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CATEGORIES OF PARTIAL ALGEBRAS FOR CRITICAL POINTS BETWEEN VARIETIES OF ALGEBRAS

PIERRE GILLIBERT

ABSTRACT. We denote by $\text{Con}_c A$ the $(\vee, 0)$ -semilattice of all finitely generated congruences of an algebra A . A *lifting* of a $(\vee, 0)$ -semilattice S is an algebra A such that $S \cong \text{Con}_c A$.

The aim of this work is to give a categorical theory of partial algebras endowed with a partial subalgebra together with a semilattice-valued distance, that we call *gamps*. This part of the theory is formulated in any variety of (universal) algebras.

Let \mathcal{V} and \mathcal{W} be varieties of algebras (on a finite similarity type). Let P be a finite lattice of order-dimension $d > 0$. Let \vec{A} be a P -indexed diagram of finite algebras in \mathcal{V} . If $\text{Con}_c \circ \vec{A}$ has no *partial lifting* in the category of gamps of \mathcal{W} , then there is an algebra $A \in \mathcal{V}$ of cardinality \aleph_{d-1} such that $\text{Con}_c A$ is not isomorphic to $\text{Con}_c B$ for any $B \in \mathcal{W}$.

We already knew a similar result for diagrams \vec{A} such that $\text{Con}_c \circ \vec{A}$ has no lifting in \mathcal{W} , however the algebra A constructed here has cardinality \aleph_d .

Gamps are also used to study congruence-preserving extensions. Denote by \mathcal{M}_3 the variety generated by the lattice of length two, with three atoms. We construct a lattice $A \in \mathcal{M}_3$ of cardinality \aleph_1 with no congruence n -permutable, congruence-preserving extension, for each $n \geq 2$.

1. INTRODUCTION

For an algebra A we denote by $\text{Con } A$ the lattice of all congruences of A under inclusion. Given $x, y \in A$, we denote by $\Theta_A(x, y)$ the smallest congruence of A that identifies x and y , such a congruence is called *principal*. A congruence is *finitely generated* if it is a finite join of principal congruences. The lattice $\text{Con } A$ is algebraic and the compact element of $\text{Con } A$ are the finitely generated congruences.

The lattice $\text{Con } A$ is determined by the $(\vee, 0)$ -semilattice $\text{Con}_c A$ of compact congruences of A . In this paper we mostly refer to $\text{Con}_c A$ instead of $\text{Con } A$. If $\text{Con}_c A$ is isomorphic to a $(\vee, 0)$ -semilattice S , we call A a *lifting* of S .

Given a class of algebras \mathcal{K} we denote by $\text{Con}_c \mathcal{K}$ the class of all $(\vee, 0)$ -semilattices with a lifting in \mathcal{K} . In general, even if \mathcal{K} is a variety of algebras, there is no good description of $\text{Con}_c \mathcal{K}$. The negative solution to the congruence lattice problem (CLP) in [16] is a good example of the difficulty to find such a description.

The study of CLP led to the following related questions. Fix two classes of algebras \mathcal{V} and \mathcal{W} .

(Q1) Given $A \in \mathcal{V}$, does there exist $B \in \mathcal{W}$ such that $\text{Con}_c A \cong \text{Con}_c B$?

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(Q2) Given $A \in \mathcal{V}$, does there exist a congruence-preserving extension $B \in \mathcal{W}$ of A ?

A positive answer to (Q1) is equivalent to $\text{Conc } \mathcal{V} \subseteq \text{Conc } \mathcal{W}$. The “containment defect” of $\text{Conc } \mathcal{V}$ into $\text{Conc } \mathcal{W}$ is measured (cf. [15, 3]) by the *critical point* between \mathcal{V} and \mathcal{W} , defined as

$$\text{crit}(\mathcal{V}; \mathcal{W}) = \begin{cases} \min\{\text{card } S \mid S \in (\text{Conc } \mathcal{V}) - (\text{Conc } \mathcal{W})\}, & \text{if } \text{Conc } \mathcal{V} \not\subseteq \text{Conc } \mathcal{W}, \\ \infty, & \text{if } \text{Conc } \mathcal{V} \subseteq \text{Conc } \mathcal{W}. \end{cases}$$

This critical point has been already studied, for different families of varieties of lattices, in [10, 11, 3].

We now give an illustration of (Q2). Every countable locally finite lattice has a relatively complemented, congruence-preserving extension (cf. [7]). In particular every countable locally finite lattice has a congruence-permutable, congruence-preserving extension. However, in every non-distributive variety of lattices, the free lattice on \aleph_1 generators has no congruence-permutable, congruence-preserving extension (cf. [5, Chapter 5]). A precise answer to (Q2) also depends on the cardinality of A .

In order to study a similar problem, Pudlák in [13] uses an approach based on liftings of diagrams. The assignment $A \mapsto \text{Conc } A$ can be extended to a functor. This leads to the following questions:

(Q1') Given a diagram \vec{A} in \mathcal{V} , does there exist a diagram \vec{B} in \mathcal{W} such that $\text{Conc } \circ \vec{A} \cong \text{Conc } \circ \vec{B}$?

(Q2') Given a diagram \vec{A} in \mathcal{V} , does there exist a diagram \vec{B} in \mathcal{W} which is a congruence-preserving extension of \vec{A} ?

The functor Conc preserves directed colimits, thus, in many cases, a positive answer for the finite case of (Q1') implies a positive answer to (Q1).

Proposition 1.1. *Assume that \mathcal{V} and \mathcal{W} are varieties of algebras. If \mathcal{V} is locally finite and for every lattice-indexed diagram \vec{A} of finite algebras in \mathcal{V} there exists a diagram \vec{B} in \mathcal{W} such that $\text{Conc } \circ \vec{A} \cong \text{Conc } \circ \vec{B}$, then (Q1) has a positive answer.*

In this proposition, we consider infinite diagrams of finite algebras. However if \mathcal{W} is finitely generated and congruence-distributive, a compactness argument makes it possible to restrict the assumptions to finite diagrams of finite algebras.

In order to study the converse of Proposition 1.1, we shall use the construction of *condensate*, introduced in [3]. This construction was introduced in order to turn diagram counterexamples to object counterexamples. We use it here to turn a diagram counterexample of (Q1') to a counterexample of (Q1).

Theorem 1.2. *Assume that \mathcal{V} and \mathcal{W} are varieties of algebras. Let P be a finite lattice. Let \vec{A} be a P -indexed diagram in \mathcal{V} . If $\text{Conc } \circ \vec{A}$ is not liftable in \mathcal{W} then there is a condensate $A \in \mathcal{V}$ of \vec{A} , such that $\text{Conc } A$ is not liftable in \mathcal{W} .*

Moreover, if \mathcal{W} has a countable similarity type and all algebras of \vec{A} are countable, then the condensate A can be chosen of cardinal \aleph_d , where d is the order-dimension of P . In particular, $\text{crit}(\mathcal{V}; \mathcal{W}) \leq \aleph_d$.

If every algebra of \vec{A} is finite and \mathcal{W} is finitely generated and congruence-distributive, then A can be chosen of cardinal \aleph_{d-1} , so $\text{crit}(\mathcal{V}; \mathcal{W}) \leq \aleph_{d-1}$.

The cardinality bound, in case \mathcal{W} is finitely generated and congruence-distributive, is optimal, in the following sense: There are finitely generated varieties \mathcal{V} and \mathcal{W}

of lattices such that every countable $(\vee, 0)$ -semilattice liftable in \mathcal{V} is liftable in \mathcal{W} and there is a square-indexed diagram of $(\vee, 0)$ -semilattices that has a lifting in \mathcal{V} but no lifting in \mathcal{W} . In particular there is a $(\vee, 0)$ -semilattice of cardinal \aleph_1 that is liftable in \mathcal{V} but not liftable in \mathcal{W} , thus $\text{crit}(\mathcal{V}; \mathcal{W}) = \aleph_1$. This example appears in [3, Section 8].

Later, we generalized the condensate construction in [5], to a larger categorical context. The best bound of Theorem 1.2 is obtained in a more general case (cf. [5, Theorem 4-9.2]), namely if \mathcal{W} is both *congruence-proper* (cf. [5, Definition 4-8.1]) and locally finite, for example \mathcal{W} is a finitely generated congruence-modular variety. Using the tools introduced in this paper, we give a new version (cf. Theorem 9.6), we assume that \mathcal{W} is congruence-proper and has finite similarity type.

This categorical version of condensate can also apply to turn a counterexample of **(Q2')** to a counterexample of **(Q2)**. For example in [5, Chapter 5] we give a square \vec{A} of finite lattices, that has no congruence-permutable, congruence-preserving extension. A condensate of this square has cardinality \aleph_1 and it has no congruence-permutable, congruence-preserving extension.

The largest part of this paper is the introduction of *pregamps* and *gamps*, it is a generalization of semilattice-metric spaces and semilattice-metric covers given in [5, Chapter 5]. The category of gamps of a variety \mathcal{V} has properties similar to a finitely generated congruence-distributive variety.

A *pregamp* is a triplet $\mathbf{A} = (A, \delta, S)$, where A is a partial algebra, S is a $(\vee, 0)$ -semilattice and $\delta: A^2 \rightarrow S$ is a *distance, compatible with the operations*. A typical example of pregamp is $(A, \Theta_A, \text{Con}_c A)$, for an algebra A . This generalizes to partial algebras the notion of a congruence.

A *gamp* \mathbf{A} is a pregamp (A, δ, S) with a partial subalgebra A^* of A . There are many natural properties that a gamp can satisfy (cf. Section 7), for example \mathbf{A} is *full* if all operations with parameters in A^* can be evaluated in A . A *morphism* of gamps is a morphism of partial algebras with a morphism of $(\vee, 0)$ -semilattices satisfying a compatibility condition with the distances (cf. Definition 6.1).

The class of all gamps (on a given type), with morphisms of gamps, forms a category. Denote by \mathbf{C} the forgetful functor from the category of gamps to the category of $(\vee, 0)$ -semilattices. A *partial lifting* of a diagram \vec{S} of $(\vee, 0)$ -semilattice is a diagram \vec{B} of gamps, with some additional properties, such that $\mathbf{C} \circ \vec{B} \cong \vec{S}$.

The category of gamps has properties similar to locally finite, congruence-proper varieties. Let S be a finite $(\vee, 0)$ -semilattice, let \mathbf{B} be a gamp such that $\mathbf{C}(\mathbf{B}) \cong S$, there are (arbitrary large) finite subgamps \mathbf{B}' of \mathbf{B} such that $\mathbf{C}(\mathbf{B}') \cong S$. There is no equivalent result for algebras: for example, the three-element chain is the congruence lattice of a modular lattice, but not the congruence lattice of any finite modular lattice.

Assume that \mathcal{V} and \mathcal{W} are varieties of algebras. Let P be a finite lattice of order-dimension d . Suppose that we find \vec{A} a P -indexed diagram of finite algebras in \mathcal{V} , such that $\text{Con}_c \circ \vec{A}$ has no partial lifting (with maybe some additional “*locally finite properties*”, cf. Section 7) in the category of gamps of \mathcal{W} , then $\text{crit}(\mathcal{V}; \mathcal{W}) \leq \aleph_{d-1}$. Hence we obtain the optimal bound, with no assumption on \mathcal{W} . However there is no (known) proof that a diagram with no lifting has no partial lifting, but no counterexample has been found.

The *dual* of a lattice L is the lattice L^d with reverse order. The *dual* of a variety of lattices \mathcal{V} is \mathcal{V}^d the variety of all duals of lattices in \mathcal{V} . Let \mathcal{V} and \mathcal{W} be varieties

of lattices, if $\mathcal{V} \subseteq \mathcal{W}$ or $\mathcal{V} \subseteq \mathcal{W}^d$, then $\text{Conc } \mathcal{V} \subseteq \text{Conc } \mathcal{W}$. In a sequel to the present paper (cf. [4]), we shall prove the following result.

Theorem. *Let \mathcal{V} and \mathcal{W} be varieties of lattices. If every simple lattice in \mathcal{W} contains a prime interval, then one of the following statements holds:*

- (1) $\text{crit}(\mathcal{V}; \mathcal{W}) \leq \aleph_2$.
- (2) $\mathcal{V} \subseteq \mathcal{W}$.
- (3) $\mathcal{V} \subseteq \mathcal{W}^d$.

The \aleph_2 bound is optimal, as there are varieties \mathcal{V} and \mathcal{W} of lattices such that $\text{crit}(\mathcal{V}; \mathcal{W}) = \aleph_2$. Without the use of gamps, we would have obtained an upper bound \aleph_3 instead of \aleph_2 .

The gamps can also be used to study congruence-preserving extensions. Denote by \mathbf{P}_{gl} the functor that maps a gamp $(A^*, A, \delta, \tilde{A})$ to the pregamp (A^*, δ, \tilde{A}) ; we also denote by \mathbf{P}_{ga} the functor that maps an algebra A to the pregamp $(A, \Theta_A, \text{Conc } A)$. Let B be a congruence-preserving extension of an algebra A , then $(A, B, \Theta_B, \text{Conc } B)$ is a gamp. Similarly, let $\vec{B} = (B_p, g_{p,q} \mid p \leq q \text{ in } P)$ be a congruence-preserving extension of a diagram $\vec{A} = (A_p, f_{p,q} \mid p \leq q \text{ in } P)$, denote by $\mathbf{B}_p = (A, B, \Theta_B, \text{Conc } B)$ and $\mathbf{g}_{p,q} = (g_{p,q}, \text{Conc } g_{p,q})$, for all $p \leq q$ in P , then $\vec{\mathbf{B}} = (\mathbf{B}_p, \mathbf{g}_{p,q} \mid p \leq q \text{ in } P)$ is a diagram of gamps. Moreover, $\mathbf{P}_{\text{gl}} \circ \vec{\mathbf{B}} = \mathbf{P}_{\text{ga}} \circ \vec{A}$, up to the identification of $\text{Conc } B$ and $\text{Conc } A$.

In Section 10, given $n \geq 2$, we construct a square \vec{A} of finite lattices in \mathcal{M}_3 , such that the diagram \vec{A} has no congruence n -permutable, congruence-preserving extension. Another condensate construction gives a result proved in [12], namely the existence of a lattice $A \in \mathcal{M}_3$ of cardinality \aleph_2 with no congruence n -permutable, congruence-preserving extension.

Hopefully, once again, the diagram \vec{A} satisfies a stronger statement, there is no *operational* (cf. Definition 10.1) diagram \vec{B} of *congruence n -permutable* gamps of lattices such that $\mathbf{P}_{\text{gl}} \circ \vec{\mathbf{B}} \cong \mathbf{P}_{\text{ga}} \circ \vec{A}$. Using a condensate, we obtain a lattice \mathcal{M}_3 of cardinality \aleph_1 with no congruence n -permutable, congruence-preserving extension.

2. BASIC CONCEPTS

We denote by 0 (resp., 1) the least (resp. largest) element of a poset if it exists. We denote by $\mathbf{2} = \{0, 1\}$ the two-element lattice, or poset (i.e., partially ordered set), or $(\vee, 0)$ -semilattice, depending of the context. Given an algebra A , we denote by $\mathbf{0}_A$ the identity congruence of A .

Given subsets P and Q of a poset R , we set

$$P \downarrow Q = \{p \in P \mid (\exists q \in Q)(p \leq q)\}.$$

If $Q = \{q\}$, we simply write $P \downarrow q$ instead of $P \downarrow \{q\}$.

Let \mathcal{V} be a variety of algebras, let κ be a cardinal, we denote by $F_{\mathcal{V}}(\kappa)$ the free algebras in \mathcal{V} with κ generators. Given an algebra A we denote by $\mathbf{Var } A$ the variety of algebras generated by A . If A is a lattice we also denote by $\mathbf{Var}^{0,1} A$ the variety of bounded lattices generated by A . We denote by \mathcal{L} the variety of lattices.

We denote the *range* of a function $f: X \rightarrow Y$ by $\text{rng } f = \{f(x) \mid x \in X\}$. We use basic set-theoretical notation, for example ω is the first infinite ordinal, and also the set of all nonnegative integers; furthermore, $n = \{0, 1, \dots, n-1\}$ for every nonnegative integer n . By “countable” we will always mean “at most countable”.

Let X, I be sets, we often denote $\vec{x} = (x_i)_{i \in I}$ an element of X^I . In particular, for $n < \omega$, we denote by $\vec{x} = (x_0, \dots, x_{n-1})$ an n -tuple of X . If $f: Y \rightarrow Z$ is a function, where $Y \subseteq X$, we denote $f(\vec{x}) = (f(x_0), \dots, f(x_{n-1}))$ whenever it is defined. Similarly, if $f: Y \rightarrow Z$ is a function, where $Y \subseteq X^n$, we denote $f(\vec{x}) = f(x_0, \dots, x_{n-1})$ whenever it is defined. We also write $f(\vec{x}, \vec{y}) = f(x_0, \dots, x_{m-1}, y_0, \dots, y_{n-1})$ in case $\vec{x} = (x_0, \dots, x_{m-1})$ and $\vec{y} = (y_0, \dots, y_{n-1})$, and so on.

For example, let A and B be algebras of the same similarity type. Let ℓ be an n -ary operation. Let $f: A \rightarrow B$ a map. The map f is *compatible* with ℓ if $f(\ell(\vec{x})) = \ell(f(\vec{x}))$ for every n -tuple \vec{x} of X . Let $m \leq n \leq \omega$. Let \vec{X} be an n -tuple of X , we denote by $\vec{x} \upharpoonright m$ the m -tuple $(x_k)_{k < m}$.

If X is a set and θ is an equivalence relation on X , we denote by X/θ the set of all equivalence classes of θ . Given $x \in X$ we denote by x/θ the equivalence class of θ containing x . Given an n -tuple \vec{x} of X , we denote $\vec{x}/\theta = (x_0/\theta, \dots, x_{n-1}/\theta)$. Given $Y \subseteq X$, we set $Y/\theta = \{x/\theta \mid x \in Y\}$.

Let $n \geq 2$ an integer. An algebra A is *congruence n -permutable* if the following equality holds:

$$\underbrace{\alpha \circ \beta \circ \alpha \circ \dots}_{n \text{ times}} = \underbrace{\beta \circ \alpha \circ \beta \circ \dots}_{n \text{ times}}, \text{ for all } \alpha, \beta \in \text{Con } A.$$

If $n = 2$ we say that A is *congruence-permutable* instead of *congruence 2-permutable*.

The following statement is folklore.

Proposition 2.1. *Let A be an algebra, let $n \geq 2$ an integer. The following conditions are equivalent:*

- (1) *The algebra A is congruence n -permutable.*
- (2) *For all $x_0, x_1, \dots, x_n \in A$, there are $x_0 = y_0, y_1, \dots, y_n = x_n \in A$ such that the following containments hold:*

$$\begin{aligned} \Theta_A(y_k, y_{k+1}) &\subseteq \bigvee (\Theta_A(x_i, x_{i+1}) \mid i < n \text{ even}), & \text{for all } k < n \text{ odd,} \\ \Theta_A(y_k, y_{k+1}) &\subseteq \bigvee (\Theta_A(x_i, x_{i+1}) \mid i < n \text{ odd}), & \text{for all } k < n \text{ even.} \end{aligned}$$

Let $n \geq 2$. The class of all congruence n -permutable algebras of a given similarity type is closed under directed colimits and quotients. Moreover the class of congruence n -permutable algebras of a congruence-distributive variety is also closed under finite products (the latter statement is known not to extend to arbitrary algebras).

3. SEMILATTICES

In this section we give some well-known facts about $(\vee, 0)$ -semilattices. Most notions and results will have later a generalization involving pregramps and gamps.

Proposition 3.1. *Let S, T be $(\vee, 0)$ -semilattices, let X be a set, and let $f: X \rightarrow S$ and $g: X \rightarrow T$ be maps. Assume that for every $x \in X$, for every positive integer n , and for every n -tuple \vec{y} of X the following implication holds:*

$$f(x) \leq \bigvee_{k < n} f(y_k) \implies g(x) \leq \bigvee_{k < n} g(y_k). \quad (3.1)$$

If S is join-generated by $f(X)$, then there exists a unique $(\vee, 0)$ -homomorphism $\phi: S \rightarrow T$ such that $\phi(f(x)) = g(x)$ for each $x \in X$.

If the converse of (3.1) also holds and $g(X)$ also join-generates T , then ϕ is an isomorphism.

Definition 3.2. An *ideal* of a $(\vee, 0)$ -semilattice S is a lower subset I of S such that $0 \in I$ and $u \vee v \in I$ for all $u, v \in I$. We denote by $\text{Id } S$ the lattice of ideals of S .

Let $\phi: S \rightarrow T$ be a $(\vee, 0)$ -homomorphism. The *0-kernel* of ϕ is $\ker_0 \phi = \{a \in S \mid \phi(a) = 0\}$; it is an ideal of S . We say that ϕ *separates zero* if $\ker_0 \phi = \{0\}$.

Let P be a poset, let $\vec{S} = (S_p, \phi_{p,q} \mid p \leq q \text{ in } P)$ be a diagram of $(\vee, 0)$ -semilattices. An *ideal* of \vec{S} is a family $(I_p)_{p \in P}$ such that I_p is an ideal of S_p and $\phi_{p,q}(I_p) \subseteq I_q$ for all $p \leq q$ in P .

Let $\vec{\phi} = (\phi_p)_{p \in P}: \vec{S} \rightarrow \vec{T}$ be a natural transformation of P -indexed diagrams of $(\vee, 0)$ -semilattices. The *0-kernel* of $\vec{\phi}$ is $\ker_0 \vec{\phi} = (\ker_0 \phi_p)_{p \in P}$, it is an ideal of \vec{S} .

Lemma 3.3. Let S be a $(\vee, 0)$ -semilattice, let $I \in \text{Id } S$. Put:

$$\theta_I = \{(x, y) \in S^2 \mid (\exists u \in I)(x \vee u = y \vee u)\}.$$

The relation θ_I is a congruence of S .

Notation 3.4. We denote by S/I the $(\vee, 0)$ -semilattice S/θ_I , where θ_I is the congruence defined in Lemma 3.3. Given $a \in S$, we denote by a/I the equivalent class of a for θ_I . The $(\vee, 0)$ -homomorphism $\phi: S \rightarrow S/I, a \mapsto a/I$ is the *canonical projection*. Notice that $\ker_0 \phi = I$.

If $I = \{0\}$, we identify S/I and S .

Lemma 3.5. Let $\phi: S \rightarrow T$ be a $(\vee, 0)$ -homomorphism, and let $I \in \text{Id } S$ and $J \in \text{Id } T$ such that $\phi(I) \subseteq J$. There exists a unique map $\psi: S/I \rightarrow T/J$ such that $\psi(a/I) = \phi(a)/J$ for each $a \in S$. Moreover, ψ is a $(\vee, 0)$ -homomorphism.

Notation 3.6. We say that ϕ *induces* the $(\vee, 0)$ -homomorphism $\psi: S/I \rightarrow T/J$ in Lemma 3.5.

Lemma 3.7. Let P be a poset, let $\vec{S} = (S_p, \phi_{p,q} \mid p \leq q \text{ in } P)$ be a diagram of $(\vee, 0)$ -semilattices, and let \vec{I} be an ideal of \vec{S} . Denote by $\psi_{p,q}: S_p/I_p \rightarrow S_q/I_q$ the $(\vee, 0)$ -homomorphism induced by $\phi_{p,q}$, then $(S_p/I_p, \psi_{p,q} \mid p \leq q \text{ in } P)$ is a diagram of $(\vee, 0)$ -semilattices.

Notation 3.8. We denote by \vec{S}/\vec{I} the diagram $(S_p/I_p, \psi_{p,q} \mid p \leq q \text{ in } P)$ introduced in Lemma 3.7.

Lemma 3.9. Let P be a poset, let $\vec{\phi}: \vec{S} \rightarrow \vec{T}$ be a natural transformation of P -indexed diagrams of $(\vee, 0)$ -semilattices, let $\vec{I} \in \text{Id } \vec{S}$ and $\vec{J} \in \text{Id } \vec{T}$ such that $\phi_p(I_p) \subseteq J_p$ for all $p \in P$. Denote by $\psi_p: S_p/I_p \rightarrow T_p/J_p$ the $(\vee, 0)$ -homomorphism induced by ϕ_p . Then $\vec{\psi}$ is a natural transformation from \vec{S}/\vec{I} to \vec{T}/\vec{J} .

Notation 3.10. We say that $\vec{\phi}$ *induces* $\vec{\psi}: \vec{S}/\vec{I} \rightarrow \vec{T}/\vec{J}$, the natural transformation defined in Lemma 3.9.

Definition 3.11. A $(\vee, 0)$ -homomorphism $\phi: S \rightarrow T$ is *ideal-induced* if ϕ is surjective and for all $x, y \in S$ with $\phi(x) = \phi(y)$ there exists $z \in S$ such that $x \vee z = y \vee z$ and $\phi(z) = 0$.

Let P be a poset, let $\vec{S} = (S_p, \phi_{p,q} \mid p \leq q \text{ in } P)$ and $\vec{T} = (T_p, \psi_{p,q} \mid p \leq q \text{ in } P)$ be P -indexed diagrams of $(\vee, 0)$ -semilattices. A natural transformation $\vec{\pi} = (\pi_p)_{p \in P}: \vec{S} \rightarrow \vec{T}$ is *ideal-induced* if π_p is ideal-induced for each $p \in P$.

Remark 3.12. Let I be an ideal of a $(\vee, 0)$ -semilattice A , denote by $\pi: A \rightarrow A/I$ the canonical projection, then π is ideal-induced.

The next lemmas give a characterization of ideal-induced $(\vee, 0)$ -homomorphisms.

Lemma 3.13. *Let $\phi: S \rightarrow T$ be a $(\vee, 0)$ -homomorphism. The following statements are equivalent*

- (1) ϕ is ideal-induced.
- (2) The $(\vee, 0)$ -homomorphism $\psi: S/\ker_0 \phi \rightarrow T$ induced by ϕ is an isomorphism.

The following lemma expresses that, given a diagram \vec{S} of $(\vee, 0)$ -semilattices, the colimits of quotients of \vec{S} are the quotients of the colimits of \vec{S} .

Lemma 3.14. *Let P be a directed poset, let $\vec{S} = (S_p, \phi_{p,q} \mid p \leq q \text{ in } P)$ be a P -indexed diagram in $\mathbf{Sem}_{\vee,0}$, and let $(S, \phi_p \mid p \in P) = \varinjlim \vec{S}$ be a directed colimit cocone in $\mathbf{Sem}_{\vee,0}$. The following statements hold:*

- (1) Let \vec{I} be an ideal of \vec{S} . Then $I = \bigcup_{p \in P} \phi_p(I_p)$ is an ideal of S . Moreover, denote by $\psi_p: S_p/I_p \rightarrow S/I$ the $(\vee, 0)$ -homomorphism induced by ϕ_p , for each $p \in P$. The following is a directed colimit cocone:

$$(S/I, \psi_p \mid p \in P) = \varinjlim \vec{S}/\vec{I} \quad \text{in } \mathbf{Sem}_{\vee,0}.$$

- (2) Let $I \in \text{Id } S$. Put $I_p = \phi_p^{-1}(I)$ for each $p \in P$. Then $\vec{I} = (I_p)_{p \in P}$ is an ideal of \vec{S} , moreover $I = \bigcup_{p \in P} \phi_p(I_p)$.

Lemma 3.15. *Let $\pi: A \twoheadrightarrow B$ be a surjective morphism of algebras. The $(\vee, 0)$ -homomorphism $\text{Con}_c \pi$ is ideal-induced. Moreover, $\ker_0(\text{Con}_c \pi) = (\text{Con}_c A) \downarrow \ker \pi$.*

Proposition 3.16. *Let S and T be $(\vee, 0)$ -semilattices with T finite, let $\phi: S \rightarrow T$ be an ideal-induced $(\vee, 0)$ -homomorphism, and let $X \subseteq S$ finite. There exists a finite $(\vee, 0)$ -subsemilattice S' of S such that $X \subseteq S'$ and $\phi \upharpoonright S': S' \rightarrow T$ is ideal-induced.*

Proof. As ϕ is surjective and X is finite, there exists a finite $(\vee, 0)$ -subsemilattice Y of S such that $X \subseteq Y$ and $\phi(Y) = T$. Given $x, y \in Y$ with $\phi(x) = \phi(y)$ we fix $u_{x,y} \in S$ such that $\phi(u_{x,y}) = 0$ and $x \vee u_{x,y} = y \vee u_{x,y}$. Let U be the $(\vee, 0)$ -subsemilattice of S generated by $\{u_{x,y} \mid x, y \in Y \text{ and } \phi(x) = \phi(y)\}$. As $\phi(u) = 0$ for all generators, $\phi(u) = 0$ for each $u \in U$.

Let S' be the $(\vee, 0)$ -subsemilattice of S generated by $Y \cup U$. As S' is finitely generated, it is finite. As $Y \subseteq S'$, $\phi(S') = T$. Let $a, b \in S'$ such that $\phi(a) = \phi(b)$. There exist $x, y \in Y$ and $u, v \in U$ such that $a = x \vee u$ and $b = y \vee v$, thus $\phi(a) = \phi(x \vee u) = \phi(x) \vee \phi(u) = \phi(x)$. Similarly, $\phi(b) = \phi(y)$, hence $\phi(x) = \phi(y)$, moreover $x, y \in Y$, so $u_{x,y} \in U$. The element $w = u \vee v \vee u_{x,y}$ belongs to U , hence $\phi(w) = 0$, moreover $w \in S'$. From $x \vee u_{x,y} = y \vee u_{x,y}$ it follows that $a \vee w = b \vee w$. Therefore, $\phi \upharpoonright S'$ is ideal-induced. \square

4. PARTIAL ALGEBRAS

In this section we introduce a few basic properties of partial algebras. We fix a similarity type \mathcal{L} . Given $\ell \in \mathcal{L}$ we denote by $\text{ar}(\ell)$ the arity of ℓ .

Definition 4.1. A *partial algebra* A is a set (the *universe* of the partial algebra), given with a set $D_\ell = \text{Def}_\ell(A) \subseteq A^{\text{ar}(\ell)}$ and a map $\ell^A: D_\ell \rightarrow A$ called a *partial operation*, for each $\ell \in \mathcal{L}$.

Let $\ell \in \mathcal{L}$ be an n -ary operation. If $\vec{x} \in \text{Def}_\ell(A)$ we say that $\ell^A(\vec{x})$ is *defined in A* . We generalize this notion to terms in the usual way. For example, given binary operations ℓ_1 and ℓ_2 of a partial algebra A and $x, y, z \in A$, $\ell_1^A(\ell_2^A(x, y), \ell_1^A(y, z))$ is defined in A if and only if $(x, y) \in \text{Def}_{\ell_2}(A)$, $(y, z) \in \text{Def}_{\ell_1}(A)$, and $(\ell_2^A(x, y), \ell_1^A(y, z)) \in \text{Def}_{\ell_1}(A)$.

Given a term t , we denote by $\text{Def}_t(A)$ the set of all tuples \vec{x} of A such that $t(\vec{x})$ is defined in A .

We denote $\ell(\vec{x})$ instead of $\ell^A(\vec{x})$ when there is no ambiguity. Any algebra A has a natural structure of partial algebra with $\text{Def}_\ell(A) = A^{\text{ar}(\ell)}$ for each $\ell \in \mathcal{L}$.

Definition 4.2. Let A, B be partial algebras. A *morphism of partial algebras* is a map $f: A \rightarrow B$ such that $f(\vec{x}) \in \text{Def}_\ell(B)$ and $\ell(f(\vec{x})) = f(\ell(\vec{x}))$, for all $\ell \in \mathcal{L}$ and all $\vec{x} \in \text{Def}_\ell(A)$.

The *category of partial algebras*, denoted by $\mathbf{PAlg}_{\mathcal{L}}$, is the category in which the objects are the partial algebras and the arrows are the above-mentioned morphisms of partial algebras.

A morphism $f: A \rightarrow B$ of partial algebras is *strong* if $(f(A))^{\text{ar}(\ell)} \subseteq \text{Def}_\ell(B)$ for each $\ell \in \mathcal{L}$.

A partial algebra A is *finite* if its universe is finite.

Remark 4.3. A morphism $f: A \rightarrow B$ of partial algebras is an isomorphism if and only if the following conditions are both satisfied

- (1) The map f is bijective.
- (2) If $\ell(f(\vec{x}))$ is defined in B then $\ell(\vec{x})$ is defined in A , for each operation $\ell \in \mathcal{L}$ and each tuple \vec{x} of A .

We remind the reader that the converse of (2) is always true.

Definition 4.4. Given a partial algebra A , a *partial subalgebra* B of A is a subset B of A endowed with a structure of partial algebra such that $\text{Def}_\ell(B) \subseteq \text{Def}_\ell(A)$ and $\ell^A(\vec{x}) = \ell^B(\vec{x})$ for all $\ell \in \mathcal{L}$ and all $\vec{x} \in \text{Def}_\ell(B)$. The inclusion map from A into B is a morphism of partial algebras called *the inclusion morphism*.

A partial subalgebra B of A is *full* if whenever $\ell \in \mathcal{L}$ and $\vec{x} \in B^{\text{ar}(\ell)}$ are such that $\ell^A(\vec{x})$ is defined and belongs to B , then $\ell(\vec{x})$ is defined in B . It is equivalent to the following equality:

$$\text{Def}_\ell(B) = \{\vec{x} \in \text{Def}_\ell(A) \cap B^{\text{ar}(\ell)} \mid \ell(\vec{x}) \in B\}, \quad \text{for each } \ell \in \mathcal{L}.$$

A partial subalgebra B of A is *strong* if the inclusion map is a strong morphism, that is, $B^{\text{ar}(\ell)} \subseteq \text{Def}_\ell(A)$ for each $\ell \in \mathcal{L}$.

An *embedding* of partial algebras is a one-to-one morphism of partial algebras.

Notation 4.5. Let $f: A \rightarrow B$ be a morphism of partial algebras, let X be a partial subalgebra of A . The set $f(X)$ can be endowed with a natural structure of partial algebra, by setting $\text{Def}_\ell(f(X)) = f(\text{Def}_\ell(X)) = \{f(\vec{x}) \mid \vec{x} \in \text{Def}_\ell(X)\}$, for each $\ell \in \mathcal{L}$. Similarly, let Y be a partial subalgebra of B . The set $f^{-1}(Y)$ can be endowed with a natural structure of partial algebra, by setting $\text{Def}_\ell(f^{-1}(Y)) = f^{-1}(\text{Def}_\ell(Y)) = \{\vec{x} \in A \mid f(\vec{x}) \in \text{Def}_\ell(Y)\}$, for each $\ell \in \mathcal{L}$.

Remark. Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be morphisms of partial algebras, let X be a sub-partial algebra of A , then $(g \circ f)(X) = g(f(X))$ as partial algebras. Let Z be a partial subalgebra of X , then $(g \circ f)^{-1}(Z) = f^{-1}(g^{-1}(Z))$ as partial algebras.

Let $f: A \rightarrow B$ be a morphism of partial algebras, let X be a partial subalgebra of A . Then X is a partial subalgebra of $f^{-1}(f(X))$. In particular $f^{-1}(f(A)) = A$ as partial algebras. Let Y be a partial subalgebra of B , then $f(f^{-1}(Y))$ is a partial subalgebra of Y .

If \mathcal{L} is infinite, then there are a finite partial algebra A (even with one element) and an infinite chain of partial subalgebras of A with union A . In particular, A is not finitely presented in the category $\mathbf{PAlg}_{\mathcal{L}}$.

A morphism $f: A \rightarrow B$ of partial algebras is strong if and only if $f(A)$ is a strong partial subalgebra of B .

Lemma 4.6. *An embedding $f: A \rightarrow B$ of partial algebras is an isomorphism if and only if $f(A) = B$ as partial algebras.*

Proof. Assume that f is an isomorphism, let g be its inverse. Notice that $f(A)$ is a partial subalgebra of B and $B = f(g(B))$ is a partial subalgebra of $f(A)$, therefore $B = f(A)$ as partial algebras.

Conversely, assume that $B = f(A)$ as partial algebras. Then f is surjective, moreover f is an embedding, so f is a bijection. Let $g = f^{-1}$ in \mathbf{Set} . Let $\vec{y} \in \text{Def}_{\ell}(B)$. As $B = f(A)$, there exists $\vec{x} \in \text{Def}_{\ell}(A)$ such that $f(\vec{x}) = \vec{y}$, thus $g(\vec{y}) = \vec{x} \in \text{Def}_{\ell}(A)$. Moreover $g(\ell(\vec{y})) = g(\ell(f(\vec{x}))) = g(f(\ell(\vec{x}))) = \ell(\vec{x}) = \ell(g(\vec{y}))$. \square

Notation 4.7. Let A be a partial algebra, let X be a subset of A . We define inductively, for each $n < \omega$,

$$\begin{aligned} \langle X \rangle_A^0 &= X \cup \{c \mid c \text{ is a constant of } \mathcal{L}\} \\ \langle X \rangle_A^{n+1} &= \langle X \rangle_A^n \cup \{\ell(\vec{x}) \mid \ell \in \mathcal{L}, \vec{x} \in \text{Def}_{\ell}(A), \vec{x} \text{ is an } \text{ar}(\ell)\text{-tuple of } \langle X \rangle_A^n\} \end{aligned}$$

We endow $\langle X \rangle_A^n$ with the induced structure of full partial subalgebra of A . If \mathcal{L} and X are both finite, then $\langle X \rangle_A^n$ is finite for each $n < \omega$. If A is understood, we shall simply denote this partial algebra by $\langle X \rangle^n$.

Definition 4.8. A partial algebra A satisfies an identity $t_1 = t_2$ if $t_1(\vec{x}) = t_2(\vec{x})$ for each tuple \vec{x} of A such that both $t_1(\vec{x})$ and $t_2(\vec{x})$ are defined in A . Otherwise we say that A fails $t_1 = t_2$.

Let \mathcal{V} be a variety of algebras, a partial algebra A is a *partial algebra of \mathcal{V}* if A satisfies all identities of \mathcal{V} .

Remark. Let A be a partial algebra, let $\ell \in \mathcal{L}$. If $\text{Def}_{\ell}(A) = \emptyset$ then A satisfies $\ell(\vec{x}) = y$, vacuously.

If A fails $t_1 = t_2$, then there exists a tuple \vec{x} of A such that $t_1(\vec{x})$ and $t_2(\vec{x})$ are both defined and $t_1(\vec{x}) \neq t_2(\vec{x})$.

Lemma 4.9. *The category $\mathbf{PAlg}_{\mathcal{L}}$ has all directed colimits. Moreover, given a directed poset P , a P -indexed diagram $\vec{A} = (A_p, f_{p,q} \mid p \leq q \text{ in } P)$ in $\mathbf{PAlg}_{\mathcal{L}}$, and a directed colimit cocone:*

$$(A, f_p \mid p \in P) = \varinjlim (A_p, f_{p,q} \mid p \leq q \text{ in } P), \quad \text{in } \mathbf{Set}, \quad (4.1)$$

the set A can be uniquely endowed with a structure of partial algebra such that:

- $\text{Def}_{\ell}(A) = \{f_p(\vec{x}) \mid p \in P \text{ and } \vec{x} \in \text{Def}_{\ell}(A_p)\}$, for each $\ell \in \mathcal{L}$;
- $\ell(f_p(\vec{x})) = f_p(\ell(\vec{x}))$ for each $p \in P$, all $\ell \in \mathcal{L}$, and all $\vec{x} \in \text{Def}_{\ell}(A_p)$.

Moreover, if A is endowed with this structure of partial algebra, the following statements hold:

- (1) $(A, f_p \mid p \in P) = \varinjlim (A_p, f_{p,q} \mid p \leq q \text{ in } P) \text{ in } \mathbf{PAlg}_{\mathcal{L}}$.
- (2) Assume that for each $\ell \in \mathcal{L}$, each $p \in P$, and each $\text{ar}(\ell)$ -tuple \vec{x} of A_p there exists $q \geq p$ such that $f_{p,q}(\vec{x}) \in \text{Def}_{\ell}(A_q)$. Then A is an algebra, that is, $\text{Def}_{\ell}(A) = A^{\text{ar}(\ell)}$ for each $\ell \in \mathcal{L}$.
- (3) If P has no maximal element and $f_{p,q}$ is strong for all $p < q$ in P , then A is an algebra.
- (4) $\text{Def}_t(A) = \{f_p(\vec{x}) \mid p \in P \text{ and } \vec{x} \in \text{Def}_t(A_p)\}$ for each term t of \mathcal{L} .
- (5) Let $t_1 = t_2$ be an identity. If A_p satisfies $t_1 = t_2$ for all $p \in P$, then A satisfies $t_1 = t_2$.

Proof. Put $\text{Def}_{\ell}(A) = \{f_p(\vec{x}) \mid p \in P \text{ and } \vec{x} \in \text{Def}_{\ell}(A_p)\}$, for each $\ell \in \mathcal{L}$.

Let $\ell \in \mathcal{L}$, let $\vec{x} \in \text{Def}_{\ell}(A)$. There exist $p \in P$ and $\vec{y} \in \text{Def}_{\ell}(A_p)$ such that $\vec{x} = f_p(\vec{y})$. We first show that $f_p(\ell(\vec{y}))$ does not depend on the choice of p and \vec{y} . Let $q \in P$ and $\vec{z} \in \text{Def}_{\ell}(A_q)$ such that $\vec{x} = f_q(\vec{z})$. As $f_p(\vec{y}) = \vec{x} = f_q(\vec{z})$, it follows from (4.1) that there exists $r \geq p, q$ such that $f_{p,r}(\vec{y}) = f_{q,r}(\vec{z})$. Therefore the following equalities hold:

$$f_p(\ell(\vec{y})) = f_r(f_{p,r}(\ell(\vec{y}))) = f_r(\ell(f_{p,r}(\vec{y}))) = f_r(\ell(f_{q,r}(\vec{z}))) = f_r(f_{q,r}(\ell(\vec{z}))) = f_q(\ell(\vec{z})).$$

Hence $\ell(f_p(\vec{y})) = f_p(\ell(\vec{y}))$ for all $p \in P$ and all $\vec{y} \in \text{Def}_{\ell}(A_p)$ uniquely define a partial operation $\ell: \text{Def}_{\ell}(A) \rightarrow A$. Moreover f_p is a morphism of partial algebras for each $p \in P$.

Let $(B, g_p \mid p \in P)$ be a cocone over $(A_p, f_{p,q} \mid p \leq q \text{ in } P)$ in $\mathbf{PAlg}_{\mathcal{L}}$. In particular, it is a cocone in \mathbf{Set} , so there exists a unique map $h: A \rightarrow B$ such that $h \circ f_p = g_p$ for each $p \in P$. Let $\ell \in \mathcal{L}$, let $\vec{x} \in \text{Def}_{\ell}(A)$. There exist $p \in P$ and $\vec{y} \in \text{Def}_{\ell}(A_p)$ such that $\vec{x} = f_p(\vec{y})$, thus $h(\vec{x}) = h(f_p(\vec{y})) = g_p(\vec{y})$. As g_p is a morphism of partial algebras and $\vec{y} \in \text{Def}_{\ell}(A_p)$, we obtain that $h(\vec{x}) \in \text{Def}_{\ell}(B)$. Moreover the following equalities hold:

$$\ell(h(\vec{x})) = \ell(g_p(\vec{y})) = g_p(\ell(\vec{y})) = h(f_p(\ell(\vec{y}))) = h(\ell(f_p(\vec{y}))) = h(\ell(\vec{x})).$$

Hence h is a morphism of partial algebras. Therefore:

$$(A, f_p \mid p \in P) = \varinjlim (A_p, f_{p,q} \mid p \leq q \text{ in } P) \text{ in } \mathbf{PAlg}_{\mathcal{L}}.$$

Assume that for each $\ell \in \mathcal{L}$, for all $p \in P$, and for all $\text{ar}(\ell)$ -tuples \vec{x} of A_p , there exists $q \geq p$ such that $f_{p,q}(\vec{x}) \in \text{Def}_{\ell}(A_q)$.

Let $\ell \in \mathcal{L}$, let \vec{x} be an $\text{ar}(\ell)$ -tuple of A . There exist $p \in P$ and a tuple \vec{y} of A_p such that $\vec{x} = f_p(\vec{y})$. Let $q \geq p$ such that $f_{p,q}(\vec{y}) \in \text{Def}_{\ell}(A_q)$. It follows that $\vec{x} = f_p(\vec{y}) = f_q(f_{p,q}(\vec{y}))$ belongs to $\text{Def}_{\ell}(A)$. Therefore A is an algebra.

The statement (3) follows directly from (2). The statement (4) is proved by a straightforward induction on terms, and (5) is an easy consequence of (4). \square

5. PREGAMPS

A *pregamp* is a partial algebra endowed with a semilattice-valued “distance” (cf. (1)-(3)) compatible with all operations of A (cf. (4)). It is a generalization of the notion of *semilattice-metric space* defined in [5, Section 5-1].

Definition 5.1. Let A be a partial algebra, let S be a $(\vee, 0)$ -semilattice. A *S-valued partial algebra distance* on A is a map $\delta: A^2 \rightarrow S$ such that:

- (1) $\delta(x, y) = 0$ if and only if $x = y$, for all $x, y \in A$.
- (2) $\delta(x, y) = \delta(y, x)$, for all $x, y \in A$.
- (3) $\delta(x, y) \leq \delta(x, z) \vee \delta(z, y)$, for all $x, y, z \in A$.

$$(4) \quad \delta(\ell(\vec{x}), \ell(\vec{y})) \leq \bigvee_{k < \text{ar}(\ell)} \delta(x_k, y_k), \text{ for all } \ell \in \mathcal{L} \text{ and all } \vec{x}, \vec{y} \in \text{Def}_\ell(A).$$

Then we say that $\mathbf{A} = (A, \delta, S)$ is a *pregamp*. We shall generally write $\delta_{\mathbf{A}} = \delta$ and $\tilde{A} = S$.

The pregamp is *distance-generated* if it satisfies the following additional property:

$$(5) \quad S \text{ is join-generated by } \delta_{\mathbf{A}}(A^2). \text{ That is, for all } \alpha \in S \text{ there are } n \geq 0 \text{ and } n\text{-tuples } \vec{x}, \vec{y} \text{ of } A \text{ such that } \alpha = \bigvee_{k < n} \delta_{\mathbf{A}}(x_k, y_k).$$

Example 5.2. Let A be an algebra. We remind the reader that $\Theta_A(x, y)$ denotes the smallest congruence that identifies x and y , for all $x, y \in A$. This defines a distance $\Theta_A: A^2 \rightarrow \text{Con}_c A$. Moreover, $(A, \Theta_A, \text{Con}_c A)$ is a distance-generated pregamp.

A straightforward induction argument on the length of the term t yields the following lemma.

Lemma 5.3. *Let \mathbf{A} be a pregamp, let t be an n -ary term, and let $\vec{x}, \vec{y} \in \text{Def}_t(A)$. The following inequality holds:*

$$\delta_{\mathbf{A}}(t(\vec{x}), t(\vec{y})) \leq \bigvee_{k < n} \delta_{\mathbf{A}}(x_k, y_k).$$

We say that $\delta_{\mathbf{A}}$ and t are compatible.

Definition 5.4. Let \mathbf{A} and \mathbf{B} be pregamps. A *morphism* from \mathbf{A} to \mathbf{B} is an ordered pair $\mathbf{f} = (f, \tilde{f})$ such that $f: A \rightarrow B$ is a morphism of partial algebras, $\tilde{f}: \tilde{A} \rightarrow \tilde{B}$ is a $(\vee, 0)$ -homomorphism, and $\delta_{\mathbf{B}}(f(x), f(y)) = \tilde{f}(\delta_{\mathbf{A}}(x, y))$ for all $x, y \in A$.

Given morphisms $\mathbf{f}: \mathbf{A} \rightarrow \mathbf{B}$ and $\mathbf{g}: \mathbf{B} \rightarrow \mathbf{C}$ of pregamps, the pair $\mathbf{g} \circ \mathbf{f} = (g \circ f, \tilde{g} \circ \tilde{f})$ is a morphism from \mathbf{A} to \mathbf{C} .

We denote by $\mathbf{PGamp}_{\mathcal{L}}$ the category of pregamps with the morphisms defined above.

We denote by \mathbf{P}_{ga} the functor from the category of \mathcal{L} -algebras to $\mathbf{PGamp}_{\mathcal{L}}$ that maps an algebra A to $(A, \Theta_A, \text{Con}_c A)$, and a morphism of algebras f to $(f, \text{Con}_c f)$. We denote by \mathbf{C}_{pg} the functor from $\mathbf{PGamp}_{\mathcal{L}}$ to $\mathbf{Sem}_{\vee, 0}$ that maps a pregamp \mathbf{A} to \tilde{A} , and maps a morphism of pregamps $\mathbf{f}: \mathbf{A} \rightarrow \mathbf{B}$ to the $(\vee, 0)$ -homomorphism \tilde{f} .

Remark 5.5. A morphism $\mathbf{f}: \mathbf{A} \rightarrow \mathbf{B}$ of pregamps is an isomorphism if and only if f is an isomorphism of partial algebras and \tilde{f} is an isomorphism of $(\vee, 0)$ -semilattices.

Notice that $\mathbf{C}_{\text{pg}} \circ \mathbf{P}_{\text{ga}} = \text{Con}_c$.

We leave to the reader the straightforward proof of the following lemma.

Lemma 5.6. *The category $\mathbf{PGamp}_{\mathcal{L}}$ has all directed colimits. Moreover, given a directed poset P , a P -indexed diagram $\vec{\mathbf{A}} = (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P)$ in $\mathbf{PGamp}_{\mathcal{L}}$, a directed colimit cocone $(A, f_p \mid p \in P) = \varinjlim (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P)$ in $\mathbf{PAlg}_{\mathcal{L}}$, and a directed colimit cocone $(\tilde{A}, \tilde{f}_p \mid p \in P) = \varinjlim (\tilde{\mathbf{A}}_p, \tilde{f}_{p,q} \mid p \leq q \text{ in } P)$ in $\mathbf{Sem}_{\vee, 0}$, there exists a unique \tilde{A} -valued partial algebra distance $\delta_{\mathbf{A}}$ on A such that $\delta_{\mathbf{A}}(f_p(x), f_p(y)) = \tilde{f}_p(\delta_{\mathbf{A}_p}(x, y))$ for all $p \in P$ and all $x, y \in A_p$.*

Furthermore $\mathbf{A} = (A, \delta_{\mathbf{A}}, \tilde{A})$ is a pregamp, $\mathbf{f}_p: \mathbf{A}_p \rightarrow \mathbf{A}$ is a morphism of pregamps for each $p \in P$, and the following is a directed colimit cocone:

$$(\mathbf{A}, \mathbf{f}_p \mid p \in P) = \varinjlim (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P), \quad \text{in } \mathbf{PGamp}_{\mathcal{L}}.$$

Moreover if \mathbf{A}_p is distance-generated for each $p \in P$, then \mathbf{A} is distance-generated.

Remark 5.7. As an immediate application of Lemma 5.6, and the fact that Con_c preserves directed colimits, we obtain that both \mathbf{C}_{pg} and \mathbf{P}_{ga} preserve directed colimits.

Definition 5.8. An embedding $\mathbf{f}: \mathbf{B} \rightarrow \mathbf{A}$ of pregamps is a morphism of pregamps such that f and \tilde{f} are both one-to-one.

A sub-pregamp of a pregamp \mathbf{A} is a pregamp \mathbf{B} such that B is a partial subalgebra of A , B is a $(\vee, 0)$ -subsemilattice of A , and $\delta_B = \delta_A \upharpoonright B^2$.

If $f: B \rightarrow A$ and $\tilde{f}: \tilde{B} \rightarrow \tilde{A}$ denote the inclusion maps, the morphism of pregamps $\mathbf{f} = (f, \tilde{f})$ is called the canonical embedding.

Notation 5.9. Let $\mathbf{f}: \mathbf{B} \rightarrow \mathbf{A}$ be a morphism of pregamps. Given a sub-pregamp \mathbf{C} of \mathbf{B} , the triple $\mathbf{f}(\mathbf{C}) = (f(C), \delta_A \upharpoonright (f(C))^2, \tilde{f}(\tilde{C}))$ (see Notation 4.5) is a sub-pregamp of \mathbf{A} .

For a sub-pregamp \mathbf{C} of \mathbf{A} , the triple $\mathbf{f}^{-1}(\mathbf{C}) = (f^{-1}(C), \delta_B \upharpoonright (f^{-1}(C))^2, \tilde{f}^{-1}(\tilde{C}))$ is a sub-pregamp of \mathbf{B} .

We leave to the reader the straightforward proof of the following description of sub-pregamps and embeddings.

Proposition 5.10. The following statements hold.

- (1) Let \mathbf{A} be a pregamp, let B be a partial subalgebra of A , let \tilde{B} be a $(\vee, 0)$ -subsemilattice of \tilde{A} that contains $\delta_A(B^2)$. Put $\delta_B = \delta_A \upharpoonright B^2$. Then (B, δ_B, \tilde{B}) is a sub-pregamp of \mathbf{A} . Moreover, all sub-pregamps of \mathbf{A} are of this form.
- (2) Let $\mathbf{f}: \mathbf{B} \rightarrow \mathbf{A}$ be a morphism of pregamps. Then f is an embedding of partial algebras if and only if \tilde{f} separates 0. Moreover \mathbf{f} is an embedding if and only if \tilde{f} is an embedding.
- (3) Let $\mathbf{f}: \mathbf{B} \rightarrow \mathbf{A}$ be an embedding of pregamps. The restriction $\mathbf{f}: \mathbf{B} \rightarrow \mathbf{f}(\mathbf{B})$ is an isomorphism of pregamps.

The following result appears in [9, Theorem 10.4]. It gives a description of finitely generated congruences of a general algebra.

Lemma 5.11. Let B be an algebra, let m be a positive integer, let $x, y \in B$, and let \vec{x}, \vec{y} be m -tuples of B . Then $\Theta_B(x, y) \leq \bigvee_{i < m} \Theta_B(x_i, y_i)$ if and only if there are a positive integer n , a list \vec{z} of parameters from B , and terms t_0, \dots, t_n such that

$$\begin{aligned} x &= t_0(\vec{x}, \vec{y}, \vec{z}), \\ y &= t_n(\vec{x}, \vec{y}, \vec{z}), \\ t_j(\vec{y}, \vec{x}, \vec{z}) &= t_{j+1}(\vec{x}, \vec{y}, \vec{z}) \quad (\text{for all } j < n). \end{aligned}$$

The following lemma shows that the obvious direction of Lemma 5.11 holds for pregamps.

Lemma 5.12. Let \mathbf{B} be a pregamp, let m be a positive integer, let $x, y \in B$, and let \vec{x}, \vec{y} be m -tuples of B . Assume that there are a positive integer n , a list \vec{z} of parameters from B , and terms t_0, \dots, t_n such that the following equalities hold and

all evaluations are defined

$$\begin{aligned} x &= t_0(\vec{x}, \vec{y}, \vec{z}), \\ y &= t_n(\vec{x}, \vec{y}, \vec{z}), \\ t_j(\vec{y}, \vec{x}, \vec{z}) &= t_{j+1}(\vec{x}, \vec{y}, \vec{z}), \quad (\text{for all } j < n). \end{aligned}$$

Then $\delta_{\mathbf{B}}(x, y) \leq \bigvee_{i < m} \delta_{\mathbf{B}}(x_i, y_i)$.

Proof. As $\delta_{\mathbf{B}}$ is compatible with terms (cf. Lemma 5.3), and $\delta_{\mathbf{B}}(u, u) = 0$ for each $u \in B$, the following inequality holds:

$$\delta_{\mathbf{B}}(t_j(\vec{x}, \vec{y}, \vec{z}), t_j(\vec{y}, \vec{x}, \vec{z})) \leq \bigvee_{k < n} \delta_{\mathbf{B}}(x_k, y_k), \quad \text{for all } j < n.$$

Hence:

$$\delta_{\mathbf{B}}(x, y) \leq \bigvee_{j < n} \delta_{\mathbf{B}}(t_j(\vec{x}, \vec{y}, \vec{z}), t_j(\vec{y}, \vec{x}, \vec{z})) \leq \bigvee_{k < n} \delta_{\mathbf{B}}(x_k, y_k). \quad \square$$

The following definition expresses that whenever two elements of A are identified by a “congruence” of A , then there is a “good reason” for this in B (cf. Lemma 5.11).

Definition 5.13. A morphism $\mathbf{f}: \mathbf{A} \rightarrow \mathbf{B}$ of preamps is *congruence-tractable* if for all $m < \omega$ and for all $x, y, x_0, y_0, \dots, x_{m-1}, y_{m-1}$ in A such that:

$$\delta_{\mathbf{B}}(x, y) \leq \bigvee_{k < m} \delta_{\mathbf{B}}(x_k, y_k),$$

there are a positive integer n , a list \vec{z} of parameters from B , and terms t_0, \dots, t_n such that the following equations are satisfied in B (in particular, all the corresponding terms are defined).

$$\begin{aligned} f(x) &= t_0(f(\vec{x}), f(\vec{y}), \vec{z}), \\ f(y) &= t_n(f(\vec{x}), f(\vec{y}), \vec{z}), \\ t_j(f(\vec{y}), f(\vec{x}), \vec{z}) &= t_{j+1}(f(\vec{x}), f(\vec{y}), \vec{z}) \quad (\text{for all } j < n). \end{aligned}$$

Lemma 5.14. Let P be directed poset and let $\vec{\mathbf{A}} = (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P)$ be a direct system of preamps. Assume that for each $p \in P$ there exists $q \geq p$ in P such that $\mathbf{f}_{p,q}$ is congruence-tractable. Let:

$$(\mathbf{A}, \mathbf{f}_p \mid p \in P) = \varinjlim (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P), \quad \text{in } \mathbf{PGamp}_{\mathcal{L}}.$$

If A is an algebra then the following statements hold:

- (1) Let $x, y \in A$, let $m < \omega$, let $x_0, y_0, \dots, x_{m-1}, y_{m-1}$ in A . The following two inequalities are equivalent:

$$\delta_{\mathbf{A}}(x, y) \leq \bigvee_{k < m} \delta_{\mathbf{A}}(x_k, y_k), \quad (5.1)$$

$$\Theta_{\mathbf{A}}(x, y) \leq \bigvee_{k < m} \Theta_{\mathbf{A}}(x_k, y_k). \quad (5.2)$$

- (2) There exists a unique $(\vee, 0)$ -homomorphism $\phi: \text{Con}_{\mathbf{c}} A \rightarrow \tilde{A}$ such that:

$$\phi(\Theta_{\mathbf{A}}(x, y)) = \delta_{\mathbf{A}}(x, y), \quad \text{for all } x, y \in A.$$

Moreover ϕ is an embedding.

- (3) If \mathbf{A} is distance-generated, then the $(\vee, 0)$ -homomorphism ϕ above is an isomorphism.

Proof. Lemma 5.6 and Lemma 4.9 imply that the following are directed colimits cocones

$$(A, f_p \mid p \in P) = \varinjlim (A_p, f_{p,q} \mid p \leq q \text{ in } P), \quad \text{in } \mathbf{PAIg}_{\mathcal{L}}. \quad (5.3)$$

$$(A, f_p \mid p \in P) = \varinjlim (A_p, f_{p,q} \mid p \leq q \text{ in } P), \quad \text{in } \mathbf{Set}. \quad (5.4)$$

$$(\tilde{A}, \tilde{f}_p \mid p \in P) = \varinjlim (\tilde{A}_p, \tilde{f}_{p,q} \mid p \leq q \text{ in } P), \quad \text{in } \mathbf{Sem}_{\vee,0}. \quad (5.5)$$

- (1) Let $x, y \in A$, let $m < \omega$, and let \vec{x}, \vec{y} be m -tuples of A .

Assume that (5.1) holds. It follows from (5.4) that there are $p \in P$, $x', y' \in A$, and m -tuples \vec{x}', \vec{y}' of A_p , such that $x = f_p(x')$, $y = f_p(y')$, $\vec{x} = f_p(\vec{x}')$, and $\vec{y} = f_p(\vec{y}')$. The inequality (5.1) can be written

$$\delta_{\mathbf{A}}(f_p(x'), f_p(y')) \leq \bigvee_{k < m} \delta_{\mathbf{A}}(f_p(x'_k), f_p(y'_k)).$$

This implies:

$$\tilde{f}_p(\delta_{\mathbf{A}_p}(x', y')) \leq \tilde{f}_p\left(\bigvee_{k < m} \delta_{\mathbf{A}_p}(x'_k, y'_k)\right).$$

Hence, it follows from (5.5) that there exists $q \geq p$ with:

$$\tilde{f}_{p,q}(\delta_{\mathbf{A}_p}(x', y')) \leq \tilde{f}_{p,q}\left(\bigvee_{k < m} \delta_{\mathbf{A}_p}(x'_k, y'_k)\right),$$

so, changing p to q , x' to $f_{p,q}(x')$, y' to $f_{p,q}(y')$, \vec{x}' to $f_{p,q}(\vec{x}')$, and \vec{y}' to $f_{p,q}(\vec{y}')$, we can assume that:

$$\delta_{\mathbf{A}_p}(x', y') \leq \bigvee_{k < m} \delta_{\mathbf{A}_p}(x'_k, y'_k).$$

Let $q \geq p$ in P such that $f_{p,q}$ is congruence-tractable. There are a positive integer n , a list \vec{z} of parameters from A_q , and terms t_0, \dots, t_n such that the following equations are satisfied in A_q :

$$\begin{aligned} f_{p,q}(x') &= t_0(f_{p,q}(\vec{x}'), f_{p,q}(\vec{y}'), \vec{z}), \\ f_{p,q}(y') &= t_n(f_{p,q}(\vec{x}'), f_{p,q}(\vec{y}'), \vec{z}), \\ t_k(f_{p,q}(\vec{y}'), f_{p,q}(\vec{x}'), \vec{z}) &= t_{k+1}(f_{p,q}(\vec{x}'), f_{p,q}(\vec{y}'), \vec{z}), \quad (\text{for all } k < n). \end{aligned}$$

Hence, applying f_q , we obtain

$$\begin{aligned} x &= t_0(\vec{x}, \vec{y}, f_q(\vec{z})), \\ y &= t_n(\vec{x}, \vec{y}, f_q(\vec{z})), \\ t_k(\vec{y}, \vec{x}, f_q(\vec{z})) &= t_{k+1}(\vec{x}, \vec{y}, f_q(\vec{z})), \quad (\text{for all } k < n). \end{aligned}$$

Therefore, it follows from Lemma 5.11 that (5.2) holds.

Conversely, assume that (5.2) holds. It follows from Lemma 5.11 that there are a positive integer n , a list \vec{z} of parameters from A , and terms t_0, \dots, t_n such that

$$\begin{aligned} x &= t_0(\vec{x}, \vec{y}, \vec{z}), \\ y &= t_n(\vec{x}, \vec{y}, \vec{z}), \\ t_j(\vec{y}, \vec{x}, \vec{z}) &= t_{j+1}(\vec{x}, \vec{y}, \vec{z}), \quad (\text{for all } j < n). \end{aligned}$$

We conclude, using Lemma 5.12, that (5.1) holds.

As $\text{Conc } A$ is generated by $\{\Theta_A(x, y) \mid x, y \in A\}$, the statement (2) follows from Proposition 3.1. Moreover if we assume that A is distance-generated, that is \tilde{A} is join-generated by $\{\delta_A(x, y) \mid x, y \in A\}$, then ϕ is an isomorphism. \square

As an immediate application, we obtain that a “true” directed colimit of “good” pregamps is an algebra together with its congruences.

Corollary 5.15. *Let P be directed poset with no maximal element and let $\tilde{A} = (A_p, f_{p,q} \mid p \leq q \text{ in } P)$ be a P -indexed diagram of distance-generated pregamps. If $f_{p,q}$ is congruence-tractable and $f_{p,q}$ is strong for all $p < q$ in P , then there exists a unique $(\vee, 0)$ -homomorphism $\phi: \text{Conc } A \rightarrow \tilde{A}$ such that:*

$$\phi(\Theta_A(x, y)) = \delta_A(x, y) \quad \text{for all } x, y \in A.$$

Moreover, ϕ is an isomorphism.

Definition 5.16. An ideal of a pregamp A is an ideal of \tilde{A} . Denote by $\text{Id } A = \text{Id } \tilde{A}$ the set of all ideals of A .

Let P be a poset, let $\tilde{A} = (A_p, f_{p,q} \mid p \leq q \text{ in } P)$ be a P -indexed diagram in $\mathbf{PGamp}_{\mathcal{L}}$. An ideal of \tilde{A} is an ideal of $(\tilde{A}_p, \tilde{f}_{p,q} \mid p \leq q \text{ in } P)$ (cf. Definition 3.2).

Definition 5.17. Let $\pi: A \rightarrow B$ be a morphism of pregamps. The 0-kernel of π , denoted by $\ker_0 \pi$, is the 0-kernel of $\tilde{\pi}$ (cf. Definition 3.2).

Let P be a poset and let $\tilde{\pi} = (\pi_p)_{p \in P}: \tilde{A} \rightarrow \tilde{B}$ be a natural transformation of P -indexed diagrams of pregamps. The 0-kernel of $\tilde{\pi}$ is $\tilde{I} = (\ker_0 \pi_p)_{p \in P}$.

Remark 5.18. The 0-kernel of π is an ideal of A . Similarly the 0-kernel of $\tilde{\pi}$ is an ideal of \tilde{A} .

If $\pi: A \rightarrow B$ is a morphism of algebras, then $\ker_0 P_{\text{ga}}(\pi)$ is the set of all compact congruences of A below $\ker \pi$, that is, $\ker_0 P_{\text{ga}}(\pi) = (\text{Conc } A) \downarrow \ker \pi$.

Definition 5.19. A morphism $f: A \rightarrow B$ of pregamps, is *ideal-induced* if $f(A) = B$ as partial algebras and \tilde{f} is ideal-induced. In that case we say that B is an *ideal-induced image* of A .

Let P be a poset, let \tilde{A} and \tilde{B} be P -indexed diagrams of pregamps. A natural transformation $\tilde{f} = (f_p)_{p \in P}: \tilde{A} \rightarrow \tilde{B}$ is *ideal-induced* if f_p is ideal-induced for each $p \in P$.

Remark 5.20. A morphism $f: A \rightarrow B$ of pregamps is ideal-induced if \tilde{f} is ideal-induced, f is surjective, and for each $\ell \in \mathcal{L}$ and each tuple \vec{b} of B , $\ell(\vec{b})$ is defined in B if and only if there exists a tuple \vec{a} in A such that $\vec{b} = f(\vec{a})$ and $\ell(\vec{a})$ is defined in A .

If $f: A \rightarrow B$ is a surjective morphism of algebras, then $P_{\text{ga}}(f)$ is ideal-induced.

If $f: A \rightarrow B$ and $g: B \rightarrow C$ are ideal-induced morphisms of pregamps, then $g \circ f$ is ideal-induced.

The following proposition gives a description of *quotients* of pregamps.

Proposition 5.21. *Let A be a pregamp and let $I \in \text{Id } \tilde{A}$. The binary relation $\theta_I = \{(x, y) \in A^2 \mid \delta_A(x, y) \in I\}$ is an equivalence relation on A . Given $a \in A$ denote by a/I the θ_I -equivalence class containing a , and set $A/I = A/\theta_I$. We can*

define a structure of partial algebra on A/I in the following way. Given $\ell \in \mathcal{L}$, we put:

$$\begin{aligned} \text{Def}_\ell(A/I) &= \{\vec{x}/I \mid \vec{x} \in \text{Def}_\ell(A)\}, \\ \ell^{A/I}(\vec{x}/I) &= \ell^A(\vec{x})/I, \quad \text{for all } \vec{x} \in \text{Def}_\ell(A). \end{aligned}$$

Moreover $\delta_{\mathbf{A}/I}: (A/I)^2 \rightarrow \tilde{A}/I$, $(x/I, y/I) \mapsto \delta_{\mathbf{A}}(x, y)/I$ defines an \tilde{A}/I -valued partial algebra distance, and the following statements hold:

- (1) $\mathbf{A}/I = (A/I, \delta_{\mathbf{A}/I}, \tilde{A}/I)$ is a pregamp.
- (2) Put $\pi: A \rightarrow A/I$, $x \mapsto x/I$, and denote by $\tilde{\pi}: \tilde{A} \rightarrow \tilde{A}/I$, $d \mapsto d/I$ the canonical projection. Then $\boldsymbol{\pi} = (\pi, \tilde{\pi})$ is an ideal-induced morphism of pregamps from \mathbf{A} to \mathbf{A}/I .
- (3) The 0-kernel of $\boldsymbol{\pi}$ is I .
- (4) If \mathbf{A} is distance-generated, then \mathbf{A}/I is distance-generated.

Proof. The relation θ_I is reflexive (it follows from Definition 5.1(1)), symmetric (see Definition 5.1(2)) and transitive (see Definition 5.1(3)), thus it is an equivalence relation.

Let $\ell \in \mathcal{L}$, let $\vec{x}, \vec{y} \in \text{Def}_\ell(A)$ such that $x_k/I = y_k/I$ for each $k < \text{ar}(\ell)$. It follows from Definition 5.1(4) that $\delta_{\mathbf{A}}(\ell^A(\vec{x}), \ell^A(\vec{y})) \leq \bigvee_{k < \text{ar}(\ell)} \delta_{\mathbf{A}}(x_k, y_k) \in I$, so $\ell^A(\vec{x})/I = \ell^A(\vec{y})/I$. Therefore the partial operation $\ell^{A/I}: \text{Def}_\ell(A/I) \rightarrow A/I$ is well-defined.

Let $x, x', y, y' \in A$, assume that $x/I = x'/I$ and $y/I = y'/I$. The following inequality holds:

$$\delta_{\mathbf{A}}(x, y) \leq \delta_{\mathbf{A}}(x, x') \vee \delta_{\mathbf{A}}(x', y') \vee \delta_{\mathbf{A}}(y', y).$$

However, $\delta_{\mathbf{A}}(x, x')$ and $\delta_{\mathbf{A}}(y, y')$ both belong to I , hence $\delta_{\mathbf{A}}(x, y)/I \leq \delta_{\mathbf{A}}(x', y')/I$. Similarly $\delta_{\mathbf{A}}(x', y')/I \leq \delta_{\mathbf{A}}(x, y)/I$. So the map $\delta_{\mathbf{A}/I}: (A/I)^2 \rightarrow \tilde{A}/I$ is well-defined.

Let $x, y \in A$, the following equivalences hold:

$$\delta_{\mathbf{A}/I}(x/I, y/I) = 0/I \iff \delta_{\mathbf{A}}(x, y) \in I \iff x/I = y/I.$$

That is, Definition 5.1(1) holds. Each of the conditions of Definition 5.1(2)-(5) for $\delta_{\mathbf{A}}$ implies its analogue for $\delta_{\mathbf{A}/I}$.

It is easy to check that $\boldsymbol{\pi}$ is well-defined and that it is a morphism of pregamps. \square

Notation 5.22. The notations \mathbf{A}/I , A/I , and $\delta_{\mathbf{A}/I}$ used in Proposition 5.21 will be used throughout the paper. The map $\boldsymbol{\pi}$ is the canonical projection.

If $I = \{0\}$, we identify \mathbf{A}/I and \mathbf{A} .

If X is a partial subalgebra of A , then we denote $X/I = \{x/I \mid x \in X\}$, with its natural structure of partial subalgebra of A , inherited from X , with $\text{Def}_\ell(X/I) = \{\vec{x}/I \mid \vec{x} \in \text{Def}_\ell(X)\}$ for each $\ell \in \mathcal{L}$. That is $X/I = \pi(X)$ as partial algebras.

Let \mathbf{B} be a sub-pregamp of \mathbf{A} and let I be a common ideal of \mathbf{A} and \mathbf{B} . Then we identify the quotient \mathbf{B}/I with the corresponding sub-pregamp of \mathbf{A}/I .

Remark. It is easy to construct a pregamp \mathbf{A} , a term t , a tuple \vec{x} of A , and an ideal I of \tilde{A} , such that $t(\vec{x})$ is not defined in A , but $t(\vec{x}/I)$ is defined in A/I .

The following proposition gives a description of how morphisms of pregamps factorize through quotients. It is related to Lemma 3.5.

Proposition 5.23. *Let $f: \mathbf{A} \rightarrow \mathbf{B}$ be a morphism of pregamps, let $I \in \text{Id } \mathbf{A}$, and let $J \in \text{Id } \mathbf{B}$. If $\tilde{f}(I) \subseteq J$, then the following maps are well-defined:*

$$\begin{aligned} g: A/I &\rightarrow B/J \\ x/I &\mapsto f(x)/J, \\ \tilde{g}: \tilde{A}/I &\rightarrow \tilde{B}/J \\ \alpha/I &\mapsto \tilde{f}(\alpha)/J. \end{aligned}$$

Moreover, $\mathbf{g} = (g, \tilde{g})$ is a morphism of pregamps from \mathbf{A}/I to \mathbf{B}/J . If $\pi_I: \mathbf{A} \rightarrow \mathbf{A}/I$ and $\pi_J: \mathbf{B} \rightarrow \mathbf{B}/J$ denote the canonical projections, then the following diagram commutes:

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{f} & \mathbf{B} \\ \pi_I \downarrow & & \downarrow \pi_J \\ \mathbf{A}/I & \xrightarrow{g} & \mathbf{B}/J \end{array}$$

Proof. Observe that $\tilde{g}: \tilde{A}/I \rightarrow \tilde{B}/J$ is the $(\vee, 0)$ -homomorphism induced by \tilde{f} . Let $x, y \in A$ such that $x/I = y/I$, that is, $\delta_{\mathbf{A}}(x, y) \in I$. It follows that $\delta_{\mathbf{B}}(f(x), f(y)) = \tilde{f}(\delta_{\mathbf{A}}(x, y)) \in J$, so $f(x)/J = f(y)/J$. Therefore the map g is well-defined.

Let $\ell \in \mathcal{L}$, let $\vec{a} \in \text{Def}_{\ell}(A/I)$, and let $\vec{x} \in \text{Def}_{\ell}(A)$ such that $\vec{a} = \vec{x}/I$. The following equalities hold:

$$g(\ell(\vec{x}/I)) = g(\ell(\vec{x})/I) = f(\ell(\vec{x}))/J = \ell(f(\vec{x}))/J = \ell(f(\vec{x})/J) = \ell(g(\vec{x}/I)).$$

Thus $g(\ell(\vec{a})) = \ell(g(\vec{a}))$. Therefore g is a morphism of partial algebras.

Let $x, y \in A$. It is easy to check $\tilde{g}(\delta_{\mathbf{A}/I}(x/I, y/I)) = \delta_{\mathbf{B}/J}(g(x/J), g(y/J))$. Therefore $\mathbf{g}: \mathbf{A}/I \rightarrow \mathbf{B}/J$ is a morphism of pregamps. Moreover $\pi_J \circ \mathbf{f} = \mathbf{g} \circ \pi_I$ is obvious. \square

Notation 5.24. We say that \mathbf{f} induces $\mathbf{g}: \mathbf{A}/I \rightarrow \mathbf{B}/J$, the morphism of Proposition 5.23.

Let P be a poset, let $\vec{\mathbf{A}} = (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P)$ be a P -indexed diagram in $\mathbf{PGamp}_{\mathcal{L}}$, let $\vec{I} = (I_p)_{p \in P}$ be an ideal of $\vec{\mathbf{A}}$, and let $\mathbf{g}_{p,q}: \mathbf{A}_p/I_p \rightarrow \mathbf{A}_q/I_q$ the morphism induced by $\mathbf{f}_{p,q}$, for all $p \leq q$ in P . We denote by $\vec{\mathbf{A}}/\vec{I} = (\mathbf{A}_p/I_p, \mathbf{g}_{p,q} \mid p \leq q \text{ in } P)$.

The diagram $\vec{\mathbf{A}}/\vec{I}$ is a quotient of $\vec{\mathbf{A}}$.

Remark 5.25. It is easy to check that $\vec{\mathbf{A}}/\vec{I}$ is indeed a diagram. Given $p \leq q \leq r$ in P and $x \in \mathbf{A}_p$ the following equalities hold:

$$g_{q,r}(g_{p,q}(x/I_p)) = g_{q,r}(f_{p,q}(x)/I_q) = f_{q,r}(f_{p,q}(x))/I_r = f_{p,r}(x)/I_r = g_{p,r}(x/I_p).$$

Proposition 5.23 can be easily extended to diagrams in the following way. It is also related to Lemma 3.7.

Proposition 5.26. *Let P be a poset, let $\vec{\mathbf{A}} = (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P)$ and $\vec{\mathbf{B}} = (\mathbf{B}_p, \mathbf{g}_{p,q} \mid p \leq q \text{ in } P)$ be P -indexed diagrams in $\mathbf{PGamp}_{\mathcal{L}}$. Let \vec{I} be an ideal of $\vec{\mathbf{A}}$, let \vec{J} be an ideal of $\vec{\mathbf{B}}$. Let $\vec{\xi} = (\xi_p)_{p \in P}: \vec{\mathbf{A}} \rightarrow \vec{\mathbf{B}}$ be a natural transformation such that $\xi_p(I_p) \subseteq J_p$ for each $p \in P$. Denote by $\chi_p: \mathbf{A}_p/I_p \rightarrow \mathbf{B}_p/J_p$ the morphism*

induced by ξ_p , for each $p \in P$. Then $\vec{\chi} = (\chi_p)_{p \in P}$ is a natural transformation from \vec{A}/\vec{I} to \vec{B}/\vec{J} .

Notation 5.27. With the notation of Proposition 5.26. We say that $\vec{\chi}: \vec{A}/\vec{I} \rightarrow \vec{B}/\vec{J}$ is induced by $\vec{\xi}$.

The following lemma expresses that ideal-induced images of pregamps correspond, up to isomorphism, to quotients of pregamps. It is related to Lemma 3.13.

Lemma 5.28. *Let $f: A \rightarrow B$ be a morphism of pregamps. The following statements are equivalent:*

- (1) f is ideal-induced.
- (2) f induces an isomorphism $g: A/\ker_0 f \rightarrow B$.

Proof. Denote by $\pi: A \rightarrow A/\ker_0 f$ the canonical projection, so $g \circ \pi = f$.

Assume that f is ideal-induced. As $\tilde{f}: \tilde{A} \rightarrow \tilde{B}$ is ideal-induced, Lemma 3.13 implies that \tilde{f} induces an isomorphism $\tilde{g}: \tilde{A}/\ker_0 \tilde{f} \rightarrow \tilde{B}$. It follows that \tilde{g} separates 0, thus (cf. Proposition 5.10(2)) g is an embedding. Moreover $g(A/\ker_0 f) = g(\pi(A)) = f(A) = B$ as partial algebras. Therefore it follows from Lemma 4.6 that g is an isomorphism of partial algebras, thus g is an isomorphism of pregamps (cf. Remark 5.5).

Assume that g is an isomorphism. It follows that \tilde{g} is an isomorphism, so Lemma 3.13 implies that \tilde{f} is ideal-induced. Moreover g is an isomorphism, thus $f(A) = g(\pi(A)) = g(A/\ker_0 f) = B$ as partial algebras. Therefore f is ideal-induced. \square

The following proposition expresses that a quotient of a quotient is a quotient. It follows from Lemma 5.28, together with the fact that a composition of ideal-induced morphisms of pregamps is ideal-induced.

Proposition 5.29. *Let A be a pregamp, let I be an ideal of A , let J be an ideal of A/I . Then $(A/I)/J$ is isomorphic to a quotient of A .*

The following results expresses that, up to isomorphism, quotients of sub-pregamps are sub-pregamps of quotients.

Proposition 5.30. *Let A be a pregamp, let B be a sub-pregamp of A , and let $I \in \text{Id } B$. Then there exist $J \in \text{Id } A$, a sub-pregamp C of A/J , and an isomorphism $f: B/I \rightarrow C$.*

Let A be a pregamp, let $I \in \text{Id } A$, and let B be a sub-pregamp of A/I . There exists a sub-pregamp C of A such that B is isomorphic to some quotient of C .

Proof. Let A be a pregamp, let B be a sub-pregamp of A , let $I \in \text{Id } B$. Put $J = \tilde{A} \downarrow I$. As I is an ideal of \tilde{B} , it is directed, therefore J is an ideal of \tilde{A} .

Let $f: B \rightarrow A$ be the canonical embedding. Notice that $\tilde{f}(I) \subseteq J$; denote by $g: B/I \rightarrow A/J$ the morphism induced by f (cf. Proposition 5.23).

Let $d, d' \in \tilde{B}$ such that $\tilde{g}(d/I) = \tilde{g}(d'/I)$, that is, $d/J = d'/J$, so there exists $u \in J$ such that $d \vee u = d' \vee u$. As $J = \tilde{A} \downarrow I$, there exists $v \in I$ such that $u \leq v$, hence $d \vee v = d' \vee v$, that is, $d/I = d'/I$. Therefore \tilde{g} is an embedding. It follows from Proposition 5.10 that $g = (g, \tilde{g})$ is an embedding and induces an isomorphism $B/I \rightarrow g(B/I)$; the latter is a sub-pregamp of A/J .

Now let $I \in \text{Id } \mathbf{A}$ and let \mathbf{B} be a sub-pregamp of \mathbf{A}/I . Denote by $\pi: \mathbf{A} \rightarrow \mathbf{A}/I$ the canonical projection, put $\mathbf{C} = \pi^{-1}(\mathbf{B})$ (cf. Notation 5.9). As π is ideal-induced, it is easy to check that $\pi(\mathbf{C}) = \mathbf{B}$, and the restriction $\pi \upharpoonright \mathbf{C} \rightarrow \mathbf{B}$ is ideal-induced. \square

The following lemma, in conjunction with Lemma 3.14, proves that, given a direct system $\vec{\mathbf{A}}$ of pregamps, every quotient of the colimit of $\vec{\mathbf{A}}$ is the colimit of a quotient of $\vec{\mathbf{A}}$.

Lemma 5.31. *Let P be directed poset and let $\vec{\mathbf{A}} = (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P)$ be a P -indexed diagram in $\mathbf{PGamp}_{\mathcal{L}}$. Let $(\mathbf{A}, \mathbf{f}_p \mid p \in P) = \varinjlim (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P)$ be a directed colimit cocone in $\mathbf{PGamp}_{\mathcal{L}}$. Let $\vec{I} = (I_p)_{p \in P}$ be an ideal of $\vec{\mathbf{A}}$. Then $I = \bigcup_{p \in P} \tilde{f}_p(I_p)$ is an ideal of \mathbf{A} .*

Let $\mathbf{g}_p: \mathbf{A}_p/I_p \rightarrow \mathbf{A}/I$ be the morphism induced by \mathbf{f}_p , let $\mathbf{g}_{p,q}: \mathbf{A}_p/I_p \rightarrow \mathbf{A}_q/I_q$ be the morphism induced by $\mathbf{f}_{p,q}$, for all $p \leq q$ in P . The following is a directed colimit cocone:

$$(\mathbf{A}/I, \mathbf{g}_p \mid p \in P) = \varinjlim (\mathbf{A}_p/I_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P) \text{ in } \mathbf{PGamp}_{\mathcal{L}}.$$

Proof. Lemma 4.9 and Lemma 5.6 imply that the following are colimits cocones:

$$(A, f_p \mid p \in P) = \varinjlim (A_p, f_{p,q} \mid p \leq q \text{ in } P) \text{ in } \mathbf{Set}, \quad (5.6)$$

$$(\tilde{A}, \tilde{f}_p \mid p \in P) = \varinjlim (\tilde{A}_p, \tilde{f}_{p,q} \mid p \leq q \text{ in } P) \text{ in } \mathbf{Sem}_{\vee,0} \quad (5.7)$$

$$(A, f_p \mid p \in P) = \varinjlim (A_p, f_{p,q} \mid p \leq q \text{ in } P) \text{ in } \mathbf{PAlg}_{\mathcal{L}} \quad (5.8)$$

Moreover, Lemma 3.14 implies that I is an ideal of \tilde{A} and that the following is a directed colimit cocone:

$$(\tilde{A}/I, \tilde{g}_p \mid p \in P) = \varinjlim (\tilde{A}_p/I_p, \tilde{g}_{p,q} \mid p \leq q \text{ in } P) \text{ in } \mathbf{Sem}_{\vee,0} \quad (5.9)$$

Let $p \in P$, let $x, y \in A_p$ such that $g_p(x/I_p) = g_p(y/I_p)$. It follows that $f_p(x)/I = f_p(y)/I$, that is, $\delta_{\mathbf{A}}(f_p(x), f_p(y)) \in I$. So there exist $q \in P$ and $\alpha \in \tilde{A}_q$ such that $\tilde{f}_p(\delta_{\mathbf{A}_p}(x, y)) = \delta_{\mathbf{A}}(f_p(x), f_p(y)) = \tilde{f}_q(\alpha)$. It follows from (5.7) that there exists $r \geq p, q$ such that $\delta_{\mathbf{A}_r}(f_{p,r}(x), f_{p,r}(y)) = \tilde{f}_{p,r}(\delta_{\mathbf{A}_p}(x, y)) = \tilde{f}_{q,r}(\alpha)$. However, $\tilde{f}_{q,r}(\alpha) \in \tilde{f}_{q,r}(I_q) \subseteq I_r$, so $f_{p,r}(x)/I_r = f_{p,r}(y)/I_r$, and so $g_{p,r}(x/I_p) = g_{p,r}(y/I_p)$. Moreover $A/I = \bigcup_{p \in P} f_p(A_p)/I = \bigcup_{p \in P} g_p(A_p/I_p)$. Hence the following is a directed colimit cocone:

$$(A/I, g_p \mid p \in P) = \varinjlim (A_p/I_p, g_{p,q} \mid p \leq q \text{ in } P) \text{ in } \mathbf{Set}.$$

Let $\ell \in \mathcal{L}$. The following equalities hold:

$$\text{Def}_{\ell}(A/I) = \text{Def}_{\ell}(A)/I = \bigcup_{p \in P} f_p(\text{Def}_{\ell}(A_p))/I = \bigcup_{p \in P} g_p(\text{Def}_{\ell}(A_p/I_p)).$$

Let $p \in P$, let $\vec{a} \in \text{Def}_{\ell}(A_p/I)$. Then, as g_p is a morphism of partial algebras, $g_p(\ell(\vec{a})) = \ell(g_p(\vec{a}))$. So, by Lemma 4.9, the following is a directed colimit cocone:

$$(A/I, g_p \mid p \in P) = \varinjlim (A_p/I_p, g_{p,q} \mid p \leq q \text{ in } P) \text{ in } \mathbf{PAlg}_{\mathcal{L}}.$$

As \mathbf{f}_p is a morphism of pregamps, $\delta_{\mathbf{A}/I}(g_p(x), g_p(y)) = \tilde{g}_p(\delta_{\mathbf{A}_p/I_p}(x, y))$ for all $p \in P$ and all $x, y \in A_p/I_p$, thus Lemma 5.6 implies that the following is a directed colimit cocone:

$$(\mathbf{A}/I, \mathbf{g}_p \mid p \in P) = \varinjlim (\mathbf{A}_p/I_p, \mathbf{g}_{p,q} \mid p \leq q \text{ in } P) \text{ in } \mathbf{PGamp}_{\mathcal{L}}. \quad \square$$

Definition 5.32. A pregamp \mathbf{A} satisfies an identity $t_1 = t_2$ if A/I satisfies $t_1 = t_2$ for each $I \in \text{Id } \mathbf{A}$.

Let \mathcal{V} be a variety of algebras. A pregamp \mathbf{A} is a *pregamp of \mathcal{V}* if it satisfies all identities of \mathcal{V} .

Remark. It is not hard to construct a pregamp \mathbf{A} , an identity $t_1 = t_2$, and an ideal I of \tilde{A} such that A satisfies $t_1 = t_2$, but A/I fails $t_1 = t_2$.

A pregamp \mathbf{A} satisfies an identity $t_1 = t_2$ if and only if for each ideal-induced morphism $\mathbf{f}: \mathbf{A} \rightarrow \mathbf{B}$ of pregamps, the partial algebra B satisfies $t_1 = t_2$.

Definition 5.33. Let \mathcal{V} be a variety of algebras. The category of pregamps of \mathcal{V} , denoted by $\mathbf{PGamp}(\mathcal{V})$, is the full subcategory of $\mathbf{PGamp}_{\mathcal{L}}$ in which the objects are all the pregamps of \mathcal{V} .

As an immediate application of Lemma 5.31 and Lemma 3.14, we obtain that the class of all pregamps that satisfy a given identity is closed under directed colimits.

Corollary 5.34. Let \mathcal{V} be a variety of algebras and let P be a directed poset. Let $\vec{\mathbf{A}} = (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P)$ be a P -indexed diagram in $\mathbf{PGamp}(\mathcal{V})$. Let $(\mathbf{A}, \mathbf{f}_p \mid p \in P) = \varinjlim (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P)$ be a directed colimit cocone in $\mathbf{PGamp}_{\mathcal{L}}$. Then \mathbf{A} is a pregamp of \mathcal{V} .

Similarly, it follows from Proposition 5.29 and Proposition 5.30 that the class of all pregamps that satisfy a given identity is closed under ideal-induced images and sub-pregamps.

Corollary 5.35. Let \mathbf{A} be a pregamp of a variety \mathcal{V} , let \mathbf{B} be a pregamp, let $\mathbf{f}: \mathbf{A} \rightarrow \mathbf{B}$ be an ideal-induced morphism of pregamps, then \mathbf{B} is a pregamp of \mathcal{V} . Furthermore, every sub-pregamp of \mathbf{A} is a pregamp of \mathcal{V} .

6. GAMPS

A *gamp* of a variety \mathcal{V} is a pregamp that “belongs” to \mathcal{V} (cf. (1)), together with a partial subalgebra (cf. (2)). The main interest of this new notion is to express later some additional properties that reflect properties of algebras (cf. Definition 6.3). It is a generalization of the notion of a *semilattice-metric cover* as defined in [5, Section 5-1].

Definition 6.1. Let \mathcal{V} be a variety of \mathcal{L} -algebras. A *gamp* (resp., a *gamp of \mathcal{V}*) is a quadruple $\mathbf{A} = (A^*, A, \delta_{\mathbf{A}}, \tilde{A})$ such that

- (1) $(A, \delta_{\mathbf{A}}, \tilde{A})$ is a pregamp (resp., a pregamp of \mathcal{V}) (cf. Definitions 5.1 and 5.32).
- (2) A^* is a partial subalgebra of A .

A *realization* of \mathbf{A} is an ordered pair (A', χ) such that $A' \in \mathcal{V}$, A is a partial subalgebra of A' , $\chi: \tilde{A} \rightarrow \text{Con}_c A'$ is a $(\vee, 0)$ -embedding, and $\chi(\delta_{\mathbf{A}}(x, y)) = \Theta_{A'}(x, y)$ for all $x, y \in A$. A realization is *isomorphic* if χ is an isomorphism.

A gamp \mathbf{A} is *finite* if both A and \tilde{A} are finite.

Let \mathbf{A} and \mathbf{B} be gamps. A morphism $\mathbf{f}: (A, \delta_{\mathbf{A}}, \tilde{A}) \rightarrow (B, \delta_{\mathbf{B}}, \tilde{B})$ of pregamps is a *morphism of gamps* from \mathbf{A} to \mathbf{B} if $f(A^*)$ is a partial subalgebra of B^* .

The category of gamps of \mathcal{V} , denoted by $\mathbf{Gamp}(\mathcal{V})$, is the category in which the objects are the gamps of \mathcal{V} and the arrows are the morphisms of gamps.

A *subgamp* of a gamp \mathbf{A} is a gamp $\mathbf{B} = (B^*, B, \delta_{\mathbf{B}}, \tilde{B})$ such that B^* is a partial subalgebra of A^* , B is a partial subalgebra of A , $\delta_{\mathbf{B}} = \delta_{\mathbf{A}} \upharpoonright B^2$, and \tilde{B} is a $(\vee, 0)$ -subsemilattice of \tilde{A} . Let $f: B \rightarrow A$ and $\tilde{f}: \tilde{B} \rightarrow \tilde{A}$ be the inclusion maps. The

ordered pair (f, \tilde{f}) is a morphism of gamps from \mathbf{B} to \mathbf{A} , called *the canonical embedding*.

Remark. A gamp might have no realization. A realization of a finite gamp does not need to be finite.

Let $\mathbf{f}: \mathbf{A} \rightarrow \mathbf{B}$ be a morphism of gamps, let (A', χ) be a realization of \mathbf{A} , and let (B', ξ) be a realization of \mathbf{B} . There might not exist any morphism $g: A' \rightarrow B'$.

Definition 6.2. A *gamp of lattices* is a gamp of the variety of all lattices.

Let \mathbf{B} be a gamp of lattices. A *chain* of \mathbf{B} is a sequence x_0, x_1, \dots, x_{n-1} of B^* such that $x_i \wedge x_j = x_i$ in B for all $i \leq j < n$. We sometime denote such a chain as $x_0 \leq x_1 \leq \dots \leq x_{n-1}$. If $x_i \neq x_j$ for all $i < j < n$, we denote the chain as $x_0 < x_1 < \dots < x_{n-1}$.

Let $u < v$ be a chain of \mathbf{B} , we say that v is a *cover* of u , and then we write $u \prec v$, if there is no chain $u < x < v$ in \mathbf{B} .

The following properties for a gamp come from algebra. It follows from Definition 6.3(1) that there are many operations defined in A . With (2) or (3) all “congruences” have a set of “generators”. Condition (4) expresses that whenever two elements are identified by a “congruence” of A^* , then there is a “good reason” for this in A (cf. Lemma 5.11). Conditions (6) and (7) are related to the transitive closure of relations. Condition (8) is related to congruence n -permutability (cf. Proposition 2.1).

Definition 6.3. A gamp \mathbf{A} is *strong* if the following holds:

- (1) A^* is a strong partial subalgebra of A (cf. Definition 4.4).

A gamp \mathbf{A} is *distance-generated* if it satisfies the following condition:

- (2) Every element of \tilde{A} is a finite join of elements of the form $\delta_{\mathbf{A}}(x, y)$ where $x, y \in A^*$.

A gamp \mathbf{A} of lattices is *distance-generated with chains* if

- (3) For all $\alpha \in \tilde{A}$ there are a positive integer n , and chains $x_0 < y_0, x_1 < y_1, \dots, x_{n-1} < y_{n-1}$ of \mathbf{A} such that $\alpha = \bigvee_{k < n} \delta_{\mathbf{A}}(x_k, y_k)$.

A gamp \mathbf{A} is *congruence-tractable* (cf. Lemma 5.11) if

- (4) For all $x, y, x_0, y_0, \dots, x_{m-1}, y_{m-1}$ in A^* , if $\delta_{\mathbf{A}}(x, y) \leq \bigvee_{k < m} \delta_{\mathbf{A}}(x_k, y_k)$ then there are a positive integer n , a list \vec{z} of parameters from A , and terms t_0, \dots, t_n such that the following equations are satisfied in A .

$$\begin{aligned} x &= t_0(\vec{x}, \vec{y}, \vec{z}), \\ y &= t_n(\vec{x}, \vec{y}, \vec{z}), \\ t_k(\vec{y}, \vec{x}, \vec{z}) &= t_{k+1}(\vec{x}, \vec{y}, \vec{z}) \quad (\text{for all } k < n). \end{aligned}$$

A morphism $\mathbf{f}: \mathbf{A} \rightarrow \mathbf{B}$ of gamps is *strong* if

- (5) $f(A)$ is a strong partial subalgebra of B^* (cf. Definition 4.4).

A morphism $\mathbf{f}: \mathbf{A} \rightarrow \mathbf{B}$ of gamps is *congruence-cuttable* if

- (6) $f(A)$ is a partial sublattice of B^* and given a finite subset X of \tilde{B} and $x, y \in A$ with $\delta_{\mathbf{A}}(f(x), f(y)) \leq \bigvee X$, there are $n < \omega$ and $f(x) = x_0, \dots, x_n = f(y)$ in B^* such that $\delta_{\mathbf{A}}(x_k, x_{k+1}) \in \tilde{B} \downarrow X$ for all $k < n$.

A morphism $\mathbf{f}: \mathbf{A} \rightarrow \mathbf{B}$ of gamps of the variety of all lattices is *congruence-cuttable with chains* if

- (7) $f(A)$ is a partial sublattice of B^* and given a finite subset X of \tilde{B} and $x, y \in A$ with $\delta_A(f(x), f(y)) \leq \bigvee X$, there is a chain $x_0 < \dots < x_n$ of B such that $x_0 = f(x) \wedge f(y)$, $x_n = f(x) \vee f(y)$, and $\delta_A(x_k, x_{k+1}) \in \tilde{B} \downarrow X$ for all $k < n$.

Let $n \geq 1$ be an integer. A gamp A is *congruence n -permutable* if the following statement holds:

- (8) For all $x_0, x_1, \dots, x_n \in A^*$, there are $x_0 = y_0, y_1, \dots, y_n = x_n$ in A such that:

$$\delta_A(y_k, y_{k+1}) \leq \bigvee (\delta_A(x_i, x_{i+1}) \mid i < n \text{ even}), \quad \text{for all } k < n \text{ odd},$$

$$\delta_A(y_k, y_{k+1}) \leq \bigvee (\delta_A(x_i, x_{i+1}) \mid i < n \text{ odd}), \quad \text{for all } k < n \text{ even}.$$

The following lemma shows that chains in strong gamps of lattices behave the same way as chains in lattices.

Lemma 6.4. *Let $x_0 < \dots < x_n$ a chain of a strong gamp of lattices B . The equalities $x_i \wedge x_j = x_j \wedge x_i = x_i$ and $x_i \vee x_j = x_j \vee x_i = x_j$ hold in B for all $i \leq j \leq n$.*

Moreover the following statements hold:

$$\delta_B(x_k, x_{k'}) \leq \delta_B(x_i, x_j), \quad \text{for all } i \leq k \leq k' \leq j \leq n. \quad (6.1)$$

$$\delta_B(x_i, x_j) = \bigvee_{i \leq k < j} \delta_B(x_k, x_{k+1}), \quad \text{for all } i \leq j \leq n. \quad (6.2)$$

Proof. Let $i \leq j \leq n$. As $x_i, x_j \in B^*$, all the elements $x_i \wedge x_j$, $x_j \wedge x_i$, $x_i \vee x_j$, and $x_j \vee x_i$ are defined in B .

As $u \wedge v = v \wedge u$ is an identity of lattices, it follows that $x_j \wedge x_i = x_i \wedge x_j = x_i$.

As $x_j \wedge x_i = x_i$, $(x_i \wedge x_j) \vee x_j = x_i \vee x_j$ in B , and $(u \wedge v) \vee v = v$ is an identity of lattices, $x_i \vee x_j = (x_i \wedge x_j) \vee x_j = x_j$. Similarly $x_j \vee x_i = x_j$.

Let $i \leq k \leq k' \leq j \leq n$. As $x_k \wedge x_{k'} = x_k$ and $x_j \wedge x_{k'} = x_{k'}$, we obtain from Definition 5.1(4) the inequality:

$$\delta_B(x_k, x_{k'}) = \delta_B(x_k \wedge x_{k'}, x_j \wedge x_{k'}) \leq \delta_B(x_k, x_j).$$

Similarly, as $x_k = x_k \vee x_i$ and $x_j = x_j \vee x_i$, the inequality $\delta_B(x_k, x_j) \leq \delta_B(x_i, x_j)$ holds. Therefore (6.1) holds.

Let $i < j \leq n$. Definition 5.1(3) implies the inequality:

$$\delta_B(x_i, x_j) \leq \bigvee_{i \leq k < j} \delta_B(x_k, x_{k+1}).$$

Moreover (6.1) implies $\delta_B(x_k, x_{k+1}) \leq \delta_B(x_i, x_j)$ for all $i \leq k < j$, it follows that (6.2) is true. \square

The following proposition gives a description of *quotient* of gamps.

Proposition 6.5. *Let A be a gamp of \mathcal{V} and let I be an ideal of \tilde{A} . Let $(A, \delta_A, \tilde{A})/I = (A/I, \delta_{A/I}, \tilde{A}/I)$ be the quotient pregamp and set $A^*/I = \{a/I \mid a \in A^*\}$ (cf. Notation 5.22). The following statements hold:*

- (1) $A/I = (A^*/I, A/I, \delta_{A/I}, \tilde{A}/I)$ is a gamp of \mathcal{V} .
- (2) The canonical projection $\pi: (A, \delta_A, \tilde{A}) \rightarrow (A/I, \delta_{A/I}, \tilde{A}/I)$ of pregamps is a morphism of gamps from A to A/I .

- (3) If (A', χ) is a realization of \mathbf{A} in \mathcal{V} , then $(A'/\bigvee \chi(I), \chi')$ is a realization of \mathbf{A}/I in \mathcal{V} , where

$$\begin{aligned}\chi' : \tilde{A}/I &\rightarrow \text{Con}_c(A'/\bigvee \chi(I)) \\ d/I &\mapsto \chi(d)/\bigvee \chi(I).\end{aligned}$$

Moreover, if (A', χ) is an isomorphic realization of \mathbf{A} , then $(A'/\bigvee \chi(I), \chi')$ is an isomorphic realization of \mathbf{A}/I .

- (4) If \mathbf{A} is strong, then \mathbf{A}/I is strong.
 (5) If \mathbf{A} is distance-generated (resp., distance-generated with chains) then \mathbf{A}/I is distance-generated (resp., distance-generated with chains).
 (6) If \mathbf{A} is distance-generated and congruence-tractable, then \mathbf{A}/I is congruence-tractable.
 (7) Let $n \geq 2$ an integer. If \mathbf{A} is congruence n -permutable then \mathbf{A}/I is congruence n -permutable.
 (8) If \mathbf{A} is a gamp of lattices and $x_0 \leq x_1 \leq \dots \leq x_n$ is a chain of \mathbf{A} , then $x_0/I \leq x_1/I \leq \dots \leq x_n/I$ is a chain of \mathbf{A}/I .

Proof. The statement (1) follows from Corollary 5.35. Denote by $\pi : (A, \delta_{\mathbf{A}}, \tilde{A}) \rightarrow (A/I, \delta_{\mathbf{A}/I}, \tilde{A}/I)$ the canonical projection of preamps. The fact that $\pi(A^*) = A^*/I$ as partial algebras follows from the definition of A^*/I . Thus (3) holds.

Let (A', χ) be a realization of \mathbf{A} in \mathcal{V} , let $d, d' \in \tilde{A}$ such that $d/I = d'/I$. Hence there exists $u \in I$ such that $d \vee u = d' \vee u$, it follows that $\chi(d)/\chi(u) = \chi(d')/\chi(u)$, hence $\chi(d)/\bigvee \chi(I) = \chi(d')/\bigvee \chi(I)$. Therefore the map χ' is well-defined. It is easy to check that χ' is a $(\vee, 0)$ -homomorphism. Assume that $\chi'(d/I) \leq \chi'(d'/I)$ for some $d, d' \in \tilde{A}$. Hence $\chi(d)/\bigvee \chi(I) \leq \chi(d')/\bigvee \chi(I)$, so $\chi(d) \leq \chi(d') \vee \bigvee \chi(I)$. However, $\chi(d)$ is a compact congruence of A' , so there exist $u \in I$ such that $\chi(d) \leq \chi(d') \vee \chi(u) = \chi(d' \vee u)$, as χ is an embedding, it follows that $d \leq d' \vee u$, so $d/I \leq d'/I$. Therefore χ' is an embedding.

Let $x, y \in A$. The following equivalences hold:

$$\begin{aligned}x/I = y/I &\iff \delta_{\mathbf{A}}(x, y) \in I \\ &\iff \chi(\delta_{\mathbf{A}}(x, y)) \leq \bigvee \chi(I) \\ &\iff \Theta_{A'}(x, y) \leq \bigvee \chi(I) \\ &\iff x/\bigvee \chi(I) = y/\bigvee \chi(I).\end{aligned}$$

So we can identify A/I with the corresponding subset of $A'/\bigvee \chi(I)$. Moreover, given $\vec{a} \in \text{Def}_{\ell}(A/I)$, there exists $\vec{x} \in \text{Def}_{\ell}(A)$ such that $\vec{a} = \vec{x}/I$, hence $\ell(\vec{a}) = \ell(\vec{x})/I$ is identified with $\ell(\vec{x})/\bigvee \chi(I) = \ell(\vec{x}/\bigvee \chi(I))$. So this identification preserves the operations.

Now assume that the realization is isomorphic. Then χ is surjective, thus χ' is surjective, and thus bijective, hence the realization $(A'/\bigvee \chi(I), \chi')$ is isomorphic. Therefore (3) holds.

The proofs of the statements (4), (5), (7), and (8) are straightforward.

Assume that \mathbf{A} is distance-generated and congruence-tractable. Let $x, y \in A^*$, let $m < \omega$, let \vec{x}, \vec{y} be m -tuples of A^* . Assume that:

$$\delta_{\mathbf{A}/I}(x/I, y/I) \leq \bigvee_{k < m} \delta_{\mathbf{A}/I}(x_k/I, y_k/I).$$

It follows that there exists $u \in I$ with:

$$\delta_{\mathbf{A}}(x, y) \leq u \vee \bigvee_{k < m} \delta_{\mathbf{A}}(x_k, y_k).$$

However, as \mathbf{A} is distance-generated, there exist $x'_0, \dots, x'_{p-1}, y'_0, \dots, y'_{p-1}$ in A^* such that $u = \bigvee_{k < p} \delta_{\mathbf{A}}(x'_k, y'_k)$. As $\delta_{\mathbf{A}}(x'_k, y'_k) \leq u \in I$, $x'_k/I = y'_k/I$ for all $k < p$. Moreover the following inequality holds:

$$\delta_{\mathbf{A}}(x, y) \leq \bigvee_{k < m} \delta_{\mathbf{A}}(x_k, y_k) \vee \bigvee_{k < p} \delta_{\mathbf{A}}(x'_k, y'_k).$$

As \mathbf{A} is congruence-tractable, there are a positive integer n , a list \vec{z} of parameters from A , and terms t_0, \dots, t_n such that, the following equations are satisfied in A :

$$\begin{aligned} x &= t_0(\vec{x}, \vec{x}', \vec{y}, \vec{y}', \vec{z}), \\ y &= t_n(\vec{x}, \vec{x}', \vec{y}, \vec{y}', \vec{z}), \\ t_k(\vec{y}, \vec{y}', \vec{x}, \vec{x}', \vec{z}) &= t_{k+1}(\vec{x}, \vec{x}', \vec{y}, \vec{y}', \vec{z}) \quad (\text{for all } k < n). \end{aligned}$$

Put $t'_k(\vec{a}, \vec{b}, \vec{c}, \vec{d}) = t_k(\vec{a}, \vec{d}, \vec{b}, \vec{c}, \vec{z})$, for all tuples $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ and all $k \leq n$. As $\vec{x}'/I = \vec{y}'/I$ the following equations are satisfied in A/I :

$$\begin{aligned} x/I &= t'_0(\vec{x}/I, \vec{y}/I, \vec{z}/I, \vec{x}'/I), \\ y/I &= t'_n(\vec{x}/I, \vec{y}/I, \vec{z}/I, \vec{x}'/I), \\ t'_k(\vec{y}/I, \vec{x}/I, \vec{z}/I, \vec{x}'/I) &= t'_{k+1}(\vec{x}/I, \vec{y}/I, \vec{z}/I, \vec{x}'/I) \quad (\text{for all } k < n). \end{aligned}$$

Therefore \mathbf{A}/I is congruence-tractable. \square

Definition 6.6. The gamp \mathbf{A}/I described in Proposition 6.5 is a *quotient of \mathbf{A}* , the morphism π is the *canonical projection*.

The following proposition describes how morphisms factorize through quotients of gamps.

Proposition 6.7. Let $\mathbf{f}: \mathbf{A} \rightarrow \mathbf{B}$ be a morphism of gamps, let I be an ideal of \tilde{A} , and let J be an ideal of \tilde{B} . Assume that $\tilde{f}(I) \subseteq J$ and denote by

$$\mathbf{g}: (A/I, \delta_{\mathbf{A}/I}, \tilde{A}/I) \rightarrow (B/J, \delta_{\mathbf{B}/J}, \tilde{B}/J)$$

the morphism of pregamps induced by \mathbf{f} . The following statements hold.

- (1) $\mathbf{g}: \mathbf{A}/I \rightarrow \mathbf{B}/J$ is a morphism of gamps.
- (2) If \mathbf{f} is strong, then \mathbf{g} is strong.
- (3) If \mathbf{f} is congruence-cutttable then \mathbf{g} is congruence-cutttable.
- (4) If \mathbf{A} and \mathbf{B} are gamps of lattices and \mathbf{f} is congruence-cutttable with chains, then \mathbf{g} is congruence-cutttable with chains.

Proof. The equality $g(A^*/I) = f(A^*)/J$ of partial algebras holds. Moreover, as $f(A^*)$ is a partial subalgebra of B , $g(A^*/I)$ is a partial subalgebra of B/J . Therefore (1) holds.

The statement (2) follows from the definitions of a quotient gamp (cf. Proposition 6.5) and of a strong morphism (Definition 6.3).

Assume that \mathbf{f} is congruence-cutttable. As $f(A)$ is a partial subalgebra of B^* it follows that $g(A/I)$ is a partial subalgebra of B^*/J . Let X be a finite subset of \tilde{B} , let $x, y \in A$ such that $\delta_{\mathbf{B}/J}(g(x/I), g(y/I)) \leq \bigvee X/I$. If $X = \emptyset$, then $x/I = y/I$, hence the case is immediate.

If $X \neq \emptyset$, let $u \in J$ such that $\delta_B(g(x), g(y)) \leq u \vee \bigvee X$. Put $X' = X \cup \{u\}$. There are $n < \omega$ and $f(x) = x_0, \dots, x_n = f(y)$ in B^* such that $\delta_B(x_k, x_{k+1}) \in \tilde{B} \downarrow X'$ for each $k < n$. If $\delta_B(x_k, y_k) \leq u$, then $\delta_{B/J}(x_k/J, y_k/J) = \delta_B(x_k, y_k)/J = 0/J \in (\tilde{B}/J) \downarrow X/J$. Otherwise $\delta_B(x_k, y_k) \in \tilde{B} \downarrow X$, thus $\delta_{B/J}(x_k/J, y_k/J) = \delta_B(x_k, y_k)/J \in (\tilde{B}/J) \downarrow X/J$. Therefore (3) holds.

The proof of (4) is similar to the proof of (3). \square

We introduce in the following definitions a functor $\mathbf{G}: \mathcal{V} \rightarrow \mathbf{Gamp}(\mathcal{V})$, a functor $\mathbf{C}: \mathbf{Gamp}(\mathcal{V}) \rightarrow \mathbf{Sem}_{\vee,0}$ and functors $\mathbf{P}_{\text{gl}}, \mathbf{P}_{\text{gr}}: \mathbf{Gamp}(\mathcal{V}) \rightarrow \mathbf{PGamp}(\mathcal{V})$.

Definition 6.8. Let A be a member of a variety \mathcal{V} of algebras. Then the quadruple $\mathbf{G}(A) = (A, A, \Theta_A, \text{Con}_c A)$ is a gamp of \mathcal{V} (we recall that $\Theta_A(x, y)$ denotes the smallest congruence that identifies x and y). If $f: A \rightarrow B$ is a morphism of algebras, then $\mathbf{G}(f) = (f, \text{Con}_c f)$ is a morphism of gamps from $\mathbf{G}(A)$ to $\mathbf{G}(B)$. It defines a functor from the category \mathcal{V} to the category $\mathbf{Gamp}(\mathcal{V})$.

A gamp \mathbf{A} is an *algebra* if \mathbf{A} is isomorphic to $\mathbf{G}(B)$ for some B . A gamp \mathbf{A} is an *algebra* of a variety \mathcal{V} if \mathbf{A} is isomorphic to $\mathbf{G}(B)$ for some $B \in \mathcal{V}$.

Let \mathbf{A} be a gamp of \mathcal{V} , we set $\mathbf{C}(\mathbf{A}) = \tilde{\mathbf{A}}$. Let $\mathbf{f}: \mathbf{A} \rightarrow \mathbf{B}$ be a morphism of gamps of \mathcal{V} , we set $\mathbf{C}(\mathbf{f}) = \tilde{\mathbf{f}}$. This defines a functor $\mathbf{C}: \mathbf{Gamp}(\mathcal{V}) \rightarrow \mathbf{Sem}_{\vee,0}$.

Let \mathbf{A} be a gamp of \mathcal{V} , we set $\mathbf{P}_{\text{gr}}(\mathbf{A}) = (A, \delta_A, \tilde{\mathbf{A}})$. Let $\mathbf{f}: \mathbf{A} \rightarrow \mathbf{B}$ be a morphism of gamps of \mathcal{V} , we put $\mathbf{P}_{\text{gr}}(\mathbf{f}) = \mathbf{f}$ as a morphism of pregamps. This defines a functor $\mathbf{P}_{\text{gr}}: \mathbf{Gamp}(\mathcal{V}) \rightarrow \mathbf{PGamp}(\mathcal{V})$.

Let \mathbf{A} be a gamp of \mathcal{V} , we set $\mathbf{P}_{\text{gl}}(\mathbf{A}) = (A^*, \delta_A \upharpoonright (A^*)^2, \tilde{\mathbf{A}})$. Let $\mathbf{f}: \mathbf{A} \rightarrow \mathbf{B}$ be a morphism of gamps of \mathcal{V} , we denote by $\mathbf{P}_{\text{gl}}(\mathbf{f})$ the restriction $(f, \tilde{\mathbf{f}}): \mathbf{P}_{\text{gl}}(\mathbf{A}) \rightarrow \mathbf{P}_{\text{gl}}(\mathbf{B})$. This defines a functor $\mathbf{P}_{\text{gl}}: \mathbf{Gamp}(\mathcal{V}) \rightarrow \mathbf{PGamp}(\mathcal{V})$.

Remark 6.9. The following assertions hold.

- (1) The following equations, between the functors introduced in Definition 5.4 and Definition 6.8, are satisfied:

$$\begin{aligned} \mathbf{C} \circ \mathbf{G} &= \text{Con}_c = \mathbf{C}_{\text{pg}} \circ \mathbf{P}_{\text{ga}}, \\ \mathbf{P}_{\text{gr}} \circ \mathbf{G} &= \mathbf{P}_{\text{gl}} \circ \mathbf{G} = \mathbf{P}_{\text{ga}}, \\ \mathbf{C}_{\text{pg}} \circ \mathbf{P}_{\text{gr}} &= \mathbf{C}_{\text{pg}} \circ \mathbf{P}_{\text{gl}} = \mathbf{C}. \end{aligned}$$

- (2) If A is a subalgebra of B , then, in general, $\mathbf{G}(A)$ is not a subgamp of $\mathbf{G}(B)$. The different “congruences” of a subgamp can be extended in a natural way to different “congruences” of the gamp.
- (3) Let \mathbf{A} be a gamp. If \mathbf{A} is an algebra, then there is a unique, up to isomorphism, algebra B such that $\mathbf{A} \cong \mathbf{G}(B)$. Moreover if \mathbf{A} is a gamp in a variety \mathcal{V} , then $B \in \mathcal{V}$. Indeed A is an algebra and $\mathbf{A} \cong \mathbf{G}(A)$.
- (4) Let $\mathbf{f}: \mathbf{A} \rightarrow \mathbf{B}$ be a morphism of gamps. If \mathbf{A} and \mathbf{B} are algebras, then $f: A \rightarrow B$ is a morphism of algebras.
- (5) Let $\vec{\mathbf{A}} = (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P)$ be a diagram of gamps. If \mathbf{A}_p is an algebra, for all $p \in P$, then $\vec{\mathbf{A}} = (A_p, f_{p,q} \mid p \leq q \text{ in } P)$ is a diagram of algebras, moreover $\vec{\mathbf{A}} \cong \mathbf{G} \circ \vec{\mathbf{A}}$.
- (6) Let $B \in \mathcal{V}$ and let I be an ideal of $\text{Con}_c B$. There is an isomorphism $\mathbf{f}: \mathbf{G}(B)/I \cong \mathbf{G}(B/\bigvee I)$ satisfying $f(x/I) = x/\bigvee I$ and $\tilde{f}(\alpha/I) = \alpha/\bigvee I$ for each $x \in B$ and each $\alpha \in \text{Con}_c B$.

- (7) Let B be an algebra. Then B is congruence n -permutable if and only if $\mathbf{G}(B)$ is congruence n -permutable (this follows immediately from Proposition 2.1).

Lemma 6.10. *Let \mathcal{V} be a variety of algebras. The category $\mathbf{Gamp}(\mathcal{V})$ has all directed colimits. Suppose that we are given a directed poset P , a P -indexed diagram $\vec{\mathbf{A}} = (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P)$ in $\mathbf{Gamp}(\mathcal{V})$, and a directed colimit cocone:*

$$((A, \delta_{\mathbf{A}}, \tilde{A}), \mathbf{f}_p \mid p \in P) = \varinjlim ((A_p, \delta_{\mathbf{A}_p}, \tilde{A}_p), \mathbf{f}_{p,q} \mid p \leq q \text{ in } P) \quad \text{in } \mathbf{PGamp}_{\mathcal{L}}.$$

Put $A^* = \bigcup_{p \in P} f(A_p^*)$ with its natural structure of partial algebra (cf. Lemma 4.9), then $\mathbf{A} = (A^*, A, \delta_{\mathbf{A}}, \tilde{A})$ is a gamp of \mathcal{V} , $\mathbf{f}_p: \mathbf{A}_p \rightarrow \mathbf{A}$ is a morphism of gamps, and the following is a directed colimit cocone:

$$(\mathbf{A}, \mathbf{f}_p \mid p \in P) = \varinjlim (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P), \quad \text{in } \mathbf{Gamp}(\mathcal{V}).$$

Moreover the following statements hold:

- (1) If \mathbf{A}_p is distance-generated for each $p \in P$, then \mathbf{A} is distance-generated.
- (2) If \mathcal{V} is a variety of lattices and \mathbf{A}_p is distance-generated with chains for each $p \in P$, then \mathbf{A} is distance-generated with chains.
- (3) Let n be a positive integer. If \mathbf{A}_p is congruence n -permutable for each $p \in P$, then \mathbf{A} is congruence n -permutable.

Proof. It follows from Corollary 5.34 that $(A, \delta_{\mathbf{A}}, \tilde{A})$ is a pregamp of \mathcal{V} . Moreover A^* is a partial subalgebra of A . Hence $\mathbf{A} = (A^*, A, \delta_{\mathbf{A}}, \tilde{A})$ is a gamp of \mathcal{V} . As $f(A_p^*)$ is a partial subalgebra of A^* , $\mathbf{f}_p: \mathbf{A}_p \rightarrow \mathbf{A}$ is a morphism of gamps. It is easy to check that the following is a directed colimit cocone:

$$(\mathbf{A}, \mathbf{f}_p \mid p \in P) = \varinjlim (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P), \quad \text{in } \mathbf{Gamp}(\mathcal{V}).$$

Assume that \mathbf{A}_p is distance-generated for each $p \in P$. Let $\alpha \in \tilde{A}$, then there are $p \in P$ and $\beta \in \tilde{A}_p$ such that $\alpha = \tilde{f}_p(\beta)$. As \mathbf{A}_p is distance-generated, there are an integer $n \geq 0$ and n -tuple \vec{x}, \vec{y} of A_p^* such that $\beta = \bigvee_{k < n} \delta_{\mathbf{A}_p}(x_k, y_k)$. Therefore the following equalities hold:

$$\alpha = \tilde{f}_p(\beta) = \tilde{f}_p \left(\bigvee_{k < n} \delta_{\mathbf{A}_p}(x_k, y_k) \right) = \bigvee_{k < n} \tilde{f}_p(\delta_{\mathbf{A}_p}(x_k, y_k)) = \bigvee_{k < n} \delta_{\mathbf{A}}(f_p(x_k), f_p(y_k)).$$

Thus \mathbf{A} is distance-generated.

The proofs of (2) and (3) are similar. \square

As an immediate application we obtain the following corollary.

Corollary 6.11. *The functors \mathbf{G} , \mathbf{C} , \mathbf{P}_{gl} , and \mathbf{P}_{gr} preserves directed colimits.*

Proof. It follows from the description of directed colimits of gamps (cf. Lemma 6.10) and pregamps (cf. Lemma 5.6) that \mathbf{C} , \mathbf{P}_{gl} , and \mathbf{P}_{gr} preserve directed colimits. As $\text{Con}_{\mathbf{C}}$ preserves directed colimits, \mathbf{G} also preserves directed colimits. \square

Definition 6.12. Let \mathcal{V} be a variety of algebras, let P be a poset, and let $\vec{\mathbf{A}} = (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P)$ in $\mathbf{Gamp}(\mathcal{V})$. An *ideal* of $\vec{\mathbf{A}}$ is an ideal of $\mathbf{C} \circ \vec{\mathbf{A}}$. It consists of a family $\vec{I} = (I_p)_{p \in P}$ such that I_p is an ideal of \tilde{A}_p and $\tilde{f}_{p,q}(I_p) \subseteq I_q$ for all $p \leq q$ in P .

We denote by $\vec{\mathbf{A}}/\vec{I} = (\mathbf{A}_p/I_p, \mathbf{g}_{p,q} \mid p \leq q \text{ in } P)$, where $\mathbf{g}_{p,q}: \mathbf{A}_p/I_p \rightarrow \mathbf{A}_q/I_q$ is induced by $\mathbf{f}_{p,q}$, for all $p \leq q$ in P .

The diagram \vec{A}/\vec{I} is a *quotient* of \vec{A} .

Remark 6.13. In the context of Definition 6.12 the following equalities hold:

$$(P_{\text{gr}} \circ \vec{A})/\vec{I} = P_{\text{gr}} \circ (\vec{A}/\vec{I}).$$

$$(C \circ \vec{A})/\vec{I} = C \circ (\vec{A}/\vec{I}).$$

Moreover, up to a natural identification (cf. Notation 5.22)

$$(P_{\text{gl}} \circ \vec{A})/\vec{I} = P_{\text{gl}} \circ (\vec{A}/\vec{I}).$$

Definition 6.14. Let \mathcal{V} be a variety of algebras, let P be a poset. A *partial lifting* in \mathcal{V} is a diagram $\vec{A} = (A_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P)$ in $\mathbf{Gamp}(\mathcal{V})$ such that the following statements hold:

- (1) The gamp A_p is strong, congruence-tractable, distance-generated, and has an isomorphic realization (cf. Definitions 6.3 and 6.1), for each $p \in P$.
- (2) The morphisms $\mathbf{f}_{p,q}$ is strong and congruence-cuttable (cf. Definition 6.3), for all $p < q$ in P .

The partial lifting is a *lattice partial lifting* if \mathcal{V} is a variety of lattices, B_p is distance-generated with chains for each $p \in P$, and $\mathbf{f}_{p,q}$ is congruence-cuttable with chains for all $p < q$ in P .

A partial lifting \vec{A} is *congruence n -permutable* if A_p is congruence n -permutable for each $p \in P$.

Let $\vec{S} = (S_p, \sigma_{p,q})$ be a diagram in $\mathbf{Sem}_{\mathcal{V},0}$. A *partial lifting of \vec{S}* is a partial lifting of \vec{A} such that $C \circ \vec{A} \cong \vec{S}$.

Remark 6.15. If \vec{A} is a partial lifting of \vec{S} , then there exists a diagram $\vec{A}' \cong \vec{A}$ such that $C \circ \vec{A}' = \vec{S}$. Hence we can assume that $C \circ \vec{A} = \vec{S}$.

The following result expresses the fact that a subdiagram or a quotient of a partial lifting is a partial lifting.

Lemma 6.16. *Let \mathcal{V} be a variety of algebras, let P be a poset, and let \vec{A} be a partial lifting in \mathcal{V} of a diagram \vec{S} . The following statements hold:*

- (1) *Let \vec{I} be an ideal of \vec{A} ; then \vec{A}/\vec{I} is a partial lifting of \vec{S}/I .*
- (2) *Let $Q \subseteq P$; then $\vec{A} \upharpoonright Q$ is a partial lifting of $\vec{S} \upharpoonright Q$.*

Proof. The statement (1) follows from Proposition 6.5 and Proposition 6.7. The statement (2) is immediate. \square

Lemma 6.17. *Let \mathcal{V} be a variety of algebras, let P be a directed poset with no maximal element, and let $\vec{A} = (A_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P)$ be a partial lifting in \mathcal{V} . Consider a colimit cocone:*

$$(A, \mathbf{f}_p \mid p \in P) = \varinjlim \vec{A} \quad \text{in } \mathbf{Gamp}(\mathcal{V}).$$

Then A is an algebra in \mathcal{V} . Moreover, for any $n \geq 2$, if all A_p are congruence n -permutable, then the algebra corresponding to A is congruence n -permutable.

Proof. The morphism $\mathbf{f}_{p,q}: (A_p, \delta_{A_p}, \tilde{A}_p) \rightarrow (A_q, \delta_{A_q}, \tilde{A}_q)$ of pregamps is both strong and congruence-tractable for all $p < q$ in P . It follows from the description of colimits in $\mathbf{Gamp}(\mathcal{V})$ (cf. Lemma 6.10 and Corollary 5.15) that A is an algebra and there is an isomorphism $\phi: \text{Con}_c A \rightarrow \tilde{A}$ satisfying:

$$\phi(\Theta_A(x, y)) = \delta_A(x, y), \quad \text{for all } x, y \in A.$$

As $A^* = \bigcup_{p \in P} f_p(A_p^*)$ and $f_{p,q}(A_p) \subseteq A_q^*$ for all $p < q$ in P , it follows that $A^* = A$. Therefore $(\text{id}_A, \phi): \mathbf{G}(A) \rightarrow \mathbf{A}$ is an isomorphism of gamps.

Now assume that all \mathbf{A}_p are congruence n -permutable. It follows from Lemma 6.10 that \mathbf{A} is congruence n -permutable. As \mathbf{A} is an algebra, the conclusion follows from Proposition 2.1. \square

7. LOCALLY FINITE PROPERTIES

The aim of this section is to prove Lemma 7.8, which is a special version of the Buttress Lemma of [5] adapted to gamps for the functor \mathbf{C} .

We use the following generalizations of (2), (3), (4), (6), and (7) of Definition 6.3.

Definition 7.1. Fix a gamp \mathbf{A} in a variety \mathcal{V} , a $(\vee, 0)$ -semilattice S , and a $(\vee, 0)$ -homomorphism $\phi: \tilde{A} \rightarrow S$. The gamp \mathbf{A} is *distance-generated through ϕ* if the following statement holds:

(2') For each $s \in S$ there are $n < \omega$ and n -tuple \vec{x}, \vec{y} of A^* such that:

$$s = \bigvee_{k < n} \phi(\delta_{\mathbf{A}}(x_k, y_k))$$

If \mathbf{A} is a gamp of lattices, we say \mathbf{A} is *distance-generated with chains through ϕ* if

(3') For each $s \in S$ there are $n < \omega$ and chains $x_0 < y_0, \dots, x_{n-1} < y_{n-1}$ of \mathbf{A} such that:

$$s = \bigvee_{k < n} \phi(\delta_{\mathbf{A}}(x_k, y_k))$$

The gamp \mathbf{A} is *congruence-tractable through ϕ* if the following statement holds:

(4') Let $x, y, x_0, y_0, \dots, x_{m-1}, y_{m-1}$ in A^* , if $\phi(\delta_{\mathbf{A}}(x, y)) \leq \bigvee_{k < m} \phi(\delta_{\mathbf{A}}(x_k, y_k))$ then there are a positive integer n , a list \vec{z} of parameters from A , and terms t_0, \dots, t_n such that, the following equations are satisfied in A :

$$\begin{aligned} \phi(\delta_{\mathbf{A}}(x, t_0(\vec{x}, \vec{y}, \vec{z}))) &= 0, \\ \phi(\delta_{\mathbf{A}}(y, t_n(\vec{x}, \vec{y}, \vec{z}))) &= 0, \\ \phi(\delta_{\mathbf{A}}(t_j(\vec{y}, \vec{x}, \vec{z}), t_{j+1}(\vec{x}, \vec{y}, \vec{z}))) &= 0 \quad (\text{for all } j < n). \end{aligned}$$

A morphism $\mathbf{f}: \mathbf{U} \rightarrow \mathbf{A}$ of gamps is *congruence-cuttable through ϕ* if the following statement holds:

(6') Given $X \subseteq S$, given x, y in U , if $\phi(\tilde{f}(\delta_{\mathbf{A}}(x, y))) \leq \bigvee X$ then there exist $n < \omega$ and $f(x) = x_0, x_1, \dots, x_n = f(y)$ in A^* such that $\delta_{\mathbf{A}}(x_k, x_{k+1}) \in S \downarrow X$ for all $k < n$.

A morphism $\mathbf{f}: \mathbf{U} \rightarrow \mathbf{A}$ of gamps of lattices is *congruence-cuttable with chains through ϕ* if the following statement holds:

(7') Given $X \subseteq S$, given $x, y \in U$, if $\phi(\tilde{f}(\delta_{\mathbf{A}}(x, y))) \leq \bigvee X$ then there are $n < \omega$ and a chain $x_0 < x_1 < \dots < x_n$ of \mathbf{A} such that $x_0 = f(x) \wedge f(y)$, $x_n = f(x) \vee f(y)$, and $\phi(\delta_{\mathbf{A}}(x_k, x_{k+1})) \in S \downarrow X$ for all $k < n$.

Remark 7.2. A gamp \mathbf{A} is distance-generated through $\text{id}_{\tilde{A}}$ if and only if \mathbf{A} is distance-generated.

A gamp \mathbf{A} is congruence-tractable through $\text{id}_{\tilde{A}}$ if and only if \mathbf{A} is congruence-tractable.

A morphism $\mathbf{f}: \mathbf{U} \rightarrow \mathbf{A}$ of gamps is congruence-cuttable through $\text{id}_{\tilde{A}}$ if and only if \mathbf{f} is congruence-cuttable.

A morphism $\mathbf{f}: \mathbf{U} \rightarrow \mathbf{A}$ of gamps of lattices is congruence-cuttable with chains through $\text{id}_{\tilde{\mathbf{A}}}$ if and only if \mathbf{f} is congruence-cuttable with chains.

Lemma 7.3. *Let \mathbf{A} be a gamp in a variety of algebras \mathcal{V} , let $\phi: \tilde{\mathbf{A}} \rightarrow S$ an ideal-induced $(\vee, 0)$ -homomorphism, and put $I = \ker_0 \phi$.*

- (1) *If \mathbf{A} is distance-generated through ϕ , then \mathbf{A}/I is distance-generated.*
- (2) *If \mathbf{A} is distance-generated with chains through ϕ , then \mathbf{A}/I is distance-generated with chains.*
- (3) *If \mathbf{A} is congruence-tractable through ϕ then \mathbf{A}/I is congruence-tractable.*

Proof. As ϕ is ideal-induced, it induces an isomorphism $\xi: \tilde{\mathbf{A}}/I \rightarrow S$.

Assume that \mathbf{A} is distance-generated through ϕ . Let $d \in \tilde{\mathbf{A}}/I$, put $s = \xi(d)$. There are $n < \omega$ and n -tuples \vec{x}, \vec{y} in A^* such that $s = \bigvee_{k < n} \phi(\delta_{\mathbf{A}}(x_k, y_k))$. It follows that $s = \bigvee_{k < n} \xi(\delta_{\mathbf{A}/I}(x_k/I, y_k/I))$, so $d = \bigvee_{k < n} \delta_{\mathbf{A}/I}(x_k/I, y_k/I)$. Therefore \mathbf{A}/I is distance-generated.

The case where \mathbf{A} is distance-generated with chains through ϕ is similar.

Assume that \mathbf{A} is congruence-tractable through ϕ . Let $x, y, x_0, y_0, \dots, x_{m-1}, y_{m-1}$ in A^* such that:

$$\delta_{\mathbf{A}/I}(x/I, y/I) \leq \bigvee_{k < m} \delta_{\mathbf{A}/I}(x_k/I, y_k/I).$$

Thus $\phi(\delta_{\mathbf{A}/I}(x, y)) \leq \bigvee_{k < m} \phi(\delta_{\mathbf{A}}(x_k, y_k))$. Hence there are a positive integer n , a list \vec{z} of parameters from A , and terms t_0, \dots, t_n such that the following equations are satisfied in A :

$$\begin{aligned} \phi(\delta_{\mathbf{A}}(x, t_0(\vec{x}, \vec{y}, \vec{z}))) &= 0, \\ \phi(\delta_{\mathbf{A}}(y, t_n(\vec{x}, \vec{y}, \vec{z}))) &= 0, \\ \phi(\delta_{\mathbf{A}}(t_k(\vec{y}, \vec{x}, \vec{z}), t_{k+1}(\vec{x}, \vec{y}, \vec{z}))) &= 0 \quad (\text{for all } k < n). \end{aligned}$$

Those equations imply that the following equations are satisfied in \mathbf{A}/I :

$$\begin{aligned} x/I &= t_0(\vec{x}/I, \vec{y}/I, \vec{z}/I), \\ y/I &= t_n(\vec{x}/I, \vec{y}/I, \vec{z}/I), \\ t_k(\vec{y}/I, \vec{x}/I, \vec{z}/I) &= t_{k+1}(\vec{x}/I, \vec{y}/I, \vec{z}/I) \quad (\text{for all } k < n). \end{aligned}$$

Therefore \mathbf{A}/I is congruence-tractable. \square

The proof of the following lemma is similar.

Lemma 7.4. *Let $\mathbf{f}: \mathbf{U} \rightarrow \mathbf{A}$ be a morphism of gamps, let I be an ideal of \mathbf{U} , and let $\phi: \tilde{\mathbf{A}} \rightarrow S$ be an ideal-induced $(\vee, 0)$ -homomorphism. Put $J = \ker_0 \phi$. Denote by $\mathbf{g}: \mathbf{U}/I \rightarrow \mathbf{A}/J$ the morphism of gamps induced by \mathbf{f} . The following statements hold:*

- (1) *If \mathbf{f} is congruence-cuttable through ϕ , then \mathbf{g} is congruence-cuttable.*
- (2) *Assume that \mathbf{f} is a morphism of gamps of lattices. If \mathbf{f} is congruence-cuttable with chains through ϕ , then \mathbf{g} is congruence-cuttable with chains.*

We shall now define *locally finite properties* for an algebra B , as properties that are satisfied by “many” finite subgamps of $\mathbf{G}(B)$.

Definition 7.5. Let B be an algebra. A *locally finite property* for B is a property (P) in a subgamp \mathbf{A} of $\mathbf{G}(B)$ such that there exists a finite $X \subseteq B$ satisfying

that for every finite full partial subalgebra A^* of B that contains X , there exists a finite $Y \subseteq B$ such that for every finite full partial subalgebra A of B that contains $A^* \cup Y$ and every finite $(\vee, 0)$ -subsemilattice \tilde{A} of $\text{Conc } B$ that contains $\text{Conc}^A(B)$, the subgamp $(A^*, A, \Theta_B, \tilde{A})$ of $\mathbf{G}(B)$ satisfies (P).

Proposition 7.6. *Any finite conjunction of locally finite properties is locally finite.*

Lemma 7.7. *Let B be an algebra and denote by \mathcal{L} the similarity type of B . The properties (1)-(8) in \mathbf{A} are locally finite for B :*

Assume that \mathcal{L} is finite.

(1) \mathbf{A} is strong.

Fix an ideal-induced $(\vee, 0)$ -homomorphism $\phi: \text{Conc } B \rightarrow S$, with S finite.

(2) \mathbf{A} is distance-generated through $\phi \upharpoonright \tilde{A}$.

Fix an ideal-induced $(\vee, 0)$ -homomorphism $\phi: \text{Conc } B \rightarrow S$ with S finite, and assume that B is a lattice.

(3) \mathbf{A} is distance-generated with chains through $\phi \upharpoonright \tilde{A}$.

Fix an ideal-induced $(\vee, 0)$ -homomorphism $\phi: \text{Conc } B \rightarrow S$, with S finite.

(4) \mathbf{A} is congruence-tractable through $\phi \upharpoonright \tilde{A}$.

Assume that \mathcal{L} is finite, fix a finite gamp \mathbf{U} , fix a morphism $\mathbf{f}: \mathbf{U} \rightarrow \mathbf{G}(B)$ of gamps.

(5) The restriction $\mathbf{f}: \mathbf{U} \rightarrow \mathbf{A}$ is strong.

Fix an ideal-induced $(\vee, 0)$ -homomorphism $\phi: \text{Conc } B \rightarrow S$ where S is finite, a finite gamp \mathbf{U} , and a morphism $\mathbf{f}: \mathbf{U} \rightarrow \mathbf{G}(B)$ of gamps.

(6) The restriction of $\mathbf{f}: \mathbf{U} \rightarrow \mathbf{A}$ is congruence-cuttable through $\phi \upharpoonright \tilde{A}$.

Assume that B is a lattice. Fix an ideal-induced $(\vee, 0)$ -homomorphism $\phi: \text{Conc } B \rightarrow S$, where S is finite. Fix a finite gamp \mathbf{U} , fix a morphism $\mathbf{f}: \mathbf{U} \rightarrow \mathbf{G}(B)$ of gamps.

(7) The restriction of $\mathbf{f}: \mathbf{U} \rightarrow \mathbf{A}$ is congruence-cuttable with chains through $\phi \upharpoonright \tilde{A}$.

Let $n \geq 2$ be an integer. Assume that B is congruence n -permutable.

(8) \mathbf{A} is congruence n -permutable.

Proof. (1) Put $X = \emptyset$ and let A^* be a finite full partial subalgebra of B . Put $Y = \{\ell^B(\vec{x}) \mid \ell \in \mathcal{L} \text{ and } \vec{x} \text{ is an } \text{ar}(\ell)\text{-tuple of } A^*\}$. As A^* and \mathcal{L} are both finite, Y is also finite. Let A be a finite full partial subalgebra of B that contains $A^* \cup Y$, let \tilde{A} be a finite $(\vee, 0)$ -subsemilattice of $\text{Conc } B$ containing $\text{Conc}^A(B)$. As $\ell(\vec{x}) \in Y \subseteq A$, it is defined in A , for each $\ell \in \mathcal{L}$ and each $\text{ar}(\ell)$ -tuple \vec{x} of A .

(2) Let $s \in S$. As ϕ is surjective, there exists $\theta \in \text{Conc } B$ such that $s = \phi(\theta)$. So there exist $n < \omega$ and n -tuple \vec{x}, \vec{y} of B such that $s = \phi(\bigvee_{k < n} \Theta_B(x_k, y_k))$. Put $X_s = \{x_0, \dots, x_{n-1}, y_0, \dots, y_{n-1}\}$.

Put $X = \bigcup_{s \in S} X_s$. As S is finite and X_s is finite for all $s \in S$, X is finite. Let A^* be a finite full partial subalgebra of B that contains X . Put $Y = \emptyset$. Let A be a finite full partial subalgebra of B that contains $A^* \cup Y$. Let \tilde{A} be a finite $(\vee, 0)$ -subsemilattice of $\text{Conc } B$ containing $\text{Conc}^A(B)$. By construction $(A^*, A, \Theta_B, \tilde{A})$ satisfies (2).

The proof that (3) is a locally finite property is similar.

(4) Put $X = \emptyset$, let A^* be a finite full partial subalgebra of B . Denote by E the set of all quadruples (x, y, \vec{x}, \vec{y}) such that the following statements are satisfied:

- $x, y \in A^*$.
- \vec{x} and \vec{y} are m -tuples of A^* , for some $m < \omega$.
- $\phi(\Theta_B(x, y)) \leq \bigvee_{k < m} \phi(\Theta_B(x_k, y_k))$.
- $(x_i, y_i) \neq (x_j, y_j)$ for all $i < j < m$.

Put $I = \ker_0 \phi$ and let $(x, y, \vec{x}, \vec{y}) \in E$. As ϕ is ideal-induced, there exists $\alpha \in I$ such that $\Theta_B(x, y) \leq \bigvee_{k < m} \Theta_B(x_k, y_k) \vee \alpha$. Let \vec{x}', \vec{y}' be p -tuples of B such that $\alpha = \bigvee_{k < p} \Theta_B(x'_k, y'_k)$. Hence:

$$\Theta_B(x, y) \leq \bigvee_{k < m} \Theta_B(x_k, y_k) \vee \bigvee_{k < p} \Theta_B(x'_k, y'_k).$$

It follows from Lemma 5.11 that there are a positive integer n , a list \vec{z} of parameters from B , and terms t_0, \dots, t_n such that the following equalities hold in B :

$$x = t_0(\vec{x}, \vec{x}', \vec{y}, \vec{y}', \vec{z}), \quad (7.1)$$

$$y = t_n(\vec{x}, \vec{x}', \vec{y}, \vec{y}', \vec{z}), \quad (7.2)$$

$$t_j(\vec{y}, \vec{y}', \vec{x}, \vec{x}', \vec{z}) = t_{j+1}(\vec{x}, \vec{x}', \vec{y}, \vec{y}', \vec{z}), \quad \text{for all } j < n. \quad (7.3)$$

Put $t'_j(\vec{a}, \vec{b}, \vec{c}, \vec{d}) = t_j(\vec{a}, \vec{d}, \vec{b}, \vec{c}, \vec{d})$ for all tuples $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ of B of appropriate length. The following inequalities follow from the compatibility of Θ_B with terms:

$$\Theta_B(t'_j(\vec{x}, \vec{y}, \vec{z}, \vec{x}'), t_j(\vec{x}, \vec{x}', \vec{y}, \vec{y}', \vec{z})) \leq \bigvee_{k < p} \Theta_B(x'_k, y'_k) = \alpha, \quad (7.4)$$

$$\Theta_B(t'_j(\vec{y}, \vec{x}, \vec{z}, \vec{x}'), t_j(\vec{y}, \vec{y}', \vec{x}, \vec{x}', \vec{z})) \leq \bigvee_{k < p} \Theta_B(x'_k, y'_k) = \alpha. \quad (7.5)$$

As $\phi(\alpha) = 0$, it follows from (7.4) and (7.1) that:

$$\phi(\Theta_B(x, t'_0(\vec{x}, \vec{y}, \vec{z}, \vec{x}'))) = 0$$

Set $\vec{z}' = (\vec{z}, \vec{x}')$. As $\phi(\alpha) = 0$, it follows from (7.1)-(7.5) that

$$\phi(\Theta_B(x, t'_0(\vec{x}, \vec{y}, \vec{z}'))) = 0,$$

$$\phi(\Theta_B(y, t'_n(\vec{x}, \vec{y}, \vec{z}'))) = 0,$$

$$\phi(\Theta_B(t'_j(\vec{y}, \vec{x}, \vec{z}'), t'_{j+1}(\vec{x}, \vec{y}, \vec{z}'))) = 0 \quad (\text{for all } j < n).$$

Let $Y_{(x, y, \vec{x}, \vec{y})}$ be a finite partial subalgebra of B such that both $t'_j(\vec{x}, \vec{y}, \vec{z}, \vec{x}')$ and $t'_j(\vec{y}, \vec{x}, \vec{z}, \vec{x}')$ are defined in $Y_{(x, y, \vec{x}, \vec{y})}$, for each $j \leq n$.

Put $Y = \bigcup (Y_e \mid e \in E)$. As Y_e is finite for each $e \in E$ and E is finite, Y is finite. Let A be a finite full partial subalgebra of B that contains $Y \cup A^*$, and let \tilde{A} be a finite $(\vee, 0)$ -subsemilattice of $\text{Con}_c B$ containing $\text{Con}_c^A(B)$. It is not hard to verify that $(A^*, A, \Theta_B, \tilde{A})$ satisfies (4).

(5) As U is finite, the set $X_1 = \langle f(U) \rangle_B^1$ (cf. Notation 4.7) is also finite. As \tilde{U} is finite and each element of $\tilde{f}(\tilde{U})$ is a compact congruence of B , we can choose a finite subset X_2 of B such that $\tilde{f}(\tilde{U}) \subseteq \text{Con}_c^{X_2}(B)$. The set $X = X_1 \cup X_2$ is finite. Let A^* be a finite full partial subalgebra of B containing X . Put $Y = \emptyset$, let A be a full partial subalgebra of B containing $Y \cup A^*$, and let \tilde{A} be a finite $(\vee, 0)$ -subsemilattice of $\text{Con}_c B$ containing $\text{Con}_c^A(B)$. The following containments hold:

$$\tilde{f}(\tilde{U}) \subseteq \text{Con}_c^{X_2}(B) \subseteq \text{Con}_c^A(B) \subseteq \tilde{A}.$$

Moreover $\langle f(U) \rangle_B^1 \subseteq A^*$.

The proofs that (6), (7), and (8) are locally finite properties are similar. \square

The following lemma is an analogue for gamps of the *Buttress Lemma* (cf. [5]).

Lemma 7.8. *Let \mathcal{V} be a variety of algebras in a finite similarity type, let P be a lower finite poset, let $(S_p)_{p \in P}$ be a family of finite $(\vee, 0)$ -semilattices, let $B \in \mathcal{V}$, and let $(\phi_p)_{p \in P}$ be a family of $(\vee, 0)$ -homomorphisms where $\phi_p: \text{Con}_c B \rightarrow S_p$ is ideal-induced for each $p \in P$. There exists a diagram $\vec{\mathbf{A}} = (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P)$ of finite subgamps of $\mathbf{G}(B)$ such that the following statements hold:*

- (1) $\phi_p \upharpoonright \tilde{\mathbf{A}}_p$ is ideal-induced for each $p \in P$.
- (2) \mathbf{A}_p is strong, distance-generated through $\phi_p \upharpoonright \tilde{\mathbf{A}}_p$, and congruence-tractable through $\phi_p \upharpoonright \tilde{\mathbf{A}}_p$, for each $p \in P$.
- (3) \mathbf{A}_p is a subgamp of \mathbf{A}_q and $\mathbf{f}_{p,q}$ is the canonical embedding, for all $p \leq q$ in P .
- (4) $\mathbf{f}_{p,q}$ is strong and congruence-cutttable through $\phi_q \upharpoonright \tilde{\mathbf{A}}_q$, for all $p < q$ in P .

If B is a lattice, then we can construct $\vec{\mathbf{A}}$ such that $\mathbf{f}_{p,q}$ is congruence-cutttable with chains through $\phi_q \upharpoonright \tilde{\mathbf{A}}_q$ for all $p < q$ in P , and \mathbf{B}_p is distance-generated with chains through ϕ_p for each $p \in P$.

If B is congruence n -permutable (where n is a positive integer), then we can construct $\vec{\mathbf{A}}$ such that \mathbf{A}_p is congruence n -permutable for each $p \in P$.

Proof. Let $r \in P$, suppose having constructed a diagram $(\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q < r)$ of finite subgamps of $\mathbf{G}(B)$ such that the following statements hold:

- (1) $\phi_p \upharpoonright \tilde{\mathbf{A}}_p$ is ideal-induced for each $p < r$.
- (2) \mathbf{A}_p is strong, distance-generated through $\phi_p \upharpoonright \tilde{\mathbf{A}}_p$, and congruence-tractable through $\phi_p \upharpoonright \tilde{\mathbf{A}}_p$, for each $p < r$.
- (3) \mathbf{A}_p is a subgamp of \mathbf{A}_q and $\mathbf{f}_{p,q}$ is the canonical embedding, for all $p \leq q < r$.
- (4) $\mathbf{f}_{p,q}$ is strong and congruence-cutttable through $\phi_q \upharpoonright \tilde{\mathbf{A}}_q$, for all $p < q < r$.

The following property in \mathbf{A} a subgamp of $\mathbf{G}(B)$ is locally finite (see Proposition 7.6 and Lemma 7.7)

- (F) \mathbf{A} is strong, distance-generated through $\phi_r \upharpoonright \tilde{\mathbf{A}}$, congruence-tractable through $\phi_r \upharpoonright \tilde{\mathbf{A}}$, and the canonical embedding $\mathbf{f}: \mathbf{A}_p \rightarrow \mathbf{A}$ is strong and congruence-cutttable through $\phi_r \upharpoonright \tilde{\mathbf{A}}$, for all $p < r$.

Thus there exist finite partial subalgebras A_r^* and A_r of B such that for each $\tilde{\mathbf{A}}$ containing $\text{Con}_c^{A_r}(B)$, the gamp $(A_r^*, A_r, \Theta_B, \tilde{\mathbf{A}})$ satisfies (F). Moreover it follows from Proposition 3.16 that there exists a finite $(\vee, 0)$ -subsemilattice $\tilde{\mathbf{A}}_r$ of $\text{Con}_c B$, such that $\text{Con}_c^{A_r}(B) \subseteq \tilde{\mathbf{A}}_r$ and $\phi_r \upharpoonright \tilde{\mathbf{A}}_r$ is ideal-induced.

Set $\mathbf{A}_r = (A_r^*, A_r, \Theta_B, \tilde{\mathbf{A}}_r)$, and $\mathbf{f}_{p,r}$ the canonical embedding for each $p \leq r$. By construction, the diagram $(\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \leq r)$ satisfies the required conditions. The construction of $(\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P)$ follows by induction.

If B is a lattice, then we can add to the property (F) the condition $\mathbf{f}: \mathbf{A}_p \rightarrow \mathbf{A}$ is congruence-cutttable with chains through $\phi_r \upharpoonright \tilde{\mathbf{A}}$ for each $p < r$, and \mathbf{B}_r is distance-generated with chains through ϕ_r .

If B is congruence n -permutable, then we can add to the property (F) the condition \mathbf{A}_r is congruence n -permutable. \square

Remark 7.9. In the context of Lemma 7.8, if we have a locally finite property for B , then we can assume that any A_p satisfies this property.

8. NORM-COVERINGS AND LIFTERS

The aim of this section is to construct a (family) of posets, which we shall use later as an index for a diagram (in [4]). We also give a combinatorial statement that is satisfied by this poset.

We introduced the following definition in [3].

Definition 8.1. A finite subset V of a poset U is a *kernel* if for every $u \in U$, there exists a largest element $v \in V$ such that $v \leq u$. We say that U is *supported* if every finite subset of U is contained in a kernel of U .

It is not hard to verify that this definition of a supported poset is equivalent to the one used in [5].

The following definition introduced in [3] also appears in [5] in a weaker form. Nevertheless, in the context of \aleph_0 -lifters (cf. Definition 8.3), all these definitions are equivalent.

Definition 8.2. A *norm-covering* of a poset P is a pair (U, ∂) , where U is a supported poset and $\partial: U \rightarrow P$, $u \mapsto \partial u$ is an isotone map.

We say that an ideal \mathbf{u} of U is *sharp* if the set $\{\partial u \mid u \in \mathbf{u}\}$ has a largest element, which we shall then denote by $\partial \mathbf{u}$. We shall denote by $\text{Id}_s U$ the set of all sharp ideals of U , partially ordered by inclusion.

We remind the reader about the following definition introduced in [5].

Definition 8.3. Let P be a poset. An \aleph_0 -*lifter* of P is a pair (U, \mathbf{U}) , where U is a norm-covering of P and \mathbf{U} is a subset of $\text{Id}_s U$ satisfying the following properties:

- (1) The set $\mathbf{U}^= = \{\mathbf{u} \in \mathbf{U} \mid \partial \mathbf{u} \text{ is not maximal in } P\}$ is lower finite, that is, the set $\mathbf{U} \downarrow \mathbf{u}$ is finite for each $\mathbf{u} \in \mathbf{U}^=$.
- (2) For every map $S: \mathbf{U}^= \rightarrow [U]^{<\omega}$, there exists an isotone map $\sigma: P \rightarrow \mathbf{U}$ such that
 - (a) the map σ is a *section* of ∂ , that is, $\partial \sigma(p) = p$ holds for each $p \in P$;
 - (b) the containment $S(\sigma(p)) \cap \sigma(q) \subseteq \sigma(p)$ holds for all $p < q$ in P . (*Observe that $\sigma(p)$ belongs to $\mathbf{U}^=$.*)

The existence of lifters is related to the following infinite combinatorial statement introduced in [6].

Definition 8.4. For cardinals κ, λ and a poset P , let $(\kappa, <\lambda) \rightsquigarrow P$ hold if for every mapping $F: \mathfrak{P}(\kappa) \rightarrow [\kappa]^{<\lambda}$, there exists a one-to-one map $f: P \hookrightarrow \kappa$ such that

$$F(f(P \downarrow p)) \cap f(P \downarrow q) \subseteq f(P \downarrow p), \quad \text{for all } p \leq q \text{ in } P.$$

Notice that in case P is lower finite, it is sufficient to verify the conclusion above for all $F: [\kappa]^{<\omega} \rightarrow [\kappa]^{<\lambda}$ isotone and all $p \prec q$ in P .

Lemma 8.5. *The square poset has an \aleph_0 -lifter (X, \mathbf{X}) such that $\text{card } X = \aleph_1$.*

Proof. By [6, Proposition 4.7], the Kuratowski index (cf. [6, Definition 4.1]) of the square P is less than or equal to its order-dimension, which is equal to 2. Hence, by the definition of the Kuratowski index, $(\kappa^+, <\kappa) \rightsquigarrow P$ for every infinite cardinal κ . Therefore, by [5, Corollary 3-5.8], P has an \aleph_0 -lifter (X, \mathbf{X}) such that $\text{card } X = \text{card } \mathbf{X} = \aleph_1$. \square

Given a poset P , we introduce a new poset which looks like a lexicographical product of P with a tree. This construction is mainly used in [4].

Definition 8.6. Let P be a poset with a smallest element, let $X \subseteq P$, let $\vec{R} = (R_x)_{x \in X}$ be a family of sets, let $\alpha \leq \omega$. Consider the following poset:

$$T = \{(n, \vec{x}, \vec{r}) \mid n < \alpha, \vec{x} \in X^n, \text{ and } \vec{r} \in R_{x_0} \times \cdots \times R_{x_{n-1}}\},$$

ordered by $(m, \vec{x}, \vec{r}) \leq (n, \vec{y}, \vec{s})$ if and only if $m \leq n$, $\vec{x} = \vec{y} \restriction m$, and $\vec{r} = \vec{s} \restriction m$. Recall that $\vec{y} \restriction m = (y_0, \dots, y_{m-1})$. Given $t = (n, \vec{x}, \vec{r}) \in T$ and $m \leq n$, we set $t \restriction m = (m, \vec{x} \restriction m, \vec{r} \restriction m)$.

Put:

$$A = P \boxtimes_{\vec{R}} \alpha = T \times P = \bigcup_{n < \alpha} \bigcup_{\vec{x} \in X^n} \left(\{n\} \times \{\vec{x}\} \times (R_{x_0} \times \cdots \times R_{x_{n-1}}) \times P \right).$$

Any element of A can be written (n, \vec{x}, \vec{r}, p) with $n < \alpha$, $\vec{x} \in X^n$ and $\vec{r} \in R_{x_0} \times \cdots \times R_{x_{n-1}}$.

We define an order on A by $(m, \vec{x}, \vec{r}, p) \leq (n, \vec{y}, \vec{s}, q)$ if and only if the following conditions hold:

- (1) $(m, \vec{x}, \vec{r}) \leq (n, \vec{y}, \vec{s})$.
- (2) If $m = n$ then $p \leq q$.
- (3) If $m < n$ then $p \leq y_m$.

Remark 8.7. In the context of Definition 8.6, notice that T is a lower finite tree. Indeed, if $(n, \vec{x}, \vec{r}) \in T$, then $T \downarrow (n, \vec{x}, \vec{r}) = \{(m, \vec{x} \restriction m, \vec{r} \restriction m) \mid m \leq n\}$ is a chain of length n . The tree T is called *the tree associated to $P \boxtimes_{\vec{R}} \alpha$* .

The following statements hold:

- (1) If P is lower finite, then A is lower finite.
- (2) The inequality $\text{card } A \leq \aleph_0 + \text{card } P + \sum_{x \in X} \text{card } R_x$ holds.
- (3) The inequality $\text{card } T \leq \aleph_0 + \sum_{x \in X} \text{card } R_x$ holds.

Remark 8.8. In the context of Definition 8.6, if $a < b$ in A , then there are $t = (n, \vec{x}, \vec{r}) \in T$, $p, q \in P$, and $m \leq n$ such that $a = (t \restriction m, p)$ and $b = (t, q)$. Moreover, if $m < n$, then $(t \restriction m, p) < (t \restriction (m+1), 0) \leq (t, q)$. It follows easily that $a \prec b$ if and only if exactly one of the following statements holds:

- (1) $m = n$ and $p \prec q$.
- (2) $n = m + 1$, $p = x_m$, and $q = 0$.

As a consequence, we obtain immediately that *If P , X , and all R_x are finite, then each $a \in A$ has only finitely many covers.*

Lemma 8.9. Let $\kappa \geq \lambda$ be infinite cardinals, let P be a lower finite κ -small poset with a smallest element, let $X \subseteq P$, let $\vec{R} = (R_x)_{x \in X}$ be a family of κ -small sets, and let $\alpha \leq \omega$. If $(\kappa, < \lambda) \rightsquigarrow P$, then $(\kappa, < \lambda) \rightsquigarrow P \boxtimes_{\vec{R}} \alpha$.

Proof. Denote by T the tree associated to $A = P \boxtimes_{\vec{R}} \alpha$. It follows from Remark 8.7 together with the assumptions on cardinalities that the following inequalities hold:

$$\text{card } T \leq \aleph_0 + \sum_{x \in X} \text{card } R_x \leq \aleph_0 + \sum_{x \in X} \kappa \leq \kappa.$$

Thus there exists a partition $(K_t)_{t \in T}$ of κ such that $\text{card } K_t = \kappa$ for each $t \in T$.

Notice that A is lower finite. Let $F: [\kappa]^{<\omega} \rightarrow [\kappa]^{<\lambda}$ isotone, let $t \in T$. Assume having constructed, for each $s < t$, a one-to-one map $\sigma_s: P \hookrightarrow K_s$ such that setting

$$S_s = \{\sigma_s \upharpoonright m(p) \mid m < n \text{ and } p \leq x_m\}, \text{ for all } s = (n, \vec{x}, \vec{r}) \leq t,$$

the following containments hold:

$$\text{rng } \sigma_s \subseteq K_s - F(S_s), \quad \text{for each } s < t,$$

$$F(\sigma_s(P \downarrow p) \cup S_s) \cap \sigma_s(P \downarrow q) \subseteq \sigma_s(P \downarrow p), \quad \text{for all } p \leq q \text{ in } P \text{ and all } s < t.$$

Put $F_t(U) = F(U \cup S_t) - F(S_t)$ for each $U \in [K_t - F(S_t)]^{<\omega}$. As S_t is finite, this defines a map $F_t: [K_t - F(S_t)]^{<\omega} \rightarrow [K_t - F(S_t)]^{<\lambda}$. As $F(S_t)$ is λ -small, $\text{card}(K_t - F(S_t)) = \kappa$, moreover $(\kappa, <\lambda) \rightsquigarrow P$, so there exists a one-to-one map $\sigma_t: P \hookrightarrow K_t - F(S_t)$ such that:

$$F(\sigma_t(P \downarrow p) \cup S_t) \cap \sigma_t(P \downarrow q) \subseteq \sigma_t(P \downarrow p), \quad \text{for all } p \leq q \text{ in } P.$$

Therefore we construct, by induction on t , a one-to-one map $\sigma_t: P \hookrightarrow K_t$ for each $t \in T$, such that setting

$$S_t = \{\sigma_t \upharpoonright m(p) \mid m < n \text{ and } p \leq x_m\}, \text{ for all } t = (n, \vec{x}, \vec{r}) \in T, \quad (8.1)$$

the following containments hold:

$$\text{rng } \sigma_t \subseteq K_t - F(S_t), \quad \text{for each } t \in T, \quad (8.2)$$

$$F(\sigma_t(P \downarrow p) \cup S_t) \cap \sigma_t(P \downarrow q) \subseteq \sigma_t(P \downarrow p), \quad \text{for all } p \leq q \text{ in } P \text{ and all } t \in T. \quad (8.3)$$

For $(t, p) \in A$, set $\sigma(t, p) = \sigma_t(p)$. This defines a map $\sigma: A \rightarrow \kappa$. Let $a = (s, p)$ and $b = (t, q)$ in A such that $\sigma(a) = \sigma(b)$. It follows from (8.2) that $\sigma(a) \in K_s$ and $\sigma(b) \in K_t$. As $(K_u)_{u \in T}$ is a partition of κ , we obtain $s = t$. Moreover $\sigma_t(p) = \sigma(a) = \sigma(b) = \sigma_t(q)$, so, as σ_t is one-to-one, $p = q$, and so $a = b$. Therefore σ is one-to-one.

Let $a = (t, p) \in A$, with $t = (n, \vec{x}, \vec{r}) \in T$. It follows from the definition of A that:

$$A \downarrow a = \{(t \upharpoonright m, q) \in A \mid m < n \text{ and } q \in P \downarrow x_m\} \cup \{(t, q) \in A \mid q \in P \downarrow p\}.$$

Thus, from (8.1), we see that

$$\sigma(A \downarrow a) = S_t \cup \sigma_t(P \downarrow p), \quad \text{for each } a = (t, p) \in A. \quad (8.4)$$

As $(K_t)_{t \in T}$ is a partition, it follows from (8.1) and (8.2) that $K_t \cap S_t = \emptyset$, thus, by (8.4),

$$\sigma(A \downarrow a) \cap K_t = \sigma_t(P \downarrow p), \quad \text{for each } a = (t, p) \in A. \quad (8.5)$$

Let $a < b$ in A . There are two cases to consider (cf. Remark 8.8). First assume that $a = (t, p)$ and $b = (t, q)$ with $p < q$ and $t = (n, \vec{x}, \vec{r}) \in T$. Let $c \leq b$ with $\sigma(c) \in F(\sigma(A \downarrow a))$. We can write $c = (t \upharpoonright m, p')$, with $m \leq n$. Suppose first that $m < n$. As $c \leq b$, $p' \leq x_m$, thus $c < a$. So $\sigma(c) \in \sigma(A \downarrow a)$. Now suppose that $m = n$. It follows from (8.2) that

$$\sigma(c) = \sigma_t(p') \in F(\sigma(A \downarrow a)) \cap \sigma(A \downarrow b) \cap (K_t - F(S_t)).$$

So, from (8.4) and (8.5) we obtain

$$\sigma(c) \in F(\sigma_t(P \downarrow p) \cup S_t) \cap \sigma_t(P \downarrow q).$$

Thus (8.3) implies that $\sigma(c) \in \sigma_t(P \downarrow p)$, from (8.4) we obtain $\sigma(c) \in \sigma(A \downarrow a)$. Therefore the containment $F(\sigma(A \downarrow a)) \cap \sigma(A \downarrow b) \subseteq \sigma(A \downarrow a)$ holds.

Now assume that $t = (n+1, \vec{x}, \vec{r}) \in T$, $b = (t, 0)$ and $a = (t \upharpoonright n, x_n)$. Let $c \leq b$ such that $\sigma(c) \in F(\sigma(A \downarrow a))$. As $c \leq b$ there are $m \leq n+1$ and $p \in P$ such that $c = (t \upharpoonright m, p)$. If $m \leq n$, then $p \leq x_m$, thus $c \leq a$, so $\sigma(c) \in \sigma(A \downarrow a)$. If $m = n+1$, then $c = b$. From (8.1) and (8.4) we obtain $\sigma(A \downarrow a) = S_{t \upharpoonright n} \cup \sigma_{t \upharpoonright n}(P \downarrow x_n) = S_t$. Therefore $\sigma_t(0) = \sigma(c) \in F(S_t)$, in contradiction with (8.2). So the containment $F(\sigma(A \downarrow a)) \cap \sigma(A \downarrow b) \subseteq \sigma(A \downarrow a)$ holds. \square

Corollary 8.10. *For an integer $m > 1$, put*

$$B_m(\leq 2) = \{X \in \mathfrak{P}(m) \mid \text{either } \text{card } X \leq 2 \text{ or } X = m\}.$$

Let P be a $(\vee, 0)$ -semilattice embeddable, as a poset, into $B_m(\leq 2)$. Let $X \subseteq P$, let $\vec{R} = (R_x)_{x \in X}$ be a family of finite sets, and let $\alpha \leq \omega$. There exists an \aleph_0 -lifter (U, \mathbf{U}) of $A = P \boxtimes_{\vec{R}} \alpha$ such that U has cardinality \aleph_2 .

Proof. It follows from [8], see also [2, Theorem 46.2], that $(\aleph_2, 2, \aleph_0) \rightarrow m$, so [6, Proposition 5.2] implies $(\aleph_2, < \aleph_0) \rightsquigarrow B_m(\leq 2)$, and so, from [6, Lemma 3.2] we obtain $(\aleph_2, < \aleph_0) \rightsquigarrow P$. It follows from Lemma 8.9 that $(\aleph_2, < \aleph_0) \rightsquigarrow A$. The conclusion follows from Remark 8.8 together with [5, Lemma 3-5.5]. \square

9. THE CONDENSATE LIFTING LEMMA FOR GAMPS

In this section we apply the Armature Lemma from [5] together with Lemma 7.8 to prove a special case of the Condensate Lifting Lemma for gamps.

In order to use the *condensate* constructions, we need categories and functors that satisfy the following conditions.

Definition 9.1. Let \mathcal{A} and \mathcal{S} be categories, let $\Phi: \mathcal{A} \rightarrow \mathcal{S}$ be a functor. We introduce the following statements:

- (CLOS) \mathcal{A} has all small directed colimits.
- (PROD) Any two objects of \mathcal{A} have a product in \mathcal{A} .
- (CONT) The functor Φ preserves all small directed colimits.

Remark 9.2. Given a norm-covering X of a poset P and a category \mathcal{A} that satisfies both (CLOS) and (PROD), we can construct an object $\mathbf{F}(X) \otimes \vec{A}$ which is a directed colimit of finite products of objects in \vec{A} , together with morphisms

$$\pi_{\vec{x}}^X \otimes \vec{A}: \mathbf{F}(X) \otimes \vec{A} \rightarrow A_{\partial \vec{x}}$$

for each $\vec{x} \in \text{Id}_{\mathcal{S}} X$.

Moreover if \mathcal{A} is a class of algebras closed under finite products and directed colimits, then $\pi_{\vec{x}}^X \otimes \vec{A}$ is surjective, and $\text{card}(\mathbf{F}(X) \otimes \vec{A}) \leq \text{card } X + \sum_{p \in P} A_p$. For more details about this construction, we refer the reader to [5, Chapter 2].

In the following theorem, we refer the reader to Definition 6.14 for the definition of a partial lifting.

Theorem 9.3. *Let \mathcal{V} and \mathcal{W} be varieties of algebras such that \mathcal{W} has finite similarity type, let (X, \mathbf{X}) be an \aleph_0 -lifter of a poset P , let $\vec{A} = (A_p, f_{p,q} \mid p \leq q \text{ in } P)$ be a diagram in \mathcal{V} such that $\text{Conc } A_p$ is finite for each $p \in P^=$, let $B \in \mathcal{W}$ such that $\text{Conc } B \cong \text{Conc } (\mathbf{F}(X) \otimes \vec{A})$. Then there exists a partial lifting $\vec{B} = (B_p, g_{p,q} \mid p \leq q \text{ in } P)$ of $\text{Conc } \circ \vec{A}$ in \mathcal{W} such that B_p is finite for each $p \in P^=$ and B_p is a quotient of $G(B)$ for each $p \in \text{Max } P$.*

Proof. Denote by \mathcal{S} the category of all $(\vee, 0)$ -semilattices with $(\vee, 0)$ -homomorphisms. The category \mathcal{V} satisfies (CLOS) and (PROD), so $\mathbf{F}(X) \otimes \vec{A}$ is well-defined (cf. [5, Section 3-1]). The functor Conc satisfies (CONT). Let $\chi: \text{Conc } B \rightarrow \text{Conc}(\mathbf{F}(X) \otimes \vec{A})$ be an isomorphism and put $\rho_x = \text{Conc}(\pi_x^X \otimes \vec{A}) \circ \chi$ for each $x \in X$. As $\pi_x^X \otimes \vec{A}$ is surjective and χ is an isomorphism, it follows that ρ_x is ideal-induced. Notice that $\text{rng } \rho_x = \text{Conc } A_{\partial x}$ is finite.

Lemma 7.8 implies that there exists a diagram $\vec{B} = (B_x, g_{x,y} \mid x \leq y \text{ in } X^-)$ of finite subgamps of $G(B)$ in \mathcal{W} such that the following statements hold:

- (1) $\rho_x \upharpoonright \vec{B}_x$ is ideal-induced for each $x \in X^-$.
- (2) B_x is strong, distance-generated through $\rho_x \upharpoonright \vec{B}_x$, and congruence-tractable through $\rho_x \upharpoonright \vec{B}_x$, for each $x \in X^-$.
- (3) $g_{x,y}$ is the canonical embedding, for all $x \leq y$ in X^- .
- (4) $g_{x,y}$ is strong and congruence-cutttable through $\rho_y \upharpoonright \vec{B}_y$, for all $x < y$ in X^- .

We extend this diagram to an X -indexed diagram, with $B_y = G(B)$ and defining $g_{x,y}$ as the canonical embedding for each $y \in X - X^-$ and each $x \leq y$. Thus $C \circ B$ is an X -indexed diagram in the comma category $\mathcal{S} \downarrow \text{Conc } B$, moreover $C(B_x) = \vec{B}_x$ is finite for each $x \in X^-$. Therefore, it follows from the Armature Lemma [5] that there exists an isotone section $\sigma: P \hookrightarrow X$ such that the family $(\rho_{\sigma(p)} \upharpoonright \vec{B}_{\sigma(p)})_{p \in P}$ is a natural transformation from $(C(B_{\sigma(p)}), C(g_{\sigma(p), \sigma(q)})) \mid p \leq q \text{ in } P$ to $\text{Conc } \vec{A}$.

Put $\vec{B}' = (B_{\sigma(p)}, g_{\sigma(p), \sigma(q)} \mid p \leq q \text{ in } P)$ and $I_p = \ker_0 \rho_{\sigma(p)}$, for each $p \in P$. This defines an ideal $\vec{I} = (I_p)_{p \in P}$ of \vec{B}' . Moreover, as $\rho_{\sigma(p)} \upharpoonright \vec{B}_{\sigma(p)}$ is ideal-induced for each $p \in P$, these morphisms induce a natural equivalence $(C \circ \vec{B}') / \vec{I} \cong \text{Conc } \vec{A}$ (cf. Lemma 3.13).

Denote by $h_{p,q}: B_{\sigma(p)} / I_p \rightarrow B_{\sigma(q)} / I_q$ the morphism induced by $g_{\sigma(p), \sigma(q)}$. It follows from Proposition 6.5 that $B_{\sigma(p)} / I_p$ is strong for each $p \in P$, and it follows from Proposition 6.7 that $h_{p,q}$ is strong for all $p < q$ in P . Lemma 7.3 implies that $B_{\sigma(p)} / I_p$ is distance-generated and congruence-tractable for each $p \in P$. From Lemma 7.4 we obtain that $h_{p,q}$ is congruence-cutttable for all $p < q$ in P .

Let $p \in P$, let $\chi: \vec{B}_{\sigma(p)} \rightarrow \text{Conc } B$ be the inclusion map. As $B_{\sigma(p)}$ is a subgamp of $G(B)$, (B, χ) is a realization of $B_{\sigma(p)}$, thus it induces a realization $(B / \bigvee I_p, \chi')$ where $\chi': \vec{B}_p / I_p \rightarrow B / \bigvee I$ satisfies $\chi'(d / I_p) = d / \bigvee I_p$ (cf. Proposition 6.5). As $\rho_{\sigma(p)} \upharpoonright \vec{B}_{\sigma(p)}$ is ideal-induced it is easy to check that χ' is surjective, hence it defines an isomorphic realization.

Let p a maximal element of P . From $B_{\sigma(p)} = G(B)$ it follows that $B_{\sigma(p)} / I_p$ is a quotient of $G(B)$.

Therefore \vec{B}' / \vec{I} is a partial lifting of $\text{Conc } \vec{A}$ in \mathcal{W} . Moreover, if $p \in P^-$ then $B_{\sigma(p)}$ is finite, thus $B'_p / I_p = B_{\sigma(p)} / I_p$ is finite. \square

Remark 9.4. Use the notation of Theorem 9.3. A small modification of the proof above shows that if \mathcal{W} is a variety of lattices, then we can construct a lattice partial lifting \vec{B} of $\text{Conc } \vec{A}$ in \mathcal{W} . Moreover, for any integer $n \geq 2$, if B is congruence n -permutable, then all B_p can be chosen congruence n -permutable.

Corollary 9.5. *Let \mathcal{V} be a locally finite variety of algebras, let \mathcal{W} be a variety of algebras with finite similarity type. The following statements are equivalent:*

- (1) $\text{crit}(\mathcal{V}; \mathcal{W}) > \aleph_0$.

- (2) Let T be a countable lower finite tree and let \vec{A} be a T -indexed diagram of finite algebras in \mathcal{V} . Then $\text{Conc} \circ \vec{A}$ has a partial lifting in \mathcal{W} .
- (3) Let \vec{A} be a ω -indexed diagram of finite algebras in \mathcal{V} . Then $\text{Conc} \circ \vec{A}$ has a partial lifting in \mathcal{W} .

Proof. Assume that (1) holds. Let T be a countable lower finite tree and let \vec{A} be a T -indexed diagram of finite algebras in \mathcal{V} . It follows from [3, Corollary 4.7] that there exists an \aleph_0 -lifter (X, \mathbf{X}) of T such that $\text{card } X = \aleph_0$. Hence the following inequalities hold:

$$\text{card}(\mathbf{F}(X) \otimes \vec{A}) \leq \text{card } X + \sum_{p \in T} A_p \leq \aleph_0 \sum_{p \in T} \aleph_0 = \aleph_0.$$

Thus, as $\text{crit}(\mathcal{V}; \mathcal{W}) > \aleph_0$, there exists $B \in \mathcal{W}$ such that $\text{Conc } B \cong \text{Conc}(\mathbf{F}(X) \otimes \vec{A})$. It follows from Theorem 9.3 that there exists a partial lifting of $\text{Conc} \circ \vec{A}$ in \mathcal{W} .

The implication (2) \implies (3) is immediate.

Assume that (3) holds and let $A \in \mathcal{V}$ such that $\text{card } \text{Conc } A \leq \aleph_0$. By replacing A with one of its subalgebras we can assume that $\text{card } A \leq \aleph_0$ (see [3, Lemma 3.6]). As \mathcal{V} is locally finite, there exists an increasing sequence $(A_k)_{k < \omega}$ of finite subalgebras of A with union A . Denote by $f_{i,j}: A_i \rightarrow A_j$ the inclusion map, for all $i \leq j < \omega$. Put $\vec{A} = (A_i, f_{i,j} \mid i \leq j < \omega)$. Let \vec{B} be a partial lifting of $\text{Conc} \circ \vec{A}$ in \mathcal{W} , let \mathbf{B} be the directed colimit of \vec{B} in $\mathbf{Gamp}(\mathcal{V})$. As \mathbf{C} and Conc both preserve directed colimits, the following gamps are isomorphic:

$$\mathbf{C}(\mathbf{B}) \cong \mathbf{C}(\varinjlim \vec{B}) \cong \varinjlim (\mathbf{C} \circ \vec{B}) \cong \varinjlim (\text{Conc} \circ \vec{A}) \cong \text{Conc } \varinjlim \vec{A} \cong \text{Conc } A.$$

Moreover, it follows from Lemma 6.17 that \mathbf{B} is an algebra in \mathcal{W} , that is, there exists $B \in \mathcal{W}$ such that $\mathbf{B} \cong \mathbf{G}(B)$, so $\text{Conc } B = \mathbf{C}(\mathbf{G}(B)) \cong \mathbf{C}(\mathbf{B}) \cong \text{Conc } A$. Therefore $\text{crit}(\mathcal{V}; \mathcal{W}) > \aleph_0$. \square

A variety of algebras is *congruence-proper* if each of its member with a finite congruence lattice is finite (cf. [5, Definition 4-8.1]).

The following theorem is similar to Theorem 9.3 if \mathcal{W} is a congruence-proper variety with a finite similarity type, then we no longer need partial liftings in the statement of the theorem. There is a similar theorem in [5, Theorem 4-9.2]. This new version applies only to varieties (not quasivarieties), but the assumption that \mathcal{W} is locally finite is no longer needed.

Theorem 9.6. *Let \mathcal{V} be a variety of algebras. Let \mathcal{W} be a congruence-proper variety of algebras with a finite similarity type. Let (X, \mathbf{X}) be an \aleph_0 -lifter of a poset P . Let $\vec{A} = (A_p, f_{p,q} \mid p \leq q \text{ in } P)$ be a diagram in \mathcal{V} such that $\text{Conc } A_p$ is finite for each $p \in P$. Let $B \in \mathcal{W}$ such that $\text{Conc } B \cong \text{Conc}(\mathbf{F}(X) \otimes \vec{A})$. Then there exists a lifting \vec{B} of $\text{Conc} \circ \vec{A}$ in \mathcal{W} such that B_p is a quotient of B for each $p \in P$.*

Proof. We notice, as in the beginning of the proof of Theorem 9.3, that $\mathbf{F}(X) \otimes \vec{A}$ is well-defined. We also uses the same notations. Let $\chi: \text{Conc } B \rightarrow \text{Conc}(\mathbf{F}(X) \otimes \vec{A})$ be an isomorphism. Put $\rho_x = \text{Conc}(\pi_x^X \otimes \vec{A}) \circ \chi$ for each $x \in \mathbf{X}$. As $\pi_x^X \otimes \vec{A}$ is surjective and χ is an isomorphism, it follows that ρ_x is ideal-induced.

Notice that $\text{rng } \rho_x = \text{Conc } A_{\partial x}$ is finite. Hence $\text{Conc } B / \ker_0 \rho_x \cong \text{Conc } A_{\partial x}$ is finite. As \mathcal{W} is congruence-proper, it follows that $B / \ker_0 \rho_x$ is finite.

Hence, the property $\mathbf{A} / \ker_0 \rho_x = \mathbf{G}(B) / \ker_0 \rho_x$, in \mathbf{A} a subgamp of $\mathbf{G}(B)$ is a locally finite property for B . It follows from Lemma 7.8 and Remark 7.9 that there

exists a diagram $\vec{B} = (B_x, g_{x,y} \mid x \leq y \text{ in } X^-)$ of finite subgamps of $G(B)$ in \mathcal{W} , as in the proof of Theorem 9.3, but that satisfies the additional condition:

$$(5) \quad B_x / \ker_0 \rho_x = G(B) / \ker_0 \rho_x, \text{ for all } x \in X^-.$$

We continue with the same argument as the one in the proof of Theorem 9.3. We obtain B'/\vec{I} a partial lifting of $\text{Con}_c \circ \vec{A}$. The following isomorphisms hold

$$\begin{aligned} B'_p / I_p &= B_{\sigma(p)} / \ker_0 \rho_{\sigma(p)} && \text{see proof of Theorem 9.3.} \\ &= G(B) / \ker_0 \rho_{\sigma(p)} && \text{by (5).} \\ &\cong G(B / \ker_0 \rho_{\sigma(p)}) && \text{by Remark 6.9(6).} \end{aligned}$$

Hence B'_p / I_p is an algebra of \mathcal{V} (cf. Remark 6.9(3)), for all $p \in P^-$, it is also true for $p \in \text{Max } P$. It follows from Remark 6.9(5) that $\vec{B}' / \vec{I} \cong G \circ \vec{B}$, for a diagram of algebras \vec{B} . Hence:

$$\begin{aligned} \text{Con}_c \circ \vec{B} &= C \circ G \circ \vec{B} && \text{by Remark 6.9(1).} \\ &\cong C \circ \vec{B}' / \vec{I} \\ &\cong \text{Con}_c \circ \vec{A} && \text{as } \vec{B}' / \vec{I} \text{ is a partial lifting of } \text{Con}_c \circ \vec{A}. \end{aligned}$$

Hence \vec{B} is a lifting of $\text{Con}_c \circ \vec{A}$. □

The following corollary is an immediate application of Theorem 9.6.

Corollary 9.7. *Let \mathcal{V} be a variety of algebras. Let \mathcal{W} be a congruence-proper variety of algebras with a finite similarity type. Let (X, X) be an \aleph_0 -lifter of a poset P . Let $\vec{A} = (A_p, f_{p,q} \mid p \leq q \text{ in } P)$ be a diagram of finite algebras in \mathcal{V} . Assume that $\text{Con}_c \circ \vec{A}$ has no lifting in \mathcal{W} , then $\text{crit}(\mathcal{V}; \mathcal{W}) \leq \aleph_0 + \text{card } X$.*

10. AN UNLIFTABLE DIAGRAM

Each countable locally finite lattice has a congruence-permutable, congruence-preserving extension (cf. [7]). This is not true for all locally finite lattices. Given a non-distributive variety \mathcal{V} of lattices, there is no congruence-permutable algebra A such that $\text{Con}_c A \cong \text{Con}_c F_{\mathcal{V}}(\aleph_2)$ (cf. [14]). Hence $\text{Con}_c F_{\mathcal{V}}(\aleph_2)$ has no congruence-permutable, congruence-preserving extension. The latter result can be improved, by stating that the free lattice $F_{\mathcal{V}}(\aleph_1)$ has no congruence-permutable, congruence-preserving extension (cf. [5]).

Let \mathcal{V} be a nondistributive variety of lattices. There is no congruence n -permutable lattices L such that $\text{Con}_c L \cong F_{\mathcal{V}}(\aleph_2)$, for each $n \geq 2$ (cf. [12]). In particular $F_{\mathcal{V}}(\aleph_2)$ has no congruence n -permutable, congruence-preserving extension, for each $n \geq 2$. The aim of this section is to improve the cardinality bound to \aleph_1 . We use gamps to find a lattice of cardinal \aleph_1 with no congruence n -permutable, congruence-preserving extension, for each $n \geq 2$. This partially solve [5, Problem 7]. The proof is based on a square-indexed diagram of lattices with no congruence n -permutable, congruence-preserving extension.

Let $\vec{A} = (A_p, f_{p,q} \mid p \leq q \text{ in } P)$ be a diagram of algebras, let $\vec{B} = (B_p, g_{p,q} \mid p \leq q \text{ in } P)$ be a congruence-preserving extension of \vec{A} . Then $B_p = (A_p, B_p, \Theta_{B_p}, \text{Con}_c B_p)$ is a gamp and $g_{p,q} = (g_{p,q}, \text{Con}_c g_{p,q}): B_p \rightarrow B_q$ is a morphism of gamps, for all $p \leq q$ in P . This defines a diagram \vec{B} of gamps. Moreover, identifying $\text{Con}_c A_p$ and $\text{Con}_c B_p$ for all $p \in P$, we have $P_{\text{gl}} \circ \vec{B} = P_{\text{ga}} \circ \vec{A}$.

Fix $n \geq 2$ an integer. Given a diagram \vec{A} of algebras, with a congruence n -permutable, congruence-preserving extension, there exists a diagram \vec{B} of congruence n -permutable gamps such that $\mathbf{P}_{\text{gl}} \circ \vec{B} = \mathbf{P}_{\text{ga}} \circ \vec{A}$. The converse might not hold in general.

However the square-indexed diagram \vec{A} of finite lattices with no congruence n -permutable, congruence-preserving extension, mentioned above, satisfies a stronger property. There is no operational diagram \vec{B} of lattice congruence n -permutable gamps of lattices (cf. Definition 10.1) such that $\mathbf{P}_{\text{ga}} \circ \vec{A} \cong \mathbf{P}_{\text{gl}} \circ \vec{B}$ (cf. Lemma 10.6).

We conclude, in Theorem 10.7, that there is a condensate of \vec{A} , of cardinal \aleph_1 , with no congruence n -permutable, congruence-preserving extension.

For the purpose of this section, we need a stronger version of congruence n -permutable gamp, specific to gamp of lattices.

Definition 10.1. A gamp \mathbf{A} of lattices is *lattice congruence n -permutable* if for all x_0, \dots, x_n in A^* there exist y_0, \dots, y_n in A such that $y_i \wedge y_j = y_j \wedge y_i = y_i$ in A for all $i \leq j \leq n$, $y_0 = x_0 \wedge x_n = x_n \wedge x_0$ in A , $y_n = x_0 \vee x_n = x_n \vee x_0$ in A , and:

$$\delta(y_k, y_{k+1}) \leq \bigvee (\delta(x_i, x_{i+1}) \mid i < n \text{ even}), \quad \text{for all } k < n \text{ odd},$$

$$\delta(y_k, y_{k+1}) \leq \bigvee (\delta(x_i, x_{i+1}) \mid i < n \text{ odd}), \quad \text{for all } k < n \text{ even}.$$

A morphism $\mathbf{f}: \mathbf{A} \rightarrow \mathbf{B}$ of gamps is *operational* if $\ell(\vec{x})$ is defined in B for all $\ell \in \mathcal{L}$ and all $\text{ar}(\ell)$ -tuple \vec{x} in $f(A) \cup B^*$.

Let P be a poset, a diagram $\vec{\mathbf{A}} = (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P)$ of gamps is *operational* if $\mathbf{f}_{p,q}$ is operational for all $p < q$ in P .

Remark 10.2. In the context of Definition 10.1, the elements y_1, \dots, y_n do not form a chain in general, as we might have $y_i \not\leq A^*$ for some i .

In this section we simply say that the gamp is congruence n -permutable instead of lattice congruence n -permutable. We also consider only pregamps and gamps of lattices. The following lemma is immediate, the properties given in Definition 10.1 go to quotients.

Lemma 10.3. Let \mathbf{A} be a congruence n -permutable gamp, let I be an ideal of \mathbf{A} , then \mathbf{A}/I is a congruence n -permutable gamp.

Let P be a poset, let $\vec{\mathbf{A}}$ be an operational P -indexed diagram of algebras. Let \vec{I} be an ideal of $\vec{\mathbf{A}}$. The quotient \mathbf{A}/\vec{I} is operational.

The following lemma is similar to the Buttress Lemma (cf. [5]) and to Lemma 7.8, this version is specific to the functor \mathbf{P}_{gl} .

Lemma 10.4. Let $n \geq 2$. Let \mathcal{V} be a variety of lattices. Let P be a lower finite poset. Let $(\mathbf{A}_p)_{p \in P}$ be a family of finite pregamps. Let \mathbf{B} be a gamp in \mathcal{V} such that B is a congruence n -permutable lattice. Let $(\pi_p)_{p \in P}$ be a family of ideal-induced morphisms of pregamps where $\pi_p: \mathbf{P}_{\text{gl}} \mathbf{B} \rightarrow \mathbf{A}_p$ for all $p \in P$. Then there exists a diagram $\vec{\mathbf{B}} = (\mathbf{B}_p, \mathbf{g}_{p,q} \mid p \leq q \text{ in } P)$ of finite subgamps of \mathbf{B} (where $\mathbf{g}_{p,q}$ is the canonical embedding for all $p \leq q$ in P), such that the following assertions hold.

- (1) $\pi_p \upharpoonright \mathbf{P}_{\text{gl}}(\mathbf{B}_p)$ is an ideal-induced morphism of pregamps for all $p \in P$.
- (2) \mathbf{B}_p is strong and congruence n -permutable for all $p \in P$.
- (3) $\mathbf{g}_{p,q}$ is operational for all $p < q$ in P .

Proof. Let $r \in P$, assume that we have already constructed $(B_p, g_{p,q} \mid p \leq q < r)$ that satisfies the required conditions up to r .

As π_r is ideal-induced, $\pi_r(B^*) = A_r$. Moreover A_r is finite, hence there exists X a finite partial subalgebra of B^* such that $\pi_r(X) = A_r$, put $B_r^* = X \cup \bigcup_{p < r} B_p^*$ with its structure of full partial subalgebra of B^* . It follows that $\pi_r(B_r^*) \subseteq \pi_r(B^*) = A_r = \pi_r(X) \subseteq \pi_r(B_r^*)$, hence $\pi_r(B_r^*) = A_r$. Moreover B_r^* is finite.

Let x_0, x_1, \dots, x_n in B_r^* . As B is a congruence n -permutable lattice, there exist $y_0 < \dots < y_n$ in B such that $y_0 = x_0 \wedge x_n$, $y_n = x_0 \vee x_n$, and

$$\delta(y_k, y_{k+1}) \leq \bigvee (\delta(x_i, x_{i+1}) \mid i < n \text{ even}), \quad \text{for all } k < n \text{ odd},$$

$$\delta(y_k, y_{k+1}) \leq \bigvee (\delta(x_i, x_{i+1}) \mid i < n \text{ odd}), \quad \text{for all } k < n \text{ even}.$$

Put $X_{x_0, x_1, \dots, x_n} = \{y_0, \dots, y_n\}$. We consider the following finite set with its structure of full partial subalgebra of B

$$\begin{aligned} B_r = & \bigcup (X_{x_0, x_1, \dots, x_n} \mid x_0, x_1, \dots, x_n \in B_p^*) \\ & \cup \left\{ x \vee y \mid x, y \in B_r^* \cup \bigcup_{p < r} B_p \right\} \\ & \cup \left\{ x \wedge y \mid x, y \in B_r^* \cup \bigcup_{p < r} B_p \right\}. \end{aligned}$$

Put $Y = \{\delta(x, y) \mid x, y \in B_r\}$. As $\tilde{\pi}_r$ is ideal-induced, it follows from Proposition 3.16 that there is a finite $(\vee, 0)$ -subsemilattice \tilde{B}_r of \tilde{B} such that $Y \subseteq \tilde{B}_r$, and $\tilde{\pi}_r \upharpoonright \tilde{B}_r$ is ideal-induced. Put $\mathbf{B}_r = (B_r^*, B_r, \delta, \tilde{B}_r)$, denote by $g_{p,r}: B_p \rightarrow \mathbf{B}_r$ the inclusion morphism for all $p \leq r$. The conditions (1)-(3) are satisfied. The conclusion follows by induction. \square

We apply Lemma 10.4 and the Armature Lemma (cf. [5]) to obtain a new, tailor-made version of CLL. Given a diagram \vec{A} of finite lattices and a congruence-preserving, congruence n -permutable extension of a condensate of \vec{A} , we obtain a “congruence-preserving, congruence n -permutable extension” of \vec{A} .

Lemma 10.5. *Let \mathcal{V} be a variety of lattices. Let (X, \mathbf{X}) be an \aleph_0 -lifter of a poset P , let $\vec{A} = (A_p, f_{p,q})$ be a diagram of \mathcal{V} such that $\text{Con}_c A_p$ is finite for all $p \in P^\perp$. Let B be a congruence n -permutable lattice. If B is a congruence-preserving extension of $\mathbf{F}(X) \otimes \vec{A}$, then there exists an operational diagram \vec{B} of congruence n -permutable gamps such that $\mathbf{P}_{\text{gl}} \circ \vec{B} \cong \mathbf{P}_{\text{ga}} \circ \vec{A}$.*

Proof. As in the proof of Theorem 9.3, $\mathbf{F}(X) \otimes \vec{A}$ is well-defined. Denote by \mathcal{S} the category of pregamps. The functor $\mathbf{P}_{\text{ga}}: \mathcal{V} \rightarrow \mathcal{S}$ satisfies (CONT), see Remark 5.7. Put $B^* = \mathbf{F}(X) \otimes \vec{A}$, as B is a congruence-preserving extension of B^* , we can identify $\text{Con}_c B^*$ with $\text{Con}_c B$. Put $\mathbf{B} = (B^*, B, \Theta_B, \text{Con}_c B)$, hence $\mathbf{P}_{\text{gl}}(\mathbf{B}) = \mathbf{P}_{\text{ga}}(B^*)$. Put $\pi_x = \mathbf{P}_{\text{ga}}(\pi_x^X \otimes \vec{A}): \mathbf{P}_{\text{gl}}(\mathbf{B}) \rightarrow \mathbf{P}_{\text{ga}}(\mathbf{A})$ for all $x \in \mathbf{X}$. It follows from Lemma 10.4, that there exists a diagram $(B_x, g_{x,y} \mid x \leq y \text{ in } \mathbf{X}^\perp)$ of finite subgamps of \mathbf{B} such that the following assertions hold.

- (1) $\pi_x \upharpoonright \mathbf{P}_{\text{gl}}(B_x)$ is an ideal-induced morphism of pregamps for all $x \in \mathbf{X}^\perp$.
- (2) B_x is strong and congruence n -permutable for all $x \in \mathbf{X}^\perp$.
- (3) $g_{x,y}$ is operational for all $x < y$ in \mathbf{X}^\perp .

We complete the diagram with $B_y = B$ and $g_{x,y}$ the inclusion morphism for all $y \in X - X^\perp$, and $x \leq y$. It follows from the Armature Lemma (cf. [5, Lemma 3-2.2]) that there exists $\sigma: P \rightarrow X$ such that $\partial\sigma(p) = p$ for all $p \in P$ and $(\pi_{\sigma(p)} \upharpoonright P_{\text{gl}}(B_{\sigma(p)}))_{p \in P}$ is a natural transformation from $(P_{\text{gl}}(B_{\sigma(p)}), P_{\text{gl}}(g_{\sigma(p),\sigma(q)}) \mid p \leq q \text{ in } P)$ to $P_{\text{ga}} \circ \vec{A}$.

Put $\rho_p = \pi_{\sigma(p)} \upharpoonright P_{\text{gl}}(B_{\sigma(p)})$, put $I_p = \ker_0 \rho_p$ for all $p \in P$. Put $\vec{I} = (I_p)_{p \in P}$, it is an ideal of $(B_{\sigma(p)}, g_{\sigma(p),\sigma(q)} \mid p \leq q \text{ in } P)$. Put $\vec{C} = (B_{\sigma(p)}/I_p, g_{\sigma(p),\sigma(q)}/\vec{I} \mid p \leq q \text{ in } P)$. Notice that \vec{C} is an operational diagram of congruence n -permutable gamps.

Denote by $\chi_p: P_{\text{gl}}(B_{\sigma(p)})/I_p \rightarrow P_{\text{ga}}(A_p)$ the morphism induced by ρ_p . It follows from Lemma 5.28 that χ_p is an isomorphism, for all $p \in P$. Hence, from Proposition 5.26 and Remark 6.13, we obtain that $\vec{\chi} = (\chi_p)_{p \in P}: P_{\text{gl}} \circ \vec{C} \rightarrow P_{\text{ga}} \circ \vec{A}$ is a natural equivalence. \square

In the following lemma we construct a square-indexed diagram of lattices with no congruence n -permutable, congruence-preserving extension.

Lemma 10.6. *Let $n \geq 2$. Let K be a nontrivial, finite, congruence $(n+1)$ -permutable lattice, let x_1, x_2, x_3 in K such that $x_1 \wedge x_2 = 0$ and $x_3 \vee x_2 = x_3 \vee x_1 = 1$. There exists a diagram \vec{A} of finite congruence $(n+1)$ -permutable lattices in $\mathbf{Var}^{0,1}(K)$ indexed by a square, such that $P_{\text{ga}} \circ \vec{A} \not\cong P_{\text{gl}} \circ \vec{B}$ for each operational square \vec{B} of congruence n -permutable gamps.*

Proof. Put $X_0 = \{0, x_3, 1\}$, put $X_1 = \{0, x_1 \wedge x_3, x_1, x_3, 1\}$, put $X_2 = \{0, x_2 \wedge x_3, x_2, x_3, 1\}$, put $X_3 = K$. Notice that X_k , $k < 4$, are all congruence $(n+1)$ -permutable sublattices of K .

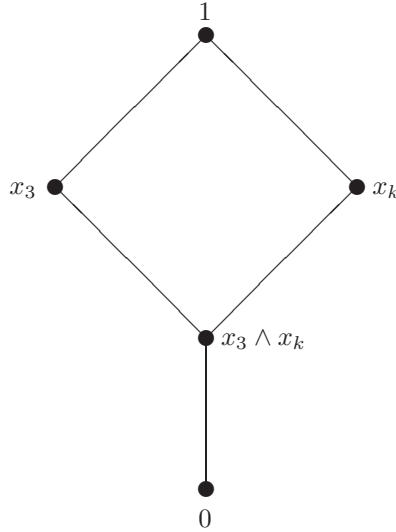


FIGURE 1. The lattice X_k , for $k \in \{1, 2\}$

We denote by $h_i: X_0 \rightarrow X_i$ and $h'_i: X_i \rightarrow X_3$ the inclusion maps, for $i = 1, 2$. Denote by \vec{X} the square on the right hand side of Figure 2.

Claim. Let \vec{B} be a square of operational gamps (as in Figure 3), let $\vec{\xi}: P_{\text{ga}} \circ \vec{X} \rightarrow P_{\text{gl}} \circ \vec{B}$ be a natural equivalence. Let $y \in B_0$ such that $\delta_{B_0}(\xi_0(0), y) \leq \delta_{B_0}(\xi_0(x_3), \xi_0(1))$, $y \wedge \xi_0(1) = y$, and $y \vee \xi_0(0) = y$ in B_0 , then $y = \xi_0(0)$.

Proof of Claim. We can assume that g_1, g_2, g'_1 , and g'_2 are inclusion maps. We can assume that $P_{\text{gl}} \circ \vec{B} = P_{\text{ga}} \circ \vec{X}$ and $\vec{\xi}$ is the identity. We denote δ_k instead of δ_{B_k} for all $k \in \{0, 1, 2, 3\}$. Notice that $\delta_k(u, v)$ is a congruence of X_k for all $u, v \in B_k$, moreover if $u, v \in X_k = B_k^*$ then $\delta_k(u, v) = \Theta_{X_k}(u, v)$. Let $y \in B_0$ such that $\delta_0(0, y) \subseteq \delta_0(x_3, 1)$, $y \wedge 1 = y$ and $y \vee 0 = y$ in B_0 .

Let $k \in \{1, 2\}$. As g_k is operational, $y \wedge x_k$ is defined in B_k . Moreover $0 \wedge x_k = 0$, hence $\delta_k(0, y \wedge x_k) \subseteq \delta_k(0, y)$. Therefore the following containments hold:

$$\delta_k(y, y \wedge x_k) \subseteq \delta_k(y, 0) \vee \delta_k(0, y \wedge x_k) \subseteq \delta_k(0, y) \subseteq \delta_k(x_3, 1) = \Theta_{X_k}(x_3, 1). \quad (10.1)$$

Moreover, as $y \wedge 1 = y$, the following containment holds

$$\delta_k(y, y \wedge x_k) = \delta_k(y \wedge 1, y \wedge x_k) \subseteq \delta_k(1, x_k) = \Theta_{X_k}(1, x_k) \quad (10.2)$$

However $\Theta_{X_k}(1, x_k) \cap \Theta_{X_k}(x_3, 1) = \mathbf{0}_{X_k}$ (see Figure 1), thus it follows from (10.1) and (10.2) that $\delta_k(y, y \wedge x_k) = \mathbf{0}_{X_k}$, hence the following equality holds

$$y = y \wedge x_k, \quad \text{for each } k \in \{1, 2\}.$$

Therefore $y = (y \wedge x_1) \wedge x_2$ in B_3 , moreover $x_1 \wedge x_2 = 0$, hence as \vec{B} is operational $y \wedge (x_1 \wedge x_2)$ is defined in B_3 , thus $y \wedge 0 = y \wedge (x_1 \wedge x_2) = (y \wedge x_1) \wedge x_2 = y$.

Moreover as $y \vee 0 = y$, it follows that $0 = (y \vee 0) \wedge 0 = y \wedge 0 = y$ (all elements are defined in B_3), hence $y = 0$. \square Claim.

Let C be an $(n+1)$ -element chain. Set $T = \{t \mid t: C \rightarrow X_0 \text{ is order preserving}\}$. Put $A_0 = C$, put $A_1 = X_1^T$, put $A_2 = X_2^T$, put $A_3 = X_3^T = K^T$. We consider the following morphisms:

$$\begin{aligned} f_i: A_0 &\rightarrow A_i \\ x &\mapsto (t(x))_{t \in T}, \quad \text{for } i = 1, 2. \end{aligned}$$

We denote by $f'_i: A_i \rightarrow A_3$ the inclusion maps, for $i = 1, 2$. We denote by \vec{A} the square in the left hand side of Figure 2.

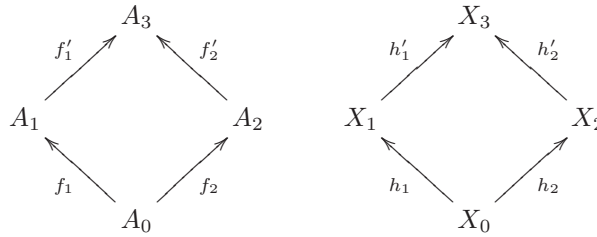


FIGURE 2. Two squares in $\mathbf{Var}^{0,1}(K)$

Given $t \in T$, denote $\pi_0^t = t$ and $\pi_k^t: A_k \rightarrow X_k$, $(a_p)_{p \in T} \mapsto a_t$ the canonical projection, for all $k \in \{1, 2, 3\}$. It defines a natural transformation $\vec{\pi}^t$ from \vec{A} to \vec{X} . We denote by $\pi_k^t = P_{\text{ga}}(\pi_k^t) = (\pi_k^t, \text{Con}_c \pi_k^t)$, for all $k \in \{0, 1, 2, 3\}$.

Assume that there exists an operational square \vec{B} , as in Figure 3, of congruence n -permutable gamps, and a natural equivalence $P_{\text{ga}} \circ \vec{A} \rightarrow P_{\text{gl}} \circ \vec{B}$. We can assume that $P_{\text{gl}} \circ \vec{B} = P_{\text{ga}} \circ \vec{A}$. Put $\delta_k = \delta_{B_k}$, the distance δ_{A_k} is a restriction of δ_k , for all $k \in \{0, 1, 2, 3\}$.

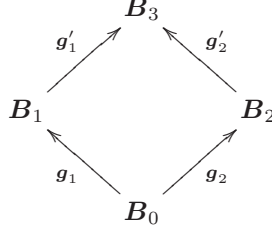


FIGURE 3. A square in the category $\mathbf{Gamp}(\mathcal{L})$

Let $a_0 < a_1 < \dots < a_n$ be the elements of $C = A_0 = B_0^*$. As B_0 is congruence n -permutable, there exist b_0, \dots, b_n in B_0 such that $b_i \wedge b_j = b_j \wedge b_i = b_i$ in B_0 for all $i \leq j \leq n$, $b_0 = a_0 \wedge a_n = a_0$ in B_0 , $b_n = a_0 \vee a_n = a_n$ in B_0 , and:

$$\delta_0(b_k, b_{k+1}) \leq \bigvee (\delta_0(a_i, a_{i+1}) \mid i < n \text{ even}), \quad \text{for all } k < n \text{ odd}, \quad (10.3)$$

$$\delta_0(b_k, b_{k+1}) \leq \bigvee (\delta_0(a_i, a_{i+1}) \mid i < n \text{ odd}), \quad \text{for all } k < n \text{ even}. \quad (10.4)$$

In particular the following inequality holds

$$\delta_0(b_k, b_{k+1}) \leq \delta_0(a_0, a_k) \vee \delta_0(a_{k+1}, a_n), \quad \text{for all } k < n. \quad (10.5)$$

As $b_n = a_n$, an immediate consequence of (10.5) is $\delta_0(b_{n-1}, a_n) \leq \delta_0(a_0, a_{n-1})$. Assume that $\delta_0(a_{n-2}, b_{n-1}) \leq \delta_0(a_0, a_{n-2})$, it follows that

$$\delta_0(a_0, a_n) \leq \delta_0(a_0, a_{n-2}) \vee \delta_0(a_{n-2}, b_{n-1}) \vee \delta_0(b_{n-1}, a_n) \leq \delta_0(a_0, a_{n-1})$$

That is $\Theta_{A_0}(a_0, a_n) \leq \Theta_{A_0}(a_0, a_{n-1})$ a contradiction, as A_0 is the chain $a_0 < \dots < a_n$. It follows that $\delta_0(a_{n-2}, b_{n-1}) \not\leq \delta_0(a_0, a_{n-2})$.

Take $i < n$ minimal such that the following inequality hold

$$\delta_0(a_i, b_{i+1}) \not\leq \delta_0(a_0, a_i). \quad (10.6)$$

If $i = 0$, then $a_0 = b_0 = b_i$, hence $\delta_0(a_0, b_i) \leq \delta_0(a_0, a_i)$. If $i > 0$, then it follows from the minimality of i that $\delta_0(a_{i-1}, b_i) \leq \delta_0(a_0, a_{i-1}) \leq \delta_0(a_0, a_i)$, thus $\delta_0(a_0, b_i) \leq \delta_0(a_0, a_{i-1}) \vee \delta_0(a_{i-1}, b_i) \leq \delta_0(a_0, a_i)$. Therefore the following inequality holds

$$\delta_0(a_0, b_i) \leq \delta_0(a_0, a_i). \quad (10.7)$$

Assume that $\delta_0(b_i, b_{i+1}) \leq \delta_0(a_0, a_i)$, this implies with (10.7) the following inequality

$$\delta_0(a_i, b_{i+1}) \leq \delta_0(a_i, a_0) \vee \delta_0(a_0, b_i) \vee \delta_0(b_i, b_{i+1}) = \delta_0(a_0, a_i),$$

which contradicts (10.6). Therefore the following statement holds

$$\delta_0(b_i, b_{i+1}) \not\leq \delta_0(a_0, a_i). \quad (10.8)$$

As $\text{Con } A_0$ is a Boolean lattice with atoms $\delta_0(a_k, a_{k+1})$ for $k < n$, it follows from (10.8) that there is $j < n$ such that

$$\delta_0(a_j, a_{j+1}) \leq \delta_0(b_i, b_{i+1}) \quad \text{and} \quad \delta_0(a_j, a_{j+1}) \not\leq \delta_0(a_0, a_i), \quad (10.9)$$

As $\delta_0(a_j, a_{j+1}) \not\leq \delta_0(a_0, a_i)$, $j \geq i$. It follows from (10.3), (10.4), and (10.9) that i and j have distinct parities, therefore $j > i$.

Put

$$t: A_0 \rightarrow \{0, x_3, 1\}$$

$$a_k \mapsto \begin{cases} 0 & \text{if } k \leq i, \\ x_3 & \text{if } i < k \leq j, \\ 1 & \text{if } j < k, \end{cases} \quad \text{for all } k \leq n.$$

As $i < j < n$ the map t is surjective, thus $t \in T$. Put $I_i = \ker_0 \pi_i^t$, for all $i \in \{0, 1, 2, 3\}$. Denote by $\vec{\chi}: \mathbf{P}_{\text{ga}} \circ \vec{A}/\vec{I} = \mathbf{P}_{\text{gl}} \circ \vec{B} \rightarrow \vec{X}$ the natural transformation induced by $\vec{\pi}$. As $\vec{\pi} = \mathbf{P}_{\text{ga}}(\vec{\pi})$ is ideal-induced, it follows from Lemma 5.28 that $\vec{\chi}$ is a natural equivalence. Put $\vec{\xi} = \vec{\chi}^{-1}$. Notice that the following inequalities hold

$$\begin{aligned} \delta_0(a_0, b_{i+1}) &\leq \delta_0(a_0, b_i) \vee \delta_0(b_i, b_{i+1}) \\ &\leq \delta_0(a_0, a_i) \vee \delta_0(a_{i+1}, a_n) \quad \text{by (10.7) and (10.5).} \end{aligned}$$

Moreover $\xi_0(0) = a_0/I_0 = a_i/I_0$, $\xi_0(x_3) = a_{i+1}/I_0$ and $\xi_0(1) = a_n/I_0$, thus $\delta_{\mathbf{B}_0/I_0}(\xi_0(0), b_{i+1}/I_0) \leq \delta_{\mathbf{B}_0/I_0}(\xi_0(x_3), \xi_0(1))$. As $b_{i+1}/I_0 \wedge \xi_0(1) = (b_{i+1} \wedge a_n)/I_0 = b_{i+1}/I_0$ and $b_{i+1}/I_0 \vee \xi_0(0) = (b_{i+1} \vee a_0)/I_0 = b_{i+1}/I_0$. It follows from the Claim that $b_{i+1}/I_0 = \xi_0(0) = a_0/I_0$, that is $\delta_0(a_0, b_{i+1}) \in I_0$. Therefore the following inequality holds:

$$\delta_0(a_0, b_{i+1}) \leq \delta_0(a_0, a_i) \vee \delta_0(a_{i+1}, a_j) \vee \delta_0(a_{j+1}, a_n) \quad (10.10)$$

Hence we obtain

$$\begin{aligned} \delta_0(a_j, a_{j+1}) &\leq \delta_0(b_i, b_{i+1}) && \text{by (10.9).} \\ &\leq \delta_0(b_i, a_0) \vee \delta_0(a_0, b_{i+1}) \\ &\leq \delta_0(a_0, a_i) \vee \delta_0(a_{i+1}, a_j) \vee \delta_0(a_{j+1}, a_n) && \text{by (10.7) and (10.10).} \\ &\leq \delta_0(a_0, a_j) \vee \delta_0(a_{j+1}, a_n). \end{aligned}$$

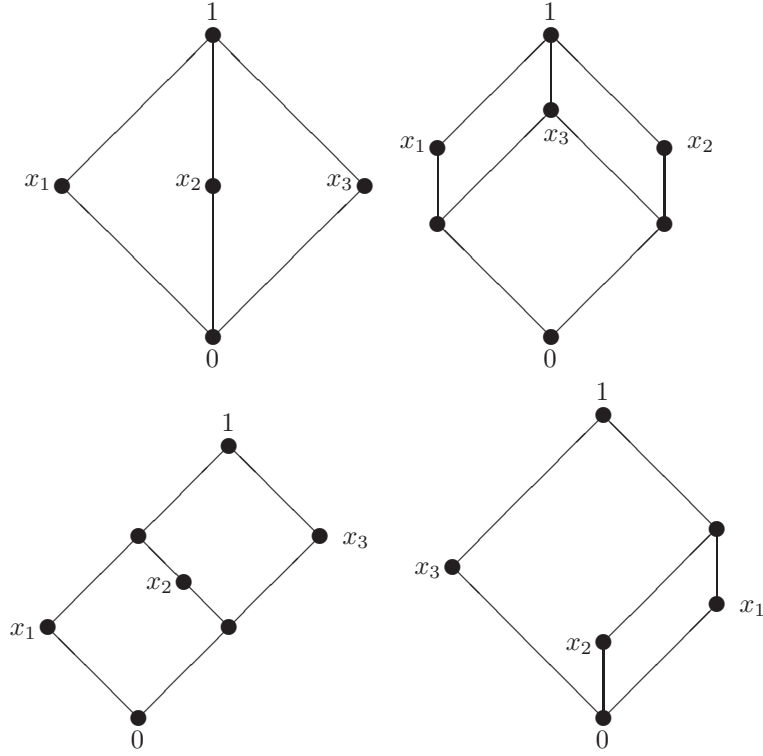
A contradiction, as A_0 is the chain $a_0 < a_1 < \dots < a_n$. \square

Theorem 10.7. *Let $n \geq 2$. Let \mathcal{V} be a variety of lattices, such that either $M_3 \in \mathcal{V}$, $L_2 \in \mathcal{V}$, $L_3 \in \mathcal{V}$, $L_4 \in \mathcal{V}$, $L_2^d \in \mathcal{V}$, or $L_4^d \in \mathcal{V}$. There exists a bounded lattice $L \in \mathcal{V}$ such that L is congruence $(n+1)$ -permutable, $\text{card } L = \aleph_1$, and L has no congruence n -permutable, congruence-preserving extension in the variety of all lattices.*

Proof. Fix (X, \mathbf{X}) an \aleph_0 -lifter of the square such that $\text{card } X = \aleph_1$ (cf. Lemma 8.5). Up to changing \mathcal{V} to its dual, we can assume that either $M_3 \in \mathcal{V}$, $L_2 \in \mathcal{V}$, $L_3 \in \mathcal{V}$, or $L_4 \in \mathcal{V}$. Let K one of those lattices such that $K \in \mathcal{V}$, let x_1, x_2, x_3 as in Figure 4. The conditions of Lemma 10.6 are satisfied. Denote by \vec{A} the diagram constructed in Lemma 10.6.

Put $L = \mathbf{F}(X) \otimes \vec{A} \in \mathbf{Var}^{0,1}(K) \subseteq \mathcal{V}$ (cf. Remark 9.2). Notice that L is a directed colimit of finite products of lattices in \vec{A} and all lattices in \vec{A} are congruence $(n+1)$ -permutable, thus L is congruence $(n+1)$ -permutable. As $\text{card } X = \aleph_1$ and each lattice in the diagram \vec{A} is finite, $\text{card } L = \aleph_1$. Moreover L cannot have a congruence n -permutable, congruence-preserving extension, as the conclusions of Lemma 10.5 and Lemma 10.6 contradict each other. \square

The following corollary is an immediate consequence of Theorem 10.7.

FIGURE 4. The lattices M_3 , L_2 , L_3 , L_4 .

Corollary 10.8. *Let \mathcal{V} be a variety of lattices such that either $M_3 \in \mathcal{V}$, $L_2 \in \mathcal{V}$, $L_3 \in \mathcal{V}$, $L_4 \in \mathcal{V}$, $L_2^d \in \mathcal{V}$, or $L_4^d \in \mathcal{V}$. The free bounded lattice on \aleph_1 generators of \mathcal{V} has no congruence n -permutable, congruence-preserving extension in the variety of all lattices, for each $n \geq 2$.*

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REFERENCES

- [1] K. A. Baker, *Equational classes of modular lattices*, Pacific J. Math. **28**, no. 1 (1969), 9–15.
- [2] P. Erdős, A. Hajnal, A. Máté, and R. Rado, “Combinatorial Set Theory: Partition Relations for Cardinals”. Studies in Logic and the Foundations of Mathematics, 106. North-Holland Publishing Co., Amsterdam, 1984. 347 p.
- [3] P. Gillibert, *Critical points of pairs of varieties of algebras*, Internat. J. Algebra Comput. **19**, no. 1 (2009), 1–40.
- [4] P. Gillibert, *The possible values of critical points between varieties of lattices*, preprint 2010, available online at <http://hal.archives-ouvertes.fr/hal-00468048>.
- [5] P. Gillibert and F. Wehrung, *From objects to diagrams for ranges of functors*, preprint 2010, available online at <http://hal.archives-ouvertes.fr/hal-00462941>.
- [6] P. Gillibert and F. Wehrung, *An infinite combinatorial statement with a poset parameter*, Combinatorica, to appear, available online at <http://hal.archives-ouvertes.fr/hal-00364329>.

- [7] G. Grätzer, H. Lakser, and F. Wehrung, *Congruence amalgamation of lattices*, Acta Sci. Math. (Szeged) **66**, no. 1-2 (2000), 3–22.
- [8] A. Hajnal and A. Máté, *Set mappings, partitions, and chromatic numbers*, Logic Colloquium '73 (Bristol, 1973), p. 347–379. Studies in Logic and the Foundations of Mathematics, Vol. **80**, North-Holland, Amsterdam, 1975.
- [9] G. Grätzer, “Universal Algebra”. D. Van Nostrand Co., Inc., Princeton, N. J.-Toronto, Ont.-London, 1968. xvi+368 p.
- [10] M. Ploščica, *Separation properties in congruence lattices of lattices*, Colloq. Math. **83** (2000), 71–84.
- [11] M. Ploščica, *Dual spaces of some congruence lattices*, Topology and its Applications **131** (2003), 1–14.
- [12] M. Ploščica, *Congruence lattices of lattices with m -permutable congruences*, Acta Sci. Math. (Szeged) **74**, no. 1-2 (2008), 23–36.
- [13] P. Pudlák, *On congruence lattices of lattices*, Algebra Universalis **20** (1985), 96–114.
- [14] P. Růžička, J. Tůma, and F. Wehrung, *Distributive congruence lattices of congruence-permutable algebras*, J. Algebra **311**, no. 1 (2007), 96–116.
- [15] J. Tůma and F. Wehrung, *A survey of recent results on congruence lattices of lattices*, Algebra Universalis **48**, no. 4 (2002), 439–471.
- [16] F. Wehrung, *A solution to Dilworth’s congruence lattice problem*, Adv. Math. **216**, no. 2 (2007), 610–625.

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