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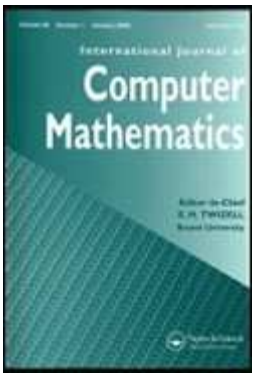

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A posteriori error estimator and indicator in Generalized Finite Differences. Application to improve the approximated solution of elliptic pdes (SI-CMMSE-2006)

J.J. BENITO[1*], F. URENA[2], L. GAVETE[3] and B. ALONSO[3]

[1] Departamento de Construcción y fabricación, Universidad Nacional de Educación a Distancia, Spain
[2] Departamento de Matemática Aplicada, Universidad de Castilla-La Mancha, Spain
[3] Departamento de Matemática Aplicada a los Recursos Naturales, Universidad politécnica de Madrid, Spain

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In this paper we compare two different procedures of approximation of the posteriori error for the GFD method, both procedures are residual based errors. The performance of the two different approximations of the error is illustrated by analysing different second order partial differential equations in 2-D problems with known analytical solutions , using irregular cloud of points. Also, in this paper an adaptive method for improving solutions using a posteriori error is shown.

Keywords: generalized finite differences, meshless methods, a posteriori error approximation, adaptive method

AMS Subject Classifications: F.1.1; F.4.2 **(Authors: Please substitute appropriate classification codes)**

1. Introduction

The classical finite difference method (FDM) emerged as an approximate discrete method of analysis of boundary-value problems given in the form of differential equations and appropriate boundary conditions. The idea of using an eight node star and weight functions to obtain finite difference formulae for irregular meshes, was first put forward by Liska and Orkisz [6, 7], using moving least squares (MLS) interpolation, and the most advanced version of the Generalized Finite Element Method (GFDM) was given by Orkisz [2, 8, 9], including: mesh generation, local approximation, generation of finite difference (FD) formulae and FD equations resulting from local (collocation) or global (Galerkin,variational,) formulations. GFDM is included in the so-called Meshless Methods (MM) [1, 2].

In this paper we compare the posteriori error indicator for the GFD method developed by Benito et al. [4, 5, 10] with a posteriori error estimator based on Orkis'z works [1, 8, 9]. Also, in this paper, an h-adaptive method is shown, for improving the approximate solution of the second order partial differential equation. The paper is organized as follows. Section 3 describes the posteriori error indicator. Section 4 describes the posteriori error estimator. Section 5 describes an adaptive method in generalized finite differences. Section 6 illustrates the posteriori error approximations at work and the efficiency of the adaptive method for the solution of elliptic equations using irregular clouds of points. Finally, in Section 7 some conclusions are given.

*Corresponding author. Email: jbenito@ind.uned.es

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2. The generalized finite difference method

Let us to consider a problem governed by

$$\alpha_1\frac{\partial U}{\partial x}+\alpha_2\frac{\partial U}{\partial y}+\alpha_3\frac{\partial^2 U}{\partial x^2}+\alpha_4\frac{\partial^2 U}{\partial y^2}+\alpha_5\frac{\partial^2 U}{\partial x\partial y}=f(x,y)\quad in\quad \Omega \tag{1}$$

with boundary condition

$$\beta\frac{\partial U}{\partial n}+\gamma U=g(x,y)\quad in\quad \Gamma \tag{2}$$

where $\Omega\subset R^2$ with boundary Γ; $\alpha_i, i=1,\cdots,5,\beta$ and γ are constant coefficients; and f,g are two known smoothed functions.
First of all, it is necesary to generate a grid of points (it can be irregular) in the domain that we call nodes. Then, every node in the domain is the central node of a set of nodes around it. We call this a star. If U_0 and U_i are approximations of the function at the central node and at the rest of nodes of the star, with $i=1..N$ (in this paper the number of star nodes is $N=8$, two nodes for each quadrants [4,5]). Then, according to the Taylor series expansion it is known that

$$U_i=U_0+h_i\frac{\partial U_0}{\partial x}+k_i\frac{\partial U_0}{\partial y}+\frac{1}{2}(h_i^2\frac{\partial^2 U_0}{\partial x^2}+k_i^2\frac{\partial^2 U_0}{\partial y^2}+2h_ik_i\frac{\partial^2 U}{\partial x\partial y})+\cdots \tag{3}$$

where (x_0,y_0) are the coordinates of the central node, (x_i,y_i) the coordinates of the i^{th} node in the star and $h_i=x_i-x_0; k_i=y_i-y_0$. If the unwritten terms in the equation (3) are ignored, an approximation of second order for the U_i function is obtained, which is indicated as u_i. It is then possible to define the function $B_5(u)$(subindex refers to the number of unknowns in the functional) as in [6,7] and [4,5]

$$B_5(u)=\sum_{i=1}^{N}[(u_0-u_i+h_i\frac{\partial u_0}{\partial x}+k_i\frac{\partial u_0}{\partial y}+\frac{h_i^2}{2}\frac{\partial^2 u_0}{\partial x^2}+\frac{k_i^2}{2}\frac{\partial^2 u_0}{\partial y^2}+h_ik_i\frac{\partial^2 u}{\partial x\partial}y)w(h_i,k_i)] \tag{4}$$

where $B_5(u)$ is norm to the second order approximation and $w(h_i,k_i)$ a weighting function.
If the norm $B_5(u)$ is minimized with respect the partial derivatives included in formulae (4) , the following linear equations systems is obtained

$$\mathbf{A}\boldsymbol{D}_u=\mathbf{b} \tag{5}$$

where the $\mathbf{A}$ is a matrix of 5×5 , the vector $\boldsymbol{D}_u$ is given by

$$D_u=\{\frac{\partial u_0}{\partial x},\frac{\partial u_0}{\partial y},\frac{\partial^2 u_0}{\partial x^2},\frac{\partial^2 u_0}{\partial y^2},\frac{\partial^2 u_0}{\partial x\partial y}\}^T \tag{6}$$

and $\mathbf{b}$ is the vector of independent terms, including the values $u_0,u_i(i=1,..,N)$ to be solved by using (GFDM) as described [4,5] and [6,7].
As the matrix of coefficients $\mathbf{A}$ is symmetrical, it is then possible to use the Cholesky method to solve the same (5). On solving the system (5), the explicit differences formulae are obtained [4,5].
On including in the equation (1) the explicit expressions obtained for the partial derivatives, the star equation (7) is obtained as

$$\lambda_0u_0+\sum_{i=1}^{N}\lambda_iu_i=f(x_0,y_0)\Leftrightarrow u_0=\sum_{i=1}^{N}m_iu_i+m_ff, with\sum_{i=1}^{N}m_i=1 \tag{7}$$

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3. A posteriori error indicator in the generalized finite differences

It is possible to take more terms of the Taylor series expansion to obtain the approximate value of the function in any node of the star. The difference between the fourth order and the second order approximations in each node of the star, is:

$$[u_i]_{order4} - [u_i]_{order2} = \frac{1}{6}(\frac{\partial^3 u_0}{\partial x^3} + \frac{\partial^3 u_0}{\partial y^3} + 3\frac{\partial^3 u_0}{\partial x^2 \partial y} + 3\frac{\partial^3 u_0}{\partial x \partial^2 y}) + \frac{1}{24}(\frac{\partial^4 u_0}{\partial x^4}$$

$$+\frac{\partial^4 u_0}{\partial y^4} + 4\frac{\partial^4 u_0}{\partial x^3 \partial y} + 6\frac{\partial^4 u_0}{\partial x^2 \partial y^2} + 4\frac{\partial^4 u_0}{\partial x \partial y^3}) \tag{8}$$

The expression(8) is valid for $i = 1, \cdots, N$, where N is the number of nodes of the star. Of the other hand, the star equation (7) indicates that the value of the function at the central node of the star, is the weighting average of the function values at the rest of nodes, plus the corresponding to the independent term of partial differential equation (1). The error at the central node of the star may then be approximated by the weighted addition of the absolute values involving the 3^{rd} and 4^{th} order partial derivatives terms:

$$Ind(u_0) = \sum_{i=1}^{N} |m_i(\frac{1}{6}(\frac{\partial^3 u_0}{\partial x^3} + \frac{\partial^3 u_0}{\partial y^3} + 3\frac{\partial^3 u_0}{\partial x^2 \partial y} + 3\frac{\partial^3 u_0}{\partial x \partial^2 y}) + \frac{1}{24}(\frac{\partial^4 u_0}{\partial x^4}$$

$$+\frac{\partial^4 u_0}{\partial y^4} + 4\frac{\partial^4 u_0}{\partial x^3 \partial y} + 6\frac{\partial^4 u_0}{\partial x^2 \partial y^2} + 4\frac{\partial^4 u_0}{\partial x \partial y^3}))| + m_f f(x_0.y_0) \tag{9}$$

where the 3^{rd} and 4^{th} order partial derivatives are calculated using the values of 1^{st} and 2^{nd} order derivatives, that were calculated before with the explicit formulas of D_U obtained in [4, 5] and

$$\frac{\partial^s u_0}{\partial x^q \partial y^r} = \frac{\partial^q}{\partial x^q}(\frac{\partial^r u_0}{\partial y^r}) = \frac{\partial^r}{\partial y^r}(\frac{\partial^q u_0}{\partial x^q}) \tag{10}$$

where $s = q + r$, $s = 3, 4$ and $q, r = 0, 1, 2$.
In this paper $Ind(u_0)$, equation (9), is denominated error indicator.

4. A posteriori error estimator in the generalized finite differences

The error indicator developed in the previous section is now compared with a posteriori error estimator based on Orkisz's works [2, 8, 9].
Let us consider the fourth order Taylor approximation and, as in (4), it is possible to define a function $B[u]$, now involving partial derivatives until 4^{th} order. Minimizing $B[u]$ with respect the partial derivatives $D_u = \{\frac{\partial u_0}{\partial x}, \frac{\partial u_0}{\partial y}, \frac{\partial^2 u_0}{\partial x^2}, \frac{\partial^2 u_0}{\partial y^2}, \frac{\partial^2 u_0}{\partial x \partial y}\}^T$, the following linear equations system is obtained

$$\mathbf{A}\boldsymbol{D}_u + \boldsymbol{b}^* = \mathbf{b} \tag{11}$$

being $\mathbf{A}$,$\boldsymbol{D}_u$, $\mathbf{b}$ the matrix and vectors considered in section 2 and $\boldsymbol{b}^*$ a new vector with the terms involving 3^{rd} and 4^{th} order partial derivatives.

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The equation (11) can be written as:

$$\boldsymbol{D}_u + \boldsymbol{A}^{-1}\boldsymbol{b}^* = \boldsymbol{A}^{-1}\boldsymbol{b} \tag{12}$$

where $\boldsymbol{A}^{-1}\boldsymbol{b}$ correspond to the second order approximation (it can be substituted by the explicit differences formulae obtained from (5)).
We can call these $u_0^{(4)}$ y $u_i^{(4)}$ to the fourth order approximations at the central node and the rest of the star nodes respectively, and $\triangle u_0$ y $\triangle u_1$ to the errors, at the same nodes, between the fourth and second order approximations. The bearing in mind the principles of linearity in the solution and superposition, we can write:

$$u_0^{(4)} = u_0 + \triangle u_0; \quad u_i^{(4)} = u_i + \triangle u_i \tag{13}$$

We can also write (1) as:

$$\boldsymbol{\alpha}\boldsymbol{D}_u = f(x,y) \tag{14}$$

where $\boldsymbol{\alpha}$ is the vector of coefficients of the equation (1), $\boldsymbol{\alpha} = \{\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5\}$.
If we substitute $\boldsymbol{D}_u$, obtained from (12), for each node of domain, into equation (14), we obtain:

$$\boldsymbol{\alpha}(\boldsymbol{A}^{-1}\boldsymbol{b} - \boldsymbol{A}^{-1}\boldsymbol{b}^*) = f(x_0,y_0) = \lambda_0 u_0^{(4)} + \sum_{i=1}^{N} \lambda_i u_i^{(4)} \tag{15}$$

According to section 2, we can write:

$$\boldsymbol{\alpha}\boldsymbol{A}^{-1}\boldsymbol{b} = \lambda_0 u_0 + \sum_{i=1}^{N} \lambda_i u_i \tag{16}$$

substituting the values of $u_0^{(4)}$ y $u_i^{(4)}$ given in (13) into equation (15), considering (16) and after operations, we have:

$$\lambda_0 \triangle u_0 + \sum_{i=1}^{N} \lambda_i \triangle u_i = -\boldsymbol{\alpha}\boldsymbol{A}^{-1}\boldsymbol{b}^* \tag{17}$$

The expression (17) is the star equation for the calculus of a estimation of the error at the nodes of the star.
If we apply (17) to the nodes of the domain, we obtain a system of equations in which the matrix of coefficients is the one obtained in section 2 for the second order approximation and in the vector of independent terms are expressions including third and fourth partial derivatives in the central node of the star. We can obtain these values as we mentioned in section 3.
In this method it is necessary to solve the system of equations twice, and also to obtain the third and fourth order derivatives, for which the method will need more time than the explained in section 3.
In this paper, $\triangle u_i$, obtained solving equation (17), is denominated error estimator.

5. An h-adaptive method

By using the posteriori error indicator as proposed in section 2, it is possible to find an algorithm to improve the approximation solution by selectively increasing the number of nodes in the domain.
In the stars whose central node had an error (obtained using the posteriori error indicator) greater than a

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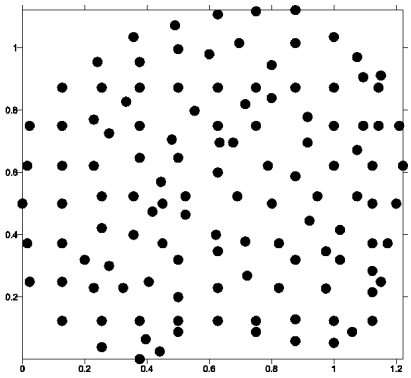

Fig. 1. First cloud of 122 nodes

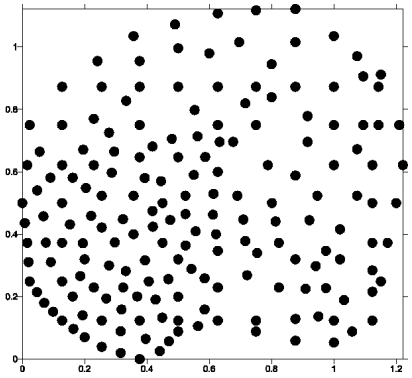

Fig. 2. Last cloud of 181 nodes

fixed value, new nodes would be added to the centre of gravity of the triangles made using the central node and the others nodes of each star one after the other to avoid overlapping of the triangles. In addition, some nodes are added to the boundary.
In order to avoid misleading clouds, a limit for the distance between nodes is used as a parameter (minimum distance) of the h-adaptive method, in such a way that if the distance between the new node and any node of the domain is smaller than this minimum distance, then, the new node should not be added. This minimum distance is given as the maximum distance between all nodes of the mesh, multiplied by a positive parameter $\alpha < 1$.

6. Numerical results

This section includes an example to illustrate the different results obtained for both posteriori error approximations and the application of h-adaptive method.
In this section both posteriori error approximations are used to solve various second order partial differential equations, with the objective of first comparing the indicator error approximations with estimator error approximations and subsequently to apply the adaptive algorithm.

6.1. *Example 1*

Application to solve Laplace equation, with Dirichlet boundary condition and the exact solution is $U(x, y) = L(x^2 + y^2)$ and the weighting function

$$\omega(x, y) = \frac{1}{(dist)^3} \tag{18}$$

where *dist* is the distance between the central node and the considered node in star. The domain is defined by figure 2 (mesh with 181 nodes).

The first cloud, figure 1, has 122 nodes (30 nodes in the boundary) and after two adaptive steps we obtain the cloud showed in figure 2, which has 181 nodes (40 nodes in the boundary, ten nodes more that the first clouds, included in the side with greater error).
Figures 3, 4, 5 show the absolute exact error and the error approximations of the solutions for each interior node (value multiplied by 10^3) (using for representation Modified Shepard's Method) using the indicator and estimator previously described for the last step (cloud of nodes of figure 2).
In table 1 we can see a summary of the global errors obtained using the adaptive method. In this table

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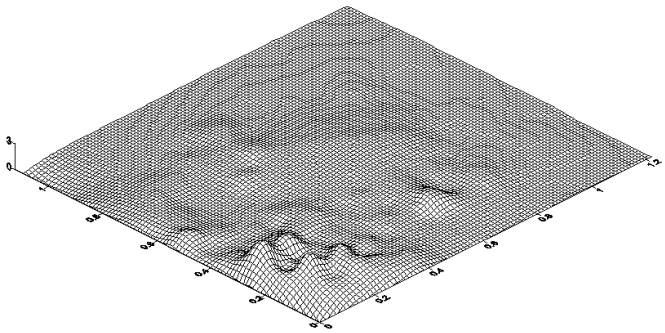

Fig. 3 Cloud of 181 nodes. Absolute error

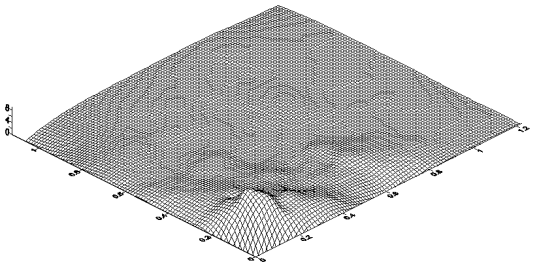

Fig. 4 Cloud of 181 nodes. Indicator error

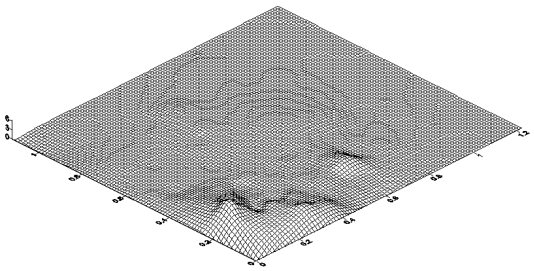

Fig. 5 Cloud of 181 nodes. Estimator error

the global error is:

$$Global \quad error = \frac{\sqrt{\frac{\sum_{i=1}^{N} e_i^2}{N}}}{exac_{max}} \tag{19}$$

where N is the number of nodes in the domain, $exac_{max}$ is the maximum exact value of function in the domain, e_i is the error in the node i. The first column includes exact error value, $e_i = sol(i) - exac(i)$; the second column, the error indicator value $e_i = Ind(u_i)$ and the last one, the error estimator value, $e_i = \triangle u_i$.

Global error			
Number of nodes	Exact error	using indicator	using estimator
122	0.3148	0.1938	1.2231
141	0.0392	0.0821	0.0614
181	0.0198	0.0485	0.0349

Table 1. Adaptive Method. Global errors.

Figures 6 and 7 show the absolute errors,$|e_i| = |sol(i) - exac(i)|$, in the nodes of domain, vertical lines, for the step 1 (cloud of 122 nodes) and for the last step (cloud of 181 nodes) respectively, as it is shown

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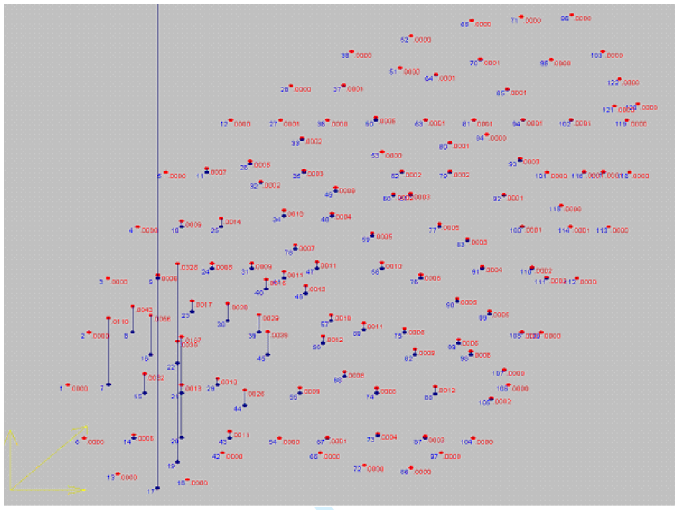

Fig.6. Absolute error (122 nodes).

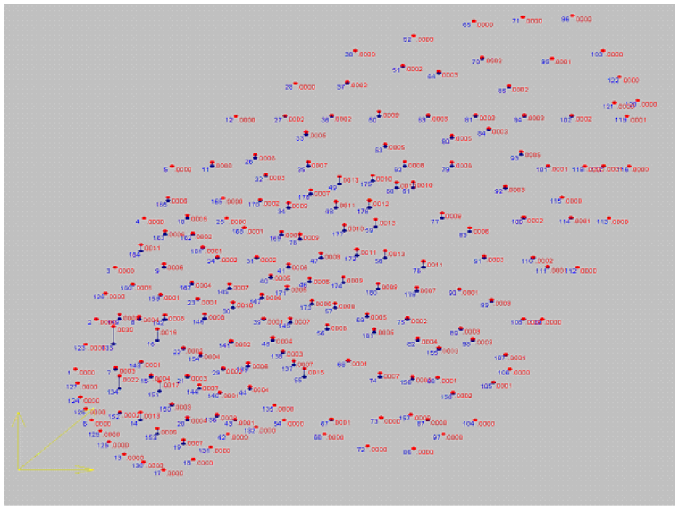

Fig.7. Absolute error (181 nodes).

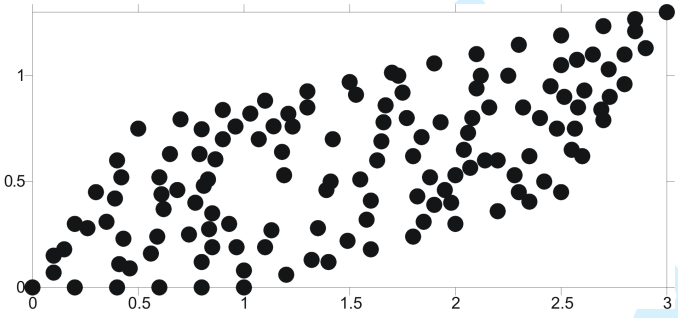

Fig. 8. First cloud of 136 nodes

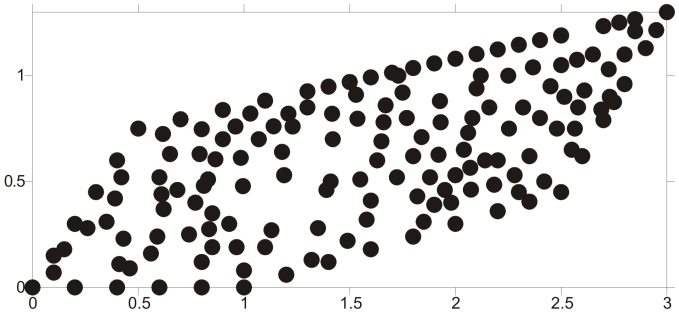

Fig. 9. Last cloud of 157 nodes

in figure 7 the error at the nodes of the domain decrease.

6.2. *Example 2*

Application to solve the equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial x \partial y} + \frac{\partial^2 U}{\partial y^2} - \frac{\partial U}{\partial x} - \frac{\partial U}{\partial y} = 0 \tag{20}$$

with Dirichlet boundary condition and the exact solution is

$$U(x, y) = ye^x \tag{21}$$

using the weighting function (18). The domain is defined by figure 8 (mesh with 136 nodes).
The first cloud, figure 8, has 136 nodes (36 nodes in the boundary) and after three steps we obtain the cloud showed in figure 9, which has 157 nodes (45 nodes in the boundary, nine nodes more that the first clouds, included in the side with greater error).
Figures 10, 11, 12 show the absolute exact error and the error approximations of the solutions for each interior node (value multiplied by 10^2) (using for representation Modified Shepard's Method) using the indicator and estimator previously described for the last step (mesh of 9).
In table 2 we can see a summary of the global errors obtained using the adaptive method

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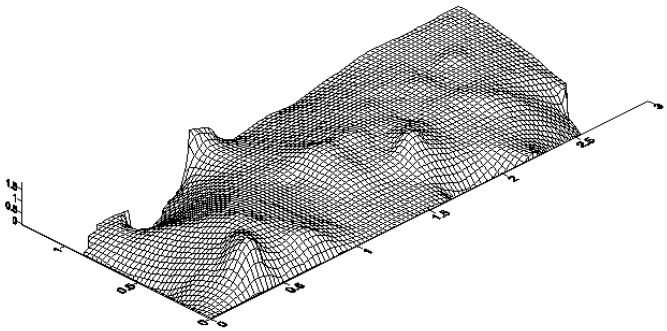

Fig. 10 Cloud of 157 nodes. Absolute error

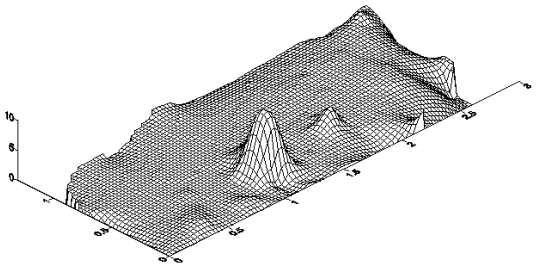

Fig. 11 Cloud of 157 nodes. Indicator error

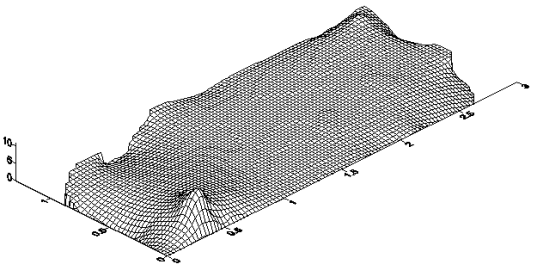

Fig. 12 Cloud of 157 nodes. Estimator error

Global error			
Number of nodes	Exact error	using indicator	using estimator
136	0.057	0.0397	0.0297
146	0.0244	0.0393	0.0277
150	0.0144	0.0232	0.0192
157	0.0124	0.0217	0.0172

Table 2. Adaptive Method. Global errors.

Figures 13 and 14 show the absolute errors in the nodes of domain for the step 1 (cloud of 136 nodes) and for the last step (cloud of 157 nodes) respectively, as it is shown in figure 14 the error at the nodes of the domain decrease.

7. Conclusions

The comparison of the results obtained using both error approximations, shows a similar efficiency. But the error indicator uses less time to get the solution since it does not need to solve the system of equations again.

The use of the error indicator and the adaptive method allows us to decrease global and local errors significantly. If we compare the results obtained here with the others obtained previously [5], it's important to add nodes looking for an improvement in the regularity of the mesh.

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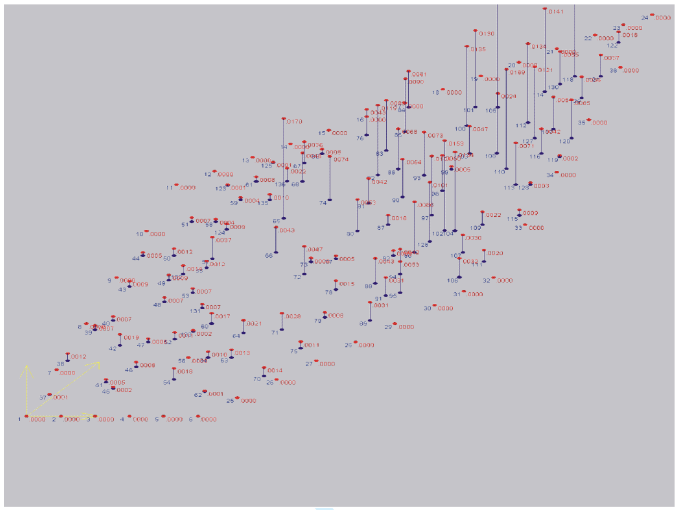

Fig.13. Absolute error (136 nodes).

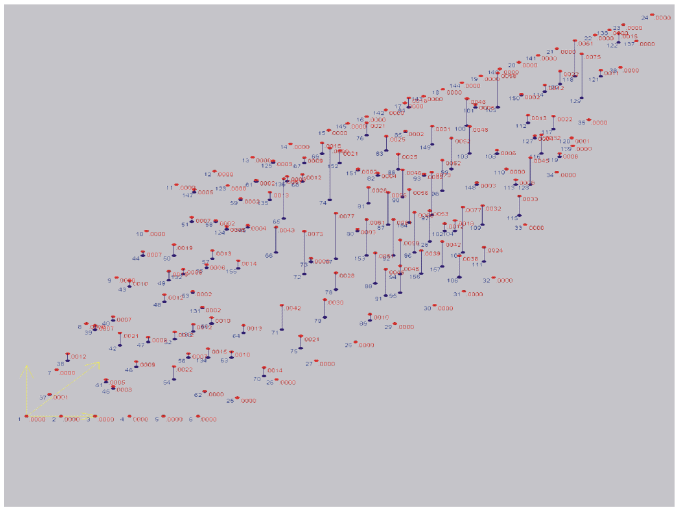

Fig.14. Absolute error (157 nodes).

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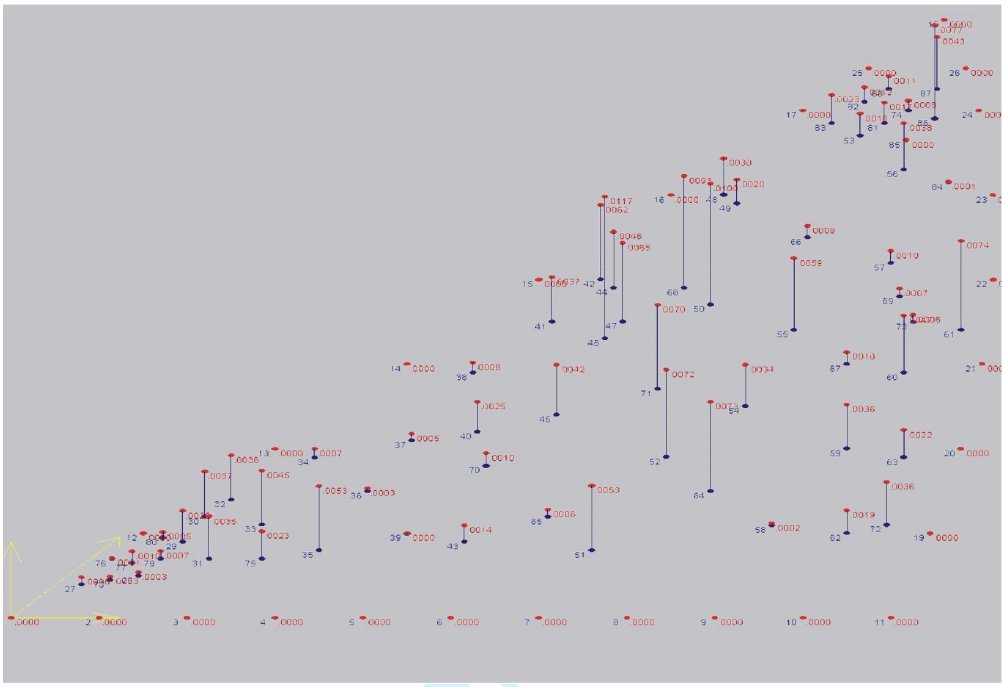

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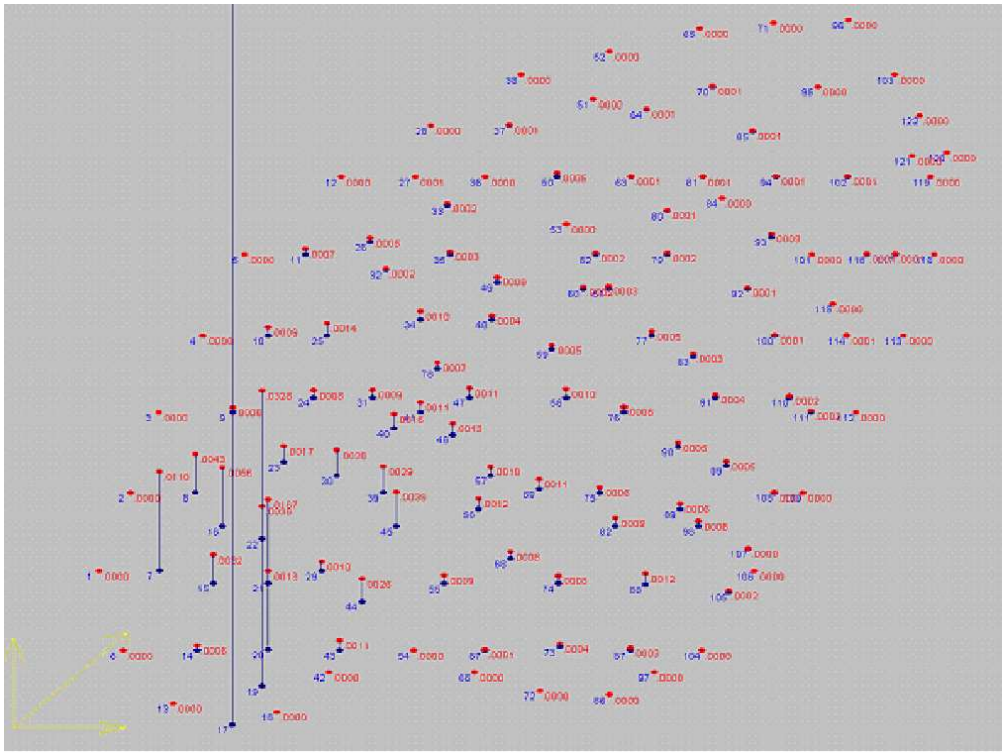

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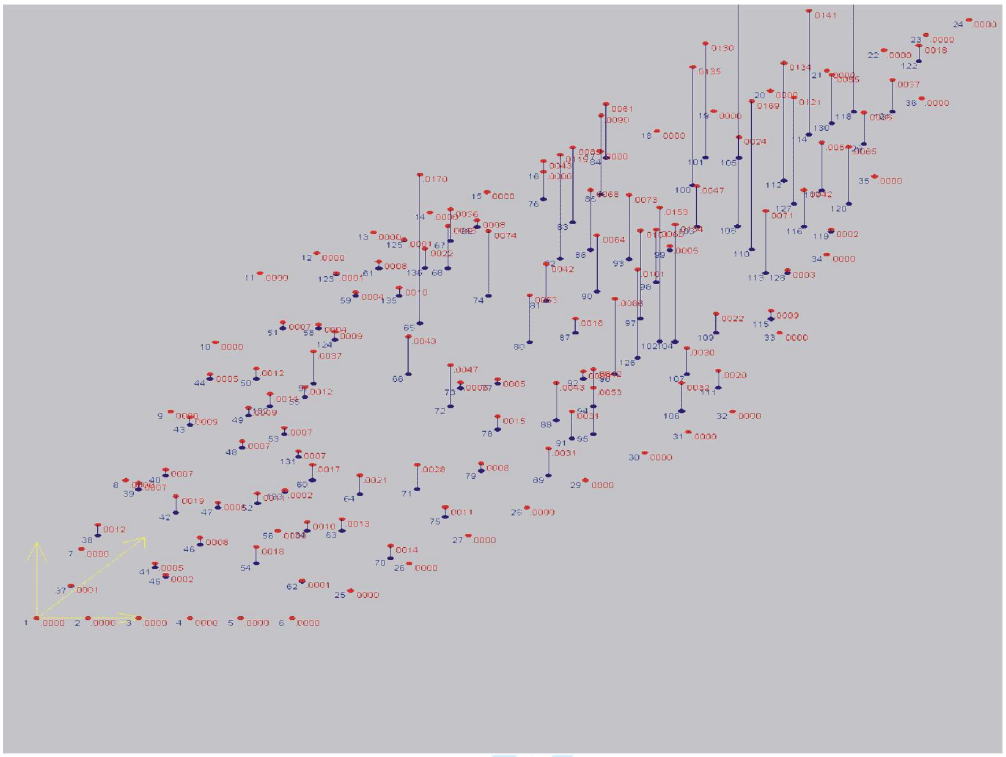

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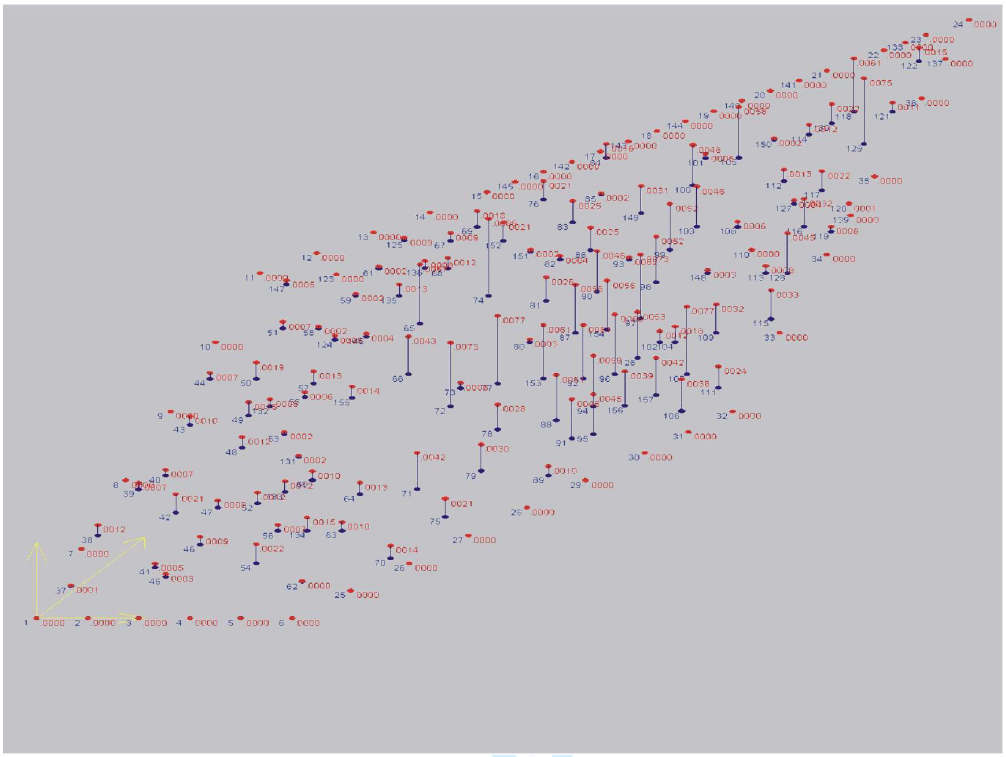

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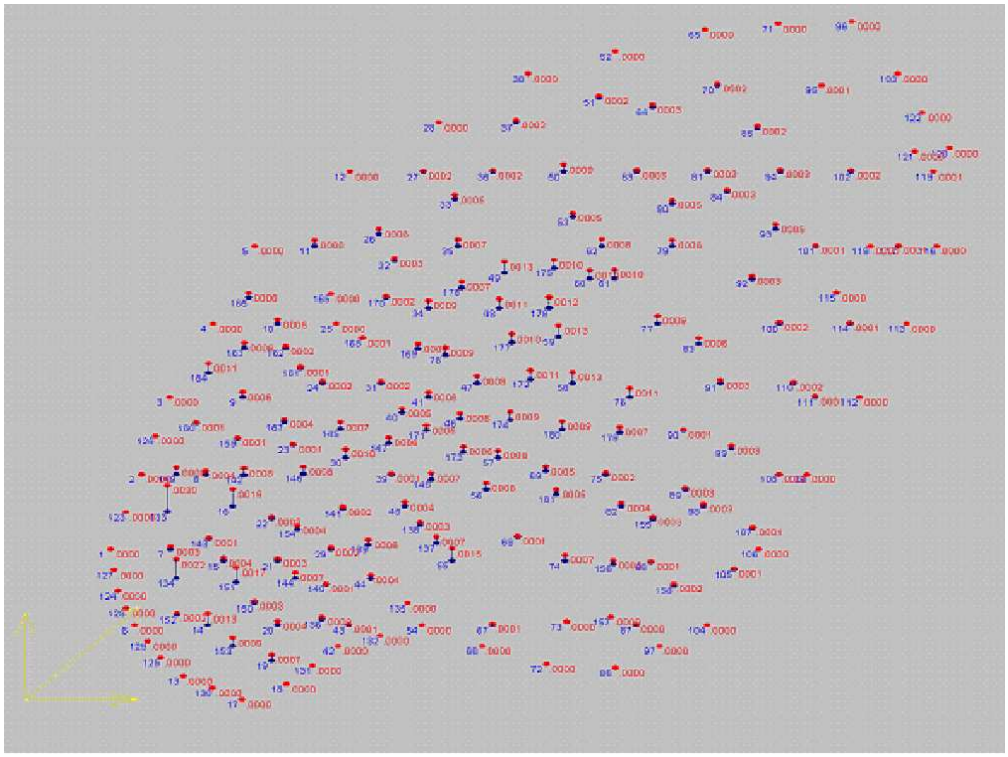

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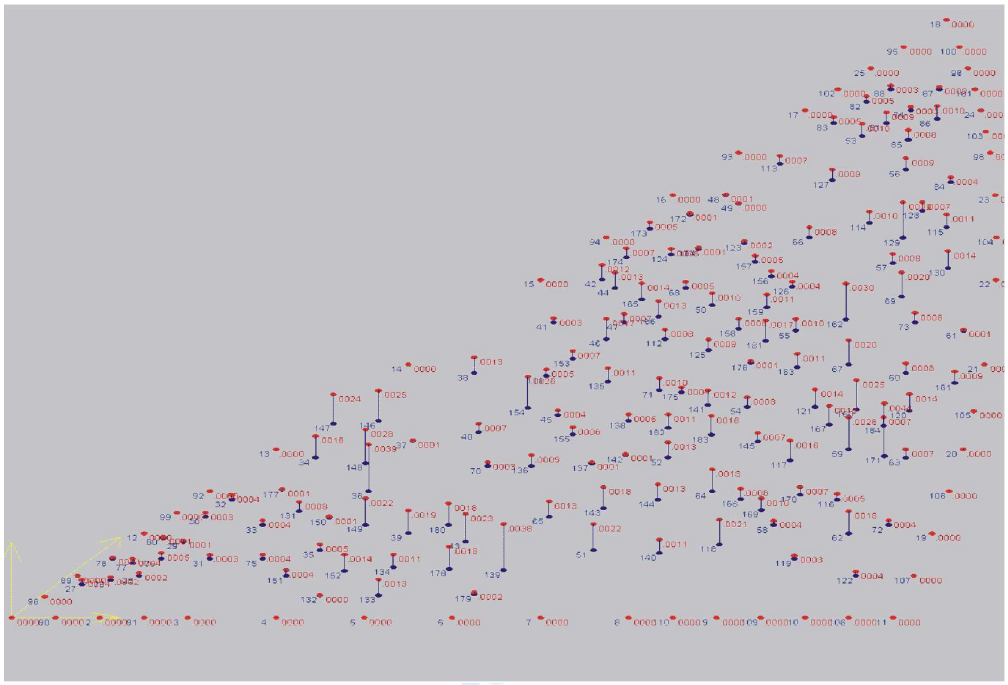

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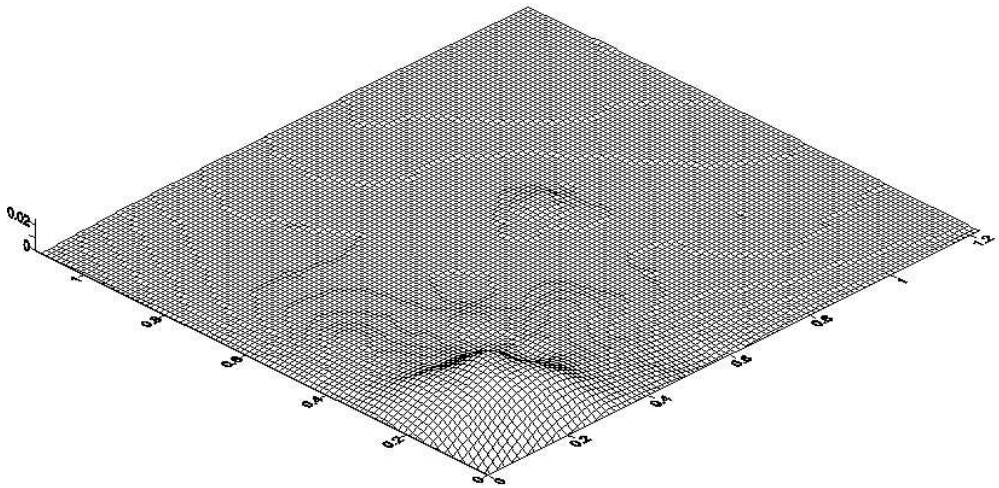

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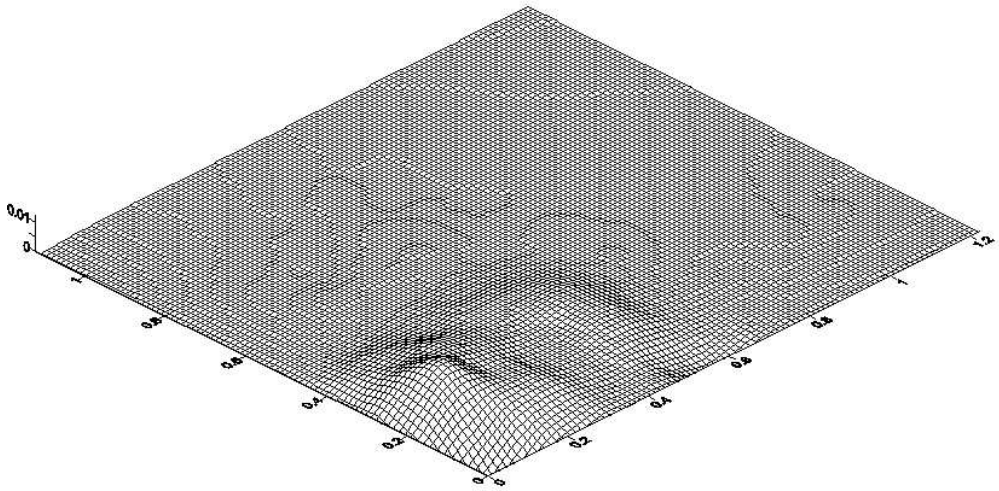

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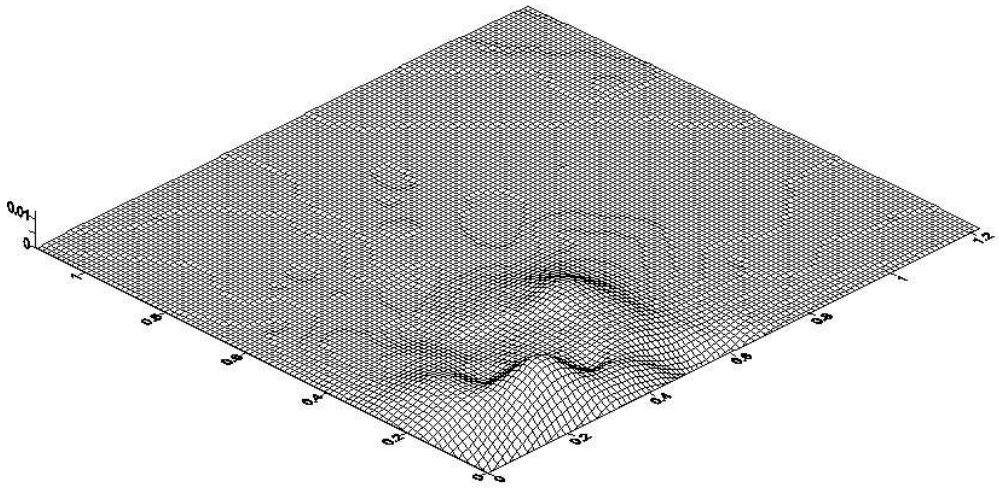

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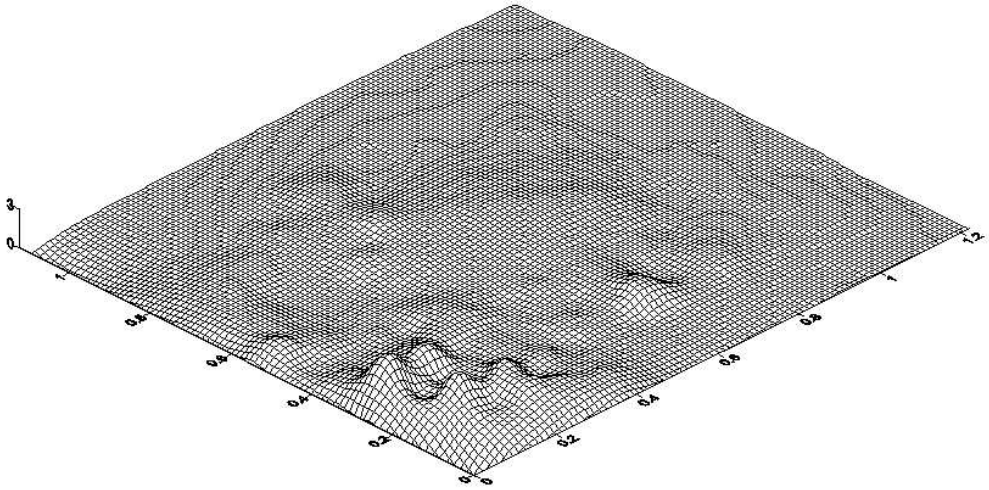

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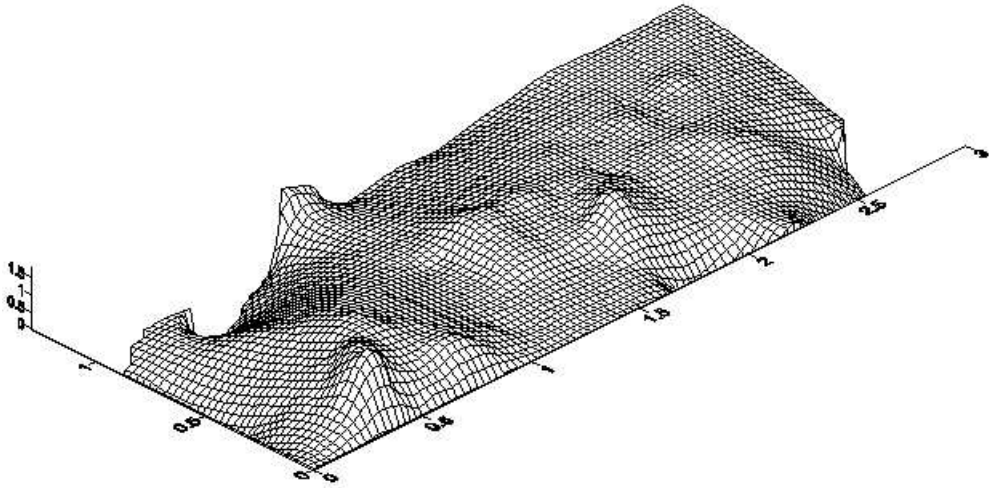

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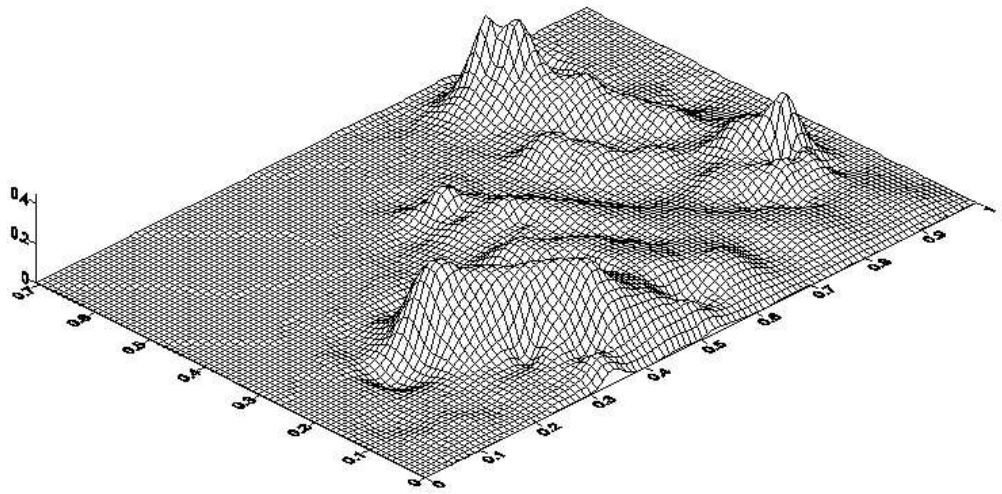

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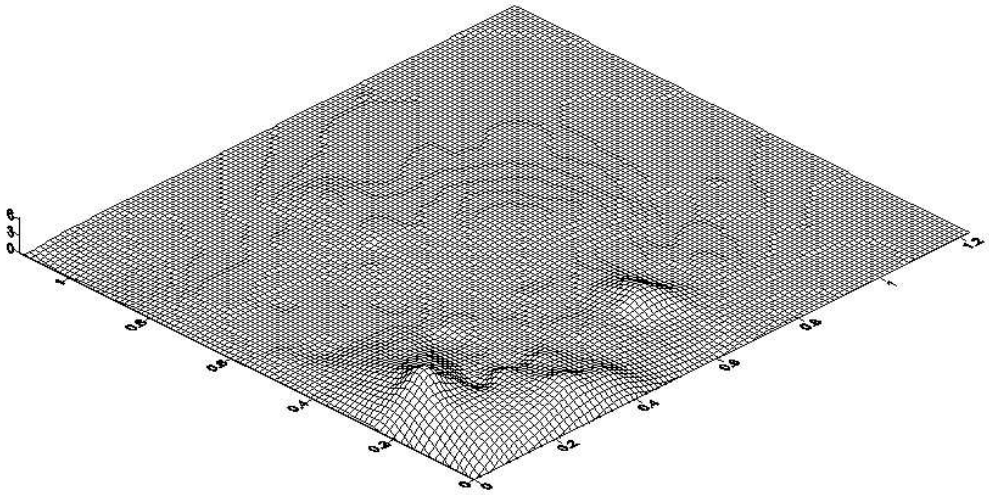

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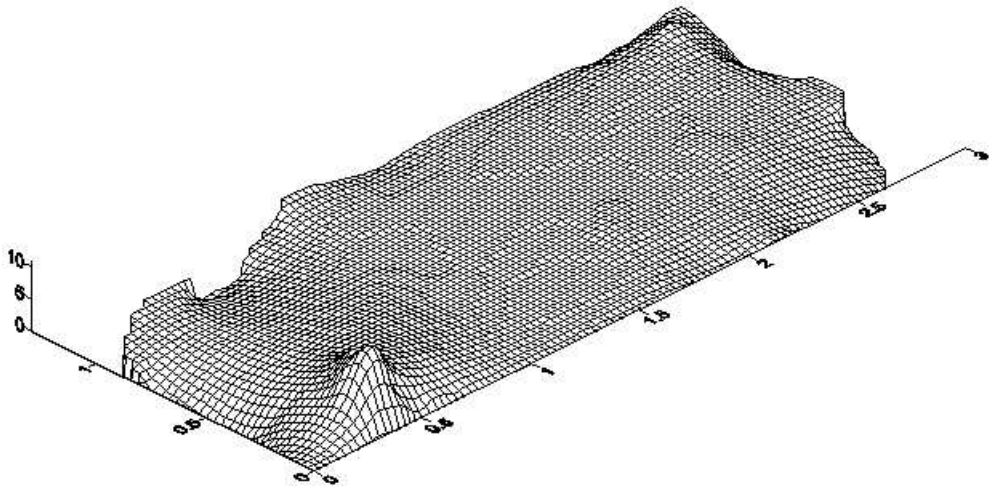

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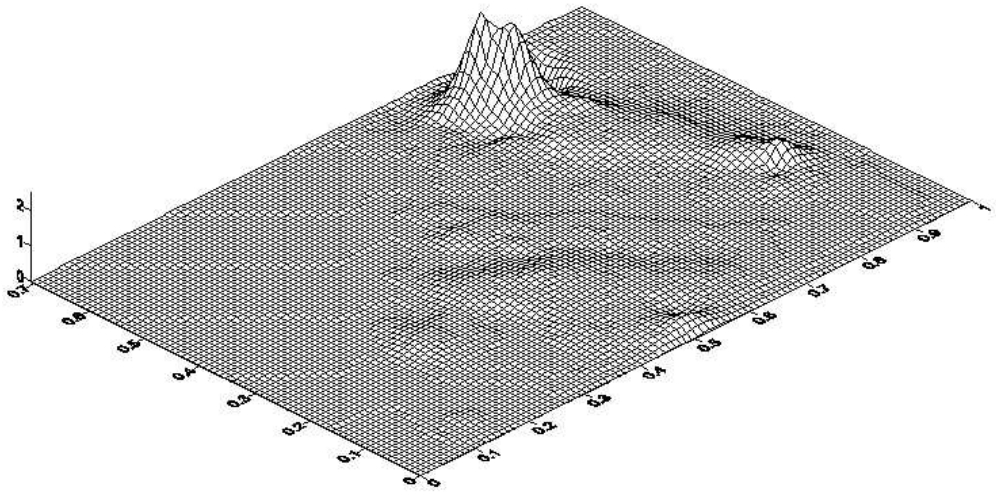

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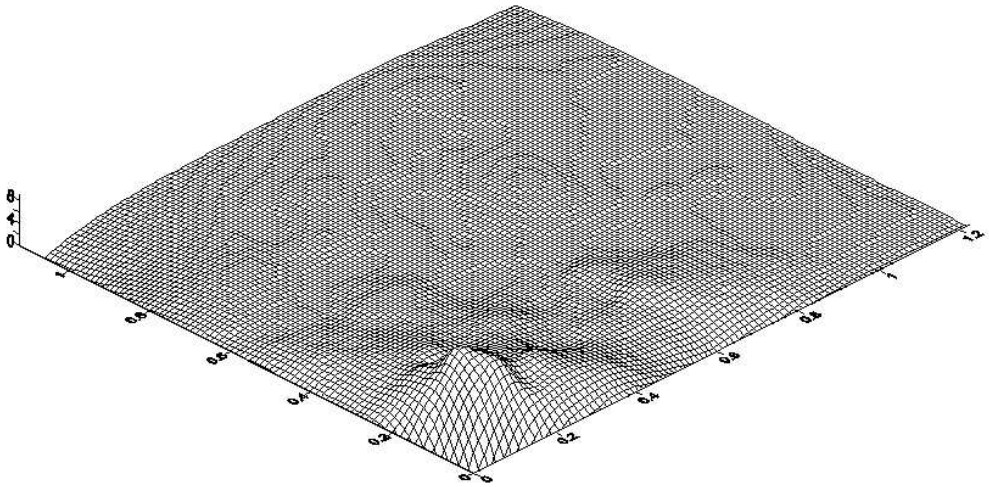

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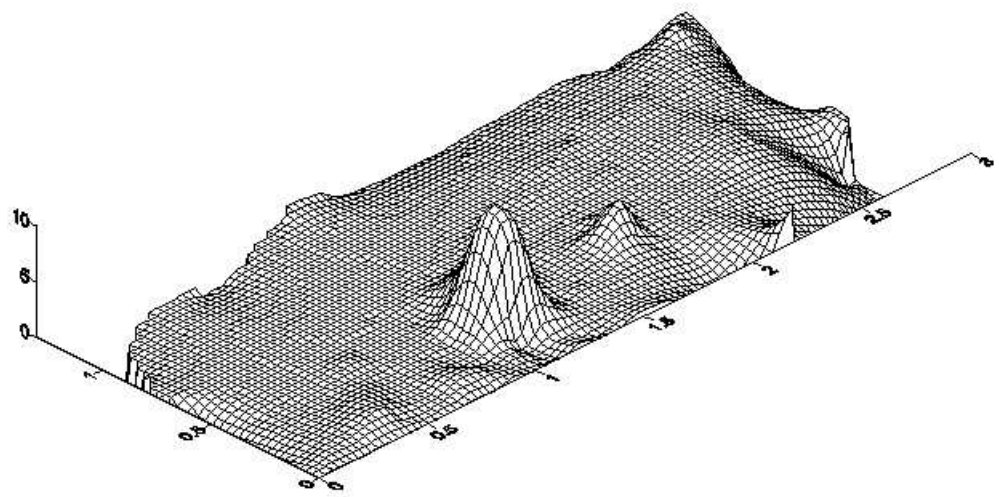

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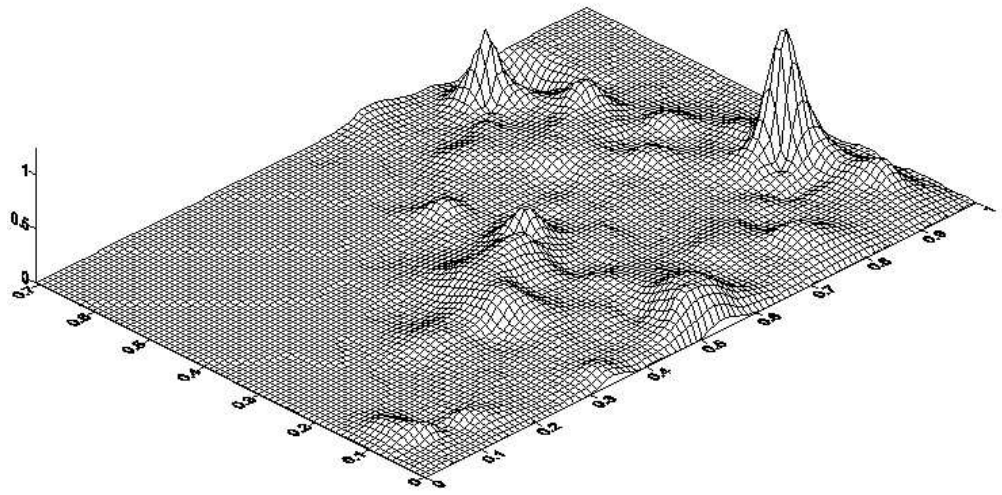

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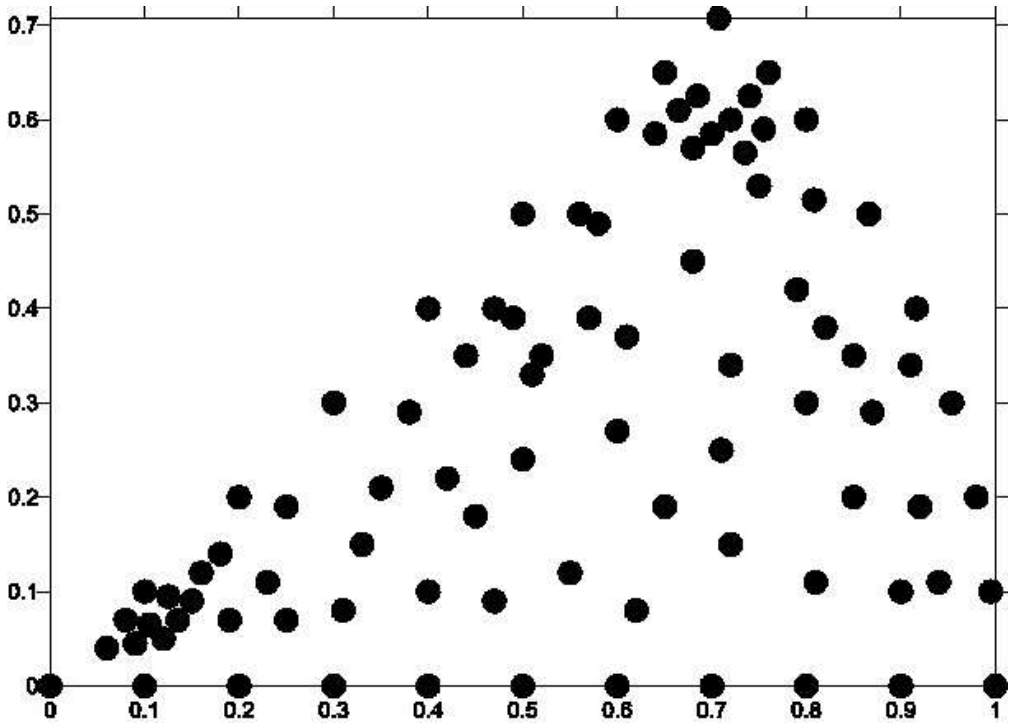

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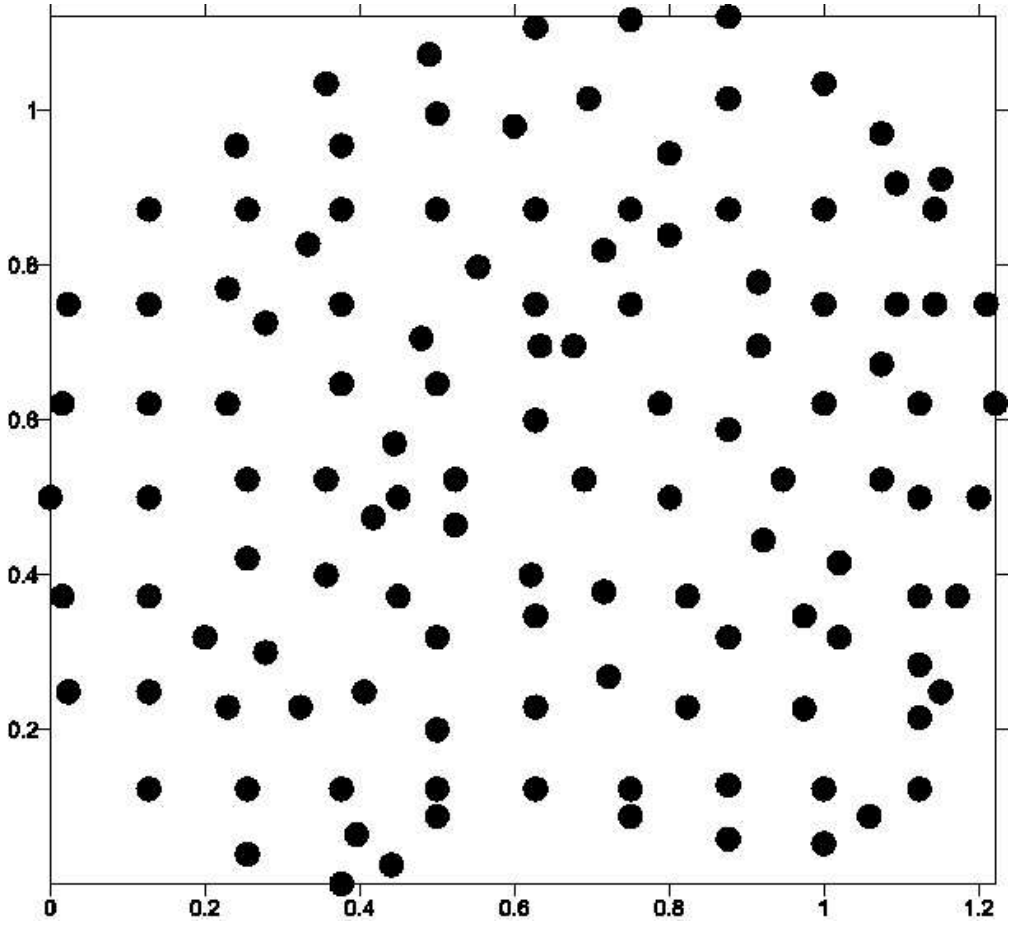

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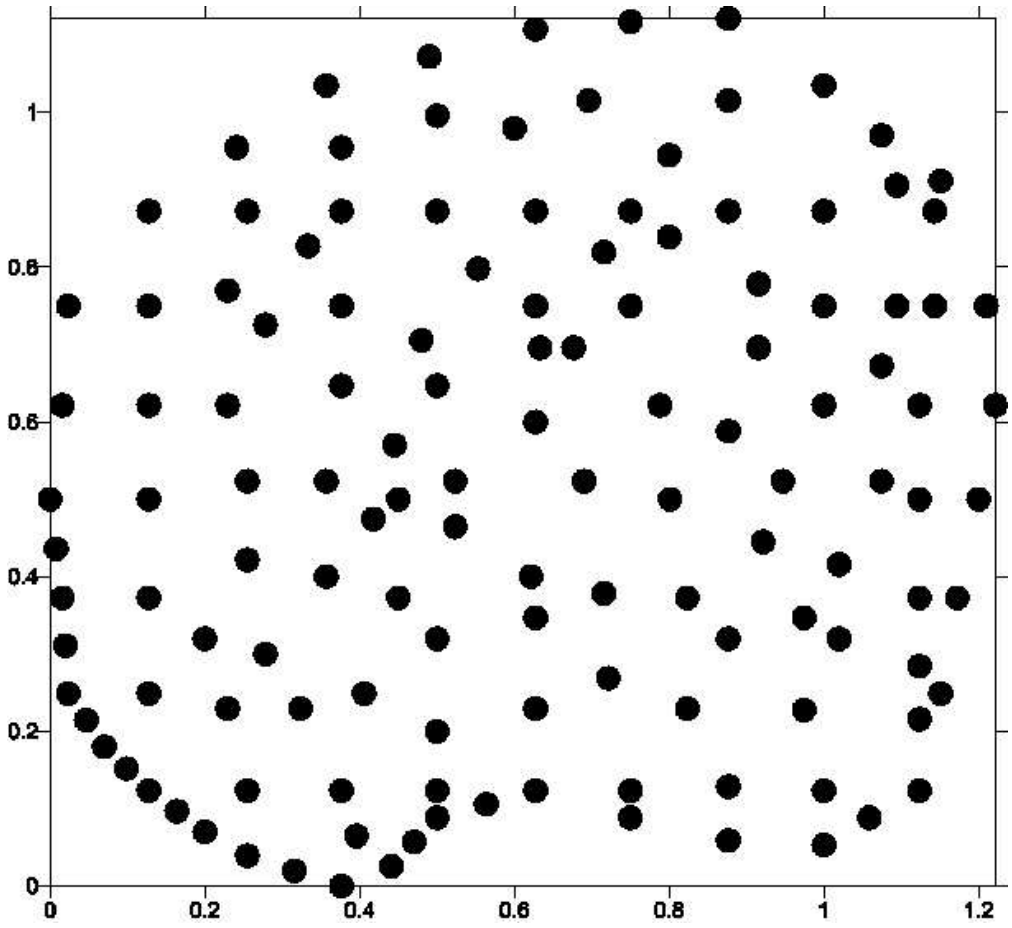

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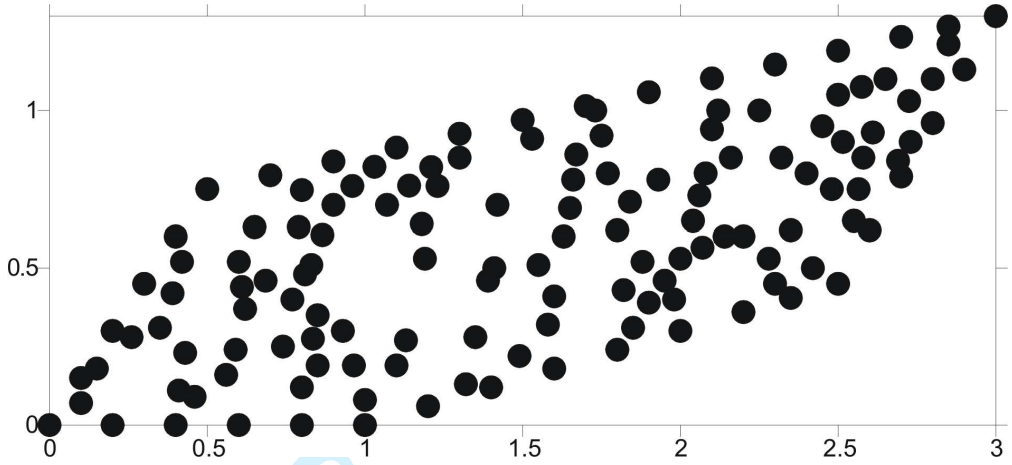

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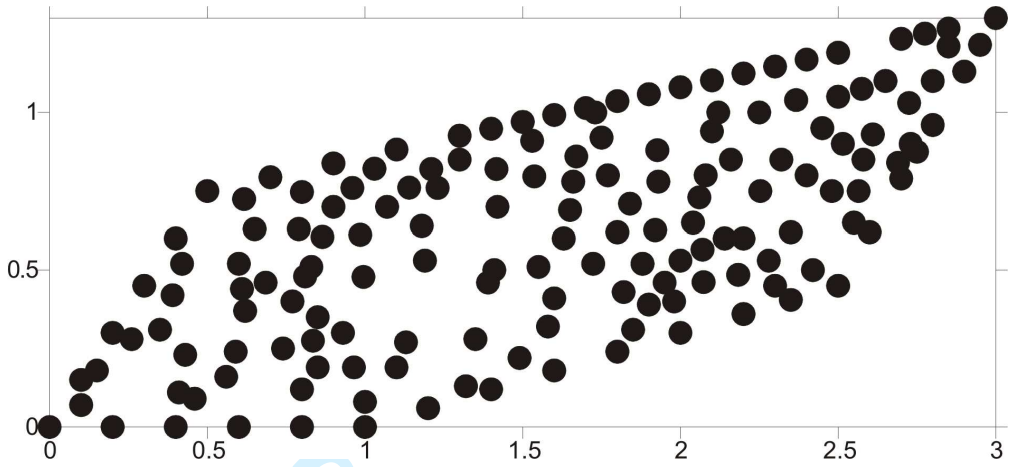

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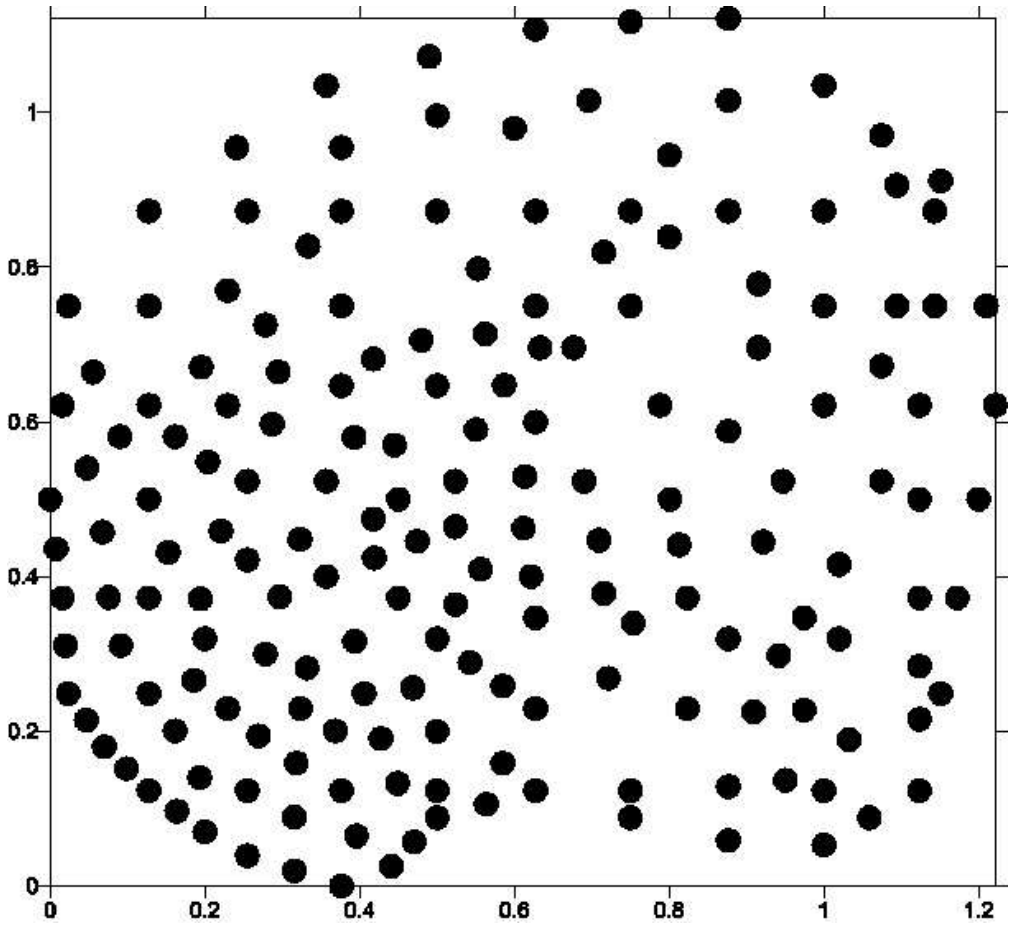

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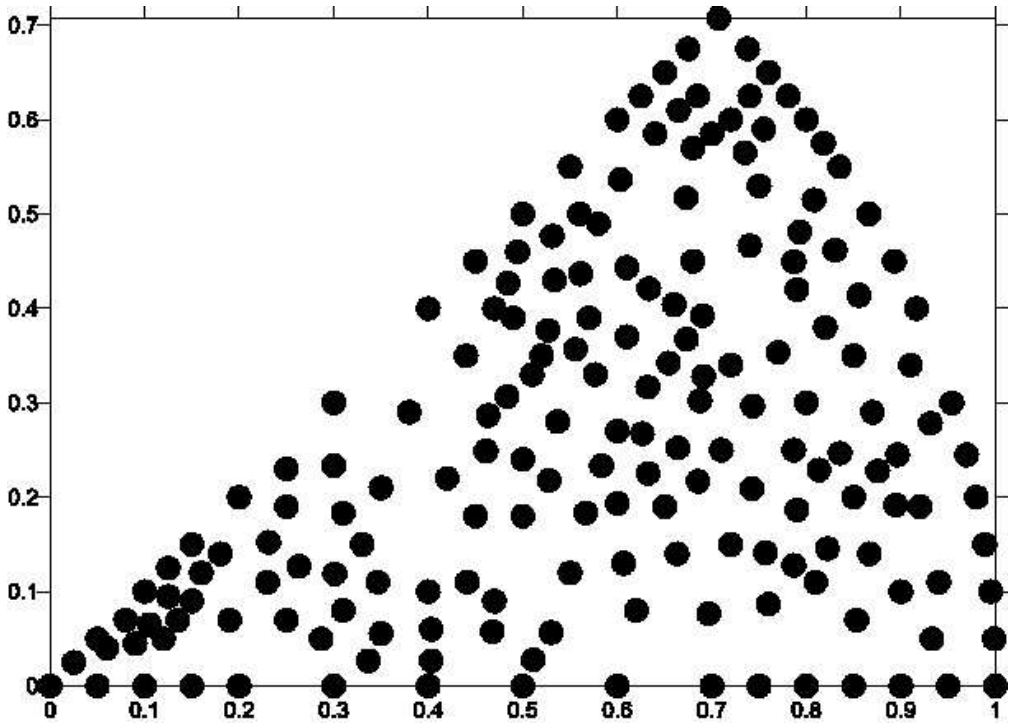

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