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Non-Cartesian Systems :
an Open Problem

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Dedicated to Marie-Louise Nykamp

Abstract

The following open problem is presented and motivated: Are there physical systems whose state spaces do not compose according to either the Cartesian product, as classical systems do, or the usual tensor product, as quantum systems do?

"History is written with the feet ..."

Ex-Chairman Mao

1. Non-Cartesian Systems

The state spaces of systems within Classical Mechanics compose according to the Cartesian product. Namely, let $X, Y$ be two such systems, each with the respective state spaces given by the sets $E, F$, then the state space of the composite system \textquotedblright$X and Y$\textquotedblright is the Cartesian product $E \times F$. 
Here one can note that composing classically $X$ and $Y$ need not necessarily mean that the two systems shall interact, or that the states of the two systems may enter into a correlation. And if they do not, then the state space of "$X$ and $Y$" will be the whole of $E \times F$. However, if they interact, and certain states of the component systems become correlated, then the state space of "$X$ and $Y$" can in fact be a strict subset of $E \times F$.

Except for that classical way of composing systems, it appears that there is only one alternative known, namely, the quantum way of system composition. Namely, let $X, Y$ be two quantum systems, each with the respective state space given by the Hilbert spaces $E, F$, then the state space of the composite quantum system "$X$ and $Y$" is the tensor product $E \otimes F$.

Now, the considerable novelty in the non-Cartesian quantum way of composing systems is that the resulting quantum composite state space $E \otimes F$ is significantly larger than the classical composite state space $E \times F$. Indeed, if for instance $E$ and $F$ are finite dimensional Hilbert spaces, with the respective dimension $n$ and $m$, then the dimension of $E \otimes F$ is no less than $n \cdot m$, while the dimension of $E \times F$ is only $n + m$. Furthermore, one has the injective mapping

\[
(1.1) \quad E \times F \ni (x, y) \mapsto x \otimes y \in E \otimes F
\]

and for convenience of notation, we shall often identify the Cartesian product $E \times F$ with its image through the injective mapping (1.1) in the tensor product $E \otimes F$.

Now, the considerable number of composite states in

\[
(1.2) \quad (E \otimes F) \setminus (E \times F)
\]

which therefore do not correspond to any pair of component states $x \in E$ and $y \in F$, and thus provide a typical quantum convenient feature that does not existent in the classical situation. This wealth of composite states
which are, therefore, not of the form
\[(1.4) \quad z = x \otimes y\]
for any \(x \in E\) and \(y \in F\), are called entangled. And as noted lately, they constitute a major and critically important resource in quantum computation and information.

In usual terms in quantum physics, entanglement is a property of composite states
\[(1.5) \quad z = \sum_{1 \leq i \leq k} x_i \otimes y_i \in (E \otimes F) \setminus (E \times F)\]
where \(x_1, \ldots, x_k \in E, y_1, \ldots, y_k \in F\), of not being of the particular simple form (1.4), thus of not admitting \(k = 1\) in (1.5), in other words, of not belonging to \(E \times F\).

As for correlations, they are about the various \(x_i\) and \(y_j\) in (1.5) which are involved in a situation that is not like in (1.4).
In other words, correlation of states \(x_i \in E\) and \(y_j \in F\) can only occur relative to a composite state \(z \in E \otimes F\) in which these states are components.

By the way, \(x \in E\) and \(y \in F\) in (1.4) are not correlated in the composite state \(z \in E \otimes F\).

2. Is Tensor Product the Only Non-Cartesian Composition of Physical Systems?

Tensor product may so far be the only known way non-Cartesian system compose, yet it has a rather strange history in quantum theory. The following citation from a recent paper, [2], of several physicists involved in state of the art research in quanta may be quite illustrative in this regard:
"In the beginning of modern quantum theory, the notion of entangle-
ment was first noted by Einstein, et.al., [3], and by Schrödinger, [8].
While in those days quantum entanglement and its predicted physical
consequences were - at least partially - considered as an un-physical
property of the formalism - a 'paradox' - the modern perspective on
this issue is very different. Now quantum entanglement is seen as an
experimentally verified property of nature, that provides a resource
for a vast variety of novel phenomena and concepts such as quantum
computation, quantum cryptography, or quantum teleportation.”

And the fact remains that, even at present, by far most of the 101
Quantum Mechanics courses do not mention entanglement ...

As for the possibility of the existence of physical systems which are
non-Cartesian and at the same time non-quantum either, a surprising
behaviour of a classical system presented in [1] may be worth attention.

In order to make a few first preparatory steps towards the study of
the possible existence of physical systems which are non-Cartesian
and at the same time non-quantum, in [4-7] a study was started on
the deeper, simpler and more general roots of tensor products, and
therefore, of entanglement as well. And as it turns out, mathemati-
cally there are indeed considerable generalizations of tensor products
and entanglement, far beyond all algebraic structures. In particular,
any finite or infinite family of sets of state spaces \( E_\lambda \), with \( \lambda \in \Lambda \),
can be composed in infinitely many different rather natural ways, and
according to one’s choice, into a generalized tensor product \( \bigotimes_{\lambda \in \Lambda} E_\lambda \),
and the generalization of (1.1) holds, with an injective mapping

\[
\prod_{\lambda \in \Lambda} E_\lambda \ni (x_\lambda)_{\lambda \in \Lambda} \mapsto \bigotimes_{\lambda \in \Lambda} x_\lambda \in \bigotimes_{\lambda \in \Lambda} E_\lambda
\]

Furthermore, as in (1.2), there is in general a large amount of com-
posite states in

\[
\left( \bigotimes_{\lambda \in \Lambda} E_\lambda \right) \setminus \left( \prod_{\lambda \in \Lambda} E_\lambda \right)
\]

which thus do not correspond to any single family of component states
\( (x_\lambda)_{\lambda \in \Lambda} \in \prod_{\lambda \in \Lambda} E_\lambda \). Therefore, the composite states in (2.2) can be
seen as the generalization of the usual entangled states in the sense of quanta.

We note that in (2.1) and (2.2), and similar to section 1, we made the identification of the Cartesian product $\prod_{\lambda \in \Lambda} E_\lambda$ with its image in $\otimes_{\lambda \in \Lambda} E_\lambda$ through the injective mapping in (2.1).

3. Open Problem

The above lead to the following

Open Problem:

Are there physical systems whose state spaces do not compose according to either the Cartesian product, as classical systems do, or the usual tensor product, as quantum systems do?

References


