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# CATEGORIES OF PARTIAL ALGEBRAS FOR CRITICAL POINTS BETWEEN VARIETIES OF ALGEBRAS 

PIERRE GILLIBERT


#### Abstract

We denote by $\operatorname{Con}_{\mathrm{c}} A$ the ( $\mathrm{V}, 0$ )-semilattice of all finitely generated congruences of an algebra $A$. A lifting of a $(\vee, 0)$-semilattice $S$ is an algebra $A$ such that $S \cong \operatorname{Con}_{\mathrm{c}} A$.

The aim of this work is to give a categorical theory of partial algebras endowed with a partial subalgebra together with a semilattice-valued distance, that we call gamps. This part of the theory is formulated in any variety of (universal) algebras.

Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of algebras (on a finite similarity type). Let $P$ be a finite lattice of order-dimension $d>0$. Let $\vec{A}$ be a $P$-indexed diagram of finite algebras in $\mathcal{V}$. If $\mathrm{Con}_{\mathrm{c}} \circ \vec{A}$ has no partial lifting in the category of gamps of $\mathcal{W}$, then there is an algebra $A \in \mathcal{V}$ of cardinality $\aleph_{d-1}$ such that $\operatorname{Con}_{\mathrm{c}} A$ is not isomorphic to $\operatorname{Con}_{\mathrm{c}} B$ for any $B \in \mathcal{W}$.

We already knew a similar result for diagrams $\vec{A}$ such that Con $_{\mathrm{c}} \circ \vec{A}$ has no lifting in $\mathcal{W}$, however the algebra $A$ constructed here has cardinality $\aleph_{d}$.

Gamps are also used to study congruence-preserving extensions. Denote by $\mathcal{N}_{3}$ the variety generated by the lattice of length two, with three atoms. We construct a lattice $A \in \mathcal{M}_{3}$ of cardinality $\aleph_{1}$ with no congruence $n$-permutable, congruence-preserving extension, for each $n \geq 2$.


## 1. Introduction

For an algebra $A$ we denote by $\operatorname{Con} A$ the lattice of all congruences of $A$ under inclusion. Given $x, y \in A$, we denote by $\Theta_{A}(x, y)$ the smallest congruence of $A$ that identifies $x$ and $y$, such a congruence is called principal. A congruence is finitely generated if it is a finite join of principal congruences. The lattice Con $A$ is algebraic and the compact element of $\operatorname{Con} A$ are the finitely generated congruences.

The lattice Con $A$ is determined by the $(V, 0)$-semilattice $\operatorname{Con}_{\mathrm{c}} A$ of compact congruences of $A$. In this paper we mostly refer to $\operatorname{Con}_{\mathrm{c}} A$ instead of Con $A$. If $\mathrm{Con}_{\mathrm{c}} A$ is isomorphic to a $(\vee, 0)$-semilattice $S$, we call $A$ a lifting of $S$.

Given a class of algebras $\mathcal{K}$ we denote by $\operatorname{Con}_{\mathrm{c}} \mathcal{K}$ the class of all $(\vee, 0)$-semilattices with a lifting in $\mathcal{K}$. In general, even if $\mathcal{K}$ is a variety of algebras, there is no good description of $\mathrm{Con}_{\mathrm{c}} \mathcal{K}$. The negative solution to the congruence lattice problem (CLP) in 16 is a good example of the difficulty to find such a description.

The study of CLP led to the following related questions. Fix two classes of algebras $\mathcal{V}$ and $\mathcal{W}$.
(Q1) Given $A \in \mathcal{V}$, does there exist $B \in \mathcal{W}$ such that $\operatorname{Con}_{\mathrm{c}} A \cong \operatorname{Con}_{\mathrm{c}} B$ ?

[^0](Q2) Given $A \in \mathcal{V}$, does there exist a congruence-preserving extension $B \in \mathcal{W}$ of $A$ ?
A positive answer to (Q1) is equivalent to $\operatorname{Con}_{\mathrm{c}} \mathcal{V} \subseteq \mathrm{Con}_{\mathrm{c}} \mathcal{W}$. The "containment defect" of $\operatorname{Con}_{c} \mathcal{V}$ into $\operatorname{Con}_{c} \mathcal{W}$ is measured (cf. [15, 3]) by the critical point between $\mathcal{V}$ and $\mathcal{W}$, defined as
\[

\operatorname{crit}(\mathcal{V} ; \mathcal{W})= $$
\begin{cases}\min \left\{\operatorname{card} S \mid S \in\left(\operatorname{Con}_{\mathrm{c}} \mathcal{V}\right)-\left(\operatorname{Con}_{\mathrm{c}} \mathcal{W}\right)\right\}, & \text { if } \operatorname{Con}_{\mathrm{c}} \mathcal{V} \nsubseteq \operatorname{Con}_{\mathrm{c}} \mathcal{W} \\ \infty, & \text { if } \operatorname{Con}_{\mathrm{c}} \mathcal{V} \subseteq \operatorname{Con}_{\mathrm{c}} \mathcal{W}\end{cases}
$$
\]

This critical point has been already studied, for different families of varieties of lattices, in 10, 11, 3].

We now give an illustration of (Q2). Every countable locally finite lattice has a relatively complemented, congruence-preserving extension (cf. [7]). In particular every countable locally finite lattice has a congruence-permutable, congruencepreserving extension. However, in every non-distributive variety of lattices, the free lattice on $\aleph_{1}$ generators has no congruence-permutable, congruence-preserving extension (cf. [5, Chapter 5]). A precise answer to (Q2) also depends on the cardinality of $A$.

In order to study a similar problem, Pudlák in (13] uses an approach based on liftings of diagrams. The assignment $A \mapsto \operatorname{Con}_{\mathrm{c}} A$ can be extended to a functor. This leads to the following questions:
(Q1') Given a diagram $\vec{A}$ in $\mathcal{V}$, does there exist a diagram $\vec{B}$ in $\mathcal{W}$ such that $\mathrm{Con}_{\mathrm{c}} \circ \vec{A} \cong \mathrm{Con}_{\mathrm{c}} \circ \vec{B}$ ?
(Q2') Given a diagram $\vec{A}$ in $\mathcal{V}$, does there exist a diagram $\vec{B}$ in $\mathcal{W}$ which is a congruence-preserving extension of $\vec{A}$ ?
The functor Con $_{c}$ preserves directed colimits, thus, in many cases, a positive answer for the finite case of ( $\mathbf{Q 1} \mathbf{1}^{\prime}$ ) implies a positive answer to (Q1).
Proposition 1.1. Assume that $\mathcal{V}$ and $\mathcal{W}$ are varieties of algebras. If $\mathcal{V}$ is locally finite and for every lattice-indexed diagram $\vec{A}$ of finite algebras in $\mathcal{V}$ there exists a diagram $\vec{B}$ in $\mathcal{W}$ such that $\mathrm{Con}_{\mathrm{c}} \circ \vec{A} \cong \mathrm{Con}_{\mathrm{c}} \circ \vec{B}$, then (Q1) has a positive answer.

In this proposition, we consider infinite diagrams of finite algebras. However if $\mathcal{W}$ is finitely generated and congruence-distributive, a compactness argument makes it possible to restrict the assumptions to finite diagrams of finite algebras.

In order to study the converse of Proposition 1.1, we shall use the construction of condensate, introduced in [3]. This construction was introduced in order to turn diagram counterexamples to object counterexamples. We use it here to turn a diagram counterexample of ( $\mathbf{Q 1}^{\prime}$ ) to a counterexample of ( $\mathbf{Q 1}$ ).
Theorem 1.2. Assume that $\mathcal{V}$ and $\mathcal{W}$ are varieties of algebras. Let $P$ be a finite lattice. Let $\vec{A}$ be a $P$-indexed diagram in $\mathcal{V}$. If $\operatorname{Con}_{\mathrm{c}} \circ \vec{A}$ is not liftable in $\mathcal{W}$ then there is a condensate $A \in \mathcal{V}$ of $\vec{A}$, such that $\operatorname{Con}_{\mathrm{c}} A$ is not liftable in $\mathcal{W}$.

Moreover, if $\mathcal{W}$ has a countable similarity type and all algebras of $\vec{A}$ are countable, then the condensate $A$ can be chosen of cardinal $\aleph_{d}$, where $d$ is the order-dimension of $P$. In particular, $\operatorname{crit}(\mathcal{V} ; \mathcal{W}) \leq \aleph_{d}$.

If every algebra of $\vec{A}$ is finite and $\mathcal{W}$ is finitely generated and congruence-distributive, then $A$ can be chosen of cardinal $\aleph_{d-1}$, so $\operatorname{crit}(\mathcal{V} ; \mathcal{W}) \leq \aleph_{d-1}$.

The cardinality bound, in case $\mathcal{W}$ is finitely generated and congruence-distributive, is optimal, in the following sense: There are finitely generated varieties $\mathcal{V}$ and $\mathcal{W}$
of lattices such that every countable $(\vee, 0)$-semilattice liftable in $\mathcal{V}$ is liftable in $\mathcal{W}$ and there is a square-indexed diagram of $(\vee, 0)$-semilattices that has a lifting in $\mathcal{V}$ but no lifting in $\mathcal{W}$. In particular there is a $(\vee, 0)$-semilattice of cardinal $\aleph_{1}$ that is liftable in $\mathcal{V}$ but not liftable in $\mathcal{W}$, thus $\operatorname{crit}(\mathcal{V} ; \mathcal{W})=\aleph_{1}$. This example appears in [3. Section 8].

Later, we generalized the condensate construction in [5] , to a larger categorical context. The best bound of Theorem 1.2 is obtained in a more general case (cf. [5] Theorem 4-9.2]), namely if $\mathcal{W}$ is both congruence-proper (cf. [5. Definition 4-8.1]) and locally finite, for example $\mathcal{W}$ is a finitely generated congruence-modular variety. Using the tools introduced in this paper, we give a new version (cf. Theorem 9.6), we assume that $\mathcal{W}$ is congruence-proper and has finite similarity type.

This categorical version of condensate can also apply to turn a counterexample of ( $\mathbf{Q 2}^{\prime}$ ) to a counterexample of (Q2). For example in [5, Chapter 5] we give a square $\vec{A}$ of finite lattices, that has no congruence-permutable, congruencepreserving extension. A condensate of this square has cardinality $\aleph_{1}$ and it has no congruence-permutable, congruence-preserving extension.

The largest part of this paper is the introduction of pregamps and gamps, it is a generalization of semilattice-metric spaces and semilattice-metric covers given in 5. Chapter 5]. The category of gamps of a variety $\mathcal{V}$ has properties similar to a finitely generated congruence-distributive variety.

A pregamp is a triplet $\boldsymbol{A}=(A, \delta, S)$, where $A$ is a partial algebra, $S$ is a $(\vee, 0)$ semilattice and $\delta: A^{2} \rightarrow S$ is a distance, compatible with the operations. A typical example of pregamp is $\left(A, \Theta_{A}, \operatorname{Con}_{\mathrm{c}} A\right)$, for an algebra $A$. This generalizes to partial algebras the notion of a congruence.

A gamp $\boldsymbol{A}$ is a pregamp $(A, \delta, S)$ with a partial subalgebra $A^{*}$ of $A$. There are many natural properties that a gamp can satisfy (cf. Section 7), for example $\boldsymbol{A}$ is full if all operations with parameters in $A^{*}$ can be evaluated in $A$. A morphism of gamps is a morphism of partial algebras with a morphism of $(\mathrm{V}, 0)$-semilattices satisfying a compatibility condition with the distances (cf. Definition 6.1).

The class of all gamps (on a given type), with morphisms of gamps, forms a category. Denote by $\boldsymbol{C}$ the forgetful functor from the category of gamps to the category of $(\vee, 0)$-semilattices. A partial lifting of a diagram $\vec{S}$ of $(\vee, 0)$-semilattice is a diagram $\overrightarrow{\boldsymbol{B}}$ of gamps, with some additional properties, such that $\boldsymbol{C} \circ \overrightarrow{\boldsymbol{B}} \cong \vec{S}$.

The category of gamps has properties similar to locally finite, congruence-proper varieties. Let $S$ be a finite $(\vee, 0)$-semilattice, let $\boldsymbol{B}$ be a gamp such that $\boldsymbol{C}(\boldsymbol{B}) \cong S$, there are (arbitrary large) finite subgamps $\boldsymbol{B}^{\prime}$ of $\boldsymbol{B}$ such that $\boldsymbol{C}\left(\boldsymbol{B}^{\prime}\right) \cong S$. There is no equivalent result for algebras: for example, the three-element chain is the congruence lattice of a modular lattice, but not the congruence lattice of any finite modular lattice.

Assume that $\mathcal{V}$ and $\mathcal{W}$ are varieties of algebras. Let $P$ be a finite lattice of orderdimension $d$. Suppose that we find $\vec{A}$ a $P$-indexed diagram of finite algebras in $\mathcal{V}$, such that $\mathrm{Con}_{\mathrm{c}} \circ \vec{A}$ has no partial lifting (with maybe some additional "locally finite properties", cf. Section (才) in the category of gamps of $\mathcal{W}$, then $\operatorname{crit}(\mathcal{V} ; \mathcal{W}) \leq \aleph_{d-1}$. Hence we obtain the optimal bound, with no assumption on $\mathcal{W}$. However there is no (known) proof that a diagram with no lifting has no partial lifting, but no counterexample has been found.

The dual of a lattice $L$ is the lattice $L^{\mathrm{d}}$ with reverse order. The dual of a variety of lattices $\mathcal{V}$ is $\mathcal{V}^{\mathrm{d}}$ the variety of all duals of lattices in $\mathcal{V}$. Let $\mathcal{V}$ and $\mathcal{W}$ be varieties
of lattices, if $\mathcal{V} \subseteq \mathcal{W}$ or $\mathcal{V} \subseteq \mathcal{W}^{\mathrm{d}}$, then $\operatorname{Con}_{\mathrm{c}} \mathcal{V} \subseteq \operatorname{Con}_{\mathrm{c}} \mathcal{W}$. In a sequel to the present paper (cf. (4) ), we shall prove the following result.
Theorem. Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of lattices. If every simple lattice in $\mathcal{W}$ contains a prime interval, then one of the following statements holds:
(1) $\operatorname{crit}(\mathcal{V} ; \mathcal{W}) \leq \aleph_{2}$.
(2) $\mathcal{V} \subseteq \mathcal{W}$.
(3) $\mathcal{V} \subseteq \mathcal{W}^{\mathrm{d}}$.

The $\aleph_{2}$ bound is optimal, as there are varieties $\mathcal{V}$ and $\mathcal{W}$ of lattices such that $\operatorname{crit}(\mathcal{V} ; \mathcal{W})=\aleph_{2}$. Without the use of gamps, we would have obtained an upper bound $\aleph_{3}$ instead of $\aleph_{2}$.

The gamps can also be used to study congruence-preserving extensions. Denote by $\boldsymbol{P}_{\mathrm{gl} 1}$ the functor that maps a gamp $\left(A^{*}, A, \delta, \widetilde{A}\right)$ to the pregamp $\left(A^{*}, \delta, \widetilde{A}\right)$; we also denote by $\boldsymbol{P}_{\mathrm{ga}}$ the functor that maps an algebra $A$ to the pregamp $\left(A, \Theta_{A}, \operatorname{Con}_{\mathrm{c}} A\right)$. Let $B$ be a congruence-preserving extension of an algebra $A$, then $\left(A, B, \Theta_{B}, \operatorname{Con}_{\mathrm{c}} B\right)$ is a gamp. Similarly, let $\vec{B}=\left(B_{p}, g_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ be a congruence-preserving extension of a diagram $\vec{A}=\left(A_{p}, f_{p, q} \mid p \leq q\right.$ in $\left.P\right)$, denote by $\boldsymbol{B}_{p}=\left(A, B, \Theta_{B}, \mathrm{Con}_{\mathrm{c}} B\right)$ and $\boldsymbol{g}_{p, q}=\left(g_{p, q}, \operatorname{Con}_{\mathrm{c}} g_{p, q}\right)$, for all $p \leq q$ in $P$, then $\overrightarrow{\boldsymbol{B}}=\left(\boldsymbol{B}_{p}, \boldsymbol{g}_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ is a diagram of gamps. Moreover, $\boldsymbol{P}_{\mathrm{gl}} \circ \overrightarrow{\boldsymbol{B}}=\boldsymbol{P}_{\mathrm{ga}} \circ \vec{A}$, up to the identification of $\mathrm{Con}_{\mathrm{c}} B$ and $\mathrm{Con}_{\mathrm{c}} A$.

In Section 10, given $n \geq 2$, we construct a square $\vec{A}$ of finite lattices in $\mathcal{M}_{3}$, such that the diagram $\vec{A}$ has no congruence $n$-permutable, congruence-preserving extension. Another condensate construction gives a result proved in 12, namely the existence of a lattice $A \in \mathcal{M}_{3}$ of cardinality $\aleph_{2}$ with no congruence $n$-permutable, congruence-preserving extension.

Hopefully, once again, the diagram $\vec{A}$ satisfies a stronger statement, there is no operational (cf. Definition 10.1) diagram $\overrightarrow{\boldsymbol{B}}$ of congruence $n$-permutable gamps of lattices such that $\boldsymbol{P}_{\mathrm{gl}} \circ \overrightarrow{\boldsymbol{B}} \cong \boldsymbol{P}_{\mathrm{ga}} \circ \vec{A}$. Using a condensate, we obtain a lattice $\mathcal{M}_{3}$ of cardinality $\aleph_{1}$ with no congruence $n$-permutable, congruence-preserving extension.

## 2. Basic Concepts

We denote by 0 (resp., 1 ) the least (resp. largest) element of a poset if it exists. We denote by $\mathbf{2}=\{0,1\}$ the two-element lattice, or poset (i.e., partially ordered set), or ( $\vee, 0$ )-semilattice, depending of the context. Given an algebra $A$, we denote by $0_{A}$ the identity congruence of $A$.

Given subsets $P$ and $Q$ of a poset $R$, we set

$$
P \downarrow Q=\{p \in P \mid(\exists q \in Q)(p \leq q)\}
$$

If $Q=\{q\}$, we simply write $P \downarrow q$ instead of $P \downarrow\{q\}$.
Let $\mathcal{V}$ be a variety of algebras, let $\kappa$ be a cardinal, we denote by $F_{\mathcal{V}}(\kappa)$ the free algebras in $\mathcal{V}$ with $\kappa$ generators. Given an algebra $A$ we denote by $\operatorname{Var} A$ the variety of algebras generated by $A$. If $A$ is a lattice we also denote by $\operatorname{Var}^{0,1} A$ the variety of bounded lattices generated by $A$. We denote by $\mathcal{L}$ the variety of lattices.

We denote the range of a function $f: X \rightarrow Y$ by $\operatorname{rng} f=\{f(x) \mid x \in X\}$. We use basic set-theoretical notation, for example $\omega$ is the first infinite ordinal, and also the set of all nonnegative integers; furthermore, $n=\{0,1, \ldots, n-1\}$ for every nonnegative integer $n$. By "countable" we will always mean "at most countable".

Let $X, I$ be sets, we often denote $\vec{x}=\left(x_{i}\right)_{i \in I}$ an element of $X^{I}$. In particular, for $n<\omega$, we denote by $\vec{x}=\left(x_{0}, \ldots, x_{n-1}\right)$ an $n$-tuple of $X$. If $f: Y \rightarrow Z$ is a function, where $Y \subseteq X$, we denote $f(\vec{x})=\left(f\left(x_{0}\right), \ldots, f\left(x_{n-1}\right)\right)$ whenever it is defined. Similarly, if $f: Y \rightarrow Z$ is a function, where $Y \subseteq X^{n}$, we denote $f(\vec{x})=f\left(x_{0}, \ldots, x_{n-1}\right)$ whenever it is defined. We also write $f(\vec{x}, \vec{y})=$ $f\left(x_{0}, \ldots, x_{m-1}, y_{0}, \ldots, y_{n-1}\right)$ in case $\vec{x}=\left(x_{0}, \ldots, x_{m-1}\right)$ and $\vec{y}=\left(y_{0}, \ldots, y_{n-1}\right)$, and so on.

For example, let $A$ and $B$ be algebras of the same similarity type. Let $\ell$ be an $n$-ary operation. Let $f: A \rightarrow B$ a map. The map $f$ is compatible with $\ell$ if $f(\ell(\vec{x}))=\ell(f(\vec{x}))$ for every $n$-tuple $\vec{x}$ of $X$. Let $m \leq n \leq \omega$. Let $\vec{X}$ be an $n$-tuple of $X$, we denote by $\vec{x} \upharpoonright m$ the $m$-tuple $\left(x_{k}\right)_{k<m}$.

If $X$ is a set and $\theta$ is an equivalence relation on $X$, we denote by $X / \theta$ the set of all equivalence classes of $\theta$. Given $x \in X$ we denote by $x / \theta$ the equivalence class of $\theta$ containing $x$. Given an $n$-tuple $\vec{x}$ of $X$, we denote $\vec{x} / \theta=\left(x_{0} / \theta, \ldots, x_{n-1} / \theta\right)$. Given $Y \subseteq X$, we set $Y / \theta=\{x / \theta \mid x \in Y\}$.

Let $n \geq 2$ an integer. An algebra $A$ is congruence $n$-permutable if the following equality holds:

$$
\underbrace{\alpha \circ \beta \circ \alpha \circ \ldots}_{n \text { times }}=\underbrace{\beta \circ \alpha \circ \beta \circ \ldots}_{n \text { times }} \text {, for all } \alpha, \beta \in \operatorname{Con} A \text {. }
$$

If $n=2$ we say that $A$ is congruence-permutable instead of congruence 2-permutable. The following statement is folklore.

Proposition 2.1. Let $A$ be an algebra, let $n \geq 2$ an integer. The following conditions are equivalent:
(1) The algebra $A$ is congruence $n$-permutable.
(2) For all $x_{0}, x_{1}, \ldots, x_{n} \in A$, there are $x_{0}=y_{0}, y_{1}, \ldots, y_{n}=x_{n} \in A$ such that the following containments hold:

$$
\begin{array}{ll}
\Theta_{A}\left(y_{k}, y_{k+1}\right) \subseteq \bigvee\left(\Theta_{A}\left(x_{i}, x_{i+1}\right) \mid i<n \text { even }\right), & \text { for all } k<n \text { odd }, \\
\Theta_{A}\left(y_{k}, y_{k+1}\right) \subseteq \bigvee\left(\Theta_{A}\left(x_{i}, x_{i+1}\right) \mid i<n \text { odd }\right), & \text { for all } k<n \text { even } .
\end{array}
$$

Let $n \geq 2$. The class of all congruence $n$-permutable algebras of a given similarity type is closed under directed colimits and quotients. Moreover the class of congruence $n$-permutable algebras of a congruence-distributive variety is also closed under finite products (the latter statement is known not to extend to arbitrary algebras).

## 3. Semilattices

In this section we give some well-known facts about ( $V, 0$ )-semilattices. Most notions and results will have later a generalization involving pregamps and gamps.

Proposition 3.1. Let $S, T$ be $(\vee, 0)$-semilattices, let $X$ be a set, and let $f: X \rightarrow S$ and $g: X \rightarrow T$ be maps. Assume that for every $x \in X$, for every positive integer $n$, and for every $n$-tuple $\vec{y}$ of $X$ the following implication holds:

$$
\begin{equation*}
f(x) \leq \bigvee_{k<n} f\left(y_{k}\right) \quad \Longrightarrow \quad g(x) \leq \bigvee_{k<n} g\left(y_{k}\right) \tag{3.1}
\end{equation*}
$$

If $S$ is join-generated by $f(X)$, then there exists a unique $(\vee, 0)$-homomorphism $\phi: S \rightarrow T$ such that $\phi(f(x))=g(x)$ for each $x \in X$.

If the converse of (3.1) also holds and $g(X)$ also join-generates $T$, then $\phi$ is an isomorphism.
Definition 3.2. An ideal of a $(\vee, 0)$-semilattice $S$ is a lower subset $I$ of $S$ such that $0 \in I$ and $u \vee v \in I$ for all $u, v \in I$. We denote by $\operatorname{Id} S$ the lattice of ideals of $S$.

Let $\phi: S \rightarrow T$ be a ( $\vee, 0)$-homomorphism. The 0 -kernel of $\phi$ is $\operatorname{ker}_{0} \phi=\{a \in S \mid$ $\phi(a)=0\}$; it is an ideal of $S$. We say that $\phi$ separates zero if $\operatorname{ker}_{0} \phi=\{0\}$.

Let $P$ be a poset, let $\vec{S}=\left(S_{p}, \phi_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ be a diagram of $(\vee, 0)$-semilattices. An ideal of $\vec{S}$ is a family $\left(I_{p}\right)_{p \in P}$ such that $I_{p}$ is an ideal of $S_{p}$ and $\phi_{p, q}\left(I_{p}\right) \subseteq I_{q}$ for all $p \leq q$ in $P$.

Let $\vec{\phi}=\left(\phi_{p}\right)_{p \in P}: \vec{S} \rightarrow \vec{T}$ be a natural transformation of $P$-indexed diagrams of $(\vee, 0)$-semilattices. The 0 -kernel of $\vec{\phi}$ is $\operatorname{ker}_{0} \vec{\phi}=\left(\operatorname{ker}_{0} \phi_{p}\right)_{p \in P}$, it is an ideal of $\vec{S}$.
Lemma 3.3. Let $S$ be a $(\vee, 0)$-semilattice, let $I \in \operatorname{Id} S$. Put:

$$
\theta_{I}=\left\{(x, y) \in S^{2} \mid(\exists u \in I)(x \vee u=y \vee u)\right\}
$$

The relation $\theta_{I}$ is a congruence of $S$.
Notation 3.4. We denote by $S / I$ the $(\vee, 0)$-semilattice $S / \theta_{I}$, where $\theta_{I}$ is the congruence defined in Lemma 3.3. Given $a \in S$, we denote by $a / I$ the equivalent class of $a$ for $\theta_{I}$. The ( $\vee, 0$ )-homomorphism $\phi: S \rightarrow S / I, a \mapsto a / I$ is the canonical projection. Notice that $\operatorname{ker}_{0} \phi=I$.

If $I=\{0\}$, we identify $S / I$ and $S$.
Lemma 3.5. Let $\phi: S \rightarrow T$ be a ( $\vee, 0)$-homomorphism, and let $I \in \operatorname{Id} S$ and $J \in \operatorname{Id} T$ such that $\phi(I) \subseteq J$. There exists a unique map $\psi: S / I \rightarrow T / J$ such that $\psi(a / I)=\phi(a) / J$ for each $a \in S$. Moreover, $\psi$ is a $(\mathrm{V}, 0)$-homomorphism.
Notation 3.6. We say that $\phi$ induces the $(\vee, 0)$-homomorphism $\psi: S / I \rightarrow T / J$ in Lemma 3.5.
Lemma 3.7. Let $P$ be a poset, let $\vec{S}=\left(S_{p}, \phi_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ be a diagram of $(\vee, 0)$-semilattices, and let $\vec{I}$ be an ideal of $\vec{S}$. Denote by $\psi_{p, q}: S_{p} / I_{p} \rightarrow S_{q} / I_{q}$ the $(\vee, 0)$-homomorphism induced by $\phi_{p, q}$, then $\left(S_{p} / I_{p}, \psi_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ is a diagram of ( $\mathrm{V}, 0)$-semilattices.
Notation 3.8. We denote by $\vec{S} / \vec{I}$ the diagram $\left(S_{p} / I_{p}, \psi_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ introduced in Lemma 3.7.
Lemma 3.9. Let $P$ be a poset, let $\vec{\phi}: \vec{S} \rightarrow \vec{T}$ be a natural transformation of $P$-indexed diagrams of $(\vee, 0)$-semilattices, let $\vec{I} \in \operatorname{Id} \vec{S}$ and $\vec{J} \in \operatorname{Id} \vec{T}$ such that $\phi_{p}\left(I_{p}\right) \subseteq J_{p}$ for all $p \in P$. Denote by $\psi_{p}: S_{p} / I_{p} \rightarrow T_{p} / J_{p}$ the ( $\left.\vee, 0\right)$-homomorphism induced by $\phi_{p}$. Then $\vec{\psi}$ is a natural transformation from $\vec{S} / \vec{I}$ to $\vec{T} / \vec{J}$.
Notation 3.10. We say that $\vec{\phi}$ induces $\vec{\psi}: \vec{S} / \vec{I} \rightarrow \vec{T} / \vec{J}$, the natural transformation defined in Lemma 3.9.
Definition 3.11. A ( $\vee, 0)$-homomorphism $\phi: S \rightarrow T$ is ideal-induced if $\phi$ is surjective and for all $x, y \in S$ with $\phi(x)=\phi(y)$ there exists $z \in S$ such that $x \vee z=y \vee z$ and $\phi(z)=0$.

Let $P$ be a poset, let $\vec{S}=\left(S_{p}, \phi_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ and $\vec{T}=\left(T_{p}, \psi_{p, q} \mid p \leq\right.$ $q$ in $P$ ) be $P$-indexed diagrams of ( $V, 0)$-semilattices. A natural transformation $\vec{\pi}=\left(\pi_{p}\right)_{p \in P}: \vec{S} \rightarrow \vec{T}$ is ideal-induced if $\pi_{p}$ is ideal-induced for each $p \in P$.

Remark 3.12. Let $I$ be an ideal of a $(\vee, 0)$-semilattice $A$, denote by $\pi: A \rightarrow A / I$ the canonical projection, then $\pi$ is ideal-induced.

The next lemmas give a characterization of ideal-induced ( $\vee, 0$ )-homomorphisms.
Lemma 3.13. Let $\phi: S \rightarrow T$ be a (V,0)-homomorphism. The following statements are equivalent
(1) $\phi$ is ideal-induced.
(2) The $(\vee, 0)$-homomorphism $\psi: S / \operatorname{ker}_{0} \phi \rightarrow T$ induced by $\phi$ is an isomorphism.

The following lemma expresses that, given a diagram $\vec{S}$ of $(\vee, 0)$-semilattices, the colimits of quotients of $\vec{S}$ are the quotients of the colimits of $\vec{S}$.

Lemma 3.14. Let $P$ be a directed poset, let $\vec{S}=\left(S_{p}, \phi_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ be a $P$ indexed diagram in $\mathbf{S e m}_{\vee, 0}$, and let $\left(S, \phi_{p} \mid p \in P\right)=\underline{\longrightarrow} \vec{S}$ be a directed colimit cocone in $\mathbf{S e m}_{\vee, 0}$. The following statements hold:
(1) Let $\vec{I}$ be an ideal of $\vec{S}$. Then $I=\bigcup_{p \in P} \phi_{p}\left(I_{p}\right)$ is an ideal of $S$. Moreover, denote by $\psi_{p}: S_{p} / I_{p} \rightarrow S / I$ the $(\vee, 0)$-homomorphism induced by $\phi_{p}$, for each $p \in P$. The following is a directed colimit cocone:

$$
\left(S / I, \psi_{p} \mid p \in P\right)=\underset{\longrightarrow}{\lim } \vec{S} / \vec{I} \quad \text { in } \mathbf{S e m}_{\vee, 0}
$$

(2) Let $I \in \operatorname{Id} S$. Put $I_{p}=\phi_{p}^{-1}(I)$ for each $p \in P$. Then $\vec{I}=\left(I_{p}\right)_{p \in P}$ is an ideal of $\vec{S}$, moreover $I=\bigcup_{p \in P} \phi_{p}\left(I_{p}\right)$.
Lemma 3.15. Let $\pi: A \rightarrow B$ be a surjective morphism of algebras. The $(\vee, 0)$-homomorphism $\operatorname{Con}_{\mathrm{c}} \pi$ is ideal-induced. Moreover, $\operatorname{ker}_{0}\left(\operatorname{Con}_{\mathrm{c}} \pi\right)=\left(\operatorname{Con}_{\mathrm{c}} A\right) \downarrow \operatorname{ker} \pi$.

Proposition 3.16. Let $S$ and $T$ be $(\vee, 0)$-semilattices with $T$ finite, let $\phi: S \rightarrow T$ be an ideal-induced $(\vee, 0)$-homomorphism, and let $X \subseteq S$ finite. There exists a finite $(\vee, 0)$-subsemilattice $S^{\prime}$ of $S$ such that $X \subseteq S^{\prime}$ and $\phi \upharpoonright S^{\prime}: S^{\prime} \rightarrow T$ is ideal-induced.

Proof. As $\phi$ is surjective and $X$ is finite, there exists a finite $(\vee, 0)$-subsemilattice $Y$ of $S$ such that $X \subseteq Y$ and $\phi(Y)=T$. Given $x, y \in Y$ with $\phi(x)=\phi(y)$ we fix $u_{x, y} \in S$ such that $\phi\left(u_{x, y}\right)=0$ and $x \vee u_{x, y}=y \vee u_{x, y}$. Let $U$ be the $(\vee, 0)$ subsemilattice of $S$ generated by $\left\{u_{x, y} \mid x, y \in Y\right.$ and $\left.\phi(x)=\phi(y)\right\}$. As $\phi(u)=0$ for all generators, $\phi(u)=0$ for each $u \in U$.

Let $S^{\prime}$ be the $(\vee, 0)$-subsemilattice of $S$ generated by $Y \cup U$. As $S^{\prime}$ is finitely generated, it is finite. As $Y \subseteq S^{\prime}, \phi\left(S^{\prime}\right)=T$. Let $a, b \in S^{\prime}$ such that $\phi(a)=\phi(b)$. There exist $x, y \in Y$ and $u, v \in U$ such that $a=x \vee u$ and $b=y \vee v$, thus $\phi(a)=\phi(x \vee u)=\phi(x) \vee \phi(u)=\phi(x)$. Similarly, $\phi(b)=\phi(y)$, hence $\phi(x)=\phi(y)$, moreover $x, y \in Y$, so $u_{x, y} \in U$. The element $w=u \vee v \vee u_{x, y}$ belongs to $U$, hence $\phi(w)=0$, moreover $w \in S^{\prime}$. From $x \vee u_{x, y}=y \vee u_{x, y}$ it follows that $a \vee w=b \vee w$. Therefore, $\phi \upharpoonright S^{\prime}$ is ideal-induced.

## 4. Partial algebras

In this section we introduce a few basic properties of partial algebras. We fix a similarity type $\mathscr{L}$. Given $\ell \in \mathscr{L}$ we denote by $\operatorname{ar}(\ell)$ the arity of $\ell$.

Definition 4.1. A partial algebra $A$ is a set (the universe of the partial algebra), given with a set $D_{\ell}=\operatorname{Def}_{\ell}(A) \subseteq A^{\operatorname{ar}(\ell)}$ and a map $\ell^{A}: D_{\ell} \rightarrow A$ called a partial operation, for each $\ell \in \mathscr{L}$.

Let $\ell \in \mathscr{L}$ be an $n$-ary operation. If $\vec{x} \in \operatorname{Def}_{\ell}(A)$ we say that $\ell^{A}(\vec{x})$ is $d e$ fined in $A$. We generalize this notion to terms in the usual way. For example, given binary operations $\ell_{1}$ and $\ell_{2}$ of a partial algebra $A$ and $x, y, z \in A$, $\ell_{1}^{A}\left(\ell_{2}^{A}(x, y), \ell_{1}^{A}(y, z)\right)$ is defined in $A$ if and only if $(x, y) \in \operatorname{Def}_{\ell_{2}}(A),(y, z) \in$ $\operatorname{Def}_{\ell_{1}}(A)$, and $\left(\ell_{2}^{A}(x, y), \ell_{1}^{A}(y, z)\right) \in \operatorname{Def}_{\ell_{1}}(A)$.

Given a term $t$, we denote by $\operatorname{Def}_{t}(A)$ the set of all tuples $\vec{x}$ of $A$ such that $t(\vec{x})$ is defined in $A$.

We denote $\ell(\vec{x})$ instead of $\ell^{A}(\vec{x})$ when there is no ambiguity. Any algebra $A$ has a natural structure of partial algebra with $\operatorname{Def}_{\ell}(A)=A^{\operatorname{ar}(\ell)}$ for each $\ell \in \mathscr{L}$.

Definition 4.2. Let $A, B$ be partial algebras. A morphism of partial algebras is a map $f: A \rightarrow B$ such that $f(\vec{x}) \in \operatorname{Def}_{\ell}(B)$ and $\ell(f(\vec{x}))=f(\ell(\vec{x}))$, for all $\ell \in \mathscr{L}$ and all $\vec{x} \in \operatorname{Def}_{\ell}(A)$.

The category of partial algebras, denoted by $\mathbf{P A l} \boldsymbol{g}_{\mathscr{L}}$, is the category in which the objects are the partial algebras and the arrows are the above-mentioned morphisms of partial algebras.

A morphism $f: A \rightarrow B$ of partial algebras is strong if $(f(A))^{\operatorname{ar}(\ell)} \subseteq \operatorname{Def}_{\ell}(B)$ for each $\ell \in \mathscr{L}$.

A partial algebra $A$ is finite if its universe is finite.
Remark 4.3. A morphism $f: A \rightarrow B$ of partial algebras is an isomorphism if and only if the following conditions are both satisfied
(1) The map $f$ is bijective.
(2) If $\ell(f(\vec{x}))$ is defined in $B$ then $\ell(\vec{x})$ is defined in $A$, for each operation $\ell \in \mathscr{L}$ and each tuple $\vec{x}$ of $A$.
We remind the reader that the converse of (2) is always true.
Definition 4.4. Given a partial algebra $A$, a partial subalgebra $B$ of $A$ is a subset $B$ of $A$ endowed with a structure of partial algebra such that $\operatorname{Def}_{\ell}(B) \subseteq \operatorname{Def}_{\ell}(A)$ and $\ell^{A}(\vec{x})=\ell^{B}(\vec{x})$ for all $\ell \in \mathscr{L}$ and all $\vec{x} \in \operatorname{Def}_{\ell}(B)$. The inclusion map from $A$ into $B$ is a morphism of partial algebras called the inclusion morphism.

A partial subalgebra $B$ of $A$ is full if whenever $\ell \in \mathscr{L}$ and $\vec{x} \in B^{\text {ar( }()}$ are such that $\ell^{A}(\vec{x})$ is defined and belongs to $B$, then $\ell(\vec{x})$ is defined in $B$. It is equivalent to the following equality:

$$
\operatorname{Def}_{\ell}(B)=\left\{\vec{x} \in \operatorname{Def}_{\ell}(A) \cap B^{\operatorname{ar}(\ell)} \mid \ell(\vec{x}) \in B\right\}, \quad \text { for each } \ell \in \mathscr{L} .
$$

A partial subalgebra $B$ of $A$ is strong if the inclusion map is a strong morphism, that is, $B^{\operatorname{ar}(\ell)} \subseteq \operatorname{Def}_{\ell}(A)$ for each $\ell \in \mathscr{L}$.

An embedding of partial algebras is a one-to-one morphism of partial algebras.
Notation 4.5. Let $f: A \rightarrow B$ be a morphism of partial algebras, let $X$ be a partial subalgebra of $A$. The set $f(X)$ can be endowed with a natural structure of partial algebra, by setting $\operatorname{Def}_{\ell}(f(X))=f\left(\operatorname{Def}_{\ell}(X)\right)=\left\{f(\vec{x}) \mid \vec{x} \in \operatorname{Def}_{\ell}(X)\right\}$, for each $\ell \in \mathscr{L}$. Similarly, let $Y$ be a partial subalgebra of $B$. The set $f^{-1}(Y)$ can be endowed with a natural structure of partial algebra, by setting $\operatorname{Def}_{\ell}\left(f^{-1}(Y)\right)=$ $f^{-1}\left(\operatorname{Def}_{\ell}(Y)\right)=\left\{\vec{x} \in A \mid f(\vec{x}) \in \operatorname{Def}_{\ell}(Y)\right\}$, for each $\ell \in \mathscr{L}$.

Remark. Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be morphisms of partial algebras, let $X$ a sub-partial algebra of $A$, then $(g \circ f)(X)=g(f(X))$ as partial algebras. Let $Z$ be a partial subalgebra of $X$, then $(g \circ f)^{-1}(Z)=f^{-1}\left(g^{-1}(Z)\right)$ as partial algebras.

Let $f: A \rightarrow B$ be a morphism of partial algebras, let $X$ be a partial subalgebra of $A$. Then $X$ is a partial subalgebra of $f^{-1}(f(X))$. In particular $f^{-1}(f(A))=A$ as partial algebras. Let $Y$ be a partial subalgebra of $B$, then $f\left(f^{-1}(Y)\right)$ is a partial subalgebra of $Y$.

If $\mathscr{L}$ is infinite, then there are a finite partial algebra $A$ (even with one element) and an infinite chain of partial subalgebras of $A$ with union $A$. In particular, $A$ is not finitely presented in the category $\mathrm{PAlg}_{\mathscr{L}}$.

A morphism $f: A \rightarrow B$ of partial algebras is strong if and only if $f(A)$ is a strong partial subalgebra of $B$.

Lemma 4.6. An embedding $f: A \rightarrow B$ of partial algebras is an isomorphism if and only if $f(A)=B$ as partial algebras.

Proof. Assume that $f$ is an isomorphism, let $g$ be its inverse. Notice that $f(A)$ is a partial subalgebra of $B$ and $B=f(g(B))$ is a partial subalgebra of $f(A)$, therefore $B=f(A)$ as partial algebras.

Conversely, assume that $B=f(A)$ as partial algebras. Then $f$ is surjective, moreover $f$ is an embedding, so $f$ is a bijection. Let $g=f^{-1}$ in Set. Let $\vec{y} \in$ $\operatorname{Def}_{\ell}(B)$. As $B=f(A)$, there exists $\vec{x} \in \operatorname{Def}_{\ell}(A)$ such that $f(\vec{x})=\vec{y}$, thus $g(\vec{y})=$ $\vec{x} \in \operatorname{Def}_{\ell}(A)$. Moreover $g(\ell(\vec{y}))=g(\ell(f(\vec{x})))=g(f(\ell(\vec{x})))=\ell(\vec{x})=\ell(g(\vec{y}))$.
Notation 4.7. Let $A$ be a partial algebra, let $X$ be a subset of $A$. We define inductively, for each $n<\omega$,

$$
\begin{aligned}
\langle X\rangle_{A}^{0} & =X \cup\{c \mid c \text { is a constant of } \mathscr{L}\} \\
\langle X\rangle_{A}^{n+1} & =\langle X\rangle_{A}^{n} \cup\left\{\ell(\vec{x}) \mid \ell \in \mathscr{L}, \vec{x} \in \operatorname{Def}_{\ell}(A), \vec{x} \text { is an } \operatorname{ar}(\ell) \text {-tuple of }\langle X\rangle_{A}^{n}\right\}
\end{aligned}
$$

We endow $\langle X\rangle_{A}^{n}$ with the induced structure of full partial subalgebra of $A$. If $\mathscr{L}$ and $X$ are both finite, then $\langle X\rangle_{A}^{n}$ is finite for each $n<\omega$. If $A$ is understood, we shall simply denote this partial algebra by $\langle X\rangle^{n}$.
Definition 4.8. A partial algebra $A$ satisfies an identity $t_{1}=t_{2}$ if $t_{1}(\vec{x})=t_{2}(\vec{x})$ for each tuple $\vec{x}$ of $A$ such that both $t_{1}(\vec{x})$ and $t_{2}(\vec{x})$ are defined in $A$. Otherwise we say that $A$ fails $t_{1}=t_{2}$.

Let $\mathcal{V}$ be a variety of algebras, a partial algebra $A$ is a partial algebra of $\mathcal{V}$ if $A$ satisfies all identities of $\mathcal{V}$.

Remark. Let $A$ be a partial algebra, let $\ell \in \mathscr{L}$. If $\operatorname{Def}_{\ell}(A)=\emptyset$ then $A$ satisfies $\ell(\vec{x})=y$, vacuously.

If $A$ fails $t_{1}=t_{2}$, then there exists a tuple $\vec{x}$ of $A$ such that $t_{1}(\vec{x})$ and $t_{2}(\vec{x})$ are both defined and $t_{1}(\vec{x}) \neq t_{2}(\vec{x})$.

Lemma 4.9. The category $\operatorname{PAlg}_{\mathscr{L}}$ has all directed colimits. Moreover, given a directed poset $P$, a $P$-indexed diagram $\vec{A}=\left(A_{p}, f_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ in $\mathbf{P A l g} \mathscr{L}_{\mathscr{L}}$, and a directed colimit cocone:

$$
\begin{equation*}
\left(A, f_{p} \mid p \in P\right)=\xrightarrow{\lim }\left(A_{p}, f_{p, q} \mid p \leq q \text { in } P\right), \quad \text { in Set, } \tag{4.1}
\end{equation*}
$$

the set $A$ can be uniquely endowed with a structure of partial algebra such that:

- $\operatorname{Def}_{\ell}(A)=\left\{f_{p}(\vec{x}) \mid p \in P\right.$ and $\left.\vec{x} \in \operatorname{Def}_{\ell}\left(A_{p}\right)\right\}$, for each $\ell \in \mathscr{L}$;
- $\ell\left(f_{p}(\vec{x})\right)=f_{p}(\ell(\vec{x}))$ for each $p \in P$, all $\ell \in \mathscr{L}$, and all $\vec{x} \in \operatorname{Def}_{\ell}\left(A_{p}\right)$.

Moreover, if $A$ is endowed with this structure of partial algebra, the following statements hold:
(1) $\left(A, f_{p} \mid p \in P\right)=\underline{\longrightarrow}\left(A_{p}, f_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ in $P A \lg _{\mathscr{L}}$.
(2) Assume that for each $\ell \in \mathscr{L}$, each $p \in P$, and each $\operatorname{ar}(\ell)$-tuple $\vec{x}$ of $A_{p}$ there exists $q \geq p$ such that $f_{p, q}(\vec{x}) \in \operatorname{Def}_{\ell}\left(A_{q}\right)$. Then $A$ is an algebra, that is, $\operatorname{Def}_{\ell}(A)=A^{\operatorname{ar}(\ell)}$ for each $\ell \in \mathscr{L}$.
(3) If $P$ has no maximal element and $f_{p, q}$ is strong for all $p<q$ in $P$, then $A$ is an algebra.
(4) $\operatorname{Def}_{t}(A)=\left\{f_{p}(\vec{x}) \mid p \in P\right.$ and $\left.\vec{x} \in \operatorname{Def}_{t}\left(A_{p}\right)\right\}$ for each term $t$ of $\mathscr{L}$.
(5) Let $t_{1}=t_{2}$ be an identity. If $A_{p}$ satisfies $t_{1}=t_{2}$ for all $p \in P$, then $A$ satisfies $t_{1}=t_{2}$.
Proof. Put $\operatorname{Def}_{\ell}(A)=\left\{f_{p}(\vec{x}) \mid p \in P\right.$ and $\left.\vec{x} \in \operatorname{Def}_{\ell}\left(A_{p}\right)\right\}$, for each $\ell \in \mathscr{L}$.
Let $\ell \in \mathscr{L}$, let $\vec{x} \in \operatorname{Def}_{\ell}(A)$. There exist $p \in P$ and $\vec{y} \in \operatorname{Def}_{\ell}\left(A_{p}\right)$ such that $\vec{x}=$ $f_{p}(\vec{y})$. We first show that $f_{p}(\ell(\vec{y}))$ does not depend on the choice of $p$ and $\vec{y}$. Let $q \in P$ and $\vec{z} \in \operatorname{Def}_{\ell}\left(A_{q}\right)$ such that $\vec{x}=f_{q}(\vec{z})$. As $f_{p}(\vec{y})=\vec{x}=f_{q}(\vec{z})$, it follows from (4.1) that there exists $r \geq p, q$ such that $f_{p, r}(\vec{y})=f_{q, r}(\vec{z})$. Therefore the following equalities hold:
$f_{p}(\ell(\vec{y}))=f_{r}\left(f_{p, r}(\ell(\vec{y}))\right)=f_{r}\left(\ell\left(f_{p, r}(\vec{y})\right)\right)=f_{r}\left(\ell\left(f_{q, r}(\vec{z})\right)\right)=f_{r}\left(f_{q, r}(\ell(\vec{z}))\right)=f_{q}(\ell(\vec{z}))$.
Hence $\ell\left(f_{p}(\vec{y})\right)=f_{p}(\ell(\vec{y}))$ for all $p \in P$ and all $\vec{y} \in \operatorname{Def}_{\ell}\left(A_{p}\right)$ uniquely define a partial operation $\ell: \operatorname{Def}_{\ell}(A) \rightarrow A$. Moreover $f_{p}$ is a morphism of partial algebras for each $p \in P$.

Let $\left(B, g_{p} \mid p \in P\right)$ be a cocone over $\left(A_{p}, f_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ in $\mathbf{P A l g}{ }_{\mathscr{L}}$. In particular, it is a cocone in Set, so there exists a unique map $h: A \rightarrow B$ such that $h \circ f_{p}=g_{p}$ for each $p \in P$. Let $\ell \in \mathscr{L}$, let $\vec{x} \in \operatorname{Def}_{\ell}(A)$. There exist $p \in P$ and $\vec{y} \in \operatorname{Def}_{\ell}\left(A_{p}\right)$ such that $\vec{x}=f_{p}(\vec{y})$, thus $h(\vec{x})=h\left(f_{p}(\vec{y})\right)=g_{p}(\vec{y})$. As $g_{p}$ is a morphism of partial algebras and $\vec{y} \in \operatorname{Def}_{\ell}\left(A_{p}\right)$, we obtain that $h(\vec{x}) \in \operatorname{Def}_{\ell}(B)$. Moreover the following equalities hold:

$$
\ell(h(\vec{x}))=\ell\left(g_{p}(\vec{y})\right)=g_{p}(\ell(\vec{y}))=h\left(f_{p}(\ell(\vec{y}))\right)=h\left(\ell\left(f_{p}(\vec{y})\right)\right)=h(\ell(\vec{x}))
$$

Hence $h$ is a morphism of partial algebras. Therefore:

$$
\left(A, f_{p} \mid p \in P\right)=\underline{\longrightarrow}\left(A_{p}, f_{p, q} \mid p \leq q \text { in } P\right) \quad \text { in } P A \lg _{\mathscr{L}}
$$

Assume that for each $\ell \in \mathscr{L}$, for all $p \in P$, and for all $\operatorname{ar}(\ell)$-tuples $\vec{x}$ of $A_{p}$, there exists $q \geq p$ such that $f_{p, q}(\vec{x}) \in \operatorname{Def}_{\ell}\left(A_{q}\right)$.

Let $\ell \in \mathscr{L}$, let $\vec{x}$ be an $\operatorname{ar}(\ell)$-tuple of $A$. There exist $p \in P$ and a tuple $\vec{y}$ of $A_{p}$ such that $\vec{x}=f_{p}(\vec{y})$. Let $q \geq p$ such that $f_{p, q}(\vec{y}) \in \operatorname{Def}_{\ell}\left(A_{q}\right)$. It follows that $\vec{x}=f_{p}(\vec{y})=f_{q}\left(f_{p, q}(\vec{y})\right)$ belongs to $\operatorname{Def}_{\ell}(A)$. Therefore $A$ is an algebra.

The statement (3) follows directly from (2). The statement (4) is proved by a straightforward induction on terms, and (5) is an easy consequence of (4).

## 5. Pregamps

A pregamp is a partial algebra endowed with a semilattice-valued "distance" (cf. (1)-(3)) compatible with all operations of $A$ (cf. (4)). It is a generalization of the notion of semilattice-metric space defined in [5, Section 5-1].
Definition 5.1. Let $A$ be a partial algebra, let $S$ be a $(\vee, 0)$-semilattice. A $S$-valued partial algebra distance on $A$ is a map $\delta: A^{2} \rightarrow S$ such that:
(1) $\delta(x, y)=0$ if and only if $x=y$, for all $x, y \in A$.
(2) $\delta(x, y)=\delta(y, x)$, for all $x, y \in A$.
(3) $\delta(x, y) \leq \delta(x, z) \vee \delta(z, y)$, for all $x, y, z \in A$.
(4) $\delta(\ell(\vec{x}), \ell(\vec{y})) \leq \bigvee_{k<\operatorname{ar}(\ell)} \delta\left(x_{k}, y_{k}\right)$, for all $\ell \in \mathscr{L}$ and all $\vec{x}, \vec{y} \in \operatorname{Def}_{\ell}(A)$.

Then we say that $\boldsymbol{A}=(A, \delta, S)$ is a pregamp. We shall generally write $\delta_{\boldsymbol{A}}=\delta$ and $\widetilde{A}=S$.

The pregamp is distance-generated if it satisfies the following additional property:
(5) $S$ is join-generated by $\delta_{\boldsymbol{A}}\left(A^{2}\right)$. That is, for all $\alpha \in S$ there are $n \geq 0$ and $n$-tuples $\vec{x}, \vec{y}$ of $A$ such that $\alpha=\bigvee_{k<n} \delta_{\boldsymbol{A}}\left(x_{k}, y_{k}\right)$.
Example 5.2. Let $A$ be an algebra. We remind the reader that $\Theta_{A}(x, y)$ denotes the smallest congruence that identifies $x$ and $y$, for all $x, y \in A$. This defines a distance $\Theta_{A}: A^{2} \rightarrow \operatorname{Con}_{\mathrm{c}} A$. Moreover, $\left(A, \Theta_{A}, \operatorname{Con}_{\mathrm{c}} A\right)$ is a distance-generated pregamp.

A straightforward induction argument on the length of the term $t$ yields the following lemma.
Lemma 5.3. Let $\boldsymbol{A}$ be a pregamp, let $t$ be an $n$-ary term, and let $\vec{x}, \vec{y} \in \operatorname{Def}_{t}(A)$. The following inequality holds:

$$
\delta_{\boldsymbol{A}}(t(\vec{x}), t(\vec{y})) \leq \bigvee_{k<n} \delta_{\boldsymbol{A}}\left(x_{k}, y_{k}\right)
$$

We say that $\delta_{\boldsymbol{A}}$ and $t$ are compatible.
Definition 5.4. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be pregamps. A morphism from $\boldsymbol{A}$ to $\boldsymbol{B}$ is an ordered pair $\boldsymbol{f}=(f, \widetilde{f})$ such that $f: A \rightarrow B$ is a morphism of partial algebras, $\widetilde{f}: \widetilde{A} \rightarrow \widetilde{B}$ is a $(\vee, 0)$-homomorphism, and $\delta_{\boldsymbol{B}}(f(x), f(y))=\widetilde{f}\left(\delta_{\boldsymbol{A}}(x, y)\right)$ for all $x, y \in A$.

Given morphisms $\boldsymbol{f}: \boldsymbol{A} \rightarrow \boldsymbol{B}$ and $\boldsymbol{g}: \boldsymbol{B} \rightarrow \boldsymbol{C}$ of pregamps, the pair $\boldsymbol{g} \circ \boldsymbol{f}=$ $(g \circ f, \widetilde{g} \circ \widetilde{f})$ is a morphism from $\boldsymbol{A}$ to $\boldsymbol{C}$.

We denote by $\mathbf{P G a m p}_{\mathscr{L}}$ the category of pregamps with the morphisms defined above.

We denote by $\boldsymbol{P}_{\text {ga }}$ the functor from the category of $\mathscr{L}$-algebras to $\mathbf{P G a m p}_{\mathscr{L}}$ that maps an algebra $A$ to $\left(A, \Theta_{A}, \operatorname{Con}_{\mathrm{c}} A\right)$, and a morphism of algebras $f$ to $\left(f, \operatorname{Con}_{\mathrm{c}} f\right)$. We denote by $\boldsymbol{C}_{\mathrm{pg}}$ the functor from $\mathbf{P G a m p}_{\mathscr{L}}$ to $\mathbf{S e m}_{\mathrm{V}, 0}$ that maps a pregamp $\boldsymbol{A}$ to $\widetilde{A}$, and maps a morphism of pregamps $\boldsymbol{f}: \boldsymbol{A} \rightarrow \boldsymbol{B}$ to the ( $\vee, 0)$ homomorphism $\tilde{f}$.

Remark 5.5. A morphism $\boldsymbol{f}: \boldsymbol{A} \rightarrow \boldsymbol{B}$ of pregamps is an isomorphism if and only if $f$ is an isomorphism of partial algebras and $\widetilde{f}$ is an isomorphism of $(\mathrm{V}, 0)$-semilattices.

Notice that $\boldsymbol{C}_{\mathrm{pg}} \circ \boldsymbol{P}_{\mathrm{ga}}=\mathrm{Con}_{\mathrm{c}}$.
We leave to the reader the straightforward proof of the following lemma.
Lemma 5.6. The category PGamp $_{\mathscr{L}}$ has all directed colimits. Moreover, given a directed poset $P$, a $P$-indexed diagram $\overrightarrow{\boldsymbol{A}}=\left(\boldsymbol{A}_{p}, \boldsymbol{f}_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ in $\mathbf{P G a m p} \mathscr{L}_{\mathscr{L}}$, a directed colimit cocone $\left(A, f_{p} \mid p \in P\right)=\xrightarrow{\lim }\left(A_{p}, f_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ in $\mathbf{P A l g} \mathscr{L}_{\mathscr{L}}$, and a directed colimit cocone $\left(\widetilde{A}, \tilde{f}_{p} \mid p \in \vec{P}\right)=\underline{\longrightarrow}\left(\widetilde{A}_{p}, \widetilde{f}_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ in Sem $_{\vee, 0}$, there exists a unique $\widetilde{A}$-valued partial algebra distance $\delta_{\boldsymbol{A}}$ on $A$ such that $\delta_{\boldsymbol{A}}\left(f_{p}(x), f_{p}(y)\right)=\widetilde{f}_{p}\left(\delta_{\boldsymbol{A}_{p}}(x, y)\right)$ for all $p \in P$ and all $x, y \in A_{p}$.

Furthermore $\boldsymbol{A}=\left(A, \delta_{\boldsymbol{A}}, \widetilde{A}\right)$ is a pregamp, $\boldsymbol{f}_{p}: \boldsymbol{A}_{p} \rightarrow \boldsymbol{A}$ is a morphism of pregamps for each $p \in P$, and the following is a directed colimit cocone:

$$
\left(\boldsymbol{A}, \boldsymbol{f}_{p} \mid p \in P\right)=\underset{\longrightarrow}{\lim }\left(\boldsymbol{A}_{p}, \boldsymbol{f}_{p, q} \mid p \leq q \text { in } P\right), \quad \text { in } \operatorname{PGamp}_{\mathscr{L}} .
$$

Moreover if $\boldsymbol{A}_{p}$ is distance-generated for each $p \in P$, then $\boldsymbol{A}$ is distance-generated.
Remark 5.7. As an immediate application of Lemma 5.6, and the fact that $\mathrm{Con}_{\mathrm{c}}$ preserves directed colimits, we obtain that both $\boldsymbol{C}_{\mathrm{pg}}$ and $\boldsymbol{P}_{\text {ga }}$ preserve directed colimits.

Definition 5.8. An embedding $\boldsymbol{f}: \boldsymbol{B} \rightarrow \boldsymbol{A}$ of pregamps is a morphism of pregamps such that $f$ and $\tilde{f}$ are both one-to-one.

A sub-pregamp of a pregamp $\boldsymbol{A}$ is a pregamp $\boldsymbol{B}$ such that $B$ is a partial subalgebra of $A, \widetilde{B}$ is a $(\vee, 0)$-subsemilattice of $\widetilde{A}$, and $\delta_{\boldsymbol{B}}=\delta_{\boldsymbol{A}} \upharpoonright B^{2}$.

If $f: B \rightarrow A$ and $\widetilde{f}: \widetilde{B} \rightarrow \widetilde{A}$ denote the inclusion maps, the morphism of pregamps $\boldsymbol{f}=(f, \widetilde{f})$ is called the canonical embedding.

Notation 5.9. Let $\boldsymbol{f}: \boldsymbol{B} \rightarrow \boldsymbol{A}$ be a morphism of pregamps. Given a sub-pregamp $\boldsymbol{C}$ of $\boldsymbol{B}$, the triple $\boldsymbol{f}(\boldsymbol{C})=\left(f(C), \delta_{\boldsymbol{A}} \upharpoonright(f(C))^{2}, \widetilde{f}(\widetilde{C})\right.$ ) (see Notation 4.5) is a subpregamp of $\boldsymbol{A}$.

For a sub-pregamp $\boldsymbol{C}$ of $\boldsymbol{A}$, the triple $\boldsymbol{f}^{-1}(\boldsymbol{C})=\left(f^{-1}(C), \delta_{\boldsymbol{B}} \upharpoonright\left(f^{-1}(C)\right)^{2}, \widetilde{f}^{-1}(\widetilde{\boldsymbol{C}})\right)$ is a sub-pregamp of $\boldsymbol{B}$.

We leave to the reader the straightforward proof of the following description of sub-pregamps and embeddings.

Proposition 5.10. The following statements hold.
(1) Let $\boldsymbol{A}$ be a pregamp, let $B$ be a partial subalgebra of $A$, let $\widetilde{B}$ be a $(\vee, 0)$ subsemilattice of $\widetilde{A}$ that contains $\delta_{\boldsymbol{A}}\left(B^{2}\right)$. Put $\delta_{\boldsymbol{B}}=\delta_{\boldsymbol{A}} \upharpoonright B^{2}$. Then $\left(B, \delta_{\boldsymbol{B}}, \widetilde{B}\right)$ is a sub-pregamp of $\boldsymbol{A}$. Moreover, all sub-pregamps of $\boldsymbol{A}$ are of this form.
(2) Let $\boldsymbol{f}: \boldsymbol{B} \rightarrow \boldsymbol{A}$ be a morphism of pregamps. Then $f$ is an embedding of partial algebras if and only if $\widetilde{f}$ separates 0 . Moreover $\boldsymbol{f}$ is an embedding if and only if $\tilde{f}$ is an embedding.
(3) Let $\boldsymbol{f}: \boldsymbol{B} \rightarrow \boldsymbol{A}$ be an embedding of pregamps. The restriction $\boldsymbol{f}: \boldsymbol{B} \rightarrow \boldsymbol{f}(\boldsymbol{B})$ is an isomorphism of pregamps.

The following result appears in [9, Theorem 10.4]. It gives a description of finitely generated congruences of a general algebra.

Lemma 5.11. Let $B$ be an algebra, let $m$ be a positive integer, let $x, y \in B$, and let $\vec{x}, \vec{y}$ be m-tuples of $B$. Then $\Theta_{B}(x, y) \leq \bigvee_{i<m} \Theta_{B}\left(x_{i}, y_{i}\right)$ if and only if there are a positive integer $n$, a list $\vec{z}$ of parameters from $B$, and terms $t_{0}, \ldots, t_{n}$ such that

$$
\begin{aligned}
x & =t_{0}(\vec{x}, \vec{y}, \vec{z}), \\
y & =t_{n}(\vec{x}, \vec{y}, \vec{z}), \\
t_{j}(\vec{y}, \vec{x}, \vec{z}) & =t_{j+1}(\vec{x}, \vec{y}, \vec{z}) \quad(\text { for all } j<n) .
\end{aligned}
$$

The following lemma shows that the obvious direction of Lemma 5.11 holds for pregamps.

Lemma 5.12. Let $\boldsymbol{B}$ be a pregamp, let $m$ be a positive integer, let $x, y \in B$, and let $\vec{x}, \vec{y}$ be m-tuples of $B$. Assume that there are a positive integer $n$, a list $\vec{z}$ of parameters from $B$, and terms $t_{0}, \ldots, t_{n}$ such that the following equalities hold and
all evaluations are defined

$$
\begin{aligned}
x & =t_{0}(\vec{x}, \vec{y}, \vec{z}) \\
y & =t_{n}(\vec{x}, \vec{y}, \vec{z}) \\
t_{j}(\vec{y}, \vec{x}, \vec{z}) & =t_{j+1}(\vec{x}, \vec{y}, \vec{z}), \quad(\text { for all } j<n) .
\end{aligned}
$$

Then $\delta_{B}(x, y) \leq \bigvee_{i<m} \delta_{B}\left(x_{i}, y_{i}\right)$.
Proof. As $\delta_{B}$ is compatible with terms (cf. Lemma 5.3), and $\delta_{\boldsymbol{B}}(u, u)=0$ for each $u \in B$, the following inequality holds:

$$
\delta_{\boldsymbol{B}}\left(t_{j}(\vec{x}, \vec{y}, \vec{z}), t_{j}(\vec{y}, \vec{x}, \vec{z})\right) \leq \bigvee_{k<n} \delta_{\boldsymbol{B}}\left(x_{k}, y_{k}\right), \quad \text { for all } j<n
$$

Hence:

$$
\delta_{\boldsymbol{B}}(x, y) \leq \bigvee_{j<n} \delta_{\boldsymbol{B}}\left(t_{j}(\vec{x}, \vec{y}, \vec{z}), t_{j}(\vec{y}, \vec{x}, \vec{z})\right) \leq \bigvee_{k<n} \delta_{\boldsymbol{B}}\left(x_{k}, y_{k}\right)
$$

The following definition expresses that whenever two elements of $A$ are identified by a "congruence" of $A$, then there is a "good reason" for this in $B$ (cf. Lemma 5.11).

Definition 5.13. A morphism $\boldsymbol{f}: \boldsymbol{A} \rightarrow \boldsymbol{B}$ of pregamps is congruence-tractable if for all $m<\omega$ and for all $x, y, x_{0}, y_{0}, \ldots, x_{m-1}, y_{m-1}$ in $A$ such that:

$$
\delta_{\boldsymbol{B}}(x, y) \leq \bigvee_{k<m} \delta_{\boldsymbol{B}}\left(x_{k}, y_{k}\right),
$$

there are a positive integer $n$, a list $\vec{z}$ of parameters from $B$, and terms $t_{0}, \ldots$, $t_{n}$ such that the following equations are satisfied in $B$ (in particular, all the corresponding terms are defined).

$$
\begin{aligned}
f(x) & =t_{0}(f(\vec{x}), f(\vec{y}), \vec{z}), \\
f(y) & =t_{n}(f(\vec{x}), f(\vec{y}), \vec{z}), \\
t_{j}(f(\vec{y}), f(\vec{x}), \vec{z}) & =t_{j+1}(f(\vec{x}), f(\vec{y}), \vec{z}) \quad(\text { for all } j<n) .
\end{aligned}
$$

Lemma 5.14. Let $P$ be directed poset and let $\overrightarrow{\boldsymbol{A}}=\left(\boldsymbol{A}_{p}, \boldsymbol{f}_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ be a direct system of pregamps. Assume that for each $p \in P$ there exists $q \geq p$ in $P$ such that $\boldsymbol{f}_{p, q}$ is congruence-tractable. Let:

$$
\left(\boldsymbol{A}, \boldsymbol{f}_{p} \mid p \in P\right)=\underset{\longrightarrow}{\lim }\left(\boldsymbol{A}_{p}, \boldsymbol{f}_{p, q} \mid p \leq q \text { in } P\right), \quad \text { in } \mathbf{P G a m p}_{\mathscr{L}} .
$$

If $A$ is an algebra then the following statements hold:
(1) Let $x, y \in A$, let $m<\omega$, let $x_{0}, y_{0}, \ldots, x_{m-1}, y_{m-1}$ in $A$. The following two inequalities are equivalent:

$$
\begin{align*}
& \delta_{\boldsymbol{A}}(x, y) \leq \bigvee_{k<m} \delta_{\boldsymbol{A}}\left(x_{k}, y_{k}\right)  \tag{5.1}\\
& \Theta_{A}(x, y) \leq \bigvee_{k<m} \Theta_{A}\left(x_{k}, y_{k}\right) \tag{5.2}
\end{align*}
$$

(2) There exists a unique $(\vee, 0)$-homomorphism $\phi: \operatorname{Con}_{\mathrm{c}} A \rightarrow \widetilde{A}$ such that:

$$
\phi\left(\Theta_{A}(x, y)\right)=\delta_{\boldsymbol{A}}(x, y), \quad \text { for all } x, y \in A
$$

Moreover $\phi$ is an embedding.
(3) If $\boldsymbol{A}$ is distance-generated, then the ( $\vee, 0)$-homomorphism $\phi$ above is an isomorphism.

Proof. Lemma 5.6 and Lemma 4.9 imply that the following are directed colimits cocones

$$
\begin{align*}
& \left(A, f_{p} \mid p \in P\right)=\underset{\longrightarrow}{\lim }\left(A_{p}, f_{p, q} \mid p \leq q \text { in } P\right), \quad \text { in } \mathbf{P A l g}_{\mathscr{L}} .  \tag{5.3}\\
& \left(A, f_{p} \mid p \in P\right)=\underset{\longrightarrow}{\lim }\left(A_{p}, f_{p, q} \mid p \leq q \text { in } P\right), \quad \text { in Set. }  \tag{5.4}\\
& \left(\widetilde{A}, \widetilde{f}_{p} \mid p \in P\right)=\underset{\longrightarrow}{\lim }\left(\widetilde{A}_{p}, \widetilde{f}_{p, q} \mid p \leq q \text { in } P\right), \quad \text { in } \operatorname{Sem}_{\vee, 0} . \tag{5.5}
\end{align*}
$$

(1) Let $x, y \in A$, let $m<\omega$, and let $\vec{x}, \vec{y}$ be $m$-tuples of $A$.

Assume that (5.1) holds. It follows from (5.4) that there are $p \in P, x^{\prime}, y^{\prime} \in A$, and $m$-tuples $\vec{x}^{\prime}, \vec{y}^{\prime}$ of $A_{p}$, such that $x=f_{p}\left(x^{\prime}\right), y=f_{p}\left(y^{\prime}\right), \vec{x}=f_{p}\left(\vec{x}^{\prime}\right)$, and $\vec{y}=$ $f_{p}\left(\vec{y}^{\prime}\right)$. The inequality (5.1) can be written

$$
\delta_{\boldsymbol{A}}\left(f_{p}\left(x^{\prime}\right), f_{p}\left(y^{\prime}\right)\right) \leq \bigvee_{k<m} \delta_{\boldsymbol{A}}\left(f_{p}\left(x_{k}^{\prime}\right), f_{p}\left(y_{k}^{\prime}\right)\right)
$$

This implies:

$$
\widetilde{f}_{p}\left(\delta_{\boldsymbol{A}_{p}}\left(x^{\prime}, y^{\prime}\right)\right) \leq \widetilde{f}_{p}\left(\bigvee_{k<m} \delta_{\boldsymbol{A}_{p}}\left(x_{k}^{\prime}, y_{k}^{\prime}\right)\right) .
$$

Hence, it follows from (5.5) that there exists $q \geq p$ with:

$$
\widetilde{f}_{p, q}\left(\delta_{\boldsymbol{A}_{p}}\left(x^{\prime}, y^{\prime}\right)\right) \leq \widetilde{f}_{p, q}\left(\bigvee_{k<m} \delta_{\boldsymbol{A}_{p}}\left(x_{k}^{\prime}, y_{k}^{\prime}\right)\right)
$$

so, changing $p$ to $q, x^{\prime}$ to $f_{p, q}\left(x^{\prime}\right), y^{\prime}$ to $f_{p, q}\left(y^{\prime}\right), \vec{x}^{\prime}$ to $f_{p, q}\left(\vec{x}^{\prime}\right)$, and $\vec{y}^{\prime}$ to $f_{p, q}\left(\vec{y}^{\prime}\right)$, we can assume that:

$$
\delta_{\boldsymbol{A}_{p}}\left(x^{\prime}, y^{\prime}\right) \leq \bigvee_{k<m} \delta_{\boldsymbol{A}_{p}}\left(x_{k}^{\prime}, y_{k}^{\prime}\right)
$$

Let $q \geq p$ in $P$ such that $f_{p, q}$ is congruence-tractable. There are a positive integer $n$, a list $\vec{z}$ of parameters from $A_{q}$, and terms $t_{0}, \ldots, t_{n}$ such that the following equations are satisfied in $A_{q}$ :

$$
\begin{aligned}
f_{p, q}\left(x^{\prime}\right) & =t_{0}\left(f_{p, q}\left(\vec{x}^{\prime}\right), f_{p, q}\left(\vec{y}^{\prime}\right), \vec{z}\right), \\
f_{p, q}\left(y^{\prime}\right) & =t_{n}\left(f_{p, q}\left(\vec{x}^{\prime}\right), f_{p, q}\left(\vec{y}^{\prime}\right), \vec{z}\right), \\
t_{k}\left(f_{p, q}\left(\vec{y}^{\prime}\right), f_{p, q}\left(\vec{x}^{\prime}\right), \vec{z}\right) & =t_{k+1}\left(f_{p, q}\left(\vec{x}^{\prime}\right), f_{p, q}\left(\vec{y}^{\prime}\right), \vec{z}\right), \quad(\text { for all } k<n) .
\end{aligned}
$$

Hence, applying $f_{q}$, we obtain

$$
\begin{aligned}
x & =t_{0}\left(\vec{x}, \vec{y}, f_{q}(\vec{z})\right), \\
y & =t_{n}\left(\vec{x}, \vec{y}, f_{q}(\vec{z})\right), \\
t_{k}\left(\vec{y}, \vec{x}, f_{q}(\vec{z})\right) & =t_{k+1}\left(\vec{x}, \vec{y}, f_{q}(\vec{z})\right), \quad(\text { for all } k<n) .
\end{aligned}
$$

Therefore, it follows from Lemma 5.11 that (5.2) holds.
Conversely, assume that (5.2) holds. It follows from Lemma 5.11 that there are a positive integer $n$, a list $\vec{z}$ of parameters from $A$, and terms $t_{0}, \ldots, t_{n}$ such that

$$
\begin{aligned}
x & =t_{0}(\vec{x}, \vec{y}, \vec{z}) \\
y & =t_{n}(\vec{x}, \vec{y}, \vec{z}) \\
t_{j}(\vec{y}, \vec{x}, \vec{z}) & =t_{j+1}(\vec{x}, \vec{y}, \vec{z}), \quad(\text { for all } j<n) .
\end{aligned}
$$

We conclude, using Lemma 5.12, that (5.1) holds.

As $\operatorname{Con}_{\mathrm{c}} A$ is generated by $\left\{\Theta_{A}(x, y) \mid x, y \in A\right\}$, the statement (2) follows from Proposition 3.1. Moreover if we assume that $\boldsymbol{A}$ is distance-generated, that is $\widetilde{A}$ is join-generated by $\left\{\delta_{\boldsymbol{A}}(x, y) \mid x, y \in A\right\}$, then $\phi$ is an isomorphism.

As an immediate application, we obtain that a "true" directed colimit of "good" pregamps is an algebra together with its congruences.
Corollary 5.15. Let $P$ be directed poset with no maximal element and let $\overrightarrow{\boldsymbol{A}}=$ $\left(\boldsymbol{A}_{p}, \boldsymbol{f}_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ be a $P$-indexed diagram of distance-generated pregamps. If $\boldsymbol{f}_{p, q}$ is congruence-tractable and $f_{p, q}$ is strong for all $p<q$ in $P$, then there exists a unique $(\vee, 0)$-homomorphism $\phi: \operatorname{Con}_{\mathrm{c}} A \rightarrow \widetilde{A}$ such that:

$$
\phi\left(\Theta_{A}(x, y)\right)=\delta_{\boldsymbol{A}}(x, y) \quad \text { for all } x, y \in A
$$

Moreover, $\phi$ is an isomorphism.
Definition 5.16. An ideal of a pregamp $\boldsymbol{A}$ is an ideal of $\widetilde{A}$. Denote by $\operatorname{Id} \boldsymbol{A}=\operatorname{Id} \widetilde{A}$ the set of all ideals of $\boldsymbol{A}$.

Let $P$ be a poset, let $\overrightarrow{\boldsymbol{A}}=\left(\boldsymbol{A}_{p}, \boldsymbol{f}_{p, q} \mid \underset{\sim}{p} \leq q\right.$ in $\left.P\right)$ be a $P$-indexed diagram in $\operatorname{PGamp}_{\mathscr{L}}$. An ideal of $\overrightarrow{\boldsymbol{A}}$ is an ideal of $\left(\widetilde{A}_{p}, \widetilde{f}_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ (cf. Definition 3.2).

Definition 5.17. Let $\boldsymbol{\pi}: \boldsymbol{A} \rightarrow \boldsymbol{B}$ be a morphism of pregamps. The 0 -kernel of $\boldsymbol{\pi}$, denoted by $\operatorname{ker}_{0} \boldsymbol{\pi}$, is the 0 -kernel of $\widetilde{\pi}$ (cf. Definition 3.2).

Let $P$ be a poset and let $\overrightarrow{\boldsymbol{\pi}}=\left(\boldsymbol{\pi}_{p}\right)_{p \in P}: \overrightarrow{\boldsymbol{A}} \rightarrow \overrightarrow{\boldsymbol{B}}$ be a natural transformation of $P$-indexed diagrams of pregamps. The 0 -kernel of $\overrightarrow{\boldsymbol{\pi}}$ is $\vec{I}=\left(\operatorname{ker}_{0} \boldsymbol{\pi}_{p}\right)_{p \in P}$.

Remark 5.18 . The 0 -kernel of $\boldsymbol{\pi}$ is an ideal of $\boldsymbol{A}$. Similarly the 0 -kernel of $\overrightarrow{\boldsymbol{\pi}}$ is an ideal of $\overrightarrow{\boldsymbol{A}}$.

If $\pi: A \rightarrow B$ is a morphism of algebras, then $\operatorname{ker}_{0} \boldsymbol{P}_{\mathrm{ga}}(\pi)$ is the set of all compact congruences of $A$ below ker $\pi$, that is, $\operatorname{ker}_{0} \boldsymbol{P}_{\mathrm{ga}}(\pi)=\left(\operatorname{Con}_{\mathrm{c}} A\right) \downarrow \operatorname{ker} \pi$.

Definition 5.19. A morphism $\boldsymbol{f}: \boldsymbol{A} \rightarrow \boldsymbol{B}$ of pregamps, is ideal-induced if $f(A)=$ $B$ as partial algebras and $\tilde{f}$ is ideal-induced. In that case we say that $\boldsymbol{B}$ is an ideal-induced image of $\boldsymbol{A}$.

Let $P$ be a poset, let $\overrightarrow{\boldsymbol{A}}$ and $\overrightarrow{\boldsymbol{B}}$ be $P$-indexed diagrams of pregamps. A natural transformation $\overrightarrow{\boldsymbol{f}}=\left(\boldsymbol{f}_{p}\right)_{p \in P}: \overrightarrow{\boldsymbol{A}} \rightarrow \overrightarrow{\boldsymbol{B}}$ is ideal-induced if $\boldsymbol{f}_{p}$ is ideal-induced for each $p \in P$.

Remark 5.20. A morphism $\boldsymbol{f}: \boldsymbol{A} \rightarrow \boldsymbol{B}$ of pregamps is ideal-induced if $\tilde{f}$ is idealinduced, $f$ is surjective, and for each $\ell \in \mathscr{L}$ and each tuple $\vec{b}$ of $B, \ell(\vec{b})$ is defined in $B$ if and only if there exists a tuple $\vec{a}$ in $A$ such that $\vec{b}=f(\vec{a})$ and $\ell(\vec{a})$ is defined in $A$.

If $f: A \rightarrow B$ is a surjective morphism of algebras, then $\boldsymbol{P}_{\mathrm{ga}}(f)$ is ideal-induced.
If $\boldsymbol{f}: \boldsymbol{A} \rightarrow \boldsymbol{B}$ and $\boldsymbol{g}: \boldsymbol{B} \rightarrow \boldsymbol{C}$ are ideal-induced morphisms of pregamps, then $\boldsymbol{g} \circ \boldsymbol{f}$ is ideal-induced.

The following proposition gives a description of quotients of pregamps.
Proposition 5.21. Let $\boldsymbol{A}$ be a pregamp and let $I \in \operatorname{Id} \widetilde{A}$. The binary relation $\theta_{I}=\left\{(x, y) \in A^{2} \mid \delta_{\boldsymbol{A}}(x, y) \in I\right\}$ is an equivalence relation on $A$. Given $a \in A$ denote by $a / I$ the $\theta_{I}$-equivalence class containing $a$, and set $A / I=A / \theta_{I}$. We can
define a structure of partial algebra on $A / I$ in the following way. Given $\ell \in \mathscr{L}$, we put:

$$
\begin{aligned}
& \operatorname{Def}_{\ell}(A / I)=\left\{\vec{x} / I \mid \vec{x} \in \operatorname{Def}_{\ell}(A)\right\} \\
& \ell^{A / I}(\vec{x} / I)=\ell^{A}(\vec{x}) / I, \quad \text { for all } \vec{x} \in \operatorname{Def}_{\ell}(A)
\end{aligned}
$$

Moreover $\delta_{\boldsymbol{A} / I}:(A / I)^{2} \rightarrow \widetilde{A} / I,(x / I, y / I) \mapsto \delta_{\boldsymbol{A}}(x, y) / I$ defines an $\widetilde{A} / I$-valued partial algebra distance, and the following statements hold:
(1) $\boldsymbol{A} / I=\left(A / I, \delta_{\boldsymbol{A} / I}, \widetilde{A} / I\right)$ is a pregamp.
(2) Put $\pi$ : $A \rightarrow A / I, x \mapsto x / I$, and denote by $\widetilde{\pi}: \widetilde{A} \rightarrow \widetilde{A} / I, d \mapsto d / I$ the canonical projection. Then $\boldsymbol{\pi}=(\pi, \widetilde{\pi})$ is an ideal-induced morphism of pregamps from $\boldsymbol{A}$ to $\boldsymbol{A} / I$.
(3) The 0-kernel of $\boldsymbol{\pi}$ is $I$.
(4) If $\boldsymbol{A}$ is distance-generated, then $\boldsymbol{A} / I$ is distance-generated.

Proof. The relation $\theta_{I}$ is reflexive (it follows from Definition 5.1(1)), symmetric (see Definition 5.1(2)) and transitive (see Definition 5.1(3)), thus it is an equivalence relation.

Let $\ell \in \mathscr{L}$, let $\vec{x}, \vec{y} \in \operatorname{Def}_{\ell}(A)$ such that $x_{k} / I=y_{k} / I$ for each $k<\operatorname{ar}(\ell)$. It follows from Definition 5.1(4) that $\delta_{\boldsymbol{A}}\left(\ell^{A}(\vec{x}), \ell^{A}(\vec{y})\right) \leq \bigvee_{k<\operatorname{ar}(\ell)} \delta_{\boldsymbol{A}}\left(x_{k}, y_{k}\right) \in I$, so $\ell^{A}(\vec{x}) / I=\ell^{A}(\vec{y}) / I$. Therefore the partial operation $\ell^{A / I}: \operatorname{Def}_{\ell}(A / I) \rightarrow A / I$ is well-defined.

Let $x, x^{\prime}, y, y^{\prime} \in A$, assume that $x / I=x^{\prime} / I$ and $y / I=y^{\prime} / I$. The following inequality holds:

$$
\delta_{\boldsymbol{A}}(x, y) \leq \delta_{\boldsymbol{A}}\left(x, x^{\prime}\right) \vee \delta_{\boldsymbol{A}}\left(x^{\prime}, y^{\prime}\right) \vee \delta_{\boldsymbol{A}}\left(y^{\prime}, y\right)
$$

However, $\delta_{\boldsymbol{A}}\left(x, x^{\prime}\right)$ and $\delta_{\boldsymbol{A}}\left(y, y^{\prime}\right)$ both belong to $I$, hence $\delta_{\boldsymbol{A}}(x, y) / I \leq \delta_{\boldsymbol{A}}\left(x^{\prime}, y^{\prime}\right) / I$. Similarly $\delta_{\boldsymbol{A}}\left(x^{\prime}, y^{\prime}\right) / I \leq \delta_{\boldsymbol{A}}(x, y) / I$. So the map $\delta_{\boldsymbol{A} / I}:(A / I)^{2} \rightarrow \widetilde{A} / I$ is welldefined.

Let $x, y \in A$, the following equivalences hold:

$$
\delta_{\boldsymbol{A} / I}(x / I, y / I)=0 / I \Longleftrightarrow \delta_{\boldsymbol{A}}(x, y) \in I \Longleftrightarrow x / I=y / I .
$$

That is, Definition 5.1(1) holds. Each of the conditions of Definition 5.1(2)-(5) for $\delta_{\boldsymbol{A}}$ implies its analogue for $\delta_{\boldsymbol{A} / I}$.

It is easy to check that $\boldsymbol{\pi}$ is well-defined and that it is a morphism of pregamps.

Notation 5.22. The notations $\boldsymbol{A} / I, A / I$, and $\delta_{\boldsymbol{A} / I}$ used in Proposition 5.21 will be used throughout the paper. The map $\boldsymbol{\pi}$ is the canonical projection.

If $I=\{0\}$, we identify $\boldsymbol{A} / I$ and $\boldsymbol{A}$.
If $X$ is a partial subalgebra of $A$, then we denote $X / I=\{x / I \mid x \in X\}$, with its natural structure of partial subalgebra of $A$, inherited from $X$, with $\operatorname{Def}_{\ell}(X / I)=$ $\left\{\vec{x} / I \mid \vec{x} \in \operatorname{Def}_{\ell}(X)\right\}$ for each $\ell \in \mathscr{L}$. That is $X / I=\pi(X)$ as partial algebras.

Let $\boldsymbol{B}$ be a sub-pregamp of $\boldsymbol{A}$ and let $I$ be a common ideal of $\boldsymbol{A}$ and $\boldsymbol{B}$. Then we identify the quotient $\boldsymbol{B} / I$ with the corresponding sub-pregamp of $\boldsymbol{A} / I$.

Remark. It is easy to construct a pregamp $\boldsymbol{A}$, a term $t$, a tuple $\vec{x}$ of $A$, and an ideal $I$ of $\widetilde{A}$, such that $t(\vec{x})$ is not defined in $A$, but $t(\vec{x} / I)$ is defined in $A / I$.

The following proposition gives a description of how morphisms of pregamps factorize through quotients. It is related to Lemma 3.5.

Proposition 5.23. Let $\boldsymbol{f}: \boldsymbol{A} \rightarrow \boldsymbol{B}$ be a morphism of pregamps, let $I \in \operatorname{Id} \boldsymbol{A}$, and let $J \in \operatorname{Id} \boldsymbol{B}$. If $\widetilde{f}(I) \subseteq J$, then the following maps are well-defined:

$$
\begin{aligned}
g: A / I & \rightarrow B / J \\
x / I & \mapsto f(x) / J, \\
\widetilde{g}: \widetilde{A} / I & \rightarrow \widetilde{B} / J \\
\alpha / I & \mapsto \widetilde{f}(\alpha) / J .
\end{aligned}
$$

Moreover, $\boldsymbol{g}=(g, \widetilde{g})$ is a morphism of pregamps from $\boldsymbol{A} / I$ to $\boldsymbol{B} / J$. If $\boldsymbol{\pi}_{I}: \boldsymbol{A} \rightarrow$ $\boldsymbol{A} / I$ and $\boldsymbol{\pi}_{J}: \boldsymbol{A} \rightarrow \boldsymbol{A} / I$ denote the canonical projections, then the following diagram commutes:


Proof. Observe that $\widetilde{g}: \widetilde{A} / I \rightarrow \widetilde{B} / J$ is the ( $\vee, 0$ )-homomorphism induced by $\widetilde{f}$. Let $x, y \in A$ such that $x / I=y / I$, that is, $\delta_{\boldsymbol{A}}(x, y) \in I$. It follows that $\delta_{\boldsymbol{B}}(f(x), f(y))=$ $\widetilde{f}\left(\delta_{\boldsymbol{A}}(x, y)\right) \in J$, so $f(x) / J=f(y) / J$. Therefore the map $g$ is well-defined.

Let $\ell \in \mathscr{L}$, let $\vec{a} \in \operatorname{Def}_{\ell}(A / I)$, and let $\vec{x} \in \operatorname{Def}_{\ell}(A)$ such that $\vec{a}=\vec{x} / I$. The following equalities hold:

$$
g(\ell(\vec{x} / I))=g(\ell(\vec{x}) / I)=f(\ell(\vec{x})) / J=\ell(f(\vec{x})) / J=\ell(f(\vec{x}) / J)=\ell(g(\vec{x} / I))
$$

Thus $g(\ell(\vec{a}))=\ell(g(\vec{a}))$. Therefore $g$ is a morphism of partial algebras.
Let $x, y \in A$. It is easy to check $\widetilde{g}\left(\delta_{\boldsymbol{A} / I}(x / I, y / I)\right)=\delta_{\boldsymbol{B} / J}(g(x / J), g(y / J))$. Therefore $\boldsymbol{g}: \boldsymbol{A} / I \rightarrow \boldsymbol{B} / J$ is a morphism of pregamps. Moreover $\boldsymbol{\pi}_{J} \circ \boldsymbol{f}=\boldsymbol{g} \circ \boldsymbol{\pi}_{I}$ is obvious.

Notation 5.24. We say that $\boldsymbol{f}$ induces $\boldsymbol{g}: \boldsymbol{A} / I \rightarrow \boldsymbol{B} / J$, the morphism of Proposition 5.23 .

Let $P$ be a poset, let $\overrightarrow{\boldsymbol{A}}=\left(\boldsymbol{A}_{p}, \boldsymbol{f}_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ be a $P$-indexed diagram in $\operatorname{PGamp}_{\mathscr{L}}$, let $\vec{I}=\left(I_{p}\right)_{p \in P}$ be an ideal of $\overrightarrow{\boldsymbol{A}}$, and let $\boldsymbol{g}_{p, q}: \boldsymbol{A}_{p} / I_{p} \rightarrow \boldsymbol{A}_{q} / I_{q}$ the morphism induced by $\boldsymbol{f}_{p, q}$, for all $p \leq q$ in $P$. We denote by $\overrightarrow{\boldsymbol{A}} / \vec{I}=\left(\boldsymbol{A}_{p} / I_{p}, \boldsymbol{g}_{p, q} \mid\right.$ $p \leq q$ in $P$ ).

The diagram $\vec{A} / \vec{I}$ is a quotient of $\vec{A}$.
Remark 5.25 . It is easy to check that $\overrightarrow{\boldsymbol{A}} / \vec{I}$ is indeed a diagram. Given $p \leq q \leq r$ in $P$ and $x \in A_{p}$ the following equalities hold:

$$
g_{q, r}\left(g_{p, q}\left(x / I_{p}\right)\right)=g_{q, r}\left(f_{p, q}(x) / I_{q}\right)=f_{q, r}\left(f_{p, q}(x)\right) / I_{r}=f_{p, r}(x) / I_{r}=g_{p, r}\left(x / I_{p}\right)
$$

Proposition 5.23 can be easily extended to diagrams in the following way. It is also related to Lemma 3.7.
Proposition 5.26. Let $P$ be a poset, let $\overrightarrow{\boldsymbol{A}}=\left(\boldsymbol{A}_{p}, \boldsymbol{f}_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ and $\overrightarrow{\boldsymbol{B}}=$ $\left(\boldsymbol{B}_{p}, \boldsymbol{g}_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ be $P$-indexed diagrams in $\mathbf{P G a m p}_{\mathscr{L}}$. Let $\vec{I}$ be an ideal of $\overrightarrow{\boldsymbol{A}}$, let $\vec{J}$ be an ideal of $\overrightarrow{\boldsymbol{B}}$. Let $\overrightarrow{\boldsymbol{\xi}}=\left(\boldsymbol{\xi}_{p}\right)_{p \in P}: \overrightarrow{\boldsymbol{A}} \rightarrow \overrightarrow{\boldsymbol{B}}$ be a natural transformation such that $\xi_{p}\left(I_{p}\right) \subseteq J_{p}$ for each $p \in P$. Denote by $\boldsymbol{\chi}_{p}: \boldsymbol{A}_{p} / I_{p} \rightarrow \boldsymbol{B}_{p} / J_{p}$ the morphism
induced by $\boldsymbol{\xi}_{p}$, for each $p \in P$. Then $\overrightarrow{\boldsymbol{\chi}}=\left(\boldsymbol{\chi}_{p}\right)_{p \in P}$ is a natural transformation from $\overrightarrow{\boldsymbol{A}} / \vec{I}$ to $\overrightarrow{\boldsymbol{B}} / \vec{J}$.

Notation 5.27. With the notation of Proposition 5.26 . We say that $\vec{\chi}: \overrightarrow{\boldsymbol{A}} / \vec{I} \rightarrow \overrightarrow{\boldsymbol{B}} / \vec{J}$ is induced by $\overrightarrow{\boldsymbol{\xi}}$.

The following lemma expresses that ideal-induced images of pregamps correspond, up to isomorphism, to quotients of pregamps. It is related to Lemma 3.13.

Lemma 5.28. Let $\boldsymbol{f}: \boldsymbol{A} \rightarrow \boldsymbol{B}$ be a morphism of pregamps. The following statements are equivalent:
(1) $\boldsymbol{f}$ is ideal-induced.
(2) $\boldsymbol{f}$ induces an isomorphism $\boldsymbol{g}: \boldsymbol{A} / \operatorname{ker}_{0} \boldsymbol{f} \rightarrow \boldsymbol{B}$.

Proof. Denote by $\boldsymbol{\pi}: \boldsymbol{A} \rightarrow \boldsymbol{A} / \operatorname{ker}_{0} \boldsymbol{f}$ the canonical projection, so $\boldsymbol{g} \circ \boldsymbol{\pi}=\boldsymbol{f}$.
Assume that $\boldsymbol{f}$ is ideal-induced. As $\widetilde{f}: \widetilde{A} \rightarrow \widetilde{B}$ is ideal-induced, Lemma 3.13 implies that $\widetilde{f}$ induces an isomorphism $\widetilde{g}: \widetilde{A} / \operatorname{ker}_{0} \boldsymbol{f} \rightarrow \widetilde{B}$. It follows that $\widetilde{g}$ separates 0 , thus (cf. Proposition $5.10(2)) g$ is an embedding. Moreover $g\left(A / \operatorname{ker}_{0} \boldsymbol{f}\right)=$ $g(\pi(A))=f(A)=B$ as partial algebras. Therefore it follows from Lemma 4.6 that $g$ is an isomorphism of partial algebras, thus $\boldsymbol{g}$ is an isomorphism of pregamps (cf. Remark 5.5).

Assume that $\boldsymbol{g}$ is an isomorphism. It follows that $\widetilde{g}$ is an isomorphism, so Lemma 3.13 implies that $\tilde{f}$ is ideal-induced. Moreover $g$ is an isomorphism, thus $f(A)=g(\pi(A))=g\left(A / \operatorname{ker}_{0} \boldsymbol{f}\right)=B$ as partial algebras. Therefore $\boldsymbol{f}$ is idealinduced.

The following proposition expresses that a quotient of a quotient is a quotient. It follows from Lemma 5.28, together with the fact that a composition of ideal-induced morphisms of pregamps is ideal-induced.

Proposition 5.29. Let $\boldsymbol{A}$ be a pregamp, let $I$ be an ideal of $\boldsymbol{A}$, let $J$ be an ideal of $\boldsymbol{A} / I$. Then $(\boldsymbol{A} / I) / J$ is isomorphic to a quotient of $\boldsymbol{A}$.

The following results expresses that, up to isomorphism, quotients of sub-pregamps are sub-pregamps of quotients.

Proposition 5.30. Let $\boldsymbol{A}$ be a pregamp, let $\boldsymbol{B}$ be a sub-pregamp of $\boldsymbol{A}$, and let $I \in \operatorname{Id} \boldsymbol{B}$. Then there exist $J \in \operatorname{Id} \boldsymbol{A}$, a sub-pregamp $\boldsymbol{C}$ of $\boldsymbol{A} / J$, and an isomorphism $\boldsymbol{f}: \boldsymbol{B} / I \rightarrow \boldsymbol{C}$.

Let $\boldsymbol{A}$ be a pregamp, let $I \in \operatorname{Id} \boldsymbol{A}$, and let $\boldsymbol{B}$ be a sub-pregamp of $\boldsymbol{A} / I$. There exists a sub-pregamp $\boldsymbol{C}$ of $\boldsymbol{A}$ such that $\boldsymbol{B}$ is isomorphic to some quotient of $\boldsymbol{C}$.

Proof. Let $\boldsymbol{A}$ be a pregamp, let $\boldsymbol{B}$ be a sub-pregamp of $\boldsymbol{A}$, let $I \in \operatorname{Id} \boldsymbol{B}$. Put $J=\widetilde{A} \downarrow I$. As $I$ is an ideal of $\widetilde{B}$, it is directed, therefore $J$ is an ideal of $\widetilde{A}$.

Let $\boldsymbol{f}: \boldsymbol{B} \rightarrow \boldsymbol{A}$ be the canonical embedding. Notice that $\widetilde{f}(I) \subseteq J$; denote by $\boldsymbol{g}: \boldsymbol{B} / I \rightarrow \boldsymbol{A} / J$ the morphism induced by $\boldsymbol{f}$ (cf. Proposition 5.23).

Let $d, d^{\prime} \in \widetilde{B}$ such that $\widetilde{g}(d / I)=\widetilde{g}\left(d^{\prime} / I\right)$, that is, $d / J=d^{\prime} / J$, so there exists $u \in J$ such that $d \vee u=d^{\prime} \vee u$. As $J=\widetilde{A} \downarrow I$, there exists $v \in I$ such that $u \leq v$, hence $d \vee v=d^{\prime} \vee v$, that is, $d / I=d^{\prime} / I$. Therefore $\widetilde{g}$ is an embedding. It follows from Proposition 5.10 that $\boldsymbol{g}=(g, \widetilde{g})$ is an embedding and induces an isomorphism $\boldsymbol{B} / I \rightarrow \boldsymbol{g}(\boldsymbol{B} / I)$; the latter is a sub-pregamp of $\boldsymbol{A} / J$.

Now let $I \in \operatorname{Id} \boldsymbol{A}$ and let $\boldsymbol{B}$ be a sub-pregamp of $\boldsymbol{A} / I$. Denote by $\boldsymbol{\pi}: \boldsymbol{A} \rightarrow \boldsymbol{A} / I$ the canonical projection, put $\boldsymbol{C}=\boldsymbol{\pi}^{-1}(\boldsymbol{B})$ (cf. Notation 5.9). As $\boldsymbol{\pi}$ is idealinduced, it is easy to check that $\boldsymbol{\pi}(\boldsymbol{C})=\boldsymbol{B}$, and the restriction $\boldsymbol{\pi} \upharpoonright \boldsymbol{C} \rightarrow \boldsymbol{B}$ is ideal-induced.

The following lemma, in conjunction with Lemma 3.14, proves that, given a direct system $\overrightarrow{\boldsymbol{A}}$ of pregamps, every quotient of the colimit of $\overrightarrow{\boldsymbol{A}}$ is the colimit of a quotient of $\vec{A}$.
Lemma 5.31. Let $P$ be directed poset and let $\overrightarrow{\boldsymbol{A}}=\left(\boldsymbol{A}_{p}, \boldsymbol{f}_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ be a $P$-indexed diagram in $\mathbf{P G a m p}_{\mathscr{L}}$. Let $\left(\boldsymbol{A}, \boldsymbol{f}_{p} \mid p \in P\right)=\xrightarrow[\longrightarrow]{\lim }\left(\boldsymbol{A}_{p}, \boldsymbol{f}_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ be a directed colimit cocone in $\mathbf{P G a m p}_{\mathscr{L}}$. Let $\vec{I}=\left(\vec{I}_{p}\right)_{p \in P}$ be an ideal of $\overrightarrow{\boldsymbol{A}}$. Then $I=\bigcup_{p \in P} \widetilde{f}_{p}\left(I_{p}\right)$ is an ideal of $\boldsymbol{A}$.

Let $\boldsymbol{g}_{p}: \boldsymbol{A}_{p} / I_{p} \rightarrow \boldsymbol{A} / I$ be the morphism induced by $\boldsymbol{f}_{p}$, let $\boldsymbol{g}_{p, q}: \boldsymbol{A}_{p} / I_{p} \rightarrow \boldsymbol{A}_{q} / I_{q}$ be the morphism induced by $\boldsymbol{f}_{p, q}$, for all $p \leq q$ in $P$. The following is a directed colimit cocone:

$$
\left(\boldsymbol{A} / I, \boldsymbol{g}_{p} \mid p \in P\right)=\underline{\longrightarrow}\left(\boldsymbol{A}_{p} / I_{p}, \boldsymbol{f}_{p, q} \mid p \leq q \text { in } P\right) \text { in } \mathbf{P G a m p}_{\mathscr{L}} .
$$

Proof. Lemma 4.9 and Lemma 5.6 imply that the following are colimits cocones:

$$
\begin{align*}
& \left(A, f_{p} \mid p \in P\right)=\underline{\longrightarrow}\left(A_{p}, f_{p, q} \mid p \leq q \text { in } P\right) \text { in Set }  \tag{5.6}\\
& \left(\widetilde{A}, \widetilde{f}_{p} \mid p \in P\right)=\underset{\longrightarrow}{\lim }\left(\widetilde{A}_{p}, \widetilde{f}_{p, q} \mid p \leq q \text { in } P\right) \text { in } \mathbf{S e m}_{\vee, 0}  \tag{5.7}\\
& \left(A, f_{p} \mid p \in P\right)=\underset{\longrightarrow}{l i m}\left(A_{p}, f_{p, q} \mid p \leq q \text { in } P\right) \text { in } \mathbf{P A l g}_{\mathscr{L}} \tag{5.8}
\end{align*}
$$

Moreover, Lemma 3.14 implies that $I$ is an ideal of $\widetilde{A}$ and that the following is a directed colimit cocone:

$$
\begin{equation*}
\left(\widetilde{A} / I, \widetilde{g}_{p} \mid p \in P\right)=\underset{\longrightarrow}{\lim }\left(\widetilde{A}_{p} / I_{p}, \widetilde{g}_{p, q} \mid p \leq q \text { in } P\right) \text { in } \mathbf{S e m}_{\vee, 0} \tag{5.9}
\end{equation*}
$$

Let $p \in P$, let $x, y \in A_{p}$ such that $g_{p}\left(x / I_{p}\right)=g_{p}\left(y / I_{p}\right)$. It follows that $f_{p}(x) / I=f_{p}(y) / I$, that is, $\delta_{\boldsymbol{A}}\left(f_{p}(x), f_{p}(y)\right) \in I$. So there exist $q \in P$ and $\alpha \in \widetilde{A}_{q}$ such that $\widetilde{f}_{p}\left(\delta_{\boldsymbol{A}_{p}}(x, y)\right)=\delta_{\boldsymbol{A}}\left(f_{p}(x), f_{p}(y)\right)=\widetilde{f}_{q}(\alpha)$. It follows from (5.7) that there exists $r \geq p, q$ such that $\delta_{\boldsymbol{A}_{r}}\left(f_{p, r}(x), f_{p, r}(y)\right)=\widetilde{f}_{p, r}\left(\delta_{\boldsymbol{A}_{p}}(x, y)\right)=\widetilde{f}_{q, r}(\alpha)$. However, $\widetilde{f}_{q, r}(\alpha) \in \widetilde{f}_{q, r}\left(I_{q}\right) \subseteq I_{r}$, so $f_{p, r}(x) / I_{r}=f_{p, r}(y) / I_{r}$, and so $g_{p, r}\left(x / I_{p}\right)=g_{p, r}\left(y / I_{p}\right)$. Moreover $A / I=\bigcup_{p \in P} f_{p}\left(A_{p}\right) / I=\bigcup_{p \in P} g_{p}\left(A_{p} / I_{p}\right)$. Hence the following is a directed colimit cocone:

$$
\left(A / I, g_{p} \mid p \in P\right)=\underline{\longrightarrow}\left(A_{p} / I_{p}, g_{p, q} \mid p \leq q \text { in } P\right) \text { in Set. }
$$

Let $\ell \in \mathscr{L}$. The following equalities hold:

$$
\operatorname{Def}_{\ell}(A / I)=\operatorname{Def}_{\ell}(A) / I=\bigcup_{p \in P} f_{p}\left(\operatorname{Def}_{\ell}\left(A_{p}\right)\right) / I=\bigcup_{p \in P} g_{p}\left(\operatorname{Def}_{\ell}\left(A_{p} / I_{p}\right)\right)
$$

Let $p \in P$, let $\vec{a} \in \operatorname{Def}_{\ell}\left(A_{p} / I\right)$. Then, as $g_{p}$ is a morphism of partial algebras, $g_{p}(\ell(\vec{a}))=\ell\left(g_{p}(\vec{a})\right)$. So, by Lemma 4.9, the following is a directed colimit cocone:

$$
\left(A / I, g_{p} \mid p \in P\right)=\underset{\longrightarrow}{\lim }\left(A_{p} / I_{p}, g_{p, q} \mid p \leq q \text { in } P\right) \text { in } P A \lg _{\mathscr{L}} .
$$

As $\boldsymbol{f}_{p}$ is a morphism of pregamps, $\delta_{\boldsymbol{A} / I}\left(g_{p}(x), g_{p}(y)\right)=\widetilde{g}_{p}\left(\delta_{\boldsymbol{A}_{p} / I_{p}}(x, y)\right)$ for all $p \in P$ and all $x, y \in A_{p} / I_{p}$, thus Lemma 5.6 implies that the following is a directed colimit cocone:

$$
\left(\boldsymbol{A} / I, \boldsymbol{g}_{p} \mid p \in P\right)=\underline{\longrightarrow}\left(\boldsymbol{A}_{p} / I_{p}, \boldsymbol{g}_{p, q} \mid p \leq q \text { in } P\right) \text { in } \mathbf{P G a m p} \mathscr{L}_{\mathscr{L}}
$$

Definition 5.32. A pregamp $\boldsymbol{A}$ satisfies an identity $t_{1}=t_{2}$ if $A / I$ satisfies $t_{1}=t_{2}$ for each $I \in \operatorname{Id} \boldsymbol{A}$.

Let $\mathcal{V}$ be a variety of algebras. A pregamp $\boldsymbol{A}$ is a pregamp of $\mathcal{V}$ if it satisfies all identities of $\mathcal{V}$.

Remark. It is not hard to construct a pregamp $\boldsymbol{A}$, an identity $t_{1}=t_{2}$, and an ideal $I$ of $\widetilde{A}$ such that $A$ satisfies $t_{1}=t_{2}$, but $A / I$ fails $t_{1}=t_{2}$.

A pregamp $\boldsymbol{A}$ satisfies an identity $t_{1}=t_{2}$ if and only if for each ideal-induced morphism $\boldsymbol{f}: \boldsymbol{A} \rightarrow \boldsymbol{B}$ of pregamps, the partial algebra $B$ satisfies $t_{1}=t_{2}$.
Definition 5.33. Let $\mathcal{V}$ be a variety of algebras. The category of pregamps of $\mathcal{V}$, denoted by $\mathbf{P G a m p}(\mathcal{V})$, is the full subcategory of $\mathbf{P G a m p}_{\mathscr{L}}$ in which the objects are all the pregamps of $\mathcal{V}$.

As an immediate application of Lemma 5.31 and Lemma 3.14, we obtain that the class of all pregamps that satisfy a given identity is closed under directed colimits.
Corollary 5.34. Let $\mathcal{V}$ be a variety of algebras and let $P$ be a directed poset. Let $\overrightarrow{\boldsymbol{A}}=\left(\boldsymbol{A}_{p}, \boldsymbol{f}_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ be a $P$-indexed diagram in $\operatorname{PGamp}(\mathcal{V})$. Let $\left(\boldsymbol{A}, \boldsymbol{f}_{p} \mid p \in P\right)=\underset{\longrightarrow}{\lim }\left(\boldsymbol{A}_{p}, \boldsymbol{f}_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ be a directed colimit cocone in $\operatorname{PGamp}_{\mathscr{L}}$. Then $\boldsymbol{A}$ is a pregamp of $\mathcal{V}$.

Similarly, it follows from Proposition 5.29 and Proposition 5.30 that the class of all pregamps that satisfy a given identity is closed under ideal-induced images and sub-pregamps.
Corollary 5.35. Let $\boldsymbol{A}$ be a pregamp of a variety $\mathcal{V}$, let $\boldsymbol{B}$ be a pregamp, let $\boldsymbol{f}: \boldsymbol{A} \rightarrow \boldsymbol{B}$ be an ideal-induced morphism of pregamps, then $\boldsymbol{B}$ is a pregamp of $\mathcal{V}$. Furthermore, every sub-pregamp of $\boldsymbol{A}$ is a pregamp of $\mathcal{V}$.

## 6. Gamps

A gamp of a variety $\mathcal{V}$ is a pregamp that "belongs" to $\mathcal{V}$ (cf. (1)), together with a partial subalgebra (cf. (2)). The main interest of this new notion is to express later some additional properties that reflect properties of algebras (cf. Definition 6.3). It is a generalization of the notion of a semilattice-metric cover as defined in Section 5-1].
Definition 6.1. Let $\mathcal{V}$ be a variety of $\mathscr{L}$-algebras. A gamp (resp., a gamp of $\mathcal{V}$ ) is a quadruple $\boldsymbol{A}=\left(A^{*}, A, \delta_{\boldsymbol{A}}, \widetilde{A}\right)$ such that
(1) $\left(A, \delta_{\boldsymbol{A}}, \widetilde{A}\right)$ is a pregamp (resp., a pregamp of $\mathcal{V}$ ) (cf. Definitions 5.1 and 5.32).
(2) $A^{*}$ is a partial subalgebra of $A$.

A realization of $\boldsymbol{A}$ is an ordered pair $\left(A^{\prime}, \chi\right)$ such that $A^{\prime} \in \mathcal{V}, A$ is a partial subalgebra of $A^{\prime}, \chi: \widetilde{A} \rightarrow \operatorname{Con}_{\mathrm{c}} A^{\prime}$ is a $(\vee, 0)$-embedding, and $\chi\left(\delta_{\boldsymbol{A}}(x, y)\right)=\Theta_{A^{\prime}}(x, y)$ for all $x, y \in A$. A realization is isomorphic if $\chi$ is an isomorphism.

A gamp $\boldsymbol{A}$ is finite if both $A$ and $\widetilde{A}$ are finite.
Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be gamps. A morphism $\boldsymbol{f}:\left(A, \delta_{\boldsymbol{A}}, \widetilde{A}\right) \rightarrow\left(B, \delta_{\boldsymbol{B}}, \widetilde{B}\right)$ of pregamps is a morphism of gamps from $\boldsymbol{A}$ to $\boldsymbol{B}$ if $f\left(A^{*}\right)$ is a partial subalgebra of $B^{*}$.

The category of gamps of $\mathcal{V}$, denoted by $\operatorname{Gamp}(\mathcal{V})$, is the category in which the objects are the gamps of $\mathcal{V}$ and the arrows are the morphisms of gamps.

A subgamp of a gamp $\boldsymbol{A}$ is a gamp $\boldsymbol{B}=\left(B^{*}, B, \delta_{\boldsymbol{B}}, \widetilde{B}\right)$ such that $B^{*}$ is a partial subalgebra of $A^{*}, \underset{\sim}{B}$ is a partial subalgebra of $A, \delta_{\boldsymbol{B}}=\delta_{\boldsymbol{A}} \upharpoonright B^{2}$, and $\widetilde{B}$ is a $(\vee, 0)$ subsemilattice of $\widetilde{A}$. Let $f: B \rightarrow A$ and $\widetilde{f}: \widetilde{B} \rightarrow \widetilde{A}$ be the inclusion maps. The
ordered pair $(f, \widetilde{f})$ is a morphism of gamps from $\boldsymbol{B}$ to $\boldsymbol{A}$, called the canonical embedding.

Remark. A gamp might have no realization. A realization of a finite gamp does not need to be finite.

Let $\boldsymbol{f}: \boldsymbol{A} \rightarrow \boldsymbol{B}$ be a morphism of gamps, let $\left(A^{\prime}, \chi\right)$ be a realization of $\boldsymbol{A}$, and let $\left(B^{\prime}, \xi\right)$ be a realization of $\boldsymbol{B}$. There might not exist any morphism $g: A^{\prime} \rightarrow B^{\prime}$.

Definition 6.2. A gamp of lattices is a gamp of the variety of all lattices.
Let $\boldsymbol{B}$ be a gamp of lattices. A chain of $\boldsymbol{B}$ is a sequence $x_{0}, x_{1}, \ldots, x_{n-1}$ of $B^{*}$ such that $x_{i} \wedge x_{j}=x_{i}$ in $B$ for all $i \leq j<n$. We sometime denote such a chain as $x_{0} \leq x_{1} \leq \cdots \leq x_{n-1}$. If $x_{i} \neq x_{j}$ for all $i<j<n$, we denote the chain as $x_{0}<x_{1}<\cdots<x_{n-1}$.

Let $u<v$ be a chain of $\boldsymbol{B}$, we say that $v$ is a cover of $u$, and then we write $u \prec v$, if there is no chain $u<x<v$ in $\boldsymbol{B}$.

The following properties for a gamp come from algebra. It follows from Definition 6.3(1) that there are many operations defined in $A$. With (2) or (3) all "congruences" have a set of "generators". Condition (4) expresses that whenever two elements are identified by a "congruence" of $A^{*}$, then there is a "good reason" for this in $A$ (cf. Lemma 5.11). Conditions (6) and (7) are related to the transitive closure of relations. Condition (8) is related to congruence $n$-permutability (cf. Proposition 2.1).

Definition 6.3. A gamp $\boldsymbol{A}$ is strong if the following holds:
(1) $A^{*}$ is a strong partial subalgebra of $A$ (cf. Definition 4.4).

A gamp $\boldsymbol{A}$ is distance-generated if it satisfies the following condition:
(2) Every element of $\widetilde{A}$ is a finite join of elements of the form $\delta_{\boldsymbol{A}}(x, y)$ where $x, y \in A^{*}$.
A gamp $\boldsymbol{A}$ of lattices is distance-generated with chains if
(3) For all $\alpha \in \widetilde{A}$ there are a positive integer $n$, and chains $x_{0}<y_{0}, x_{1}<$ $y_{1}, \ldots, x_{n-1}<y_{n-1}$ of $\boldsymbol{A}$ such that $\alpha=\bigvee_{k<n} \delta_{\boldsymbol{A}}\left(x_{k}, y_{k}\right)$.
A gamp $\boldsymbol{A}$ is congruence-tractable (cf. Lemma 5.11) if
(4) For all $x, y, x_{0}, y_{0}, \ldots, x_{m-1}, y_{m-1}$ in $A^{*}$, if $\delta_{\boldsymbol{A}}(x, y) \leq \bigvee_{k<m} \delta_{\boldsymbol{A}}\left(x_{k}, y_{k}\right)$ then there are a positive integer $n$, a list $\vec{z}$ of parameters from $A$, and terms $t_{0}, \ldots, t_{n}$ such that the following equations are satisfied in $A$.

$$
\begin{aligned}
x & =t_{0}(\vec{x}, \vec{y}, \vec{z}), \\
y & =t_{n}(\vec{x}, \vec{y}, \vec{z}), \\
t_{k}(\vec{y}, \vec{x}, \vec{z}) & =t_{k+1}(\vec{x}, \vec{y}, \vec{z}) \quad(\text { for all } k<n) .
\end{aligned}
$$

A morphism $\boldsymbol{f}: \boldsymbol{A} \rightarrow \boldsymbol{B}$ of gamps is strong if
(5) $f(A)$ is a strong partial subalgebra of $B^{*}$ (cf. Definition 4.4).

A morphism $\boldsymbol{f}: \boldsymbol{A} \rightarrow \boldsymbol{B}$ of gamps is congruence-cuttable if
(6) $f(A)$ is a partial sublattice of $B^{*}$ and given a finite subset $X$ of $\widetilde{B}$ and $x, y \in$ $A$ with $\delta_{\boldsymbol{A}}(f(x), f(y)) \leq \bigvee X$, there are $n<\omega$ and $f(x)=x_{0}, \ldots, x_{n}=$ $f(y)$ in $B^{*}$ such that $\delta_{\boldsymbol{A}}\left(x_{k}, x_{k+1}\right) \in \widetilde{B} \downarrow X$ for all $k<n$.
A morphism $\boldsymbol{f}: \boldsymbol{A} \rightarrow \boldsymbol{B}$ of gamps of the variety of all lattices is congruencecuttable with chains if
(7) $f(A)$ is a partial sublattice of $B^{*}$ and given a finite subset $X$ of $\widetilde{B}$ and $x, y \in A$ with $\delta_{\boldsymbol{A}}(f(x), f(y)) \leq \bigvee X$, there is a chain $x_{0}<\cdots<x_{n}$ of $\boldsymbol{B}$ such that $x_{0}=f(x) \wedge f(y), x_{n}=f(x) \vee f(y)$, and $\delta_{\boldsymbol{A}}\left(x_{k}, x_{k+1}\right) \in \widetilde{B} \downarrow X$ for all $k<n$.
Let $n \geq 1$ be an integer. A gamp $\boldsymbol{A}$ is congruence $n$-permutable if the following statement holds:
(8) For all $x_{0}, x_{1}, \ldots, x_{n} \in A^{*}$, there are $x_{0}=y_{0}, y_{1}, \ldots, y_{n}=x_{n}$ in $A$ such that:

$$
\begin{array}{ll}
\delta_{\boldsymbol{A}}\left(y_{k}, y_{k+1}\right) \leq \bigvee\left(\delta_{\boldsymbol{A}}\left(x_{i}, x_{i+1}\right) \mid i<n \text { even }\right), & \text { for all } k<n \text { odd } \\
\delta_{\boldsymbol{A}}\left(y_{k}, y_{k+1}\right) \leq \bigvee\left(\delta_{\boldsymbol{A}}\left(x_{i}, x_{i+1}\right) \mid i<n \text { odd }\right), & \text { for all } k<n \text { even. }
\end{array}
$$

The following lemma shows that chains in strong gamps of lattices behave the same way as chains in lattices.

Lemma 6.4. Let $x_{0}<\cdots<x_{n}$ a chain of a strong gamp of lattices $\boldsymbol{B}$. The equalities $x_{i} \wedge x_{j}=x_{j} \wedge x_{i}=x_{i}$ and $x_{i} \vee x_{j}=x_{j} \vee x_{i}=x_{j}$ hold in $B$ for all $i \leq j \leq n$.

Moreover the following statements hold:

$$
\begin{align*}
\delta_{\boldsymbol{B}}\left(x_{k}, x_{k^{\prime}}\right) & \leq \delta_{\boldsymbol{B}}\left(x_{i}, x_{j}\right), \quad \text { for all } i \leq k \leq k^{\prime} \leq j \leq n  \tag{6.1}\\
\delta_{\boldsymbol{B}}\left(x_{i}, x_{j}\right) & =\bigvee_{i \leq k<j} \delta_{\boldsymbol{B}}\left(x_{k}, x_{k+1}\right), \quad \text { for all } i \leq j \leq n \tag{6.2}
\end{align*}
$$

Proof. Let $i \leq j \leq n$. As $x_{i}, x_{j} \in B^{*}$, all the elements $x_{i} \wedge x_{j}, x_{j} \wedge x_{i}, x_{i} \vee x_{j}$, and $x_{j} \vee x_{i}$ are defined in $B$.

As $u \wedge v=v \wedge u$ is an identity of lattices, it follows that $x_{j} \wedge x_{i}=x_{i} \wedge x_{j}=x_{i}$.
As $x_{j} \wedge x_{i}=x_{i},\left(x_{i} \wedge x_{j}\right) \vee x_{j}=x_{i} \vee x_{j}$ in $B$, and $(u \wedge v) \vee v=v$ is an identity of lattices, $x_{i} \vee x_{j}=\left(x_{i} \wedge x_{j}\right) \vee x_{j}=x_{j}$. Similarly $x_{j} \vee x_{i}=x_{j}$.

Let $i \leq k \leq k^{\prime} \leq j \leq n$. As $x_{k} \wedge x_{k^{\prime}}=x_{k}$ and $x_{j} \wedge x_{k^{\prime}}=x_{k^{\prime}}$, we obtain from Definition 5.1(4) the inequality:

$$
\delta_{\boldsymbol{B}}\left(x_{k}, x_{k^{\prime}}\right)=\delta_{\boldsymbol{B}}\left(x_{k} \wedge x_{k^{\prime}}, x_{j} \wedge x_{k^{\prime}}\right) \leq \delta_{\boldsymbol{B}}\left(x_{k}, x_{j}\right)
$$

Similarly, as $x_{k}=x_{k} \vee x_{i}$ and $x_{j}=x_{j} \vee x_{i}$, the inequality $\delta_{B}\left(x_{k}, x_{j}\right) \leq \delta_{B}\left(x_{i}, x_{j}\right)$ holds. Therefore (6.1) holds.

Let $i<j \leq n$. Definition 5.1(3) implies the inequality:

$$
\delta_{\boldsymbol{B}}\left(x_{i}, x_{j}\right) \leq \bigvee_{i \leq k<j} \delta_{\boldsymbol{B}}\left(x_{k}, x_{k+1}\right)
$$

Moreover (6.1) implies $\delta_{\boldsymbol{B}}\left(x_{k}, x_{k+1}\right) \leq \delta_{\boldsymbol{B}}\left(x_{i}, x_{j}\right)$ for all $i \leq k<j$, it follows that (6.2) is true.

The following proposition gives a description of quotient of gamps.
Proposition 6.5. Let $\boldsymbol{A}$ be a gamp of $\mathcal{V}$ and let $I$ be an ideal of $\widetilde{A}$. Let $\left(A, \delta_{\boldsymbol{A}}, \widetilde{A}\right) / I=$ $\left(A / I, \delta_{\boldsymbol{A} / I}, \widetilde{A} / I\right)$ be the quotient pregamp and set $A^{*} / I=\left\{a / I \mid a \in A^{*}\right\}$ (cf. Notation 5.22). The following statements hold:
(1) $\boldsymbol{A} / I=\left(A^{*} / I, A / I, \delta_{\boldsymbol{A} / I}, \widetilde{A} / I\right)$ is a gamp of $\mathcal{V}$.
(2) The canonical projection $\boldsymbol{\pi}:\left(A, \delta_{\boldsymbol{A}}, \widetilde{A}\right) \rightarrow\left(A / I, \delta_{\boldsymbol{A} / I}, \widetilde{A} / I\right)$ of pregamps is a morphism of gamps from $\boldsymbol{A}$ to $\boldsymbol{A} / I$.
(3) If $\left(A^{\prime}, \chi\right)$ is a realization of $\boldsymbol{A}$ in $\mathcal{V}$, then $\left(A^{\prime} / \bigvee \chi(I), \chi^{\prime}\right)$ is a realization of $\boldsymbol{A} / I$ in $\mathcal{V}$, where

$$
\begin{aligned}
& \chi^{\prime}: \widetilde{A} / I \rightarrow \operatorname{Con}_{\mathrm{c}}\left(A^{\prime} / \bigvee \chi(I)\right) \\
& d / I \mapsto \chi(d) / \bigvee \chi(I)
\end{aligned}
$$

Moreover, if $\left(A^{\prime}, \chi\right)$ is an isomorphic realization of $\boldsymbol{A}$, then $\left(A^{\prime} / \bigvee \chi(I), \chi^{\prime}\right)$ is an isomorphic realization of $\boldsymbol{A} / I$.
(4) If $\boldsymbol{A}$ is strong, then $\boldsymbol{A} / I$ is strong.
(5) If $\boldsymbol{A}$ is distance-generated (resp., distance-generated with chains) then $\boldsymbol{A} / I$ is distance-generated (resp., distance-generated with chains).
(6) If $\boldsymbol{A}$ is distance-generated and congruence-tractable, then $\boldsymbol{A} / I$ is congruencetractable.
(7) Let $n \geq 2$ an integer. If $\boldsymbol{A}$ is congruence $n$-permutable then $\boldsymbol{A} / I$ is congruence $n$-permutable.
(8) If $\boldsymbol{A}$ is a gamp of lattices and $x_{0} \leq x_{1} \leq \cdots \leq x_{n}$ is a chain of $\boldsymbol{A}$, then $x_{0} / I \leq x_{1} / I \leq \cdots \leq x_{n} / I$ is a chain of $\boldsymbol{A} / I$.
Proof. The statement (1) follows from Corollary 5.35. Denote by $\boldsymbol{\pi}:\left(A, \delta_{\boldsymbol{A}}, \widetilde{A}\right) \rightarrow$ $\left(A / I, \delta_{\boldsymbol{A} / I}, \widetilde{A} / I\right)$ the canonical projection of pregamps. The fact that $\pi\left(A^{*}\right)=A^{*} / I$ as partial algebras follows from the definition of $A^{*} / I$. Thus (3) holds.

Let $\left(A^{\prime}, \chi\right)$ be a realization of $\boldsymbol{A}$ in $\mathcal{V}$, let $d, d^{\prime} \in \widetilde{A}$ such that $d / I=d^{\prime} / I$. Hence there exists $u \in I$ such that $d \vee u=d^{\prime} \vee u$, it follows that $\chi(d) / \chi(u)=\chi\left(d^{\prime}\right) / \chi(u)$, hence $\chi(d) / \bigvee \chi(I)=\chi\left(d^{\prime}\right) / \bigvee \chi(I)$. Therefore the map $\chi^{\prime}$ is well-defined. It is easy to check that $\chi^{\prime}$ is a $(\vee, 0)$-homomorphism. Assume that $\chi^{\prime}(d / I) \leq \chi^{\prime}\left(d^{\prime} / I\right)$ for some $d, d^{\prime} \in \widetilde{A}$. Hence $\chi(d) / \bigvee \chi(I) \leq \chi\left(d^{\prime}\right) / \bigvee \chi(I)$, so $\chi(d) \leq \chi\left(d^{\prime}\right) \vee \bigvee \chi(I)$. However, $\chi(d)$ is a compact congruence of $A^{\prime}$, so there exist $u \in I$ such that $\chi(d) \leq \chi\left(d^{\prime}\right) \vee \chi(u)=\chi\left(d^{\prime} \vee u\right)$, as $\chi$ is an embedding, it follows that $d \leq d^{\prime} \vee u$, so $d / I \leq d^{\prime} / I$. Therefore $\chi^{\prime}$ is an embedding.

Let $x, y \in A$. The following equivalences hold:

$$
\begin{aligned}
x / I=y / I & \Longleftrightarrow \delta_{\boldsymbol{A}}(x, y) \in I \\
& \Longleftrightarrow \chi\left(\delta_{\boldsymbol{A}}(x, y)\right) \leq \bigvee \chi(I) \\
& \Longleftrightarrow \Theta_{A^{\prime}}(x, y) \leq \bigvee \chi(I) \\
& \Longleftrightarrow x / \bigvee \chi(I)=y / \bigvee \chi(I)
\end{aligned}
$$

So we can identify $A / I$ with the corresponding subset of $A^{\prime} / \bigvee \chi(I)$. Moreover, given $\vec{a} \in \operatorname{Def}_{\ell}(A / I)$, there exists $\vec{x} \in \operatorname{Def}_{\ell}(A)$ such that $\vec{a}=\vec{x} / I$, hence $\ell(\vec{a})=$ $\ell(\vec{x}) / I$ is identified with $\ell(\vec{x}) / \bigvee \chi(I)=\ell(\vec{x} / \bigvee \chi(I))$. So this identification preserves the operations.

Now assume that the realization is isomorphic. Then $\chi$ is surjective, thus $\chi^{\prime}$ is surjective, and thus bijective, hence the realization $\left(A^{\prime} / \bigvee \chi(I), \chi^{\prime}\right)$ is isomorphic. Therefore (3) holds.

The proofs of the statements (4), (5), (7), and (8) are straightforward.
Assume that $\boldsymbol{A}$ is distance-generated and congruence-tractable. Let $x, y \in A^{*}$, let $m<\omega$, let $\vec{x}, \vec{y}$ be $m$-tuples of $A^{*}$. Assume that:

$$
\delta_{\boldsymbol{A} / I}(x / I, y / I) \leq \bigvee_{k<m} \delta_{\boldsymbol{A} / I}\left(x_{k} / I, y_{k} / I\right)
$$

It follows that there exists $u \in I$ with:

$$
\delta_{\boldsymbol{A}}(x, y) \leq u \vee \underset{k<m}{\bigvee} \delta_{\boldsymbol{A}}\left(x_{k}, y_{k}\right)
$$

However, as $\boldsymbol{A}$ is distance-generated, there exist $x_{0}^{\prime}, \ldots, x_{p-1}^{\prime}, y_{0}^{\prime}, \ldots, y_{p-1}^{\prime}$ in $A^{*}$ such that $u=\bigvee_{k<p} \delta_{\boldsymbol{A}}\left(x_{k}^{\prime}, y_{k}^{\prime}\right)$. As $\delta_{\boldsymbol{A}}\left(x_{k}^{\prime}, y_{k}^{\prime}\right) \leq u \in I, x_{k}^{\prime} / I=y_{k}^{\prime} / I$ for all $k<p$. Moreover the following inequality holds:

$$
\delta_{\boldsymbol{A}}(x, y) \leq \bigvee_{k<m} \delta_{\boldsymbol{A}}\left(x_{k}, y_{k}\right) \vee \bigvee_{k<p} \delta_{\boldsymbol{A}}\left(x_{k}^{\prime}, y_{k}^{\prime}\right)
$$

As $\boldsymbol{A}$ is congruence-tractable, there are a positive integer $n$, a list $\vec{z}$ of parameters from $A$, and terms $t_{0}, \ldots, t_{n}$ such that, the following equations are satisfied in $A$ :

$$
\begin{aligned}
x & =t_{0}\left(\vec{x}, \vec{x}^{\prime}, \vec{y}, \vec{y}^{\prime}, \vec{z}\right), \\
y & =t_{n}\left(\vec{x}, \vec{x}^{\prime}, \vec{y}, \vec{y}^{\prime}, \vec{z}\right), \\
t_{k}\left(\vec{y}, \vec{y}^{\prime}, \vec{x}, \vec{x}^{\prime}, \vec{z}\right) & =t_{k+1}\left(\vec{x}, \vec{x}^{\prime}, \vec{y}, \vec{y}^{\prime}, \vec{z}\right) \quad(\text { for all } k<n) .
\end{aligned}
$$

Put $t_{k}^{\prime}(\vec{a}, \vec{b}, \vec{c}, \vec{d})=t_{k}(\vec{a}, \vec{d}, \vec{b}, \vec{d}, \vec{c})$, for all tuples $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ and all $k \leq n$. As $\vec{x}^{\prime} / I=$ $\vec{y}^{\prime} / I$ the following equations are satisfied in $A / I$ :

$$
\begin{aligned}
x / I & =t_{0}^{\prime}\left(\vec{x} / I, \vec{y} / I, \vec{z} / I, \vec{x}^{\prime} / I\right), \\
y / I & =t_{n}^{\prime}\left(\vec{x} / I, \vec{y} / I, \vec{z} / I, \vec{x}^{\prime} / I\right), \\
t_{k}^{\prime}\left(\vec{y} / I, \vec{x} / I, \vec{z} / I, \vec{x}^{\prime} / I\right) & =t_{k+1}^{\prime}\left(\vec{x} / I, \vec{y} / I, \vec{z} / I, \vec{x}^{\prime} / I\right) \quad(\text { for all } k<n) .
\end{aligned}
$$

Therefore $\boldsymbol{A} / I$ is congruence-tractable.
Definition 6.6. The gamp $\boldsymbol{A} / I$ described in Proposition 6.5 is a quotient of $\boldsymbol{A}$, the morphism $\boldsymbol{\pi}$ is the canonical projection.

The following proposition describes how morphisms factorize through quotients of gamps.
Proposition 6.7. Let $\boldsymbol{f}: \boldsymbol{A} \rightarrow \boldsymbol{B}$ be a morphism of gamps, let $I$ be an ideal of $\widetilde{A}$, and let $J$ be an ideal of $\widetilde{B}$. Assume that $\widetilde{f}(I) \subseteq J$ and denote by

$$
\boldsymbol{g}:\left(A / I, \delta_{\boldsymbol{A} / I}, \widetilde{A} / I\right) \rightarrow\left(B / J, \delta_{\boldsymbol{B} / J}, \widetilde{B} / J\right)
$$

the morphism of pregamps induced by $\boldsymbol{f}$. The following statements hold.
(1) $\boldsymbol{g}: \boldsymbol{A} / I \rightarrow \boldsymbol{B} / J$ is a morphism of gamps.
(2) If $\boldsymbol{f}$ is strong, then $\boldsymbol{g}$ is strong.
(3) If $\boldsymbol{f}$ is congruence-cuttable then $\boldsymbol{g}$ is congruence-cuttable.
(4) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are gamps of lattices and $\boldsymbol{f}$ is congruence-cuttable with chains, then $\boldsymbol{g}$ is congruence-cuttable with chains.

Proof. The equality $g\left(A^{*} / I\right)=f\left(A^{*}\right) / J$ of partial algebras holds. Moreover, as $f\left(A^{*}\right)$ is a partial subalgebra of $B, g\left(A^{*} / I\right)$ is a partial subalgebra of $B / J$. Therefore (1) holds.

The statement (2) follows from the definitions of a quotient gamp (cf. Proposition (6.5) and of a strong morphism (Definition 6.3).

Assume that $\boldsymbol{f}$ is congruence-cuttable. As $f(A)$ is a partial subalgebra of $B^{*}$ it follows that $g(A / I)$ is a partial subalgebra of $B^{*} / J$. Let $X$ be a finite subset of $\widetilde{B}$, let $x, y \in A$ such that $\delta_{B / J}(g(x / I), g(y / I)) \leq \bigvee X / I$. If $X=\emptyset$, then $x / I=y / I$, hence the case is immediate.

If $X \neq \emptyset$, let $u \in J$ such that $\delta_{\boldsymbol{B}}(g(x), g(y)) \leq u \vee \bigvee X$. Put $X^{\prime}=X \cup\{u\}$. There are $n<\omega$ and $f(x)=x_{0}, \ldots, x_{n}=f(y)$ in $B^{*}$ such that $\delta_{B}\left(x_{k}, x_{k+1}\right) \in \widetilde{B} \downarrow X^{\prime}$ for each $k<n$. If $\delta_{\boldsymbol{B}}\left(x_{k}, y_{k}\right) \leq u$, then $\delta_{\boldsymbol{B} / J}\left(x_{k} / J, y_{k} / J\right)=\delta_{\boldsymbol{B}}\left(x_{k}, y_{k}\right) / J=$ $0 / J \in(\widetilde{B} / J) \downarrow X / J$. Otherwise $\delta_{\boldsymbol{B}}\left(x_{k}, y_{k}\right) \in \widetilde{B} \downarrow X$, thus $\delta_{\boldsymbol{B} / J}\left(x_{k} / J, y_{k} / J\right)=$ $\delta_{\boldsymbol{B}}\left(x_{k}, y_{k}\right) / J \in(\widetilde{B} / J) \downarrow X / J$. Therefore (3) holds.

The proof of (4) is similar to the proof of (3).
We introduce in the following definitions a functor $G: \mathcal{V} \rightarrow \operatorname{Gamp}(\mathcal{V})$, a functor $C: \operatorname{Gamp}(\mathcal{V}) \rightarrow \operatorname{Sem}_{\vee, 0}$ and functors $\boldsymbol{P}_{\mathrm{gl}}, \boldsymbol{P}_{\mathrm{gr}}: \operatorname{Gamp}(\mathcal{V}) \rightarrow \mathbf{P G a m p}(\mathcal{V})$.

Definition 6.8. Let $A$ be a member of a variety $\mathcal{V}$ of algebras. Then the quadruple $\boldsymbol{G}(A)=\left(A, A, \Theta_{A}, \operatorname{Con}_{\mathrm{c}} A\right)$ is a gamp of $\mathcal{V}$ (we recall that $\Theta_{A}(x, y)$ denotes the smallest congruence that identifies $x$ and $y$ ). If $f: A \rightarrow B$ is a morphism of algebras, then $\boldsymbol{G}(f)=\left(f, \operatorname{Con}_{\mathrm{c}} f\right)$ is a morphism of gamps from $\boldsymbol{G}(A)$ to $\boldsymbol{G}(B)$. It defines a functor from the category $\mathcal{V}$ to the category $\operatorname{Gamp}(\mathcal{V})$.

A gamp $\boldsymbol{A}$ is an algebra if $\boldsymbol{A}$ is isomorphic to $\boldsymbol{G}(B)$ for some $B$. A gamp $\boldsymbol{A}$ is an algebra of a variety $\mathcal{V}$ if $\boldsymbol{A}$ is isomorphic to $\boldsymbol{G}(B)$ for some $B \in \mathcal{V}$.

Let $\boldsymbol{A}$ be a gamp of $\mathcal{V}$, we set $\boldsymbol{C}(\boldsymbol{A})=\widetilde{A}$. Let $\boldsymbol{f}: \boldsymbol{A} \rightarrow \boldsymbol{B}$ be a morphism of gamps of $\mathcal{V}$, we set $\boldsymbol{C}(\boldsymbol{f})=\widetilde{f}$. This defines a functor $\boldsymbol{C}: \operatorname{Gamp}(\mathcal{V}) \rightarrow \mathbf{S e m}_{\vee, 0}$.

Let $\boldsymbol{A}$ be a gamp of $\mathcal{V}$, we set $\boldsymbol{P}_{\mathrm{gr}}(\boldsymbol{A})=\left(A, \delta_{\boldsymbol{A}}, \widetilde{A}\right)$. Let $\boldsymbol{f}: \boldsymbol{A} \rightarrow \boldsymbol{B}$ be a morphism of gamps of $\mathcal{V}$, we put $\boldsymbol{P}_{\mathrm{gr}}(\boldsymbol{f})=\boldsymbol{f}$ as a morphism of pregamps. This defines a functor $\boldsymbol{P}_{\mathrm{gr}}: \operatorname{Gamp}(\mathcal{V}) \rightarrow \mathbf{P G a m p}(\mathcal{V})$.

Let $\boldsymbol{A}$ be a gamp of $\mathcal{V}$, we set $\boldsymbol{P}_{\mathrm{gl}}(\boldsymbol{A})=\left(A^{*}, \delta_{\boldsymbol{A}} \upharpoonright\left(A^{*}\right)^{2}, \widetilde{A}\right)$. Let $\underset{\sim}{\boldsymbol{f}}: \boldsymbol{A} \rightarrow \boldsymbol{B}$ be a morphism of gamps of $\mathcal{V}$, we denote by $\boldsymbol{P}_{\mathrm{gl}}(\boldsymbol{f})$ the restriction $(f, \widetilde{f}): \boldsymbol{P}_{\mathrm{gl}}(\boldsymbol{A}) \rightarrow$ $\boldsymbol{P}_{\mathrm{gl}}(\boldsymbol{B})$. This defines a functor $\boldsymbol{P}_{\mathrm{gl}}: \operatorname{Gamp}(\mathcal{V}) \rightarrow \mathbf{P G a m p}(\mathcal{V})$.
Remark 6.9. The following assertions hold.
(1) The following equations, between the functors introduced in Definition 5.4 and Definition 6.8, are satisfied:

$$
\begin{aligned}
\boldsymbol{C} \circ \boldsymbol{G} & =\mathrm{Con}_{\mathrm{c}}=\boldsymbol{C}_{\mathrm{pg}} \circ \boldsymbol{P}_{\mathrm{ga}}, \\
\boldsymbol{P}_{\mathrm{gr}} \circ \boldsymbol{G} & =\boldsymbol{P}_{\mathrm{gl}} \circ \boldsymbol{G}=\boldsymbol{P}_{\mathrm{ga}}, \\
\boldsymbol{C}_{\mathrm{pg}} \circ \boldsymbol{P}_{\mathrm{gr}} & =\boldsymbol{C}_{\mathrm{pg}} \circ \boldsymbol{P}_{\mathrm{gl}}=\boldsymbol{C} .
\end{aligned}
$$

(2) If $A$ is a subalgebra of $B$, then, in general, $\boldsymbol{G}(A)$ is not a subgamp of $\boldsymbol{G}(B)$. The different "congruences" of a subgamp can be extended in a natural way to different "congruences" of the gamp.
(3) Let $\boldsymbol{A}$ be a gamp. If $\boldsymbol{A}$ is an algebra, then there is a unique, up to isomorphism, algebra $B$ such that $\boldsymbol{A} \cong \boldsymbol{G}(B)$. Moreover if $\boldsymbol{A}$ is a gamp in a variety $\mathcal{V}$, then $B \in \mathcal{V}$. Indeed $A$ is an algebra and $\boldsymbol{A} \cong \boldsymbol{G}(A)$.
(4) Let $\boldsymbol{f}: \boldsymbol{A} \rightarrow \boldsymbol{B}$ be a morphism of gamps. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are algebras, then $f: A \rightarrow B$ is a morphism of algebras.
(5) Let $\overrightarrow{\boldsymbol{A}}=\left(\boldsymbol{A}_{p}, \boldsymbol{f}_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ be a diagram of gamps. If $\boldsymbol{A}_{p}$ is an algebra, for all $p \in P$, then $\vec{A}=\left(A_{p}, f_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ is a diagram of algebras, moreover $\vec{A} \cong G \circ \vec{A}$.
(6) Let $B \in \mathcal{V}$ and let $I$ be an ideal of $\mathrm{Con}_{\mathrm{c}} B$. There is an isomorphism $\boldsymbol{f}: \boldsymbol{G}(B) / I \cong \boldsymbol{G}(B / \bigvee I)$ satisfying $f(x / I)=x / \bigvee I$ and $\widetilde{f}(\alpha / I)=\alpha / \bigvee I$ for each $x \in B$ and each $\alpha \in \operatorname{Con}_{\mathrm{c}} B$.
(7) Let $B$ be an algebra. Then $B$ is congruence $n$-permutable if and only if $\boldsymbol{G}(B)$ is congruence $n$-permutable (this follows immediately from Proposition 2.1.).
Lemma 6.10. Let $\mathcal{V}$ be a variety of algebras. The category $\operatorname{Gamp}(\mathcal{V})$ has all directed colimits. Suppose that we are given a directed poset $P$, a $P$-indexed diagram $\overrightarrow{\boldsymbol{A}}=\left(\boldsymbol{A}_{p}, \boldsymbol{f}_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ in $\operatorname{Gamp}(\mathcal{V})$, and a directed colimit cocone:

$$
\left(\left(A, \delta_{\boldsymbol{A}}, \widetilde{A}\right), \boldsymbol{f}_{p} \mid p \in P\right)=\underset{\longrightarrow}{\lim }\left(\left(A_{p}, \delta_{\boldsymbol{A}_{p}}, \widetilde{A}_{p}\right), \boldsymbol{f}_{p, q} \mid p \leq q \text { in } P\right) \quad \text { in } \mathbf{P G a m p}_{\mathscr{L}}
$$

Put $A^{*}=\bigcup_{p \in P} f\left(A_{p}^{*}\right)$ with its natural structure of partial algebra (cf. Lemma 4.9), then $\boldsymbol{A}=\left(A^{*}, A, \delta_{\boldsymbol{A}}, \widetilde{A}\right)$ is a gamp of $\mathcal{V}, \boldsymbol{f}_{p}: \boldsymbol{A}_{p} \rightarrow \boldsymbol{A}$ is a morphism of gamps, and the following is a directed colimit cocone:

$$
\left(\boldsymbol{A}, \boldsymbol{f}_{p} \mid p \in P\right)=\underline{\longrightarrow}\left(\boldsymbol{A}_{p}, \boldsymbol{f}_{p, q} \mid p \leq q \text { in } P\right), \quad \text { in } \operatorname{Gamp}(\mathcal{V}) .
$$

Moreover the following statements hold:
(1) If $\boldsymbol{A}_{p}$ is distance-generated for each $p \in P$, then $\boldsymbol{A}$ is distance-generated.
(2) If $\mathcal{V}$ is a variety of lattices and $\boldsymbol{A}_{p}$ is distance-generated with chains for each $p \in P$, then $\boldsymbol{A}$ is distance-generated with chains.
(3) Let $n$ be a positive integer. If $\boldsymbol{A}_{p}$ is congruence $n$-permutable for each $p \in P$, then $\boldsymbol{A}$ is congruence $n$-permutable.
Proof. It follows from Corollary 5.34 that $\left(A, \delta_{\boldsymbol{A}}, \widetilde{A}\right)$ is a pregamp of $\mathcal{V}$. Moreover $A^{*}$ is a partial subalgebra of $A$. Hence $\boldsymbol{A}=\left(A^{*}, A, \delta_{\boldsymbol{A}}, \widetilde{A}\right)$ is a gamp of $\mathcal{V}$. As $f\left(A_{p}^{*}\right)$ is a partial subalgebra of $A^{*}, \boldsymbol{f}_{p}: \boldsymbol{A}_{p} \rightarrow \boldsymbol{A}$ is a morphism of gamps. It is easy to check that the following is a directed colimit cocone:

$$
\left(\boldsymbol{A}, \boldsymbol{f}_{p} \mid p \in P\right)=\underset{\longrightarrow}{\lim }\left(\boldsymbol{A}_{p}, \boldsymbol{f}_{p, q} \mid p \leq q \text { in } P\right), \quad \text { in } \operatorname{Gamp}(\mathcal{V}) .
$$

Assume that $\boldsymbol{A}_{p}$ is distance-generated for each $p \in P$. Let $\alpha \in \widetilde{A}$, then there are $p \in P$ and $\beta \in \widetilde{A}_{p}$ such that $\alpha=\widetilde{f}_{p}(\beta)$. As $\boldsymbol{A}_{p}$ is distance-generated, there are an integer $n \geq 0$ and $n$-tuple $\vec{x}, \vec{y}$ of $A_{p}^{*}$ such that $\beta=\bigvee_{k<n} \delta_{\boldsymbol{A}_{p}}\left(x_{k}, y_{k}\right)$. Therefore the following equalities hold:
$\alpha=\widetilde{f}_{p}(\beta)=\widetilde{f}_{p}\left(\bigvee_{k<n} \delta_{\boldsymbol{A}_{p}}\left(x_{k}, y_{k}\right)\right)=\bigvee_{k<n} \widetilde{f}_{p}\left(\delta_{\boldsymbol{A}_{p}}\left(x_{k}, y_{k}\right)\right)=\bigvee_{k<n} \delta_{\boldsymbol{A}}\left(f_{p}\left(x_{k}\right), f_{p}\left(y_{k}\right)\right)$.
Thus $\boldsymbol{A}$ is distance-generated.
The proofs of (2) and (3) are similar.
As an immediate application we obtain the following corollary.
Corollary 6.11. The functors $\boldsymbol{G}, \boldsymbol{C}, \boldsymbol{P}_{\mathrm{gl}}$, and $\boldsymbol{P}_{\mathrm{gr}}$ preserves directed colimits.
Proof. It follows from the description of directed colimits of gamps (cf. Lemma 6.10) and pregamps (cf. Lemma 5.6) that $\boldsymbol{C}, \boldsymbol{P}_{\mathrm{gl}}$, and $\boldsymbol{P}_{\mathrm{gr}}$ preserve directed colimits. As $\mathrm{Con}_{\mathrm{c}}$ preserves directed colimits, $\boldsymbol{G}$ also preserves directed colimits.
Definition 6.12. Let $\mathcal{V}$ be a variety of algebras, let $P$ be a poset, and let $\vec{A}=$ $\left(\boldsymbol{A}_{p}, \boldsymbol{f}_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ in $\operatorname{Gamp}(\mathcal{V})$. An ideal of $\overrightarrow{\boldsymbol{A}}$ is an ideal of $\boldsymbol{C} \circ \overrightarrow{\boldsymbol{A}}$. It consists of a family $\vec{I}=\left(I_{p}\right)_{p \in P}$ such that $I_{p}$ is an ideal of $\widetilde{A}_{p}$ and $\widetilde{f}_{p, q}\left(I_{p}\right) \subseteq I_{q}$ for all $p \leq q$ in $P$.

We denote by $\overrightarrow{\boldsymbol{A}} / \vec{I}=\left(\boldsymbol{A}_{p} / I_{p}, \boldsymbol{g}_{p, q} \mid p \leq q\right.$ in $\left.P\right)$, where $\boldsymbol{g}_{p, q}: \boldsymbol{A}_{p} / I_{p} \rightarrow \boldsymbol{A}_{q} / I_{q}$ is induced by $\boldsymbol{f}_{p, q}$, for all $p \leq q$ in $P$.

The diagram $\vec{A} / \vec{I}$ is a quotient of $\vec{A}$.
Remark 6.13. In the context of Definition 6.12 the following equalities hold:

$$
\begin{aligned}
\left(\boldsymbol{P}_{\mathrm{gr}} \circ \overrightarrow{\boldsymbol{A}}\right) / \vec{I} & =\boldsymbol{P}_{\mathrm{gr}} \circ(\overrightarrow{\boldsymbol{A}} / \vec{I}) . \\
(\boldsymbol{C} \circ \overrightarrow{\boldsymbol{A}}) / \vec{I} & =\boldsymbol{C} \circ(\overrightarrow{\boldsymbol{A}} / \vec{I}) .
\end{aligned}
$$

Moreover, up to a natural identification (cf. Notation 5.22 )

$$
\left(\boldsymbol{P}_{\mathrm{gl}} \circ \overrightarrow{\boldsymbol{A}}\right) / \vec{I}=\boldsymbol{P}_{\mathrm{g} 1} \circ(\overrightarrow{\boldsymbol{A}} / \vec{I})
$$

Definition 6.14. Let $\mathcal{V}$ be a variety of algebras, let $P$ be a poset. A partial lifting in $\mathcal{V}$ is a diagram $\overrightarrow{\boldsymbol{A}}=\left(\boldsymbol{A}_{p}, \boldsymbol{f}_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ in $\operatorname{Gamp}(\mathcal{V})$ such that the following statements hold:
(1) The gamp $\boldsymbol{A}_{p}$ is strong, congruence-tractable, distance-generated, and has an isomorphic realization (cf. Definitions 6.3 and 6.1), for each $p \in P$.
(2) The morphisms $f_{p, q}$ is strong and congruence-cuttable (cf. Definition 6.3), for all $p<q$ in $P$.
The partial lifting is a lattice partial lifting if $\mathcal{V}$ is a variety of lattices, $\boldsymbol{B}_{p}$ is distance-generated with chains for each $p \in P$, and $\boldsymbol{f}_{p, q}$ is congruence-cuttable with chains for all $p<q$ in $P$.

A partial lifting $\overrightarrow{\boldsymbol{A}}$ is congruence $n$-permutable if $\boldsymbol{A}_{p}$ is congruence $n$-permutable for each $p \in P$.

Let $\vec{S}=\left(S_{p}, \sigma_{p, q}\right)$ be a diagram in Sem $_{\vee, 0}$. A partial lifting of $\vec{S}$ is a partial lifting of $\overrightarrow{\boldsymbol{A}}$ such that $\boldsymbol{C} \circ \overrightarrow{\boldsymbol{A}} \cong \vec{S}$.
Remark 6.15. If $\overrightarrow{\boldsymbol{A}}$ is a partial lifting of $\vec{S}$, then there exists a diagram $\vec{A}^{\prime} \cong \overrightarrow{\boldsymbol{A}}$ such that $\boldsymbol{C} \circ \overrightarrow{\boldsymbol{A}}^{\prime}=\vec{S}$. Hence we can assume that $\boldsymbol{C} \circ \overrightarrow{\boldsymbol{A}}=\vec{S}$.

The following result expresses the fact that a subdiagram or a quotient of a partial lifting is a partial lifting.
Lemma 6.16. Let $\mathcal{V}$ be a variety of algebras, let $P$ be a poset, and let $\overrightarrow{\boldsymbol{A}}$ be a partial lifting in $\mathcal{V}$ of a diagram $\vec{S}$. The following statements hold:
(1) Let $\vec{I}$ be an ideal of $\overrightarrow{\boldsymbol{A}}$; then $\overrightarrow{\boldsymbol{A}} / \vec{I}$ is a partial lifting of $\vec{S} / I$.
(2) Let $Q \subseteq P$; then $\overrightarrow{\boldsymbol{A}} \upharpoonright Q$ is a partial lifting of $\vec{S} \upharpoonright Q$.

Proof. The statement (1) follows from Proposition 6.5 and Proposition 6.7. The statement (2) is immediate.
Lemma 6.17. Let $\mathcal{V}$ be a variety of algebras, let $P$ be a directed poset with no maximal element, and let $\overrightarrow{\boldsymbol{A}}=\left(\boldsymbol{A}_{p}, \boldsymbol{f}_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ be a partial lifting in $\mathcal{V}$. Consider a colimit cocone:

$$
\left(\boldsymbol{A}, \boldsymbol{f}_{p} \mid p \in P\right)=\underset{\longrightarrow}{\lim } \overrightarrow{\boldsymbol{A}} \quad \text { in } \operatorname{Gamp}(\mathcal{V}) .
$$

Then $\boldsymbol{A}$ is an algebra in $\mathcal{V}$. Moreover, for any $n \geq 2$, if all $\boldsymbol{A}_{p}$ are congruence n-permutable, then the algebra corresponding to $\boldsymbol{A}$ is congruence n-permutable.
Proof. The morphism $\boldsymbol{f}_{p, q}:\left(A_{p}, \delta_{\boldsymbol{A}_{p}}, \widetilde{A}_{p}\right) \rightarrow\left(A_{q}, \delta_{\boldsymbol{A}_{q}}, \widetilde{A}_{q}\right)$ of pregamps is both strong and congruence-tractable for all $p<q$ in $P$. It follows from the description of colimits in $\operatorname{Gamp}(\mathcal{V})(c f$. Lemma 6.10 and Corollary 5.15) that $A$ is an algebra and there is an isomorphism $\phi: \operatorname{Con}_{\mathrm{c}} A \rightarrow \widetilde{A}$ satisfying:

$$
\phi\left(\Theta_{A}(x, y)\right)=\delta_{\boldsymbol{A}}(x, y), \quad \text { for all } x, y \in A
$$

As $A^{*}=\bigcup_{p \in P} f_{p}\left(A_{p}^{*}\right)$ and $f_{p, q}\left(A_{p}\right) \subseteq A_{q}^{*}$ for all $p<q$ in $P$, it follows that $A^{*}=A$. Therefore $\left(\mathrm{id}_{A}, \phi\right): \boldsymbol{G}(A) \rightarrow \boldsymbol{A}$ is an isomorphism of gamps.

Now assume that all $\boldsymbol{A}_{p}$ are congruence $n$-permutable. It follows from Lemma 6.10 that $\boldsymbol{A}$ is congruence $n$-permutable. As $\boldsymbol{A}$ is an algebra, the conclusion follows from Proposition 2.1.

## 7. Locally finite properties

The aim of this section is to prove Lemma 7.8, which is a special version of the Buttress Lemma of [5] adapted to gamps for the functor $\boldsymbol{C}$.

We use the following generalizations of (2), (3), (4), (6), and (7) of Definition 6.3 .
Definition 7.1. Fix a gamp $\boldsymbol{A}$ in a variety $\mathcal{V}$, a $(\vee, 0)$-semilattice $S$, and a $(\vee, 0)$ homomorphism $\phi: \widetilde{A} \rightarrow S$. The gamp $\boldsymbol{A}$ is distance-generated through $\phi$ if the following statement holds:
(2') For each $s \in S$ there are $n<\omega$ and $n$-tuple $\vec{x}, \vec{y}$ of $A^{*}$ such that:

$$
s=\bigvee_{k<n} \phi\left(\delta_{\boldsymbol{A}}\left(x_{k}, y_{k}\right)\right)
$$

If $\boldsymbol{A}$ is a gamp of lattices, we say $\boldsymbol{A}$ is distance-generated with chains through $\phi$ if
(3') For each $s \in S$ there are $n<\omega$ and chains $x_{0}<y_{0}, \ldots, x_{n-1}<y_{n-1}$ of $\boldsymbol{A}$ such that:

$$
s=\bigvee_{k<n} \phi\left(\delta_{\boldsymbol{A}}\left(x_{k}, y_{k}\right)\right)
$$

The gamp $\boldsymbol{A}$ is congruence-tractable through $\phi$ if the following statement holds:
(4') Let $x, y, x_{0}, y_{0}, \ldots, x_{m-1}, y_{m-1}$ in $A^{*}$, if $\phi\left(\delta_{\boldsymbol{A}}(x, y)\right) \leq \bigvee_{k<m} \phi\left(\delta_{\boldsymbol{A}}\left(x_{k}, y_{k}\right)\right)$ then there are a positive integer $n$, a list $\vec{z}$ of parameters from $A$, and terms $t_{0}, \ldots, t_{n}$ such that, the following equations are satisfied in $A$ :

$$
\begin{aligned}
\phi\left(\delta_{\boldsymbol{A}}\left(x, t_{0}(\vec{x}, \vec{y}, \vec{z})\right)\right) & =0 \\
\phi\left(\delta_{\boldsymbol{A}}\left(y, t_{n}(\vec{x}, \vec{y}, \vec{z})\right)\right) & =0 \\
\phi\left(\delta_{\boldsymbol{A}}\left(t_{j}(\vec{y}, \vec{x}, \vec{z}), t_{j+1}(\vec{x}, \vec{y}, \vec{z})\right)\right) & =0 \quad(\text { for all } j<n) .
\end{aligned}
$$

A morphism $\boldsymbol{f}: \boldsymbol{U} \rightarrow \boldsymbol{A}$ of gamps is congruence-cuttable through $\phi$ if the following statement holds:
( $6^{\prime}$ ) Given $X \subseteq S$, given $x, y$ in $U$, if $\phi\left(\widetilde{f}\left(\delta_{\boldsymbol{A}}(x, y)\right)\right) \leq \bigvee X$ then there exist $n<$ $\omega$ and $f(x)=x_{0}, x_{1}, \ldots, x_{n}=f(y)$ in $A^{*}$ such that $\delta_{\boldsymbol{A}}\left(x_{k}, x_{k+1}\right) \in S \downarrow X$ for all $k<n$.
A morphism $\boldsymbol{f}: \boldsymbol{U} \rightarrow \boldsymbol{A}$ of gamps of lattices is congruence-cuttable with chains through $\phi$ if the following statement holds:
( $7^{\prime}$ ) Given $X \subseteq S$, given $x, y \in U$, if $\phi\left(\tilde{f}\left(\delta_{\boldsymbol{A}}(x, y)\right)\right) \leq \bigvee X$ then there are $n<\omega$ and a chain $x_{0}<x_{1}<\cdots<x_{n}$ of $\boldsymbol{A}$ such that $x_{0}=f(x) \wedge f(y)$, $x_{n}=f(x) \vee f(y)$, and $\phi\left(\delta_{\boldsymbol{A}}\left(x_{k}, x_{k+1}\right)\right) \in S \downarrow X$ for all $k<n$.

Remark 7.2. A gamp $\boldsymbol{A}$ is distance-generated through $\operatorname{id}_{\widetilde{A}}$ if and only if $\boldsymbol{A}$ is distance-generated.

A gamp $\boldsymbol{A}$ is congruence-tractable through $\operatorname{id}_{\tilde{A}}$ if and only if $\boldsymbol{A}$ is congruencetractable.

A morphism $\boldsymbol{f}: \boldsymbol{U} \rightarrow \boldsymbol{A}$ of gamps is congruence-cuttable through $\mathrm{id}_{\widetilde{A}}$ if and only if $f$ is congruence-cuttable.

A morphism $\boldsymbol{f}: \boldsymbol{U} \rightarrow \boldsymbol{A}$ of gamps of lattices is congruence-cuttable with chains through id $\tilde{A}$ if and only if $f$ is congruence-cuttable with chains.
Lemma 7.3. Let $\boldsymbol{A}$ be a gamp in a variety of algebras $\mathcal{V}$, let $\phi: \widetilde{A} \rightarrow S$ an idealinduced ( $\mathrm{V}, 0$ )-homomorphism, and put $I=\operatorname{ker}_{0} \phi$.
(1) If $\boldsymbol{A}$ is distance-generated through $\phi$, then $\boldsymbol{A} / I$ is distance-generated.
(2) If $\boldsymbol{A}$ is distance-generated with chains through $\phi$, then $\boldsymbol{A} / I$ is distancegenerated with chains.
(3) If $\boldsymbol{A}$ is congruence-tractable through $\phi$ then $\boldsymbol{A} / I$ is congruence-tractable.

Proof. As $\phi$ is ideal-induced, it induces an isomorphism $\xi: \widetilde{A} / I \rightarrow S$.
Assume that $\boldsymbol{A}$ is distance-generated through $\phi$. Let $d \in \widetilde{A} / I$, put $s=\xi(d)$. There are $n<\omega$ and $n$-tuples $\vec{x}, \vec{y}$ in $A^{*}$ such that $s=\bigvee_{k<n} \phi\left(\delta_{\boldsymbol{A}}\left(x_{k}, y_{k}\right)\right)$. It follows that $s=\bigvee_{k<n} \xi\left(\delta_{\boldsymbol{A} / I}\left(x_{k} / I, y_{k} / I\right)\right)$, so $d=\bigvee_{k<n} \delta_{\boldsymbol{A} / I}\left(x_{k} / I, y_{k} / I\right)$. Therefore $\boldsymbol{A} / I$ is distance-generated.

The case where $\boldsymbol{A}$ is distance-generated with chains through $\phi$ is similar.
Assume that $\boldsymbol{A}$ is congruence-tractable through $\phi$. Let $x, y, x_{0}, y_{0}, \ldots, x_{m-1}, y_{m-1}$ in $A^{*}$ such that:

$$
\delta_{\boldsymbol{A} / I}(x / I, y / I) \leq \bigvee_{k<m} \delta_{\boldsymbol{A} / I}\left(x_{k} / I, y_{k} / I\right)
$$

Thus $\phi\left(\delta_{\boldsymbol{A} / I}(x, y)\right) \leq \bigvee_{k<m} \phi\left(\delta_{\boldsymbol{A}}\left(x_{k}, y_{k}\right)\right)$. Hence there are a positive integer $n$, a list $\vec{z}$ of parameters from $A$, and terms $t_{0}, \ldots, t_{n}$ such that the following equations are satisfied in $A$ :

$$
\begin{aligned}
\phi\left(\delta_{\boldsymbol{A}}\left(x, t_{0}(\vec{x}, \vec{y}, \vec{z})\right)\right) & =0 \\
\phi\left(\delta_{\boldsymbol{A}}\left(y, t_{n}(\vec{x}, \vec{y}, \vec{z})\right)\right) & =0 \\
\phi\left(\delta_{\boldsymbol{A}}\left(t_{k}(\vec{y}, \vec{x}, \vec{z}), t_{k+1}(\vec{x}, \vec{y}, \vec{z})\right)\right) & =0 \quad(\text { for all } k<n) .
\end{aligned}
$$

Those equations imply that the following equations are satisfied in $A / I$ :

$$
\begin{aligned}
x / I & =t_{0}(\vec{x} / I, \vec{y} / I, \vec{z} / I), \\
y / I & =t_{n}(\vec{x} / I, \vec{y} / I, \vec{z} / I), \\
t_{k}(\vec{y} / I, \vec{x} / I, \vec{z} / I) & =t_{k+1}(\vec{x} / I, \vec{y} / I, \vec{z} / I) \quad(\text { for all } k<n) .
\end{aligned}
$$

Therefore $\boldsymbol{A} / I$ is congruence-tractable.
The proof of the following lemma is similar.
Lemma 7.4. Let $\boldsymbol{f}: \boldsymbol{U} \rightarrow \boldsymbol{A}$ be a morphism of gamps, let $I$ be an ideal of $\boldsymbol{U}$, and let $\phi: \widetilde{A} \rightarrow S$ be an ideal-induced $(\vee, 0)$-homomorphism. Put $J=\operatorname{ker}_{0} \phi$. Denote by $\boldsymbol{g}: \boldsymbol{U} / I \rightarrow \boldsymbol{A} / J$ the morphism of gamps induced by $\boldsymbol{f}$. The following statements hold:
(1) If $\boldsymbol{f}$ is congruence-cuttable through $\phi$, then $\boldsymbol{g}$ is congruence-cuttable.
(2) Assume that $\boldsymbol{f}$ is a morphism of gamps of lattices. If $\boldsymbol{f}$ is congruencecuttable with chains through $\phi$, then $\boldsymbol{g}$ is congruence-cuttable with chains.

We shall now define locally finite properties for an algebra $B$, as properties that are satisfied by "many" finite subgamps of $\boldsymbol{G}(B)$.

Definition 7.5. Let $B$ be an algebra. A locally finite property for $B$ is a property $(P)$ in a subgamp $\boldsymbol{A}$ of $\boldsymbol{G}(B)$ such that there exists a finite $X \subseteq B$ satisfying
that for every finite full partial subalgebra $A^{*}$ of $B$ that contains $X$, there exists a finite $Y \subseteq B$ such that for every finite full partial subalgebra $A$ of $B$ that contains $A^{*} \cup Y$ and every finite $(\vee, 0)$-subsemilattice $\widetilde{A}$ of $\operatorname{Con}_{\mathrm{c}} B$ that contains $\operatorname{Con}_{\mathrm{c}}^{A}(B)$, the subgamp $\left(A^{*}, A, \Theta_{B}, \widetilde{A}\right)$ of $\boldsymbol{G}(B)$ satisfies $(P)$.
Proposition 7.6. Any finite conjunction of locally finite properties is locally finite.
Lemma 7.7. Let $B$ be an algebra and denote by $\mathscr{L}$ the similarity type of $B$. The properties (1)-(8) in $\boldsymbol{A}$ are locally finite for $B$ :

Assume that $\mathscr{L}$ is finite.
(1) $\boldsymbol{A}$ is strong.

Fix an ideal-induced $(\vee, 0)$-homomorphism $\phi: \operatorname{Con}_{\mathrm{c}} B \rightarrow S$, with $S$ finite.
(2) $\boldsymbol{A}$ is distance-generated through $\phi \upharpoonright \widetilde{A}$.
 sume that $B$ is a lattice.
(3) $\boldsymbol{A}$ is distance-generated with chains through $\phi \upharpoonright \widetilde{A}$.

Fix an ideal-induced $(\vee, 0)$-homomorphism $\phi: \operatorname{Con}_{\mathrm{c}} B \rightarrow S$, with $S$ finite.
(4) $\boldsymbol{A}$ is congruence-tractable through $\phi \upharpoonright \widetilde{A}$.

Assume that $\mathscr{L}$ is finite, fix a finite gamp $\boldsymbol{U}$, fix a morphism $\boldsymbol{f}: \boldsymbol{U} \rightarrow \boldsymbol{G}(B)$ of gamps.
(5) The restriction $\boldsymbol{f}: \boldsymbol{U} \rightarrow \boldsymbol{A}$ is strong.

Fix an ideal-induced $(\vee, 0)$-homomorphism $\phi: \operatorname{Con}_{\mathrm{c}} B \rightarrow S$ where $S$ is finite, a finite gamp $\boldsymbol{U}$, and a morphism $\boldsymbol{f}: \boldsymbol{U} \rightarrow \boldsymbol{G}(B)$ of gamps.
(6) The restriction of $\boldsymbol{f}: \boldsymbol{U} \rightarrow \boldsymbol{A}$ is congruence-cuttable through $\phi \upharpoonright \widetilde{A}$.

Assume that $B$ is a lattice. Fix an ideal-induced $(\vee, 0)$-homomorphism $\phi: \operatorname{Con}_{\mathrm{c}} B \rightarrow$ $S$, where $S$ is finite. Fix a finite gamp $\boldsymbol{U}$, fix a morphism $\boldsymbol{f}: \boldsymbol{U} \rightarrow \boldsymbol{G}(B)$ of gamps.
(7) The restriction of $\boldsymbol{f}: \boldsymbol{U} \rightarrow \boldsymbol{A}$ is congruence-cuttable with chains through $\phi \upharpoonright \widetilde{A}$.
Let $n \geq 2$ be an integer. Assume that $B$ is congruence $n$-permutable.
(8) $\boldsymbol{A}$ is congruence $n$-permutable.

Proof. (1) Put $X=\emptyset$ and let $A^{*}$ be a finite full partial subalgebra of $B$. Put $Y=$ $\left\{\ell^{B}(\vec{x}) \mid \ell \in \mathscr{L}\right.$ and $\vec{X}$ is an $\operatorname{ar}(\ell)$-tuple of $\left.A^{*}\right\}$. As $A^{*}$ and $\mathscr{L}$ are both finite, $Y$ is also finite. Let $A$ be a finite full partial subalgebra of $B$ that contains $A^{*} \cup Y$, let $\widetilde{A}$ be a finite $(\vee, 0)$-subsemilattice of $\operatorname{Con}_{\mathrm{c}} B$ containing $\operatorname{Con}_{\mathrm{c}}^{A}(B)$. As $\ell(\vec{x}) \in Y \subseteq A$, it is defined in $A$, for each $\ell \in \mathscr{L}$ and each $\operatorname{ar}(\ell)$-tuple $\vec{X}$ of $A$.
(2) Let $s \in S$. As $\phi$ is surjective, there exists $\theta \in \operatorname{Con}_{\mathrm{c}} B$ such that $s=\phi(\theta)$. So there exist $n<\omega$ and $n$-tuple $\vec{x}, \vec{y}$ of $B$ such that $s=\phi\left(\bigvee_{k<n} \Theta_{B}\left(x_{k}, y_{k}\right)\right)$. Put $X_{s}=\left\{x_{0}, \ldots, x_{n-1}, y_{0}, \ldots, y_{n-1}\right\}$.

Put $X=\bigcup_{s \in S} X_{s}$. As $S$ is finite and $X_{s}$ is finite for all $s \in S, X$ is finite. Let $A^{*}$ be a finite full partial subalgebra of $B$ that contains $X$. Put $Y=\emptyset$. Let $A$ be a finite full partial subalgebra of $B$ that contains $A^{*} \cup Y$. Let $\widetilde{A}$ be a finite $(\vee, 0)$ subsemilattice of $\operatorname{Con}_{\mathrm{c}} B$ containing $\operatorname{Con}_{\mathrm{c}}^{A}(B)$. By construction $\left(A^{*}, A, \Theta_{B}, \widetilde{A}\right)$ satisfies (2).

The proof that (3) is a locally finite property is similar.
(4) Put $X=\emptyset$, let $A^{*}$ be a finite full partial subalgebra of $B$. Denote by $E$ the set of all quadruples $(x, y, \vec{x}, \vec{y})$ such that the following statements are satisfied:

- $x, y \in A^{*}$.
- $\vec{X}$ and $\vec{y}$ are $m$-tuples of $A^{*}$, for some $m<\omega$.
- $\phi\left(\Theta_{B}(x, y)\right) \leq \bigvee_{k<m} \phi\left(\Theta_{B}\left(x_{k}, y_{k}\right)\right)$.
- $\left(x_{i}, y_{i}\right) \neq\left(x_{j}, y_{j}\right)$ for all $i<j<m$.

Put $I=\operatorname{ker}_{0} \phi$ and let $(x, y, \vec{x}, \vec{y}) \in E$. As $\phi$ is ideal-induced, there exists $\alpha \in I$ such that $\Theta_{B}(x, y) \leq \bigvee_{k<m} \Theta_{B}\left(x_{k}, y_{k}\right) \vee \alpha$. Let $\vec{x}^{\prime}, \vec{y}^{\prime}$ be $p$-tuples of $B$ such that $\alpha=\bigvee_{k<p} \Theta\left(x_{k}^{\prime}, y_{k}^{\prime}\right)$. Hence:

$$
\Theta_{B}(x, y) \leq \bigvee_{k<m} \Theta_{B}\left(x_{k}, y_{k}\right) \vee \bigvee_{k<p} \Theta_{B}\left(x_{k}^{\prime}, y_{k}^{\prime}\right)
$$

It follows from Lemma 5.11 that there are a positive integer $n$, a list $\vec{z}$ of parameters from $B$, and terms $t_{0}, \ldots, t_{n}$ such that the following equalities hold in $B$ :

$$
\begin{align*}
x & =t_{0}\left(\vec{x}, \vec{x}^{\prime}, \vec{y}, \vec{y}^{\prime}, \vec{z}\right),  \tag{7.1}\\
y & =t_{n}\left(\vec{x}, \vec{x}^{\prime}, \vec{y}, \vec{y}^{\prime}, \vec{z}\right),  \tag{7.2}\\
t_{j}\left(\vec{y}, \vec{y}^{\prime}, \vec{x}, \vec{x}^{\prime}, \vec{z}\right) & =t_{j+1}\left(\vec{x}, \vec{x}^{\prime}, \vec{y}, \vec{y}^{\prime}, \vec{z}\right), \quad \text { for all } j<n . \tag{7.3}
\end{align*}
$$

Put $t_{j}^{\prime}(\vec{a}, \vec{b}, \vec{c}, \vec{d})=t_{j}(\vec{a}, \vec{d}, \vec{b}, \vec{d}, \vec{c})$ for all tuples $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ of $B$ of appropriate length. The following inequalities follow from the compatibility of $\Theta_{B}$ with terms:

$$
\begin{align*}
& \Theta_{B}\left(t_{j}^{\prime}\left(\vec{x}, \vec{y}, \vec{z}, \vec{x}^{\prime}\right), t_{j}\left(\vec{x}, \vec{x}^{\prime}, \vec{y}, \vec{y}^{\prime}, \vec{z}\right)\right) \leq \bigvee_{k<p} \Theta_{B}\left(x_{k}^{\prime}, y_{k}^{\prime}\right)=\alpha  \tag{7.4}\\
& \Theta_{B}\left(t_{j}^{\prime}\left(\vec{y}, \vec{x}, \vec{z}, \vec{x}^{\prime}\right), t_{j}\left(\vec{y}, \vec{y}^{\prime}, \vec{x}, \vec{x}^{\prime}, \vec{z}\right)\right) \leq \bigvee_{k<p} \Theta_{B}\left(x_{k}^{\prime}, y_{k}^{\prime}\right)=\alpha \tag{7.5}
\end{align*}
$$

As $\phi(\alpha)=0$, it follows from (7.4) and (7.1) that:

$$
\phi\left(\Theta_{B}\left(x, t_{0}^{\prime}\left(\vec{x}, \vec{y}, \vec{z}, \vec{x}^{\prime}\right)\right)\right)=0
$$

Set $\vec{z}^{\prime}=\left(\vec{z}, \vec{x}^{\prime}\right)$. As $\phi(\alpha)=0$, it follows from (7.1)-(7.5) that

$$
\begin{aligned}
\phi\left(\Theta_{B}\left(x, t_{0}^{\prime}\left(\vec{x}, \vec{y}, \vec{z}^{\prime}\right)\right)\right) & =0 \\
\phi\left(\Theta_{B}\left(y, t_{n}^{\prime}\left(\vec{x}, \vec{y}, \vec{z}^{\prime}\right)\right)\right) & =0 \\
\phi\left(\Theta_{B}\left(t_{j}^{\prime}\left(\vec{y}, \vec{x}, \vec{z}^{\prime}\right), t_{j+1}^{\prime}\left(\vec{x}, \vec{y}, \vec{z}^{\prime}\right)\right)\right) & =0 \quad(\text { for all } j<n) .
\end{aligned}
$$

Let $Y_{(x, y, \vec{x}, \vec{y})}$ be a finite partial subalgebra of $B$ such that both $t_{j}^{\prime}\left(\vec{x}, \vec{y}, \vec{z}, \vec{x}^{\prime}\right)$ and $t_{j}^{\prime}\left(\vec{y}, \vec{x}, \vec{z}, \vec{x}^{\prime}\right)$ are defined in $Y_{(x, y, \vec{x}, \vec{y})}$, for each $j \leq n$.

Put $Y=\bigcup\left(Y_{e} \mid e \in E\right)$. As $Y_{e}$ is finite for each $e \in E$ and $E$ is finite, $Y$ is finite. Let $A$ be a finite full partial subalgebra of $B$ that contains $Y \cup A^{*}$, and let $\widetilde{A}$ be a finite $(\vee, 0)$-subsemilattice of $\operatorname{Con}_{\mathrm{c}} B$ containing $\operatorname{Con}_{\mathrm{c}}^{A}(B)$. It is not hard to verify that $\left(A^{*}, A, \Theta_{B}, \widetilde{A}\right)$ satisfies (4).
(5) As $U$ is finite, the set $X_{1}=\langle f(U)\rangle_{B}^{1}$ (cf. Notation 4.7) is also finite. As $\widetilde{U}$ is finite and each element of $\tilde{f}(\widetilde{U})$ is a compact congruence of $B$, we can choose a finite subset $X_{2}$ of $B$ such that $\widetilde{f}(\widetilde{U}) \subseteq \operatorname{Con}_{\mathrm{c}}^{X_{2}}(B)$. The set $X=X_{1} \cup X_{2}$ is finite. Let $A^{*}$ be a finite full partial subalgebra of $B$ containing $X$. Put $Y=\emptyset$, let $A$ be a full partial subalgebra of $B$ containing $Y \cup A^{*}$, and let $\widetilde{A}$ be a finite $(\vee, 0)$-subsemilattice of $\operatorname{Con}_{\mathrm{c}} B$ containing $\operatorname{Con}_{\mathrm{c}}^{A}(B)$. The following containments hold:

$$
\widetilde{f}(\widetilde{U}) \subseteq \operatorname{Con}_{\mathrm{c}}^{X_{2}}(B) \subseteq \operatorname{Con}_{\mathrm{c}}^{A}(B) \subseteq \widetilde{A}
$$

Moreover $\langle f(U)\rangle_{B}^{1} \subseteq A^{*}$.

The proofs that (6), (7), and (8) are locally finite properties are similar.
The following lemma is an analogue for gamps of the Buttress Lemma (cf. ${ }^{6}$ ).
Lemma 7.8. Let $\mathcal{V}$ be a variety of algebras in a finite similarity type, let $P$ be a lower finite poset, let $\left(S_{p}\right)_{p \in P}$ be a family of finite $(\vee, 0)$-semilattices, let $B \in \mathcal{V}$, and let $\left(\phi_{p}\right)_{p \in P}$ be a family of $(\vee, 0)$-homomorphisms where $\phi_{p}: \operatorname{Con}_{\mathrm{c}} B \rightarrow S_{p}$ is ideal-induced for each $p \in P$. There exists a diagram $\overrightarrow{\boldsymbol{A}}=\left(\boldsymbol{A}_{p}, \boldsymbol{f}_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ of finite subgamps of $\boldsymbol{G}(B)$ such that the following statements hold:
(1) $\phi_{p} \upharpoonright \widetilde{A}_{p}$ is ideal-induced for each $p \in P$.
(2) $\boldsymbol{A}_{p}$ is strong, distance-generated through $\phi_{p} \upharpoonright \widetilde{A}_{p}$, and congruence-tractable through $\phi_{p} \upharpoonright \widetilde{A}_{p}$, for each $p \in P$.
(3) $\boldsymbol{A}_{p}$ is a subgamp of $\boldsymbol{A}_{q}$ and $\boldsymbol{f}_{p, q}$ is the canonical embedding, for all $p \leq q$ in $P$.
(4) $\boldsymbol{f}_{p, q}$ is strong and congruence-cuttable through $\phi_{q} \upharpoonright \widetilde{A}_{q}$, for all $p<q$ in $P$.

If $B$ is a lattice, then we can construct $\overrightarrow{\boldsymbol{A}}$ such that $\boldsymbol{f}_{p, q}$ is congruence-cuttable with chains through $\phi_{q} \upharpoonright \widetilde{A}_{q}$ for all $p<q$ in $P$, and $\boldsymbol{B}_{p}$ is distance-generated with chains through $\phi_{p}$ for each $p \in P$.

If $B$ is congruence $n$-permutable (where $n$ is a positive integer), then we can construct $\overrightarrow{\boldsymbol{A}}$ such that $\boldsymbol{A}_{p}$ is congruence $n$-permutable for each $p \in P$.

Proof. Let $r \in P$, suppose having constructed a diagram ( $\boldsymbol{A}_{p}, \boldsymbol{f}_{p, q} \mid p \leq q<r$ ) of finite subgamps of $\boldsymbol{G}(B)$ such that the following statements hold:
(1) $\phi_{p} \upharpoonright \widetilde{A}_{p}$ is ideal-induced for each $p<r$.
(2) $\boldsymbol{A}_{p}$ is strong, distance-generated through $\phi_{p} \upharpoonright \widetilde{A}_{p}$, and congruence-tractable through $\phi_{p} \upharpoonright \widetilde{A}_{p}$, for each $p<r$.
(3) $\boldsymbol{A}_{p}$ is a subgamp of $\boldsymbol{A}_{q}$ and $\boldsymbol{f}_{p, q}$ is the canonical embedding, for all $p \leq$ $q<r$.
(4) $\boldsymbol{f}_{p, q}$ is strong and congruence-cuttable through $\phi_{q} \upharpoonright \widetilde{A}_{q}$, for all $p<q<r$.

The following property in $\boldsymbol{A}$ a subgamp of $\boldsymbol{G}(B)$ is locally finite (see Proposition 7.6 and Lemma 7.7)
( $F$ ) $\boldsymbol{A}$ is strong, distance-generated through $\phi_{r} \upharpoonright \widetilde{A}$, congruence-tractable through $\phi_{r} \upharpoonright \widetilde{A}$, and the canonical embedding $\boldsymbol{f}: \boldsymbol{A}_{p} \rightarrow \boldsymbol{A}$ is strong and congruencecuttable through $\phi_{r} \upharpoonright \widetilde{A}$, for all $p<r$.
Thus there exist finite partial subalgebras $A_{r}^{*}$ and $A_{r}$ of $B$ such that for each $\widetilde{A}$ containing $\operatorname{Con}_{\mathrm{c}}^{A_{r}}(B)$, the $\operatorname{gamp}\left(A_{r}^{*}, A_{r}, \Theta_{B}, \widetilde{A}\right)$ satisfies $(F)$. Moreover it follows from Proposition 3.16 that there exists a finite $(\mathrm{V}, 0)$-subsemilattice $\widetilde{A}_{r}$ of $\mathrm{Con}_{\mathrm{c}} B$, such that $\operatorname{Con}_{\mathrm{c}}^{A_{r}}(B) \subseteq \widetilde{A}_{r}$ and $\phi_{r} \upharpoonright \widetilde{A}_{r}$ is ideal-induced.

Set $\boldsymbol{A}_{r}=\left(A_{r}^{*}, A_{r}, \Theta_{B}, \widetilde{A}_{r}\right)$, and $\boldsymbol{f}_{p, r}$ the canonical embedding for each $p \leq r$. By construction, the diagram $\left(\boldsymbol{A}_{p}, \boldsymbol{f}_{p, q} \mid p \leq q \leq r\right)$ satisfies the required conditions. The construction of ( $\boldsymbol{A}_{p}, \boldsymbol{f}_{p, q} \mid p \leq q$ in $P$ ) follows by induction.

If $B$ is a lattice, then we can add to the property $(F)$ the condition $\boldsymbol{f}: \boldsymbol{A}_{p} \rightarrow \boldsymbol{A}$ is congruence-cuttable with chains through $\phi_{r} \upharpoonright \widetilde{A}$ for each $p<r$, and $\boldsymbol{B}_{r}$ is distancegenerated with chains through $\phi_{r}$.

If $B$ is congruence $n$-permutable, then we can add to the property $(F)$ the condition $\boldsymbol{A}_{r}$ is congruence $n$-permutable.

Remark 7.9. In the context of Lemma 7.8, if we have a locally finite property for $B$, then we can assume that any $A_{p}$ satisfies this property.

## 8. Norm-Coverings and lifters

The aim of this section is to construct a (family) of posets, which we shall use later as an index for a diagram (in (4). We also give a combinatorial statement that is satisfied by this poset.

We introduced the following definition in [3].
Definition 8.1. A finite subset $V$ of a poset $U$ is a kernel if for every $u \in U$, there exists a largest element $v \in V$ such that $v \leq u$. We say that $U$ is supported if every finite subset of $U$ is contained in a kernel of $U$.

It is not hard to verify that this definition of a supported poset is equivalent to the one used in [5].

The following definition introduced in [3] also appears in [5] in a weaker form. Nevertheless, in the context of $\aleph_{0}$-lifters (cf. Definition 8.3), all these definitions are equivalent.
Definition 8.2. A norm-covering of a poset $P$ is a pair $(U, \partial)$, where $U$ is a supported poset and $\partial: U \rightarrow P, u \mapsto \partial u$ is an isotone map.

We say that an ideal $\boldsymbol{u}$ of $U$ is sharp if the set $\{\partial u \mid u \in \boldsymbol{u}\}$ has a largest element, which we shall then denote by $\partial \boldsymbol{u}$. We shall denote by $\mathrm{Id}_{\mathrm{s}} U$ the set of all sharp ideals of $U$, partially ordered by inclusion.

We remind the reader about the following definition introduced in [5].
Definition 8.3. Let $P$ be a poset. An $\aleph_{0}$-lifter of $P$ is a pair $(U, \boldsymbol{U})$, where $U$ is a norm-covering of $P$ and $\boldsymbol{U}$ is a subset of $\operatorname{Id}_{\mathrm{s}} U$ satisfying the following properties:
(1) The set $\boldsymbol{U}^{=}=\{\boldsymbol{u} \in \boldsymbol{U} \mid \partial \boldsymbol{u}$ is not maximal in $P\}$ is lower finite, that is, the set $\boldsymbol{U} \downarrow \boldsymbol{u}$ is finite for each $\boldsymbol{u} \in \boldsymbol{U}^{=}$.
(2) For every map $S: \boldsymbol{U}^{=} \rightarrow[U]^{<\omega}$, there exists an isotone map $\sigma: P \rightarrow \boldsymbol{U}$ such that
(a) the map $\sigma$ is a section of $\partial$, that is, $\partial \sigma(p)=p$ holds for each $p \in P$;
(b) the containment $S(\sigma(p)) \cap \sigma(q) \subseteq \sigma(p)$ holds for all $p<q$ in $P$. (Observe that $\sigma(p)$ belongs to $\boldsymbol{U}^{=}$.)
The existence of lifters is related to the following infinite combinatorial statement introduced in [6].
Definition 8.4. For cardinals $\kappa, \lambda$ and a poset $P$, let $(\kappa,<\lambda) \rightsquigarrow P$ hold if for every mapping $F: \mathfrak{P}(\kappa) \rightarrow[\kappa]^{<\lambda}$, there exists a one-to-one map $f: P \hookrightarrow \kappa$ such that

$$
F(f(P \downarrow p)) \cap f(P \downarrow q) \subseteq f(P \downarrow p), \quad \text { for all } p \leq q \text { in } P
$$

Notice that in case $P$ is lower finite, it is sufficient to verify the conclusion above for all $F:[\kappa]^{<\omega} \rightarrow[\kappa]^{<\lambda}$ isotone and all $p \prec q$ in $P$.
Lemma 8.5. The square poset has an $\aleph_{0}$-lifter $(X, \boldsymbol{X})$ such that $\operatorname{card} X=\aleph_{1}$.
Proof. By 6, Proposition 4.7], the Kuratowski index (cf. [6, Definition 4.1]) of the square $P$ is less than or equal to its order-dimension, which is equal to 2 . Hence, by the definition of the Kuratowski index, $\left(\kappa^{+},<\kappa\right) \rightsquigarrow P$ for every infinite cardinal $\kappa$. Therefore, by [5, Corollary 3-5.8], $P$ has an $\aleph_{0}$-lifter $(X, \boldsymbol{X})$ such that $\operatorname{card} X=\operatorname{card} \boldsymbol{X}=\aleph_{1}$.

Given a poset $P$, we introduce a new poset which looks like a lexicographical product of $P$ with a tree. This construction is mainly used in 4 .
Definition 8.6. Let $P$ be a poset with a smallest element, let $X \subseteq P$, let $\vec{R}=$ $\left(R_{x}\right)_{x \in X}$ be a family of sets, let $\alpha \leq \omega$. Consider the following poset:

$$
T=\left\{(n, \vec{x}, \vec{r}) \mid n<\alpha, \vec{x} \in X^{n}, \text { and } \vec{r} \in R_{x_{0}} \times \cdots \times R_{x_{n-1}}\right\}
$$

ordered by $(m, \vec{x}, \vec{r}) \leq(n, \vec{y}, \vec{s})$ if and only if $m \leq n, \vec{x}=\vec{y} \upharpoonright m$, and $\vec{r}=\vec{s} \upharpoonright m$. Recall that $\vec{y} \upharpoonright m=\left(y_{0}, \ldots, y_{m-1}\right)$. Given $t=(n, \vec{x}, \vec{r}) \in T$ and $m \leq n$, we set $t \upharpoonright m=(m, \vec{x} \upharpoonright m, \vec{r} \upharpoonright m)$.

Put:

$$
A=P \boxtimes_{\vec{R}} \alpha=T \times P=\bigcup_{n \in \alpha} \bigcup_{\vec{x} \in X^{n}}\left(\{n\} \times\{\vec{x}\} \times\left(R_{x_{0}} \times \cdots \times R_{x_{n-1}}\right) \times P\right)
$$

Any element of $A$ can be written $(n, \vec{x}, \vec{r}, p)$ with $n<\alpha, \vec{x} \in X^{n}$ and $\vec{r} \in R_{x_{0}} \times$ $\cdots \times R_{x_{n-1}}$.

We define an order on $A$ by $(m, \vec{x}, \vec{r}, p) \leq(n, \vec{y}, \vec{s}, q)$ if and only if the following conditions hold:
(1) $(m, \vec{x}, \vec{r}) \leq(n, \vec{y}, \vec{s})$.
(2) If $m=n$ then $p \leq q$.
(3) If $m<n$ then $p \leq y_{m}$.

Remark 8.7. In the context of Definition 8.6, notice that $T$ is a lower finite tree. Indeed, if $(n, \vec{x}, \vec{r}) \in T$, then $T \downarrow(n, \vec{x}, \vec{r})=\{(m, \vec{x} \upharpoonright m, \vec{r} \upharpoonright m) \mid m \leq n\}$ is a chain of length $n$. The tree $T$ is called the tree associated to $P \boxtimes_{\vec{R}} \alpha$.

The following statements hold:
(1) If $P$ is lower finite, then $A$ is lower finite.
(2) The inequality card $A \leq \aleph_{0}+\operatorname{card} P+\sum_{x \in X}$ card $R_{x}$ holds.
(3) The inequality card $T \leq \aleph_{0}+\sum_{x \in X}$ card $R_{x}$ holds.

Remark 8.8. In the context of Definition 8.6, if $a<b$ in $A$, then there are $t=$ $(n, \vec{x}, \vec{r}) \in T, p, q \in P$, and $m \leq n$ such that $a=(t \upharpoonright m, p)$ and $b=(t, q)$. Moreover, if $m<n$, then $(t \upharpoonright m, p)<(t \upharpoonright(m+1), 0) \leq(t, q)$. It follows easily that $a \prec b$ if and only if exactly one of the following statements holds:
(1) $m=n$ and $p \prec q$.
(2) $n=m+1, p=x_{m}$, and $q=0$.

As a consequence, we obtain immediately that If $P, X$, and all $R_{x}$ are finite, then each $a \in A$ has only finitely many covers.

Lemma 8.9. Let $\kappa \geq \lambda$ be infinite cardinals, let $P$ be a lower finite $\kappa$-small poset with a smallest element, let $X \subseteq P$, let $\vec{R}=\left(R_{x}\right)_{x \in X}$ be a family of $\kappa$-small sets, and let $\alpha \leq \omega$. If $(\kappa,<\lambda) \rightsquigarrow P$, then $(\kappa,<\lambda) \rightsquigarrow P \boxtimes_{\vec{R}} \alpha$.
Proof. Denote by $T$ the tree associated to $A=P \boxtimes_{\vec{R}} \alpha$. It follows from Remark 8.7 together with the assumptions on cardinalities that the following inequalities hold:

$$
\operatorname{card} T \leq \aleph_{0}+\sum_{x \in X} \operatorname{card} R_{x} \leq \aleph_{0}+\sum_{x \in X} \kappa \leq \kappa
$$

Thus there exists a partition $\left(K_{t}\right)_{t \in T}$ of $\kappa$ such that $\operatorname{card} K_{t}=\kappa$ for each $t \in T$.

Notice that $A$ is lower finite. Let $F:[\kappa]^{<\omega} \rightarrow[\kappa]^{<\lambda}$ isotone, let $t \in T$. Assume having constructed, for each $s<t$, a one-to-one map $\sigma_{s}: P \hookrightarrow K_{s}$ such that setting

$$
S_{s}=\left\{\sigma_{s \backslash m}(p) \mid m<n \text { and } p \leq x_{m}\right\}, \text { for all } s=(n, \vec{x}, \vec{r}) \leq t
$$

the following containments hold:

$$
\begin{aligned}
\operatorname{rng} \sigma_{s} & \subseteq K_{s}-F\left(S_{s}\right), & & \text { for each } s<t, \\
F\left(\sigma_{s}(P \downarrow p) \cup S_{s}\right) \cap \sigma_{s}(P \downarrow q) & \subseteq \sigma_{s}(P \downarrow p), & & \text { for all } p \leq q \text { in } P \text { and all } s<t .
\end{aligned}
$$

Put $F_{t}(U)=F\left(U \cup S_{t}\right)-F\left(S_{t}\right)$ for each $U \in\left[K_{t}-F\left(S_{t}\right)\right]^{<\omega}$. As $S_{t}$ is finite, this defines a map $F_{t}:\left[K_{t}-F\left(S_{t}\right)\right]^{<\omega} \rightarrow\left[K_{t}-F\left(S_{t}\right)\right]^{<\lambda}$. As $F\left(S_{t}\right)$ is $\lambda$-small, $\operatorname{card}\left(K_{t}-F\left(S_{t}\right)\right)=\kappa$, moreover $(\kappa,<\lambda) \rightsquigarrow P$, so there exists a one-to-one map $\sigma_{t}: P \hookrightarrow K_{t}-F\left(S_{t}\right)$ such that:

$$
F\left(\sigma_{t}(P \downarrow p) \cup S_{t}\right) \cap \sigma_{t}(P \downarrow q) \subseteq \sigma_{t}(P \downarrow p), \quad \text { for all } p \leq q \text { in } P
$$

Therefore we construct, by induction on $t$, a one-to-one map $\sigma_{t}: P \hookrightarrow K_{t}$ for each $t \in T$, such that setting

$$
\begin{equation*}
S_{t}=\left\{\sigma_{t \mid m}(p) \mid m<n \text { and } p \leq x_{m}\right\}, \text { for all } t=(n, \vec{x}, \vec{r}) \in T \tag{8.1}
\end{equation*}
$$

the following containments hold:

$$
\begin{align*}
\operatorname{rng} \sigma_{t} & \subseteq K_{t}-F\left(S_{t}\right), & & \text { for each } t \in T,  \tag{8.2}\\
F\left(\sigma_{t}(P \downarrow p) \cup S_{t}\right) \cap \sigma_{t}(P \downarrow q) & \subseteq \sigma_{t}(P \downarrow p), & & \text { for all } p \leq q \text { in } P \text { and all } t \in T . \tag{8.3}
\end{align*}
$$

For $(t, p) \in A$, set $\sigma(t, p)=\sigma_{t}(p)$. This defines a $\operatorname{map} \sigma: A \rightarrow \kappa$. Let $a=(s, p)$ and $b=(t, q)$ in $A$ such that $\sigma(a)=\sigma(b)$. It follows from (8.2) that $\sigma(a) \in K_{s}$ and $\sigma(b) \in K_{t}$. As $\left(K_{u}\right)_{u \in T}$ is a partition of $\kappa$, we obtain $s=t$. Moreover $\sigma_{t}(p)=\sigma(a)=\sigma(b)=\sigma_{t}(q)$, so, as $\sigma_{t}$ is one-to-one, $p=q$, and so $a=b$. Therefore $\sigma$ is one-to-one.

Let $a=(t, p) \in A$, with $t=(n, \vec{x}, \vec{r}) \in T$. It follows from the definition of $A$ that:

$$
A \downarrow a=\left\{(t \upharpoonright m, q) \in A \mid m<n \text { and } q \in P \downarrow x_{m}\right\} \cup\{(t, q) \in A \mid q \in P \downarrow p\}
$$

Thus, from (8.1), we see that

$$
\begin{equation*}
\sigma(A \downarrow a)=S_{t} \cup \sigma_{t}(P \downarrow p), \quad \text { for each } a=(t, p) \in A \tag{8.4}
\end{equation*}
$$

As $\left(K_{t}\right)_{t \in T}$ is a partition, it follows from (8.1) and (8.2) that $K_{t} \cap S_{t}=\emptyset$, thus, by (8.4),

$$
\begin{equation*}
\sigma(A \downarrow a) \cap K_{t}=\sigma_{t}(P \downarrow p), \quad \text { for each } a=(t, p) \in A \tag{8.5}
\end{equation*}
$$

Let $a \prec b$ in $A$. There are two cases to consider (cf. Remark 8.8). First assume that $a=(t, p)$ and $b=(t, q)$ with $p \prec q$ and $t=(n, \vec{x}, \vec{r}) \in T$. Let $c \leq b$ with $\sigma(c) \in F(\sigma(A \downarrow a))$. We can write $c=\left(t \upharpoonright m, p^{\prime}\right)$, with $m \leq n$. Suppose first that $m<n$. As $c \leq b, p^{\prime} \leq x_{m}$, thus $c<a$. So $\sigma(c) \in \sigma(A \downarrow a)$. Now suppose that $m=n$. It follows from (8.2) that

$$
\sigma(c)=\sigma_{t}\left(p^{\prime}\right) \in F(\sigma(A \downarrow a)) \cap \sigma(A \downarrow b) \cap\left(K_{t}-F\left(S_{t}\right)\right) .
$$

So, from (8.4) and (8.5) we obtain

$$
\sigma(c) \in F\left(\sigma_{t}(P \downarrow p) \cup S_{t}\right) \cap \sigma_{t}(P \downarrow q)
$$

Thus (8.3) implies that $\sigma(c) \in \sigma_{t}(P \downarrow p)$, from (8.4) we obtain $\sigma(c) \in \sigma(A \downarrow a)$. Therefore the containment $F(\sigma(A \downarrow a)) \cap \sigma(A \downarrow b) \subseteq \sigma(A \downarrow a)$ holds.

Now assume that $t=(n+1, \vec{x}, \vec{r}) \in T, b=(t, 0)$ and $a=\left(t \upharpoonright n, x_{n}\right)$. Let $c \leq b$ such that $\sigma(c) \in F(\sigma(A \downarrow a))$. As $c \leq b$ there are $m \leq n+1$ and $p \in P$ such that $c=(t \upharpoonright m, p)$. If $m \leq n$, then $p \leq x_{m}$, thus $c \leq a$, so $\sigma(c) \in \sigma(A \downarrow a)$. If $m=n+1$, then $c=b$. From (8.1) and (8.4) we obtain $\sigma(A \downarrow a)=S_{t i n} \cup \sigma_{t i n}\left(P \downarrow x_{n}\right)=S_{t}$. Therefore $\sigma_{t}(0)=\sigma(c) \in F\left(S_{t}\right)$, in contradiction with (8.2). So the containment $F(\sigma(A \downarrow a)) \cap \sigma(A \downarrow b) \subseteq \sigma(A \downarrow a)$ holds.
Corollary 8.10. For an integer $m>1$, put

$$
\mathrm{B}_{m}(\leq 2)=\{X \in \mathfrak{P}(m) \mid \text { either } \operatorname{card} X \leq 2 \text { or } X=m\}
$$

Let $P$ be a $(\vee, 0)$-semilattice embeddable, as a poset, into $\mathrm{B}_{m}(\leq 2)$. Let $X \subseteq P$, let $\vec{R}=\left(R_{x}\right)_{x \in X}$ be a family of finite sets, and let $\alpha \leq \omega$. There exists an $\aleph_{0}$-lifter $(U, \boldsymbol{U})$ of $A=P \boxtimes_{\vec{R}}$ a such that $U$ has cardinality $\aleph_{2}$.
Proof. It follows from [8], see also [2, Theorem 46.2], that $\left(\aleph_{2}, 2, \aleph_{0}\right) \rightarrow m$, so [6, Proposition 5.2] implies $\left(\aleph_{2},<\aleph_{0}\right) \rightsquigarrow \mathrm{B}_{m}(\leq 2)$, and so, from [6, Lemma 3.2] we obtain $\left(\aleph_{2},<\aleph_{0}\right) \rightsquigarrow P$. It follows from Lemma 8.9 that $\left(\aleph_{2},<\aleph_{0}\right) \rightsquigarrow A$. The conclusion follows from Remark 8.8 together with [5, Lemma 3-5.5].

## 9. The Condensate Lifting Lemma for gamps

In this section we apply the Armature Lemma from together with Lemma 7.8 to prove a special case of the Condensate Lifting Lemma for gamps.

In order to use the condensate constructions, we need categories and functors that satisfy the following conditions.
Definition 9.1. Let $\mathcal{A}$ and $\mathcal{S}$ be categories, let $\Phi: \mathcal{A} \rightarrow \mathcal{S}$ be a functor. We introduce the following statements:
(CLOS) $\mathcal{A}$ has all small directed colimits.
(PROD) Any two objects of $\mathcal{A}$ have a product in $\mathcal{A}$.
(CONT) The functor $\Phi$ preserves all small directed colimits.
Remark 9.2. Given a norm-covering $X$ of a poset $P$ and a category $\mathcal{A}$ that satisfies both (CLOS) and (PROD), we can construct an object $\mathbf{F}(X) \otimes \vec{A}$ which is a directed colimit of finite products of objects in $\vec{A}$, together with morphisms

$$
\pi_{x}^{X} \otimes \vec{A}: \mathbf{F}(X) \otimes \vec{A} \rightarrow A_{\partial \boldsymbol{x}}
$$

for each $\boldsymbol{x} \in \operatorname{Id}_{\mathrm{s}} X$.
Moreover if $\mathcal{A}$ is a class of algebras closed under finite products and directed colimits, then $\pi_{\boldsymbol{x}}^{X} \otimes \vec{A}$ is surjective, and $\operatorname{card}(\mathbf{F}(X) \otimes \vec{A}) \leq \operatorname{card} X+\sum_{p \in P} A_{p}$. For more details about this construction, we refer the reader to [5, Chapter 2].

In the following theorem, we refer the reader to Definition 6.14 for the definition of a partial lifting.
Theorem 9.3. Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of algebras such that $\mathcal{W}$ has finite similarity type, let $(X, \boldsymbol{X})$ be an $\aleph_{0}$-lifter of a poset $P$, let $\vec{A}=\left(A_{p}, f_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ be a diagram in $\mathcal{V}$ such that $\operatorname{Con}_{\mathrm{c}} A_{p}$ is finite for each $p \in P^{=}$, let $B \in \mathcal{W}$ such that $\operatorname{Con}_{\mathrm{c}} B \cong \operatorname{Con}_{\mathrm{c}}(\mathbf{F}(X) \otimes \vec{A})$. Then there exists a partial lifting $\overrightarrow{\boldsymbol{B}}=\left(\boldsymbol{B}_{p}, \boldsymbol{g}_{p, q} \mid\right.$ $p \leq q$ in $P)$ of $\operatorname{Con}_{\mathrm{c}} \circ \vec{A}$ in $\mathcal{W}$ such that $\boldsymbol{B}_{p}$ is finite for each $p \in P^{=}$and $\boldsymbol{B}_{p}$ is a quotient of $\boldsymbol{G}(B)$ for each $p \in \operatorname{Max} P$.

Proof. Denote by $\mathcal{S}$ the category of all $(\vee, 0)$-semilattices with ( $\vee, 0)$-homomorphisms. The category $\mathcal{V}$ satisfies (CLOS) and (PROD), so $\mathbf{F}(X) \otimes \vec{A}$ is well-defined (cf. [5, Section 3-1]). The functor Con $_{c}$ satisfies (CONT). Let $\chi: \operatorname{Con}_{\mathrm{c}} B \rightarrow$ $\operatorname{Con}_{\mathrm{c}}(\mathbf{F}(X) \otimes \vec{A})$ be an isomorphism and put $\rho_{\boldsymbol{x}}=\operatorname{Con}_{\mathrm{c}}\left(\pi_{\boldsymbol{x}}^{X} \otimes \vec{A}\right) \circ \chi$ for each $\boldsymbol{x} \in \boldsymbol{X}$. As $\pi_{\boldsymbol{x}}^{X} \otimes \vec{A}$ is surjective and $\chi$ is an isomorphism, it follows that $\rho_{\boldsymbol{x}}$ is ideal-induced. Notice that rng $\rho_{\boldsymbol{x}}=\operatorname{Con}_{\mathrm{c}} A_{\partial \boldsymbol{x}}$ is finite.

Lemma 7.8 implies that there exists a diagram $\overrightarrow{\boldsymbol{B}}=\left(\boldsymbol{B}_{\boldsymbol{x}}, \boldsymbol{g}_{\boldsymbol{x}, \boldsymbol{y}} \mid \boldsymbol{x} \leq \boldsymbol{y}\right.$ in $\left.\boldsymbol{X}^{=}\right)$ of finite subgamps of $\boldsymbol{G}(B)$ in $\mathcal{W}$ such that the following statements hold:
(1) $\rho_{\boldsymbol{x}} \upharpoonright \widetilde{B}_{\boldsymbol{x}}$ is ideal-induced for each $\boldsymbol{x} \in \boldsymbol{X}^{=}$.
(2) $\boldsymbol{B}_{\boldsymbol{x}}$ is strong, distance-generated through $\rho_{\boldsymbol{x}} \upharpoonright \widetilde{B}_{\boldsymbol{x}}$, and congruence-tractable through $\rho_{\boldsymbol{x}} \upharpoonright \widetilde{B}_{\boldsymbol{x}}$, for each $\boldsymbol{x} \in \boldsymbol{X}^{=}$.
(3) $\boldsymbol{g}_{\boldsymbol{x}, \boldsymbol{y}}$ is the canonical embedding, for all $\boldsymbol{x} \leq \boldsymbol{y}$ in $\boldsymbol{X}^{=}$.
(4) $\boldsymbol{g}_{\boldsymbol{x}, \boldsymbol{y}}$ is strong and congruence-cuttable through $\rho_{\boldsymbol{y}} \mid \widetilde{B}_{\boldsymbol{y}}$, for all $\boldsymbol{x}<\boldsymbol{y}$ in $\boldsymbol{X}^{=}$. We extend this diagram to an $\boldsymbol{X}$-indexed diagram, with $\boldsymbol{B}_{\boldsymbol{y}}=\boldsymbol{G}(B)$ and defining $\boldsymbol{g}_{\boldsymbol{x}, \boldsymbol{y}}$ as the canonical embedding for each $\boldsymbol{y} \in \boldsymbol{X}-\boldsymbol{X}^{=}$and each $\boldsymbol{x} \leq \boldsymbol{y}$. Thus $\boldsymbol{C} \circ \boldsymbol{B}$ is an $\boldsymbol{X}$-indexed diagram in the comma category $\mathcal{S} \downarrow \operatorname{Con}_{\mathrm{c}} B$, moreover $\boldsymbol{C}\left(\boldsymbol{B}_{\boldsymbol{x}}\right)=\widetilde{B}_{\boldsymbol{x}}$ is finite for each $\boldsymbol{x} \in \boldsymbol{X}^{=}$. Therefore, it follows from the Armature Lemma [5] that there exists an isotone section $\sigma: P \hookrightarrow \boldsymbol{X}$ such that the family $\left(\rho_{\sigma(p)} \upharpoonright \widetilde{B}_{\sigma(p)}\right)_{p \in P}$ is a natural transformation from $\left(\boldsymbol{C}\left(\boldsymbol{B}_{\sigma(p)}\right), \boldsymbol{C}\left(\boldsymbol{g}_{\sigma(p), \sigma(q)}\right) \mid p \leq q\right.$ in $\left.P\right)$ to $\mathrm{Con}_{\mathrm{c}} \circ \vec{A}$.

Put $\overrightarrow{\boldsymbol{B}}^{\prime}=\left(\boldsymbol{B}_{\sigma(p)}, \boldsymbol{g}_{\sigma(p), \sigma(q)} \mid p \leq q\right.$ in $\left.P\right)$ and $I_{p}=\operatorname{ker}_{0} \rho_{\sigma(p)}$, for each $p \in P$. This defines an ideal $\vec{I}=\left(I_{p}\right)_{p \in P}$ of $\overrightarrow{\boldsymbol{B}}^{\prime}$. Moreover, as $\rho_{\sigma(p)} \upharpoonright \widetilde{B}_{\sigma(p)}$ is ideal-induced for each $p \in P$, these morphisms induce a natural equivalence $\left(\boldsymbol{C} \circ \overrightarrow{\boldsymbol{B}}^{\prime}\right) / \vec{I} \cong \operatorname{Con}_{\mathrm{c}} \circ \overrightarrow{\boldsymbol{A}}$ (cf. Lemma 3.13).

Denote by $\boldsymbol{h}_{p, q}: \boldsymbol{B}_{\sigma(p)} / I_{p} \rightarrow \boldsymbol{B}_{\sigma(q)} / I_{q}$ the morphism induced by $\boldsymbol{g}_{\sigma(p), \sigma(q)}$. It follows from Proposition 6.5 that $\boldsymbol{B}_{\sigma(p)} / I_{p}$ is strong for each $p \in P$, and it follows from Proposition 6.7 that $\boldsymbol{h}_{p, q}$ is strong for all $p<q$ in $P$. Lemma 7.3 implies that $\boldsymbol{B}_{\sigma(p)} / I_{p}$ is distance-generated and congruence-tractable for each $p \in P$. From Lemma 7.4 we obtain that $\boldsymbol{h}_{p, q}$ is congruence-cuttable for all $p<q$ in $P$.

Let $p \in P$, let $\chi: \widetilde{B}_{\sigma(p)} \rightarrow \operatorname{Con}_{\mathrm{c}} B$ be the inclusion map. As $\boldsymbol{B}_{\sigma(p)}$ is a subgamp of $\boldsymbol{G}(B),(B, \chi)$ is a realization of $\boldsymbol{B}_{\sigma(p)}$, thus it induces a realization $\left(B / \bigvee I_{p}, \chi^{\prime}\right)$ where $\chi^{\prime}: \widetilde{B}_{p} / I_{p} \rightarrow B / \bigvee I$ satisfies $\chi^{\prime}\left(d / I_{p}\right)=d / \bigvee I_{p}$ (cf. Proposition 6.5). As $\rho_{\sigma(p)} \upharpoonright \widetilde{B}_{\sigma(p)}$ is ideal-induced it is easy to check that $\chi^{\prime}$ is surjective, hence it defines an isomorphic realization.

Let $p$ a maximal element of $P$. From $\boldsymbol{B}_{\sigma(p)}=\boldsymbol{G}(B)$ it follows that $\boldsymbol{B}_{\sigma(p)} / I_{p}$ is a quotient of $\boldsymbol{G}(B)$.

Therefore $\overrightarrow{\boldsymbol{B}}^{\prime} / \vec{I}$ is a partial lifting of $\operatorname{Con}_{\mathrm{c}} \circ \vec{A}$ in $\mathcal{W}$. Moreover, if $p \in P^{=}$then $\boldsymbol{B}_{\sigma(p)}$ is finite, thus $\boldsymbol{B}_{p}^{\prime} / I_{p}=\boldsymbol{B}_{\sigma(p)} / I_{p}$ is finite.
Remark 9.4. Use the notation of Theorem 9.3. A small modification of the proof above shows that if $\mathcal{W}$ is a variety of lattices, then we can construct a lattice partial lifting $\overrightarrow{\boldsymbol{B}}$ of $\mathrm{Con}_{\mathrm{c}} \circ \vec{A}$ in $\mathcal{W}$. Moreover, for any integer $n \geq 2$, if $B$ is congruence $n$-permutable, then all $\boldsymbol{B}_{p}$ can be chosen congruence $n$-permutable.

Corollary 9.5. Let $\mathcal{V}$ be a locally finite variety of algebras, let $\mathcal{W}$ be a variety of algebras with finite similarity type. The following statements are equivalent:
(1) $\operatorname{crit}(\mathcal{V} ; \mathcal{W})>\aleph_{0}$.
(2) Let $T$ be a countable lower finite tree and let $\vec{A}$ be a $T$-indexed diagram of finite algebras in $\mathcal{V}$. Then $\mathrm{Con}_{\mathrm{c}} \circ \vec{A}$ has a partial lifting in $\mathcal{W}$.
(3) Let $\vec{A}$ be a $\omega$-indexed diagram of finite algebras in $\mathcal{V}$. Then $\operatorname{Con}_{\mathrm{c}} \circ \vec{A}$ has a partial lifting in $\mathcal{W}$.
Proof. Assume that (1) holds. Let $T$ be a countable lower finite tree and let $\vec{A}$ be a $T$-indexed diagram of finite algebras in $\mathcal{V}$. It follows from [3, Corollary 4.7] that there exists an $\aleph_{0}$-lifter $(X, \boldsymbol{X})$ of $T$ such that card $X=\aleph_{0}$. Hence the following inequalities hold:

$$
\operatorname{card}(\mathbf{F}(X) \otimes \vec{A}) \leq \operatorname{card} X+\sum_{p \in T} A_{p} \leq \aleph_{0} \sum_{p \in T} \aleph_{0}=\aleph_{0}
$$

Thus, as $\operatorname{crit}(\mathcal{V} ; \mathcal{W})>\aleph_{0}$, there exists $B \in \mathcal{W}$ such that $\operatorname{Con}_{\mathrm{c}} B \cong \operatorname{Con}_{\mathrm{c}}(\mathbf{F}(X) \otimes \vec{A})$. It follows from Theorem 9.3 that there exists a partial lifting of $\operatorname{Con}_{\mathrm{c}} \circ \vec{A}$ in $\mathcal{W}$.

The implication $(2) \Longrightarrow(3)$ is immediate.
Assume that (3) holds and let $A \in \mathcal{V}$ such that card $\operatorname{Con}_{\mathrm{c}} A \leq \aleph_{0}$. By replacing $A$ with one of its subalgebras we can assume that card $A \leq \aleph_{0}$ (see [3, Lemma 3.6]). As $\mathcal{V}$ is locally finite, there exists an increasing sequence $\left(A_{k}\right)_{k<\omega}$ of finite subalgebras of $A$ with union $A$. Denote by $f_{i, j}: A_{i} \rightarrow A_{j}$ the inclusion map, for all $i \leq j<\omega$. Put $\vec{A}=\left(A_{i}, f_{i, j} \mid i \leq j<\omega\right)$. Let $\overrightarrow{\boldsymbol{B}}$ be a partial lifting of $\mathrm{Con}_{\mathrm{c}} \circ \vec{A}$ in $\mathcal{W}$, let $\boldsymbol{B}$ be the directed colimit of $\overrightarrow{\boldsymbol{B}}$ in $\operatorname{Gamp}(\mathcal{V})$. As $\boldsymbol{C}$ and $\mathrm{Con}_{\mathrm{c}}$ both preserve directed colimits, the following gamps are isomorphic:

$$
\boldsymbol{C}(\boldsymbol{B}) \cong \boldsymbol{C}(\underset{\longrightarrow}{\lim } \overrightarrow{\boldsymbol{B}}) \cong \underset{\longrightarrow}{\lim }(\boldsymbol{C} \circ \overrightarrow{\boldsymbol{B}}) \cong \lim _{\longrightarrow}\left(\operatorname{Con}_{\mathrm{c}} \circ \vec{A}\right) \cong \operatorname{Con}_{\mathrm{c}} \xrightarrow[\longrightarrow]{\lim } \vec{A} \cong \operatorname{Con}_{\mathrm{c}} A .
$$

Moreover, it follows from Lemma 6.17 that $\boldsymbol{B}$ is an algebra in $\mathcal{W}$, that is, there exists $B \in \mathcal{W}$ such that $\boldsymbol{B} \cong \boldsymbol{G}(B)$, so $\operatorname{Con}_{\mathrm{c}} B=\boldsymbol{C}(\boldsymbol{G}(B)) \cong \boldsymbol{C}(\boldsymbol{B}) \cong \operatorname{Con}_{\mathrm{c}} A$. Therefore $\operatorname{crit}(\mathcal{V} ; \mathcal{W})>\aleph_{0}$.

A variety of algebras is congruence-proper if each of its member with a finite congruence lattice is finite (cf. E, Definition 4-8.1]).

The following theorem is similar to Theorem 9.3 if $\mathcal{W}$ is a congruence-proper variety with a finite similarity type, then we no longer need partial liftings in the statement of the theorem. There is a similar theorem in [5, Theorem 4-9.2]. This new version applies only to varieties (not quasivarieties), but the assumption that $\mathcal{W}$ is locally finite is no longer needed.

Theorem 9.6. Let $\mathcal{V}$ be a variety of algebras. Let $\mathcal{W}$ be a congruence-proper variety of algebras with a finite similarity type. Let $(X, \boldsymbol{X})$ be an $\aleph_{0}$-lifter of a poset $P$. Let $\vec{A}=\left(A_{p}, f_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ be a diagram in $\mathcal{V}$ such that $\operatorname{Con}_{\mathrm{c}} A_{p}$ is finite for each $p \in P^{=}$. Let $B \in \mathcal{W}$ such that $\operatorname{Con}_{\mathrm{c}} B \cong \operatorname{Con}_{\mathrm{c}}(\mathbf{F}(X) \otimes \vec{A})$. Then there exists a lifting $\vec{B}$ of $\mathrm{Con}_{\mathrm{c}} \circ \vec{A}$ in $\mathcal{W}$ such that $B_{p}$ is a quotient of $B$ for each $p \in P$.
Proof. We notice, as in the beginning of the proof of Theorem 9.3, that $\mathbf{F}(X) \otimes \vec{A}$ is well-defined. We also uses the same notations. Let $\chi: \operatorname{Con}_{\mathrm{c}} B \rightarrow \operatorname{Con}_{\mathrm{c}}(\mathbf{F}(X) \otimes \vec{A})$ be an isomorphism. Put $\rho_{\boldsymbol{x}}=\operatorname{Con}_{\mathrm{c}}\left(\pi_{\boldsymbol{x}}^{X} \otimes \vec{A}\right) \circ \chi$ for each $\boldsymbol{x} \in \boldsymbol{X}$. As $\pi_{\boldsymbol{x}}^{X} \otimes \vec{A}$ is surjective and $\chi$ is an isomorphism, it follows that $\rho_{\boldsymbol{x}}$ is ideal-induced.

Notice that $\operatorname{rng} \rho_{\boldsymbol{x}}=\operatorname{Con}_{\mathrm{c}} A_{\partial \boldsymbol{x}}$ is finite. Hence $\operatorname{Con}_{\mathrm{c}} B / \operatorname{ker}_{0} \rho_{\boldsymbol{x}} \cong \operatorname{Con}_{\mathrm{c}} A_{\partial \boldsymbol{x}}$ is finite. As $\mathcal{W}$ is congruence-proper, it follows that $B / \operatorname{ker}_{0} \rho_{\boldsymbol{x}}$ is finite.

Hence, the property $\boldsymbol{A} / \operatorname{ker}_{0} \rho_{\boldsymbol{x}}=\boldsymbol{G}(B) / \operatorname{ker}_{0} \rho_{\boldsymbol{x}}$, in $\boldsymbol{A}$ a subgamp of $\boldsymbol{G}(B)$ is a locally finite property for $B$. It follows from Lemma 7.8 and Remark 7.9 that there
exists a diagram $\overrightarrow{\boldsymbol{B}}=\left(\boldsymbol{B}_{\boldsymbol{x}}, \boldsymbol{g}_{\boldsymbol{x}, \boldsymbol{y}} \mid \boldsymbol{x} \leq \boldsymbol{y}\right.$ in $\left.\boldsymbol{X}^{=}\right)$of finite subgamps of $\boldsymbol{G}(B)$ in $\mathcal{W}$, as in the proof of Theorem 9.3 , but that satisfies the additional condition:
(5) $\boldsymbol{B}_{\boldsymbol{x}} / \operatorname{ker}_{0} \rho_{\boldsymbol{x}}=\boldsymbol{G}(B) / \operatorname{ker}_{0} \rho_{\boldsymbol{x}}$, for all $\boldsymbol{x} \in \boldsymbol{X}^{=}$.

We continue with the same argument as the one in the proof of Theorem 9.3. We obtain $\boldsymbol{B}^{\prime} / \vec{I}$ a partial lifting of $\mathrm{Con}_{\mathrm{c}} \circ \vec{A}$. The following isomorphisms hold

$$
\begin{aligned}
\boldsymbol{B}_{p}^{\prime} / I_{p} & =\boldsymbol{B}_{\sigma(p)} / \operatorname{ker}_{0} \rho_{\sigma(p)} & & \text { see proof of Theorem } 9.3 . \\
& =\boldsymbol{G}(B) / \operatorname{ker}_{0} \rho_{\sigma(p)} & & \text { by }(5) . \\
& \cong \boldsymbol{G}\left(B / \operatorname{ker}_{0} \rho_{\sigma(p)}\right) & & \text { by Remark } 6.9(6) .
\end{aligned}
$$

Hence $\boldsymbol{B}_{p}^{\prime} / I_{p}$ is an algebra of $\mathcal{V}$ (cf. Remark 6.9(3)), for all $p \in P^{=}$, it is also true for $p \in \operatorname{Max} P$. It follows from Remark $6.9(5)$ that $\overrightarrow{\boldsymbol{B}}^{\prime} / \vec{I} \cong \boldsymbol{G} \circ \vec{B}$, for a diagram of algebras $\vec{B}$. Hence:

$$
\begin{array}{rlrl}
\operatorname{Con}_{\mathrm{c}} \circ \vec{B} & =\boldsymbol{C} \circ \boldsymbol{G} \circ \vec{B} & & \text { by Remark } 6.9(1) . \\
& \cong \boldsymbol{C} \circ \overrightarrow{\boldsymbol{B}}^{\prime} / \vec{I} & \\
& \cong \operatorname{Con}_{\mathrm{c}} \circ \vec{A} & & \text { as } \overrightarrow{\boldsymbol{B}}^{\prime} / \vec{I} \text { is a partial lifting of } \operatorname{Con}_{\mathrm{c}} \circ \vec{A} .
\end{array}
$$

Hence $\vec{B}$ is a lifting of $\mathrm{Con}_{\mathrm{c}} \circ \vec{A}$.
The following corollary is an immediate application of Theorem 9.6.
Corollary 9.7. Let $\mathcal{V}$ be a variety of algebras. Let $\mathcal{W}$ be a congruence-proper variety of algebras with a finite similarity type. Let $(X, \boldsymbol{X})$ be an $\aleph_{0}$-lifter of a poset $P$. Let $\vec{A}=\left(A_{p}, f_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ be a diagram of finite algebras in $\mathcal{V}$. Assume that $\operatorname{Con}_{\mathrm{c}} \circ \vec{A}$ has no lifting in $\mathcal{W}$, then $\operatorname{crit}(\mathcal{V} ; \mathcal{W}) \leq \aleph_{0}+\operatorname{card} X$.

## 10. An unliftable diagram

Each countable locally finite lattice has a congruence-permutable, congruencepreserving extension (cf. [7]). This is not true for all locally finite lattices. Given a non-distributive variety $\mathcal{V}$ of lattices, there is no congruence-permutable algebra $A$ such that $\operatorname{Con}_{\mathrm{c}} A \cong \operatorname{Con}_{\mathrm{c}} F_{\mathcal{V}}\left(\aleph_{2}\right)$ (cf. [14). Hence $\operatorname{Con}_{\mathrm{c}} F_{\mathcal{V}}\left(\aleph_{2}\right)$ has no congruencepermutable, congruence-preserving extension. The latter result can be improved, by stating that the free lattice $F_{\mathcal{V}}\left(\aleph_{1}\right)$ has no congruence-permutable, congruencepreserving extension (cf. [5]).

Let $\mathcal{V}$ be a nondistributive variety of lattices. There is no congruence $n$-permutable lattices $L$ such that $\operatorname{Con}_{\mathrm{c}} L \cong F_{\mathcal{V}}\left(\aleph_{2}\right)$, for each $n \geq 2$ (cf. 12]). In particular $F_{\mathcal{V}}\left(\aleph_{2}\right)$ has no congruence $n$-permutable, congruence-preserving extension, for each $n \geq 2$. The aim of this section is to improve the cardinality bound to $\aleph_{1}$. We use gamps to find a lattice of cardinal $\aleph_{1}$ with no congruence $n$-permutable, congruencepreserving extension, for each $n \geq 2$. This partially solve [5, Problem 7]. The proof is based on a square-indexed diagram of lattices with no congruence $n$-permutable, congruence-preserving extension.

Let $\vec{A}=\left(A_{p}, f_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ be a diagram of algebras, let $\vec{B}=\left(B_{p}, g_{p, q} \mid p \leq\right.$ $q$ in $P$ ) be a congruence-preserving extension of $\vec{A}$. Then $\boldsymbol{B}_{p}=\left(A_{p}, B_{p}, \Theta_{B_{p}}, \mathrm{Con}_{\mathrm{c}} B_{p}\right)$ is a gamp and $\boldsymbol{g}_{p, q}=\left(g_{p, q}, \operatorname{Con}_{\mathrm{c}} g_{p, q}\right): \boldsymbol{B}_{p} \rightarrow \boldsymbol{B}_{q}$ is a morphism of gamps, for all $p \leq q$ in $P$. This defines a diagram $\overrightarrow{\boldsymbol{B}}$ of gamps. Moreover, identifying $\operatorname{Con}_{\mathrm{c}} A_{p}$ and $\mathrm{Con}_{\mathrm{c}} B_{p}$ for all $p \in P$, we have $\boldsymbol{P}_{\mathrm{gl}} \circ \overrightarrow{\boldsymbol{B}}=\boldsymbol{P}_{\mathrm{ga}} \circ \vec{A}$.

Fix $n \geq 2$ an integer. Given a diagram $\vec{A}$ of algebras, with a congruence $n$ permutable, congruence-preserving extension, there exists a diagram $\overrightarrow{\boldsymbol{B}}$ of congruence $n$-permutable gamps such that $\boldsymbol{P}_{\mathrm{gl}} \circ \overrightarrow{\boldsymbol{B}}=\boldsymbol{P}_{\mathrm{ga}} \circ \vec{A}$. The converse might not hold in general.

However the square-indexed diagram $\vec{A}$ of finite lattices with no congruence $n$ permutable, congruence-preserving extension, mentioned above, satisfies a stronger property. There is no operational diagram $\overrightarrow{\boldsymbol{B}}$ of lattice congruence $n$-permutable gamps of lattices (cf. Definition 10.1) such that $\boldsymbol{P}_{\mathrm{ga}} \circ \vec{A} \cong \boldsymbol{P}_{\mathrm{gl}} \circ \overrightarrow{\boldsymbol{B}}$ (cf. Lemma 10.6).

We conclude, in Theorem 10.7, that there is a condensate of $\vec{A}$, of cardinal $\aleph_{1}$, with no congruence $n$-permutable, congruence-preserving extension.

For the purpose of this section, we need a stronger version of congruence $n$ permutable gamp, specific to gamp of lattices.
Definition 10.1. A gamp $\boldsymbol{A}$ of lattices is lattice congruence $n$-permutable if for all $x_{0}, \ldots, x_{n}$ in $A^{*}$ there exist $y_{0}, \ldots, y_{n}$ in $A$ such that $y_{i} \wedge y_{j}=y_{j} \wedge y_{i}=y_{i}$ in $A$ for all $i \leq j \leq n, y_{0}=x_{0} \wedge x_{n}=x_{n} \wedge x_{0}$ in $A, y_{n}=x_{0} \vee x_{n}=x_{n} \vee x_{0}$ in $A$, and:

$$
\begin{aligned}
& \delta\left(y_{k}, y_{k+1}\right) \leq \bigvee\left(\delta\left(x_{i}, x_{i+1}\right) \mid i<n \text { even }\right), \quad \text { for all } k<n \text { odd, } \\
& \delta\left(y_{k}, y_{k+1}\right) \leq \bigvee\left(\delta\left(x_{i}, x_{i+1}\right) \mid i<n \text { odd }\right), \quad \text { for all } k<n \text { even. }
\end{aligned}
$$

A morphism $\boldsymbol{f}: \boldsymbol{A} \rightarrow \boldsymbol{B}$ of gamps is operational if $\ell(\vec{x})$ is defined in $B$ for all $\ell \in \mathscr{L}$ and all $\operatorname{ar}(\ell)$-tuple $\vec{X}$ in $f(A) \cup B^{*}$.

Let $P$ be a poset, a diagram $\overrightarrow{\boldsymbol{A}}=\left(\boldsymbol{A}_{p}, \boldsymbol{f}_{p, q} \mid p \leq q\right.$ in $\left.P\right)$ of gamps is operational if $\boldsymbol{f}_{p, q}$ is operational for all $p<q$ in $P$.
Remark 10.2. In the context of Definition 10.1, the elements $y_{1}, \ldots, y_{n}$ doe not form a chain in general, as we might have $y_{i} \notin A^{*}$ for some $i$.

In this section we simply say that the gamp is congruence $n$-permutable instead of lattice congruence $n$-permutable. We also consider only pregamps and gamps of lattices. The following lemma is immediate, the properties given in Definition 10.1 go to quotients.
Lemma 10.3. Let $\boldsymbol{A}$ be a congruence n-permutable gamp, let $I$ be an ideal of $\boldsymbol{A}$, then $\boldsymbol{A} / I$ is a congruence $n$-permutable gamp.

Let $P$ be a poset, let $\overrightarrow{\boldsymbol{A}}$ be an operational $P$-indexed diagram of algebras. Let $\vec{I}$ be an ideal of $\overrightarrow{\boldsymbol{A}}$. The quotient $\boldsymbol{A} / \vec{I}$ is operational.

The following lemma is similar to the Buttress Lemma (cf. [5]) and to Lemma 7.8, this version is specific to the functor $\boldsymbol{P}_{\mathrm{gl}}$.
Lemma 10.4. Let $n \geq 2$. Let $\mathcal{V}$ be a variety of lattices. Let $P$ be a lower finite poset. Let $\left(\boldsymbol{A}_{p}\right)_{p \in P}$ be a family of finite pregamps. Let $\boldsymbol{B}$ be a gamp in $\mathcal{V}$ such that $B$ is a congruence n-permutable lattice. Let $\left(\boldsymbol{\pi}_{p}\right)_{p \in P}$ be a family of idealinduced morphisms of pregamps where $\boldsymbol{\pi}_{p}: \boldsymbol{P}_{\mathrm{gl}} \boldsymbol{B} \rightarrow \boldsymbol{A}_{p}$ for all $p \in P$. Then there exists a diagram $\overrightarrow{\boldsymbol{B}}=\left(\boldsymbol{B}_{p}, \boldsymbol{g}_{p, q} \mid p \leq q\right.$ in $P$ ) of finite subgamps of $\boldsymbol{B}$ (where $\boldsymbol{g}_{p, q}$ is the canonical embedding for all $p \leq q$ in $P$ ), such that the following assertions hold.
(1) $\boldsymbol{\pi}_{p} \upharpoonright \boldsymbol{P}_{\mathrm{gl}}\left(\boldsymbol{B}_{p}\right)$ is an ideal-induced morphism of pregamps for all $p \in P$.
(2) $\boldsymbol{B}_{p}$ is strong and congruence n-permutable for all $p \in P$.
(3) $\boldsymbol{g}_{p, q}$ is operational for all $p<q$ in $P$.

Proof. Let $r \in P$, assume that we have already constructed ( $\left.\boldsymbol{B}_{p}, \boldsymbol{g}_{p, q} \mid p \leq q<r\right)$ that satisfies the required conditions up to $r$.

As $\boldsymbol{\pi}_{r}$ is ideal-induced, $\pi_{r}\left(B^{*}\right)=A_{r}$. Moreover $A_{r}$ is finite, hence there exists $X$ a finite partial subalgebra of $B^{*}$ such that $\pi_{r}(X)=A_{r}$, put $B_{r}^{*}=X \cup \bigcup_{p<r} B_{p}^{*}$ with its structure of full partial subalgebra of $B^{*}$. It follows that $\pi_{r}\left(B_{r}^{*}\right) \subseteq \pi_{r}\left(B^{*}\right)=$ $A_{r}=\pi_{r}(X) \subseteq \pi_{r}\left(B_{r}^{*}\right)$, hence $\pi_{r}\left(B_{r}^{*}\right)=A_{r}$. Moreover $B_{r}^{*}$ is finite.

Let $x_{0}, x_{1}, \ldots, x_{n}$ in $B_{r}^{*}$. As $B$ is a congruence $n$-permutable lattice, there exist $y_{0}<\cdots<y_{n}$ in $B$ such that $y_{0}=x_{0} \wedge x_{n}, y_{n}=x_{0} \vee x_{n}$, and

$$
\begin{aligned}
& \delta\left(y_{k}, y_{k+1}\right) \leq \bigvee\left(\delta\left(x_{i}, x_{i+1}\right) \mid i<n \text { even }\right), \quad \text { for all } k<n \text { odd, } \\
& \delta\left(y_{k}, y_{k+1}\right) \leq \bigvee\left(\delta\left(x_{i}, x_{i+1}\right) \mid i<n \text { odd }\right), \quad \text { for all } k<n \text { even. }
\end{aligned}
$$

Put $X_{x_{0}, x_{1}, \ldots, x_{n}}=\left\{y_{0}, \ldots, y_{n}\right\}$. We consider the following finite set with its structure of full partial subalgebra of $B$

$$
\begin{aligned}
B_{r}= & \bigcup\left(X_{x_{0}, x_{1}, \ldots, x_{n}} \mid x_{0}, x_{1}, \ldots, x_{n} \in B_{p}^{*}\right) \\
& \cup\left\{x \vee y \mid x, y \in B_{r}^{*} \cup \bigcup_{p<r} B_{p}\right\} \\
& \cup\left\{x \wedge y \mid x, y \in B_{r}^{*} \cup \bigcup_{p<r} B_{p}\right\} .
\end{aligned}
$$

Put $Y=\left\{\delta(x, y) \mid x, y \in B_{r}\right\}$. As $\widetilde{\pi}_{r}$ is ideal-induced, it follows from Proposition 3.16 that there is a finite $(\vee, 0)$-subsemilattice $\widetilde{B}_{r}$ of $\widetilde{B}$ such that $Y \subseteq \widetilde{B}_{r}$, and $\widetilde{\pi}_{r} \upharpoonright \widehat{B}_{r}$ is ideal-induced. Put $\boldsymbol{B}_{r}=\left(B_{r}^{*}, B_{r}, \delta, \widetilde{B}_{r}\right)$, denote by $\boldsymbol{g}_{p, r}: \boldsymbol{B}_{p} \rightarrow \boldsymbol{B}_{r}$ the inclusion morphism for all $p \leq r$. The conditions (1)-(3) are satisfied. The conclusion follows by induction.

We apply Lemma 10.4 and the Armature Lemma (cf. 5) to obtain a new, tailor-made version of CLL. Given a diagram $\vec{A}$ of finite lattices and a congruencepreserving, congruence $n$-permutable extension of a condensate of $\vec{A}$, we obtain a "congruence-preserving, congruence $n$-permutable extension" of $\vec{A}$.

Lemma 10.5. Let $\mathcal{V}$ be a variety of lattices. Let $(X, \boldsymbol{X})$ be an $\aleph_{0}$-lifter of a poset $P$, let $\vec{A}=\left(A_{p}, f_{p, q}\right)$ be a diagram of $\mathcal{V}$ such that $\operatorname{Con}_{\mathrm{c}} A_{p}$ is finite for all $p \in P^{=}$. Let $B$ be a congruence n-permutable lattice. If $B$ is a congruence-preserving extension of $\mathbf{F}(X) \otimes \vec{A}$, then there exists an operational diagram $\overrightarrow{\boldsymbol{B}}$ of congruence $n$-permutable gamps such that $\boldsymbol{P}_{\mathrm{gl}} \circ \overrightarrow{\boldsymbol{B}} \cong \boldsymbol{P}_{\mathrm{ga}} \circ \vec{A}$.

Proof. As in the proof of Theorem 9.3, $\mathbf{F}(X) \otimes \vec{A}$ is well-defined. Denote by $\mathcal{S}$ the category of pregamps. The functor $\boldsymbol{P}_{\text {ga }}: \mathcal{V} \rightarrow \mathcal{S}$ satisfies (CONT), see Remark 5.7 . Put $B^{*}=\mathbf{F}(X) \otimes \vec{A}$, as $B$ is a congruence-preserving extension of $B^{*}$, we can identify $\operatorname{Con}_{\mathrm{c}} B^{*}$ with $\operatorname{Con}_{\mathrm{c}} B$. Put $\boldsymbol{B}=\left(B^{*}, B, \Theta_{B}, \operatorname{Con}_{\mathrm{c}} B\right)$, hence $\boldsymbol{P}_{\mathrm{gl}}(\boldsymbol{B})=$ $\boldsymbol{P}_{\mathrm{ga}}\left(B^{*}\right)$. Put $\boldsymbol{\pi}_{\boldsymbol{x}}=\boldsymbol{P}_{\mathrm{ga}}\left(\pi_{\boldsymbol{x}}^{X} \otimes \vec{A}\right): \boldsymbol{P}_{\mathrm{gl}}(\boldsymbol{B}) \rightarrow \boldsymbol{P}_{\mathrm{ga}}(\boldsymbol{A})$ for all $\boldsymbol{x} \in \boldsymbol{X}$. It follows from Lemma 10.4, that there exists a diagram $\left(\boldsymbol{B}_{\boldsymbol{x}}, \boldsymbol{g}_{\boldsymbol{x}, \boldsymbol{y}} \mid \boldsymbol{x} \leq \boldsymbol{y}\right.$ in $\left.\boldsymbol{X}^{=}\right)$of finite subgamps of $\boldsymbol{B}$ such that the following assertions hold.
(1) $\boldsymbol{\pi}_{\boldsymbol{x}} \upharpoonright \boldsymbol{P}_{\mathrm{gl}}\left(\boldsymbol{B}_{\boldsymbol{x}}\right)$ is an ideal-induced morphism of pregamps for all $\boldsymbol{x} \in \boldsymbol{X}^{=}$.
(2) $\boldsymbol{B}_{\boldsymbol{x}}$ is strong and congruence $n$-permutable for all $\boldsymbol{x} \in \boldsymbol{X}^{=}$.
(3) $\boldsymbol{g}_{\boldsymbol{x}, \boldsymbol{y}}$ is operational for all $\boldsymbol{x}<\boldsymbol{y}$ in $\boldsymbol{X}^{=}$.

We complete the diagram with $\boldsymbol{B}_{\boldsymbol{y}}=\boldsymbol{B}$ and $\boldsymbol{g}_{\boldsymbol{x}, \boldsymbol{y}}$ the inclusion morphism for all $\boldsymbol{y} \in \boldsymbol{X}-\boldsymbol{X}^{=}$, and $\boldsymbol{x} \leq \boldsymbol{y}$. It follows from the Armature Lemma (cf. 氖, Lemma 32.2]) that there exists $\sigma: P \rightarrow \boldsymbol{X}$ such that $\partial \sigma(p)=p$ for all $p \in P$ and $\left(\boldsymbol{\pi}_{\sigma(p)} \upharpoonright\right.$ $\left.\boldsymbol{P}_{\mathrm{gl}}\left(\boldsymbol{B}_{\sigma(p)}\right)\right)_{p \in P}$ is a natural transformation from $\left(\boldsymbol{P}_{\mathrm{gl}}\left(\boldsymbol{B}_{\sigma(p)}\right), \boldsymbol{P}_{\mathrm{gl}}\left(\boldsymbol{g}_{\sigma(p), \sigma(q)}\right) \mid p \leq\right.$ $q$ in $P$ ) to $\boldsymbol{P}_{\mathrm{ga}} \circ \vec{A}$.

Put $\boldsymbol{\rho}_{p}=\boldsymbol{\pi}_{\sigma(p)} \upharpoonright \boldsymbol{P}_{\mathrm{gl}}\left(\boldsymbol{B}_{\sigma(p)}\right)$, put $I_{p}=\operatorname{ker}_{0} \boldsymbol{\rho}_{p}$ for all $p \in P$. Put $\vec{I}=\left(I_{p}\right)_{p \in P}$, it is an ideal of $\left(\boldsymbol{B}_{\sigma(p)}, \boldsymbol{g}_{\sigma(p), \sigma(q)} \mid p \leq q\right.$ in $\left.P\right)$. Put $\overrightarrow{\boldsymbol{C}}=\left(\boldsymbol{B}_{\sigma(p)} / I_{p}, \boldsymbol{g}_{\sigma(p), \sigma(q)} / \vec{I} \mid\right.$ $p \leq q$ in $P$ ). Notice that $\overrightarrow{\boldsymbol{C}}$ is an operational diagram of congruence $n$-permutable gamps.

Denote by $\boldsymbol{\chi}_{p}: \boldsymbol{P}_{\mathrm{gl}}\left(\boldsymbol{B}_{\sigma(p)}\right) / I_{p} \rightarrow \boldsymbol{P}_{\mathrm{ga}}\left(A_{p}\right)$ the morphism induced by $\boldsymbol{\rho}_{p}$. It follows from Lemma 5.28 that $\chi_{p}$ is an isomorphism, for all $p \in P$. Hence, from Proposition 5.26 and Remark 6.13, we obtain that $\overrightarrow{\boldsymbol{\chi}}=\left(\boldsymbol{\chi}_{p}\right)_{p \in P}: \boldsymbol{P}_{\mathrm{g} 1} \circ \overrightarrow{\boldsymbol{C}} \rightarrow \boldsymbol{P}_{\mathrm{ga}} \circ \vec{A}$ is a natural equivalence.

In the following lemma we construct a square-indexed diagram of lattices with no congruence $n$-permutable, congruence-preserving extension.

Lemma 10.6. Let $n \geq 2$. Let $K$ be a nontrivial, finite, congruence $(n+1)$ permutable lattice, let $x_{1}, x_{2}, x_{3}$ in $K$ such that $x_{1} \wedge x_{2}=0$ and $x_{3} \vee x_{2}=x_{3} \vee x_{1}=$ 1. There exists a diagram $\vec{A}$ of finite congruence $(n+1)$-permutable lattices in $\operatorname{Var}^{0,1}(\mathbb{K})$ indexed by a square, such that $\boldsymbol{P}_{\mathrm{ga}} \circ \vec{A} \neq \boldsymbol{P}_{\mathrm{gl}} \circ \overrightarrow{\boldsymbol{B}}$ for each operational square $\overrightarrow{\boldsymbol{B}}$ of congruence n-permutable gamps.
Proof. Put $X_{0}=\left\{0, x_{3}, 1\right\}$, put $X_{1}=\left\{0, x_{1} \wedge x_{3}, x_{1}, x_{3}, 1\right\}$, put $X_{2}=\left\{0, x_{2} \wedge\right.$ $\left.x_{3}, x_{2}, x_{3}, 1\right\}$, put $X_{3}=K$. Notice that $X_{k}, k<4$, are all congruence $(n+1)$ permutable sublattices of $K$.


Figure 1. The lattice $X_{k}$, for $k \in\{1,2\}$
We denote by $h_{i}: X_{0} \rightarrow X_{i}$ and $h_{i}^{\prime}: X_{i} \rightarrow X_{3}$ the inclusion maps, for $i=1,2$. Denote by $\vec{X}$ the square on the right hand side of Figure 2.

Claim. Let $\overrightarrow{\boldsymbol{B}}$ be a square of operational gamps (as in Figure (3), let $\overrightarrow{\boldsymbol{\xi}}: \boldsymbol{P}_{\text {ga }} \circ$ $\vec{X} \rightarrow \boldsymbol{P}_{\mathrm{gl}} \circ \overrightarrow{\boldsymbol{B}}$ be a natural equivalence. Let $y \in B_{0}$ such that $\delta_{\boldsymbol{B}_{0}}\left(\xi_{0}(0), y\right) \leq$ $\delta_{\boldsymbol{B}_{0}}\left(\xi_{0}\left(x_{3}\right), \xi_{0}(1)\right), y \wedge \xi_{0}(1)=y$, and $y \vee \xi_{0}(0)=y$ in $B_{0}$, then $y=\xi_{0}(0)$.
Proof of Claim. We can assume that $g_{1}, g_{2}, g_{1}^{\prime}$, and $g_{2}^{\prime}$ are inclusion maps. We can assume that $\boldsymbol{P}_{\mathrm{gl}} \circ \overrightarrow{\boldsymbol{B}}=\boldsymbol{P}_{\mathrm{ga}} \circ \vec{X}$ and $\overrightarrow{\boldsymbol{\xi}}$ is the identity. We denote $\delta_{k}$ instead of $\delta_{\boldsymbol{B}_{k}}$ for all $k \in\{0,1,2,3\}$. Notice that $\delta_{k}(u, v)$ is a congruence of $X_{k}$ for all $u, v \in B_{k}$, moreover if $u, v \in X_{k}=B_{k}^{*}$ then $\delta_{k}(u, v)=\Theta_{X_{k}}(u, v)$. Let $y \in B_{0}$ such that $\delta_{0}(0, y) \subseteq \delta_{0}\left(x_{3}, 1\right), y \wedge 1=y$ and $y \vee 0=y$ in $B_{0}$.

Let $k \in\{1,2\}$. As $\boldsymbol{g}_{k}$ is operational, $y \wedge x_{k}$ is defined in $B_{k}$. Moreover $0 \wedge x_{k}=0$, hence $\delta_{k}\left(0, y \wedge x_{k}\right) \subseteq \delta_{k}(0, y)$. Therefore the following containments hold:

$$
\begin{equation*}
\delta_{k}\left(y, y \wedge x_{k}\right) \subseteq \delta_{k}(y, 0) \vee \delta_{k}\left(0, y \wedge x_{k}\right) \subseteq \delta_{k}(0, y) \subseteq \delta_{k}\left(x_{3}, 1\right)=\Theta_{X_{k}}\left(x_{3}, 1\right) \tag{10.1}
\end{equation*}
$$

Moreover, as $y \wedge 1=y$, the following containment holds

$$
\begin{equation*}
\delta_{k}\left(y, y \wedge x_{k}\right)=\delta_{k}\left(y \wedge 1, y \wedge x_{k}\right) \subseteq \delta_{k}\left(1, x_{k}\right)=\Theta_{X_{k}}\left(1, x_{k}\right) \tag{10.2}
\end{equation*}
$$

However $\Theta_{X_{k}}\left(1, x_{k}\right) \cap \Theta_{X_{k}}\left(x_{3}, 1\right)=\mathbf{0}_{X_{k}}$ (see Figure 1), thus it follows from (10.1) and (10.2) that $\delta_{k}\left(y, y \wedge x_{k}\right)=\mathbf{0}_{X_{k}}$, hence the following equality holds

$$
y=y \wedge x_{k}, \quad \text { for each } k \in\{1,2\}
$$

Therefore $y=\left(y \wedge x_{1}\right) \wedge x_{2}$ in $B_{3}$, moreover $x_{1} \wedge x_{2}=0$, hence as $\overrightarrow{\boldsymbol{B}}$ is operational $y \wedge\left(x_{1} \wedge x_{2}\right)$ is defined in $B_{3}$, thus $y \wedge 0=y \wedge\left(x_{1} \wedge x_{2}\right)=\left(y \wedge x_{1}\right) \wedge x_{2}=y$.

Moreover as $y \vee 0=y$, it follows that $0=(y \vee 0) \wedge 0=y \wedge 0=y$ (all elements are defined in $B_{3}$ ), hence $y=0$.Claim.

Let $C$ be an $(n+1)$-element chain. Set $T=\left\{t \mid t: C \rightarrow X_{0}\right.$ is order preserving $\}$. Put $A_{0}=C$, put $A_{1}=X_{1}^{T}$, put $A_{2}=X_{2}^{T}$, put $A_{3}=X_{3}^{T}=K^{T}$. We consider the following morphisms:

$$
\begin{aligned}
f_{i}: A_{0} & \rightarrow A_{i} \\
x & \mapsto(t(x))_{t \in T}, \quad \text { for } i=1,2 .
\end{aligned}
$$

We denote by $f_{i}^{\prime}: A_{i} \rightarrow A_{3}$ the inclusion maps, for $i=1,2$. We denote by $\vec{A}$ the square in the left hand side of Figure 2.


Figure 2. Two squares in $\operatorname{Var}^{0,1}(K)$
Given $t \in T$, denote $\pi_{0}^{t}=t$ and $\pi_{k}^{t}: A_{k} \rightarrow X_{k},\left(a_{p}\right)_{p \in T} \mapsto a_{t}$ the canonical projection, for all $k \in\{1,2,3\}$. It defines a natural transformation $\vec{\pi}^{t}$ from $\vec{A}$ to $\vec{X}$. We denote by $\boldsymbol{\pi}_{k}^{t}=\boldsymbol{P}_{\mathrm{ga}}\left(\pi_{k}^{t}\right)=\left(\pi_{k}^{t}, \operatorname{Con}_{\mathrm{c}} \pi_{k}^{t}\right)$, for all $k \in\{0,1,2,3\}$.

Assume that there exists an operational square $\overrightarrow{\boldsymbol{B}}$, as in Figure 3 , of congruence $n$-permutable gamps, and a natural equivalence $\boldsymbol{P}_{\mathrm{ga}} \circ \vec{A} \rightarrow \boldsymbol{P}_{\mathrm{gl}} \circ \overrightarrow{\boldsymbol{B}}$. We can assume that $\boldsymbol{P}_{\mathrm{gl}} \circ \overrightarrow{\boldsymbol{B}}=\boldsymbol{P}_{\mathrm{ga}} \circ \vec{A}$. Put $\delta_{k}=\delta_{\boldsymbol{B}_{k}}$, the distance $\delta_{\boldsymbol{A}_{k}}$ is a restriction of $\delta_{k}$, for all $k \in\{0,1,2,3\}$.


Figure 3. A square in the category $\operatorname{Gamp}(\mathcal{L})$
Let $a_{0}<a_{1}<\cdots<a_{n}$ be the elements of $C=A_{0}=B_{0}^{*}$. As $\boldsymbol{B}_{0}$ is congruence $n$-permutable, there exist $b_{0}, \ldots, b_{n}$ in $B_{0}$ such that $b_{i} \wedge b_{j}=b_{j} \wedge b_{i}=b_{i}$ in $B_{0}$ for all $i \leq j \leq n, b_{0}=a_{0} \wedge a_{n}=a_{0}$ in $B_{0}, b_{n}=a_{0} \vee a_{n}=a_{n}$ in $B_{0}$, and:

$$
\begin{align*}
& \delta_{0}\left(b_{k}, b_{k+1}\right) \leq \bigvee\left(\delta_{0}\left(a_{i}, a_{i+1}\right) \mid i<n \text { even }\right), \quad \text { for all } k<n \text { odd, }  \tag{10.3}\\
& \delta_{0}\left(b_{k}, b_{k+1}\right) \leq \bigvee\left(\delta_{0}\left(a_{i}, a_{i+1}\right) \mid i<n \text { odd }\right), \quad \text { for all } k<n \text { even. } \tag{10.4}
\end{align*}
$$

In particular the following inequality holds

$$
\begin{equation*}
\delta_{0}\left(b_{k}, b_{k+1}\right) \leq \delta_{0}\left(a_{0}, a_{k}\right) \vee \delta_{0}\left(a_{k+1}, a_{n}\right), \quad \text { for all } k<n . \tag{10.5}
\end{equation*}
$$

As $b_{n}=a_{n}$, an immediate consequence of (10.5) is $\delta_{0}\left(b_{n-1}, a_{n}\right) \leq \delta_{0}\left(a_{0}, a_{n-1}\right)$. Assume that $\delta_{0}\left(a_{n-2}, b_{n-1}\right) \leq \delta_{0}\left(a_{0}, a_{n-2}\right)$, it follows that

$$
\delta_{0}\left(a_{0}, a_{n}\right) \leq \delta_{0}\left(a_{0}, a_{n-2}\right) \vee \delta_{0}\left(a_{n-2}, b_{n-1}\right) \vee \delta_{0}\left(b_{n-1}, a_{n}\right) \leq \delta_{0}\left(a_{0}, a_{n-1}\right)
$$

That is $\Theta_{A_{0}}\left(a_{0}, a_{n}\right) \leq \Theta_{A_{0}}\left(a_{0}, a_{n-1}\right)$ a contradiction, as $A_{0}$ is the chain $a_{0}<\cdots<$ $a_{n}$. It follows that $\delta_{0}\left(a_{n-2}, b_{n-1}\right) \not \leq \delta_{0}\left(a_{0}, a_{n-2}\right)$.

Take $i<n$ minimal such that the following inequality hold

$$
\begin{equation*}
\delta_{0}\left(a_{i}, b_{i+1}\right) \not \leq \delta_{0}\left(a_{0}, a_{i}\right) . \tag{10.6}
\end{equation*}
$$

If $i=0$, then $a_{0}=b_{0}=b_{i}$, hence $\delta_{0}\left(a_{0}, b_{i}\right) \leq \delta_{0}\left(a_{0}, a_{i}\right)$. If $i>0$, then it follows from the minimality of $i$ that $\delta_{0}\left(a_{i-1}, b_{i}\right) \leq \delta_{0}\left(a_{0}, a_{i-1}\right) \leq \delta_{0}\left(a_{0}, a_{i}\right)$, thus $\delta_{0}\left(a_{0}, b_{i}\right) \leq \delta_{0}\left(a_{0}, a_{i-1}\right) \vee \delta_{0}\left(a_{i-1}, b_{i}\right) \leq \delta_{0}\left(a_{0}, a_{i}\right)$. Therefore the following inequality holds

$$
\begin{equation*}
\delta_{0}\left(a_{0}, b_{i}\right) \leq \delta_{0}\left(a_{0}, a_{i}\right) \tag{10.7}
\end{equation*}
$$

Assume that $\delta_{0}\left(b_{i}, b_{i+1}\right) \leq \delta_{0}\left(a_{0}, a_{i}\right)$, this implies with (10.7) the following inequality

$$
\delta_{0}\left(a_{i}, b_{i+1}\right) \leq \delta_{0}\left(a_{i}, a_{0}\right) \vee \delta_{0}\left(a_{0}, b_{i}\right) \vee \delta_{0}\left(b_{i}, b_{i+1}\right)=\delta_{0}\left(a_{0}, a_{i}\right)
$$

which contradicts (10.6). Therefore the following statement holds

$$
\begin{equation*}
\delta_{0}\left(b_{i}, b_{i+1}\right) \not \leq \delta_{0}\left(a_{0}, a_{i}\right) . \tag{10.8}
\end{equation*}
$$

As Con $A_{0}$ is a Boolean lattice with atoms $\delta_{0}\left(a_{k}, a_{k+1}\right)$ for $k<n$, it follows from (10.8) that there is $j<n$ such that

$$
\begin{equation*}
\delta_{0}\left(a_{j}, a_{j+1}\right) \leq \delta_{0}\left(b_{i}, b_{i+1}\right) \text { and } \delta_{0}\left(a_{j}, a_{j+1}\right) \not \leq \delta_{0}\left(a_{0}, a_{i}\right) \tag{10.9}
\end{equation*}
$$

As $\delta_{0}\left(a_{j}, a_{j+1}\right) \not \leq \delta_{0}\left(a_{0}, a_{i}\right), j \geq i$. It follows from (10.3), (10.4), and (10.9) that $i$ and $j$ have distinct parities, therefore $j>i$.

Put

$$
\begin{aligned}
t: A_{0} & \rightarrow\left\{0, x_{3}, 1\right\} \\
a_{k} & \mapsto \begin{cases}0 & \text { if } k \leq i, \\
x_{3} & \text { if } i<k \leq j, \quad \text { for all } k \leq n . \\
1 & \text { if } j<k,\end{cases}
\end{aligned}
$$

As $i<j<n$ the map $t$ is surjective, thus $t \in T$. Put $I_{i}=\operatorname{ker}_{0} \boldsymbol{\pi}_{i}^{t}$, for all $i \in\{0,1,2,3\}$. Denote by $\vec{\chi}: \boldsymbol{P}_{\mathrm{ga}} \circ \vec{A} / \vec{I}=\boldsymbol{P}_{\mathrm{g} 1} \circ \overrightarrow{\boldsymbol{B}} \rightarrow \overrightarrow{\boldsymbol{X}}$ the natural transformation induced by $\overrightarrow{\boldsymbol{\pi}}$. As $\overrightarrow{\boldsymbol{\pi}}=\boldsymbol{P}_{\text {ga }}(\vec{\pi})$ is ideal-induced, it follows from Lemma 5.28 that $\overrightarrow{\boldsymbol{\chi}}$ is a natural equivalence. Put $\overrightarrow{\boldsymbol{\xi}}=\vec{\chi}^{-1}$. Notice that the following inequalities hold

$$
\begin{aligned}
\delta_{0}\left(a_{0}, b_{i+1}\right) & \leq \delta_{0}\left(a_{0}, b_{i}\right) \vee \delta_{0}\left(b_{i}, b_{i+1}\right) \\
& \leq \delta_{0}\left(a_{0}, a_{i}\right) \vee \delta_{0}\left(a_{i+1}, a_{n}\right) \quad \text { by (10.7) and (10.5). }
\end{aligned}
$$

Moreover $\xi_{0}(0)=a_{0} / I_{0}=a_{i} / I_{0}, \xi_{0}\left(x_{3}\right)=a_{i+1} / I_{0}$ and $\xi_{0}(1)=a_{n} / I_{0}$, thus $\delta_{\boldsymbol{B}_{0} / I_{0}}\left(\xi_{0}(0), b_{i+1} / I_{0}\right) \leq \delta_{\boldsymbol{B}_{0} / I_{0}}\left(\xi_{0}\left(x_{3}\right), \xi_{0}(1)\right)$. As $b_{i+1} / I_{0} \wedge \xi_{0}(1)=\left(b_{i+1} \wedge a_{n}\right) / I_{0}=$ $b_{i+1} / I_{0}$ and $b_{i+1} / I_{0} \vee \xi_{0}(0)=\left(b_{i+1} \vee a_{0}\right) / I_{0}=b_{i+1} / I_{0}$. It follows from the Claim that $b_{i+1} / I_{0}=\xi_{0}(0)=a_{0} / I_{0}$, that is $\delta_{0}\left(a_{0}, b_{i+1}\right) \in I_{0}$. Therefore the following inequality holds:

$$
\begin{equation*}
\delta_{0}\left(a_{0}, b_{i+1}\right) \leq \delta_{0}\left(a_{0}, a_{i}\right) \vee \delta_{0}\left(a_{i+1}, a_{j}\right) \vee \delta_{0}\left(a_{j+1}, a_{n}\right) \tag{10.10}
\end{equation*}
$$

Hence we obtain

$$
\begin{array}{rlrl}
\delta_{0}\left(a_{j}, a_{j+1}\right) & \leq \delta_{0}\left(b_{i}, b_{i+1}\right) & & \text { by (10.9). } \\
& \leq \delta_{0}\left(b_{i}, a_{0}\right) \vee \delta_{0}\left(a_{0}, b_{i+1}\right) & & \\
& \leq \delta_{0}\left(a_{0}, a_{i}\right) \vee \delta_{0}\left(a_{i+1}, a_{j}\right) \vee \delta_{0}\left(a_{j+1}, a_{n}\right) & & \text { by (10.7) and (10.10). } \\
& \leq \delta_{0}\left(a_{0}, a_{j}\right) \vee \delta_{0}\left(a_{j+1}, a_{n}\right) . &
\end{array}
$$

A contradiction, as $A_{0}$ is the chain $a_{0}<a_{1}<\cdots<a_{n}$.
Theorem 10.7. Let $n \geq 2$. Let $\mathcal{V}$ be a variety of lattices, such that either $M_{3} \in \mathcal{V}$, $L_{2} \in \mathcal{V}, L_{3} \in \mathcal{V}, L_{4} \in \mathcal{V}, L_{2}{ }^{\mathrm{d}} \in \mathcal{V}$, or $L_{4}{ }^{\mathrm{d}} \in \mathcal{V}$. There exists a bounded lattice $L \in \mathcal{V}$ such that $L$ is congruence $(n+1)$-permutable, card $L=\aleph_{1}$, and $L$ has no congruence $n$-permutable, congruence-preserving extension in the variety of all lattices.
Proof. Fix $(X, \boldsymbol{X})$ an $\aleph_{0}$-lifter of the square such that card $X=\aleph_{1}$ (cf. Lemma 8.5). Up to changing $\mathcal{V}$ to its dual, we can assume that either $M_{3} \in \mathcal{V}, L_{2} \in \mathcal{V}, L_{3} \in \mathcal{V}$, or $L_{4} \in \mathcal{V}$. Let $K$ one of those lattices such that $K \in \mathcal{V}$, let $x_{1}, x_{2}, x_{3}$ as in Figure The conditions of Lemma 10.6 are satisfied. Denote by $\vec{A}$ the diagram constructed in Lemma 10.6 .

Put $L=\mathbf{F}(X) \otimes \vec{A} \in \operatorname{Var}^{0,1}(K) \subseteq \mathcal{V}$ (cf. Remark 9.2). Notice that $L$ is a directed colimit of finite products of lattices in $\vec{A}$ and all lattices in $\vec{A}$ are congruence $(n+1)$-permutable, thus $L$ is congruence $(n+1)$-permutable. As card $X=\aleph_{1}$ and each lattice in the diagram $\vec{A}$ is finite, card $L=\aleph_{1}$. Moreover $L$ cannot have a congruence $n$-permutable, congruence-preserving extension, as the conclusions of Lemma 10.5 and Lemma 10.6 contradict each other.

The following corollary is an immediate consequence of Theorem 10.7 .


Figure 4. The lattices $M_{3}, L_{2}, L_{3}, L_{4}$.

Corollary 10.8. Let $\mathcal{V}$ be a variety of lattices such that either $M_{3} \in \mathcal{V}, L_{2} \in$ $\mathcal{V}, L_{3} \in \mathcal{V}, L_{4} \in \mathcal{V}, L_{2}{ }^{\mathrm{d}} \in \mathcal{V}$, or $L_{4}{ }^{\mathrm{d}} \in \mathcal{V}$. The free bounded lattice on $\aleph_{1}$ generators of $\mathcal{V}$ has no congruence n-permutable, congruence-preserving extension in the variety of all lattices, for each $n \geq 2$.

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