need some auxiliary statements. We exclude some proofs of known facts.

Proposition 2.1. *Let H be a finite subgroup of $\text{O}(2, \mathbb{R})$. If $H \subset \text{SO}(2, \mathbb{R})$ then H is a cyclic group. If $H \not\subset \text{SO}(2, \mathbb{R})$, then $H = D_k$ for some k .*

Proposition 2.2. *Suppose that a finite group G is a subgroup of a direct product of two cyclic groups $H = \langle a_1 \rangle_{k_1} \times \langle a_2 \rangle_{k_2}$ of orders k_1 and k_2 respectively. Denote by $\nu_i : G \rightarrow \langle a_i \rangle_{k_i}$ the natural projection and suppose that ν_1, ν_2 are surjective. Then $(a_1^{s_1}, a_2)$, $(a_1, a_2^{s_2}) \in G$ for some $s_i \in \mathbb{Z}$ and therefore $a_i^{s_1 s_2 - 1} \in G \cap \langle a_i \rangle_{k_i}$. If $d = (k_1, k_2)$ is the greatest common divisor of k_1, k_2 then $a_i^d \in G \cap \langle a_i \rangle_{k_i}$ for $i = 1, 2$. In particular $G \cap \langle a_i \rangle_{k_i}$ is a cyclic group $\langle a_i^{l_i} \rangle$, where $l_i \mid d$ and $l_i \mid (s_1 s_2 - 1)$.*

Proof. Since ν_1 is surjective, then there exists an element $(a_1, b) \in G$ for some $b \in \langle a_2 \rangle$. Then $(a_1, b)^{k_2} = (a_1^{k_2}, b^{k_2}) = (a_1^{k_2}, 1) \in G \cap \langle a_1 \rangle$. Since the order of a_1 is k_1 by [2, Theorem 1.6, § 1.4] we can conclude that $a_1^d \in G \cap \langle a_1 \rangle$. Apply [2, Theorem 1.5, § 1.4]. Similarly if $(a_1^{s_1}, a_2), (a_1, a_2^{s_2}) \in G$, then

$$(a_1^{s_1}, a_2)^{s_2} (a_1, a_2^{s_2})^{-1} = (a_1^{s_1 s_2 - 1}, 1) \in G \cap \langle a_1 \rangle.$$

Similarly $a_2^{s_1 s_2 - 1} \in G \cap \langle a_2 \rangle$. \square

Recall that the *Euler function* $\phi(m)$, $m \in \mathbb{Z}$, is the number of all integers $1 \leq k < m$ which are coprime with m . Recall that if $\zeta = \exp \frac{2\pi i}{m}$ then $\phi(m)$ coincides with the degree of the minimal integer cyclotomic polynomial $\Phi_m(X) \in \mathbb{Z}[X]$ such that $\Phi_m(\zeta) = 0$ and with the order of the Galois group $\text{Gal}(\mathbb{Q}[\zeta]/\mathbb{Q})$.

Proposition 2.3. *Let m be a positive integer such that $\phi(m) \leq 4$. Then m is one of the numbers*

m	10	5	12	6	3	8	4	2	1
$\phi(m)$	4	4	4	2	2	4	2	1	1

Proposition 2.4 ([2], Theorem 2.16, § 2.4). *Let Δ be a finite subgroup in $\text{SO}(2, \mathbb{R})$. Then $\Delta = \langle a \rangle_k$ is a cyclic group of order k , where*

$$a = \begin{pmatrix} \cos \frac{2\pi}{k} & -\sin \frac{2\pi}{k} \\ \sin \frac{2\pi}{k} & \cos \frac{2\pi}{k} \end{pmatrix}.$$

Moreover, the following are equivalent

- 1) Δ is conjugate in $\text{GL}(2, \mathbb{R})$ to a subgroup of $\text{SL}(2, \mathbb{Z})$;
- 2) the trace $\text{tr } a = 2 \cos \frac{2\pi}{k} \in \mathbb{Z}$;
- 3) $k = 1, 2, 3, 4, 6$,
- 4) $\phi(k) \leq 2$.

Recall that a subgroup G of a direct product of groups $G_1 \times G_2$ is a *subdirect product* of G_1, G_2 if each projection $G \rightarrow G_i$, $i = 1, 2$, is surjective. In this case we also say that G is a subdirect subgroup of $G_1 \times G_2$.

Theorem 2.5. *Let G be a finite subgroup in $\text{Sym } Q$ where $\dim U = 2 = \dim V$. Then G is a subdirect product of either $C_{k_1} \times C_{k_2}$, $C_{k_1} \times D_{k_2}$, $D_{k_1} \times D_{k_2}$ (C_k : cyclic group of order k) The integers k_1, k_2 satisfy one of the following conditions*

- 1) $k_1, k_2 = 1, 2, 3, 4, 6$;
- 2) (k_1, k_2) is either **(5,10) or (10,5)**
- 3) $k_1 = k_2 = 12$.
- 4) $k_1 = k_2 = 8$.

Proof. By [2, Theorem 3.1, § 3.1] there exists a scalar product $[x, y]$ in E such that each linear operator from G is orthogonal with respect to the new product $[x, y]$. The physical space U is G -invariant. Let

W be an orthogonal complement of U with respect to $[x, y]$. Then W is also G -invariant. It means that there exists an invertible matrix C such that (1) holds with $A_2 = 0$. In this case G is a subgroup of a direct product $G \subseteq \nu_1(G) \times \nu_2(G)$, where ν_1, ν_2 are projections and each $\nu_i(G)$ is a finite subgroup of $O(2, R)$.

According to Proposition 2.1 each $\nu_j(G)$ is either a cyclic group $\langle a_j \rangle_{k_j}$ generated by the matrix a_j of order k_j from (2) or a dihedral group D_{k_j} generated by matrices a_j, b from (2). A subgroup $G \cap \langle a_j \rangle$ of a cyclic group $\langle a_j \rangle$ is cyclic generated by the element $a_j^{l_j}$ where l_j is the least exponent of elements from $G \cap \langle a_j \rangle$. Note that if, say $k_1 = 1$, then $G = G_2$ and applying Proposition 2.4 we deduce the statement of the theorem. Hence in what follows we can assume that $k_1, k_2 > 1$. Let $d = (k_1, k_2)$ be the greatest common divisor of k_1 and k_2 .

Without loss of generality we can assume that ν_1, ν_2 are surjective that is G is a subdirect subgroup of $\nu_1(G) \times \nu_2(G)$ and one of the required cases takes place.

Lemma 2.6. $\phi(k_1), \phi(k_2) \leq 4$.

Proof. Since G is a subdirect subgroup it contains a matrix

$$F = \begin{pmatrix} a_1 & 0 \\ 0 & b_2^i a_2^j \end{pmatrix}, \quad j \in \mathbb{Z}, \quad i = 0, 1.$$

where $a_1, b_2^i a_2^j$ are from (2), (3). Since F is conjugate in $GL(4, \mathbb{R})$ to an integer matrix the characteristic polynomial $\det(F - tE)$ has integer coefficients.

Let $\zeta = \exp \frac{2\pi i}{k_1} \in \mathbb{C}$. Then ζ is a root of the integer polynomial $\det(F - tE)$ of degree 4 because

$$\det(F - tE) = \det(a_1 - tE) \det(b_2^i a_2^j - tE).$$

The minimal integer polynomial of $\exp \frac{2\pi i}{k_1}$ is the cyclotomic integral polynomial $\Phi_{k_1}(t)$ of degree $\phi(k_1)$. Hence $\Phi_{k_1}(t)$ divides $\det(F - tE)$ and therefore $\phi(k_1) \leq 4$. The case of k_2 is similar. \square

Lemma 2.7. *If G contains a matrix*

$$F = \begin{pmatrix} a_1 & 0 \\ 0 & b_2 a_1^j \end{pmatrix}, \quad j \in \mathbb{Z}, \quad (4)$$

then $\phi(k_1) \leq 2$. Similarly if

$$F_1 = \begin{pmatrix} b_1 a_1^j & 0 \\ 0 & a_1 \end{pmatrix} \in G, \quad j \in \mathbb{Z},$$

then $\phi(k_2) \leq 2$.

Proof. Note that $\text{tr } F = \text{tr } a_1 + \text{tr}(b_2 a_1^j) = \text{tr } a_1 \in \mathbb{Z}$ by (3). Apply Proposition 2.4. \square

Corollary 2.8. *One of the cases takes place:*

6

V.ARTAMONOV, S.SÀNCHEZ

- 1) $\phi(k_1), \phi(k_2) \leq 2$;
- 2) $\phi(k_1) = \phi(k_2) = 4$.

Proof. Suppose that $\phi(k_2) \leq 2$. If G contains a matrix F_1 then we apply Lemma 2.7. Suppose that G contains a matrix

$$B = \begin{pmatrix} a_1 & 0 \\ 0 & a_2^{s_2} \end{pmatrix} = \begin{pmatrix} \cos \frac{2\pi}{k_1} & -\sin \frac{2\pi}{k_1} & 0 & 0 \\ \sin \frac{2\pi}{k_1} & \cos \frac{2\pi}{k_1} & 0 & 0 \\ 0 & 0 & \cos \frac{2\pi s_2}{k_2} & -\sin \frac{2\pi s_2}{k_2} \\ 0 & 0 & \sin \frac{2\pi s_2}{k_2} & \cos \frac{2\pi s_2}{k_2} \end{pmatrix} \in G \subseteq G_1 \times G_2. \quad (5)$$

Then $\text{tr } B = 2 \left(\cos \frac{2\pi}{k_1} + \cos \frac{2\pi s_2}{k_2} \right) \in \mathbb{Z}$. By the assumption $2 \cos \frac{2\pi}{k_1} \in \mathbb{Z}$. Apply Proposition 2.4. \square

Starting from now we shall assume that $\phi(k_1) = \phi(k_2) = 4$. By Proposition 2.3 it means that $k_1, k_2 = 10, 5, 12, 8$ and by Lemma 2.7 the group G contains the matrix B and a matrix

$$B_2 = \begin{pmatrix} a_1^{s_1} & 0 \\ 0 & a_2 \end{pmatrix}. \quad (6)$$

Let $\xi_n = \exp \frac{2\pi i}{n} \in \mathbb{C}$. As in the proof of Lemma 2.6 we know that $\Phi_4(t) = \det(B - tE)$. Hence the roots of $\det(B - tE)$ are $\xi_{k_1}^r$, where $1 \leq r < k_1$ and $(r, k_1) = 1$. Similarly the roots of $\det(B - tE)$ are $\xi_{k_2}^{s_2 t}$ where $1 \leq t < \frac{k_2}{(s, k_2)}$ and $(t, \frac{k_2}{(s, k_2)}) = 1$. Hence $\xi_{k_2}^{s_2} = \xi_{k_1}^r$, $\xi_{k_1} = \xi_{k_2}^{s_2 t}$. In particular $\mathbb{Z}[\xi_{k_1}] = \mathbb{Z}[\xi_{k_2}^{s_2}] \subseteq \mathbb{Z}[\xi_{k_2}]$. Similarly $\mathbb{Z}[\xi_{k_2}] = \mathbb{Z}[\xi_{k_1}^{s_1}] \subseteq \mathbb{Z}[\xi_{k_1}]$. Hence

$$\mathbb{Z}[\xi_{k_1}] = \mathbb{Z}[\xi_{k_1}^{s_1}] = \mathbb{Z}[\xi_{k_2}] = \mathbb{Z}[\xi_{k_2}^{s_2}]. \quad (7)$$

Lemma 2.9. *The following cases take place*

- a) $k_1, k_2 = 10, 5$ and $s_1 s_2 \equiv 1 \pmod{5}$;
- b) $k_1 = k_2 = 12$ and $s_1, s_2 = 1, 5, 7, 11$;
- c) $k_1 = k_2 = 8$ and $s_1, s_2 = 1, 3, 5, 7$.

Proof. Recall that $\mathbb{Z}[\xi_{10}] = \mathbb{Z}[\xi_5]$ and

$$\mathbb{Z}[\xi_{10}] \neq \mathbb{Z}[\xi_{12}] \neq \mathbb{Z}[\xi_8] \neq \mathbb{Z}[\xi_{10}].$$

Apply (7). Note also that $a_i^{s_1 s_2 - 1} \in G$ by Proposition 2.2. Hence by Proposition 2.4 $s_1 s_2 - 1$ in the first case should be divisible by 5. \square

By Lemma 2.7 we complete the proof of Theorem 2.5. \square

In order to present more detailed classification of groups in Theorem 2.5 we need

Proposition 2.10. *Let G, k_1, k_2 be from Theorem 2.5 and $\phi(k_1) = \phi(k_2) = 4$. Suppose that the group G contains a matrix*

$$F = \begin{pmatrix} a_1^t & 0 \\ 0 & b_2 \end{pmatrix}.$$

Then G contains

$$a_1^2 = \begin{pmatrix} a_1^2 & 0 \\ 0 & 1 \end{pmatrix}$$

and a matrix F with $t = 0$, that is b_2 . In this case $G \cap (\langle a_1 \rangle \times D_{k_2})$ is a semidirect product of a normal subgroup $\langle B \rangle \times \langle a_2^2 \rangle$ by a cyclic group $\langle F \rangle$ of order 2 with $t = 0$, where B is from (5) with $s_2 = 1$.

Proof. We know that G contains a matrix B from (5). Then

$$U = BFBF^{-1} = a_1^2 \in G.$$

Multiplying F and B_2 from (6) by powers of U we can assume that $t = 0, 1$ and $s_1 = 1$. The case $t = 1$ is impossible because otherwise $\text{tr } a_1^2 \in \mathbb{Z}$ which contradicts Proposition 2.4. Thus we can assume that $B_2 = B$ and $t = 0$.

Note that $G_1 = \langle B \rangle \times \langle a_2^2 \rangle$ is a subgroup in G containing a_1^2 and therefore a_2^2 . Hence the index of G_1 in $H = \langle a_1 \rangle \times \langle a_2 \rangle$ is dividing 4. Since $a_1, a_2 \notin G$ we can conclude that $G_1 = G \cap H$. Now the proof follows. \square

Theorem 2.11. *Let a group G be a subdirect product from Theorem 2.5 where $k_1 = k_2 = 10$. Then G is one of the following groups.*

a) *If G is a subdirect product of two cyclic groups $\langle a_1 \rangle_{10} \times \langle a_2 \rangle_{10}$ then G is direct product of two cyclic groups*

$$\langle B \rangle \times \langle a_1^l \rangle \quad (8)$$

where B is from (5) with $1 \leq s_2 \leq 9$, $s_2 \neq 5$ and $l = 0, 5$. If s_2 is even then $l = 5$.

b) *G is a subdirect subgroup of $D_{10} \times D_{10}$ and a semidirect product of the normal group (8) by a cyclic group $\langle W \rangle$ of order 2, where*

$$W = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 a_2^j \end{pmatrix}, \quad j = 0, \dots, 9. \quad (9)$$

Proof. Case **a)**. We know that some matrices B, B_2 from (5), (6) belong to G where s_i are from Lemma 2.9, case a). The group G contains the cyclic group $\langle B \rangle$ of order 10 which projects onto $\langle a_1 \rangle_{10}$. Thus $G = \langle B \rangle \times (G \cap \langle a_2 \rangle_{10})$. But by Proposition 2.4 the order of the group $G \cap \langle a_2 \rangle_{10}$ is equal to 1, 2, 3, 4 or 6. Hence $G \cap \langle a_2 \rangle_{10} = \langle a_1^l \rangle$ where $l = 0$ or 5. Suppose that s_2 is even and $l = 0$. Then the cyclic group $\langle B \rangle$ does not contain B_2 which is impossible.

Suppose that G is a subdirect product of $\langle a_1 \rangle_{10} \times D_{10}$. Then $G \cap \langle a_1 \rangle_{10} \times \langle a_2 \rangle_{10}$ is a group (8). Also G contains the matrix F from

Proposition 2.10 which contradicts Proposition 2.4. So this case is impossible.

Case **b**). Using previous argument we can show that G does not contain the matrix F from Proposition 2.10. Thus $G \cap (\langle a_1 \rangle_{10} \times \langle a_2 \rangle_{10})$ is a subdirect product and therefore as in the case **a**) it has the form (8). As above G does not contain the matrix F from Proposition 2.10. Hence G contains a matrix $W \in G$ as in (9). Then $WgW^{-1} = g^{-1}$ for all $g \in \langle a_1 \rangle_{10} \times \langle a_2 \rangle_{10}$ and $W^2 = E$. Thus G has the required form. \square

Theorem 2.12. *Let $k_1 = 10, k_2 = 5$. Then there are only two cases.*

- a)** *If G is a subdirect product $\langle a_1 \rangle_{10} \times \langle a_2 \rangle_5$ then $G = \langle B \rangle$ where $s_2 = 1, 2, 3, 4$.*
- b)** *If G is a subdirect product of $D_{10} \times D_5$ then G is generated by B and by W from (9), where $j = 0, 1, 2, 3, 4$. Thus $G \simeq D_{10}$.*

Proof. The group G contains a matrix B from (5) where s_2 is invertible modulo 5 by Lemma 2.9. In the case **a**) either G is generated by B or G has a nontrivial intersection with $\langle a_2 \rangle$ which is impossible by Proposition 2.4.

The case when G is a subdirect product $\langle a_1 \rangle_{10} \times D_5$ is impossible by Proposition 2.10.

The proof in the case **b**) is similar to those in the previous Theorem. \square

The proof of the next Theorem is similar.

Theorem 2.13. *Let $k_1 = 5, k_2 = 10$. Then there are only two cases.*

- a)** *If G is a subdirect product $\langle a_1 \rangle_5 \times \langle a_2 \rangle_{10}$ then $G = \langle B_2 \rangle$ from (6) where $s_1 = 1, 2, 3, 4$.*
- b)** *If G is a subdirect product of $D_5 \times D_{10}$ then G is generated by B_2 and by W from (9). Thus $G \simeq D_{10}$.*

Theorem 2.14. *Let group G be a subdirect product from Theorem 2.5, where $k_1 = k_2 = 8$.*

- A)** *If G is a subdirect product of cyclic groups $\langle a_1 \rangle_8 \times \langle a_2 \rangle_8$, then G is a direct product of two cyclic groups (8), where $l = 0, 2, 4$, and s_2 being odd in B from (5).*
- B)** *Suppose that G is a subdirect product of $\langle a_1 \rangle \times D_8$. Then G is a semidirect product of the normal subgroup (8) and a cyclic group $\langle b_2 \rangle$ of order 2, where $l = 0, 2, 4$, and s_2 being odd in B from (5).*
- C)** *Suppose that G is a subdirect subgroup of $D_8 \times D_8$. Then G is one of the groups*
 - a)** *a semidirect product of a normal subgroup (8) with $l = 2$ and s_2 odd by a direct product $\langle b_1 \rangle_2 \times \langle b_2 \rangle_2$.*
 - b)** *G is semidirect product of the normal subgroup (8) and a cyclic group $\langle W \rangle_2$, where $l = 0, 2, 4$, s_2 being odd in B from (5) and W is from (9).*

Proof. In the case **A)** the group G contains a matrix B from (5) with odd s_2 . If $G \neq \langle B \rangle$ then $G = \langle B \rangle \times (G \cap \langle a_2 \rangle)$, where $G \cap \langle a_2 \rangle = \langle a_2^l \rangle$ and $l = 0, 2, 4$ by Proposition 2.4.

The case **B)** follows from Proposition 2.10.

In the case **C)** by Lemma 2.9 the group G contains a matrix B from (5) with odd s_2 . Hence $G \cap (\langle a_1 \rangle \times \langle a_2 \rangle)$ is a subdirect product from the case **A)**. If F from Proposition 2.10 belongs to G then $t = 0$ and

$$G \cap (\langle a_1 \rangle \times \langle a_2 \rangle) = \langle B \rangle \times \langle a_2^2 \rangle.$$

Symmetrically G contains a matrix

$$b_1 = \begin{pmatrix} b_1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence G has the form **Ca)**.

Suppose that $U, U' \notin G$. Then G contains a matrix W from (9) where $j = 0, \dots, 8$. Also

$$G \cap (\langle a_1 \rangle \times \langle a_2 \rangle) = \langle B \rangle \times \langle a_2^l \rangle, \quad l = 0, 2, 4.$$

Then G has the form **Cb)**. □

Note that the case B) with $l = 0$ and $s_2 = 3$ corresponds to perfect 8-fold dihedral symmetry in a certain member of the LI-class of Ammann-Beenker tilings [4]. The generating matrices of the group are

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Using the same argument one can prove

Theorem 2.15. *Let $k_1 = k_2 = 12$. Then G is one of the following groups.*

A) *If G is subdirect product of $\langle a_1 \rangle_{12} \times \langle a_2 \rangle_{12}$ then G is the group*

$$G = \langle B \rangle \times \langle a_2^l \rangle \quad (10)$$

where B is from (5) with $s_2 = 1, 5, 7, 11$ and $l = 0, 2, 3, 4, 6$.

B) *If G is a subdirect product of $\langle a_1 \rangle_{12} \times D_{12}$ then G is a semidirect product of the normal subgroup (10) with $l = 2$ and $\langle b_2 \rangle_2$.*

C) *If G is a subdirect product of $D_{12} \times D_{12}$ then G is one of groups:*

a) *a semidirect product of a normal subgroup (10) with $l = 2$ and s_2 odd by a direct product $\langle b_1 \rangle_2 \times \langle b_2 \rangle_2$.*

b) *a semidirect product of the normal subgroup (10) and a cyclic group $\langle W \rangle_2$, where $l = 0, 2, 4, s_2 = 1, 5, 7, 11$ in B from (5) and where W is from (9).*

Applying previous results to the window W which is a closure of a projection of A into V we obtain

Corollary 2.16. *The symmetry group of the window of any plane module set with $2 \dim$ internal space is one of the following: C_i or D_i , $i = 1, 2, 3, 4, 5, 6, 8, 10, 12$.*

3. REALIZATION

In this section we shall show that two exceptional cases 2), 3) in Theorem 2.5 can be realized. The idea of this realization comes from [10], [7], [4]. Let $m \geq 3$ be a positive integer. Take the real vector space $E = \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}[\xi]$ of dimension $\phi(m)$ where $\xi = \exp\left(\frac{2\pi i}{m}\right)$. Then **basis** of E consists of $\phi(m)$ vectors $e_j = 1 \otimes \xi^j$, $0 \leq j < \phi(m)$, [6, Chapter IV, § 1]. Define in E two linear operators

$$a(1 \otimes \xi^j) = 1 \otimes \xi^{j+1}, \quad b(1 \otimes \xi^j) = 1 \otimes \xi^{-j}$$

for all $j \in \mathbb{Z}$. These definitions are correct since $\mathbb{Z}[\xi]$ is a left $\mathbb{Z}[\xi]$ -module and the map b is a Galois automorphism of $\mathbb{Z}[\xi]$. It is easy to see that $b^2 = (ba)^2 = 1$. Hence the group of invertible operators generated by a, b is the dihedral group D_m and we have a representation of D_m in E of dimension $\phi(m)$. It is decomposed into direct sum of irreducible ones. Since dimensions of irreducible representations of dihedral groups are 1 or 2 [2, chapter 3], it suffices to show that ± 1 are not eigenvalues of operators a and b . Note that the characteristic polynomial of the operator a is equal to $\Phi_m(t)$, and **in fact** ± 1 are not roots of $\Phi_m(t)$, provided $m \geq 3$.

We have proved

Theorem 3.1. *Let $m \geq 3$ be an integer. Then the dihedral group D_m is a subgroup of a symmetry group $\text{Sym } Q$ of some 2D-quasicrystal Q which can be constructed by a cut and project method using a hyperspace E of dimension $\phi(m) \geq 2$.*

There is another way of constructing symmetries of quasicrystals based on algebraic integers. Let $\xi = \exp\left(\frac{2\pi i}{m}\right)$ be as above. Then again $\mathbb{Z}[\xi]$ is a left $\mathbb{Z}[\xi]$ -module and therefore the Abelian group $\mathbb{Z}[\xi]^*$ of invertible elements of the ring $\mathbb{Z}[\xi]$ acts on $E = \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}[\xi]$. By [2, Theorem 3.3, Chapter 3] the complex space $\mathbb{C} \otimes_{\mathbb{R}} E$ contains one-dimensional subspace which is invariant under the action of the **Abelian** group $\mathbb{Z}[\xi]^*$. Hence E contains 2-dimensional real invariant subspace U . So taking U as a physical space and the lattice $\mathbb{Z}[\xi]$ in E as M we can construct a quasicrystal Q in which the symmetry group $\text{Sym } Q$ contains $\mathbb{Z}[\xi]^*$ as a subgroup. By [3, page 561] the Abelian group $\mathbb{Z}[\xi]^*$ has rank $\frac{\phi(m)}{2} - 1$. Applying Proposition 2.3 as in [7] we obtain

Theorem 3.2. *Let m be an integer and either $m = 5$ or $m \geq 7$. Consider 2D-quasicrystal Q constructed by a cut and project method*

using a hyperspace E of dimension $\phi(m) \geq 2$ and $M = \mathbb{Z}[\xi]$ as a lattice. The symmetry group $\text{Sym } Q$ contains an Abelian group isomorphic to $\mathbb{Z}[\xi]^*$ and having rank $\frac{\phi(m)}{2} - 1 \geq 1$. In particular $\text{Sym } Q$ contains elements of infinite order.

For example take $E = \mathbb{R} \otimes \mathbb{Z}[\xi]$, $\xi = \frac{1+i}{\sqrt{2}}$. Then $\dim E = \phi(8) = 4$ and E is a left module over $\mathbb{Z}[\xi]$. Therefore the group $\mathbb{Z}[\xi]^*$ of invertible elements of the ring $\mathbb{Z}[\xi]$ acts on E . The rank of the group $\mathbb{Z}[\xi]^*$ is equal to $\frac{\phi(8)}{2} - 1 = 1$ [3, page 561] and therefore it contains an element g of infinite order.

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