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## Emergence, reduction and quantum mechanics

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# EMERGENCE, REDUCTION, AND QUANTUM MECHANICS

THIERRY PAUL, SÉBASTIEN POINAT

## CONTENTS

|  |    |
|--|----|
| 1. Emergence and Quantum Mechanics                                       | 1  |
| 1.1. Emergence in general  | 1  |
| 1.2. Link with Quantum Mechanics   | 4  |
| 2. Quantum Mechanics   | 4  |
| 3. Entangled systems are they emergent systems?                          | 7  |
| 4. A notion of decomposition into subsystems inside the quantum paradigm | 8  |
| 4.1. Decomposable states   | 8  |
| 4.2. Link with Entanglement  | 10 |
| 5. Link with "classical" partition                                       | 10 |
| 6. Back to emergence and reduction                                       | 13 |
| Appendix A. On the condition 1   | 14 |
| Appendix B. Proofs   | 16 |
| References   | 23 |

## 1. EMERGENCE AND QUANTUM MECHANICS

1.1. **Emergence in general.** Emergence is a notion which has been introduced as an intermediate notion between reduction and vitalism. The historical debate between these two last doctrines concerned the living bodies and the following question : can we conceive the living bodies as a purely combination of non-living parts, without anything special? Or must we recognize that living bodies contains something special, something that the non-living bodies doesn't have? The reductionism choses the first part of the alternative, the vitalism the second one.

This historical debate concerns the relation between a whole and its parts. For the reductionist, the behaviour of the whole are deducible from the behaviour of the parts: there is nothing in the whole that is not in the parts. Reductionism is thus the doctrine according to which the behaviour of a

compound system is explainable in terms of the behaviour of its parts. In a very approximate manner, we can say that, for the reductionist, the whole is nothing more than the sum of the parts. On the contrary, vitalism asserts that the whole is more than the sum of the parts, because living bodies are not only made of physical stuff but also of something called "vital principle", "vital force", or sometimes "entelechy", which is absolutely necessary to understand the behaviour of the living body but is not physical. According to the vitalist, the "vital principle" belongs to the entire body, and not to any of its parts.

Although vitalism is not a priori impossible (it isn't self-contradictory), it suffers from a bad reputation : it has been accused of obscurantism, of being anti-scientific, because the "vital principle" is an hidden principle and doesn't appear directly to our senses (we can only see its effects but not the principle itself). The "vital principle" seems to be an unjustified hypothesis and an obstruction to scientific work.

Is reductionism the unique doctrine that can be accepted to understand the composed systems? The answer is no since the doctrine of emergentism has been constituted. Emergentism conflicts reductionism in the sense that the behaviour of an emergent whole is supposed to be not reducible to the behaviour of the parts. According to emergentism, there is something new in the behaviour of an emergent whole that is not in its parts and that is not a priori deducible from the part. For example, the behaviour of a living body isn't deducible from the behaviour of purely physical matter. But emergentism conflicts also vitalism because it supposes that a living body is entirely made of physical matter and refutes that it has a "vital principle" which would be made of a non-physical stuff.

The common intuition that emergentism tries to grasp is that certain composed systems present something new that were not in the isolated parts. When these last ones are combined, something new emerges. More precisely emergentism supposes that the behaviour of certain wholes cannot be deduced from the behaviour of its parts taken in isolation and the general law of association. It supposes on the contrary that the behaviour of an emergent whole is new compared to those of its parts and to those of other composed systems that haven't the same parts but not the same structure. For example the behaviour of a living body is supposed to have a kind of autonomy in comparison with the behaviour of its physical parts.

There are different forms of emergence. In particular it is common to distinguish a strong and a weak form of emergence. The weak form of emergence corresponds to the case where the behaviour of a whole appears to

us to be emergent, while in fact it could be reduced to those of the parts. The weak emergence is due only to a waste of knowledge and is often called "epistemological emergence". The strong emergence is called "ontological emergence". A whole is emergent in the ontological sense when its behaviour cannot be reducible (to those of the parts) in principle and not only according to our present knowledge. The strong form of emergence is obviously the most interesting form of emergence and will be the one we'll talk in this paper.

Some models of emergence have been proposed and can be discussed <sup>1</sup>. It is not easy to determine exactly a definition of emergence: the common intuition seems to be not precisely enough to lead us to a model that everyone can except. But we'll not discuss them in this paper because we simply need to remark that all the models of emergentism or reductionism concern composed systems and only them. This point is common to all the models of emergence and reductionism for the simple reason that the debate between these two doctrines concerns the relations between a whole and its parts. If a system doesn't have any parts, that is to say if a system is not combined, there is no sense to ask if this system is emergent or not. That why we won't discuss all these models and won't ask which one is the best. In this paper we'll just propose a simple model of emergence in order to be easily understood.

The simple model of emergence is this one :

*The behaviour of a system can be said emergent if and only if:*

- (a) *this system is a composed system, with the structure  $[A_1, \dots, A_n; R]$  (which means : the system is composed by  $n$  entities  $A_i$  and these entities are in the relation  $R$ ).*
- (b) *there is a law (called  $P_L$ ) that says that: for all  $x$ , when  $x$  has the structure  $[A_1, \dots, A_n; R]$  then  $x$  has the behaviour  $C$ .*
- (c)  *$P_L$  cannot be deduced from the laws concerning the isolated entities  $A_1, \dots, A_n$ , nor from the laws concerning combined systems that contains some of (but not all) these entities  $A_1, \dots, A_n$ .*

From this model, it is easy to construct a model of reduction. In order to do that we only have to change point (c) and to suppose that  $P_L$  can be deduced from the laws concerning the isolated entities  $A_1, \dots, A_n$ , or from the laws concerning combined systems that contains some of (but not all) these entities  $A_1, \dots, A_n$ .

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<sup>1</sup>For a complete review, see [1]

Let us say it again: in this article we don't want to discuss points (b) and (c). We only need to remark that point (a) is necessary to all the models of emergence or of reduction.

**1.2. Link with Quantum Mechanics.** Why is quantum mechanics involved in this debate? Our answer is that quantum mechanics seems to solve the two kinds of difficulties that emergentism meets. The first kind of problems concerns only the purely conceptual aspect of emergentism. This part of the discussion can be summed up in this question: "is emergentism, when correctly defined, an acceptable metaphysical doctrine?" Or: "is it a priori possible for a system to be emergent and in what sense?" We'll call this question "Q1". The principal criticisms to emergentism came from Jaegwon Kim<sup>2</sup> who gave raise to a big literature.

The second kind of problems is less general and concerns the existing systems. The question (called "Q2") is the following: does it exist at least one real system that is emergent? This question can then be asked in the different domains of scientific investigation: in biology (do the living bodies emerge from their physical basis?), in philosophy of mind (are the mental properties reducible to the physical properties?), in sociology (must we say that a social fact is emergent on the base constituted by individuals?), in economy (does the behaviour of the market emerge from the behaviour of an isolated economical agent?), etc.

The great hope of the advocates of emergentism is that Quantum Mechanics allows us to demonstrate rigorously that some quantum systems are emergent. The stake in this point is big: if it is possible to find in Quantum Mechanics some systems that are emergent and if it is possible to demonstrate this result, then we will answer the two questions Q1 and Q2 by 'yes'. It would allow us to prove that emergentism is not only an acceptable metaphysical doctrine (Q1), but also that it describes some part of our experience (Q2). The mathematical formalism of Quantum Mechanics and the particular property of entanglement would justify emergentism in all its aspects and in a way that would not be questionable. If we can prove that entangled systems are emergent, then we prove emergence! That why so much attention has been given to Quantum Mechanics.

## 2. QUANTUM MECHANICS

We will not describe precisely in this article the axiomatics of Quantum mechanics, but rather illustrate the different points we need in the sequel

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<sup>2</sup>see in particular:[2]

for developing our argument. The reader is referred to [4] for a mathematical description, [5] for a more physical one and [6] for a non technical presentation of the mathematical formalism of Quantum Mechanics.

One point will be important for our demonstration and must be emphasized: it is the role of space-state in measurement. This point is rarely mentioned while it can cause confusion. see also [6] for a discussion of this point related to the so-called “teleportation” phenomenon.

A measurement in Quantum Mechanics, as in Physics in general, supposes the interaction between two physical systems : the systems  $S$  that we want to measure and the measurement apparatus. We must be able to describe this interaction in the physical space. The physical space represents some new degrees of freedom of any system and as every degrees of freedom, we have to use an Hilbert Space to describe them. If we want to describe the physical space in a three-dimension space, we’ll use the Hilbert Space of the square integrable functions (that is noted  $L^2(\mathbb{R}^3)$ ).

Let us suppose for example that a system  $S$  is a  $\frac{1}{2}$ -spin particle. We need two Hilbert Spaces: one for the spin and one for the space-state. It comes from it that the global Hilbert Space will be:  $H = \mathbb{C}^2 \otimes L^2(\mathbb{R}^3)$ . Thus the general form of the state vector will be any vector of  $H$ . For example if the particle is localised in a region of the physical space and is in an entangled state for the values of spin, the state vector will be of this form:  $|\Psi\rangle = \frac{1}{\sqrt{2}}[|+\rangle + |-\rangle] \otimes \alpha(X)$ , where  $\alpha(X)$  is a function of  $\mathbb{R}^3$  that is equal to zero everywhere excepted in the region where the particle is localised.

If we want now to describe the operators that can be used for a measurement, we have to specify also the space-state of the operator. In other terms the operators that we have to use must be defined in the Hilbert Space  $H$ . For example if  $\sigma$  is an operator of  $\mathbb{C}^2$ , the operator that we have to write is:  $\Sigma = \sigma_1 \otimes \beta(X)$ , where  $\beta(X)$  describes the space-state of the apparatus.

It is now simply to describe basic situations.

- When the apparatus is off,  $\beta(X) = 0$ .
- If the supports of the two functions  $\alpha(X)$  and  $\beta(X)$  are disjoint, no measurement can be done. It means simply that there isn’t any interaction between  $S$  and the apparatus.
- If  $\beta(X) = 1$  in the region when  $\alpha(X)$  isn’t equal to zero, we have  $\alpha(X).\beta(X) = \alpha(X)$ . Then:

$$\Sigma|\Psi\rangle = [\sigma.\left(\frac{1}{\sqrt{2}}[|+\rangle + |-\rangle]\right)] \otimes \alpha(X).$$

In this expression the term  $\sigma.\left(\frac{1}{\sqrt{2}}[|+\rangle + |-\rangle]\right)$  represents the action of the operator on the system  $S$  when we don’t mention the space-state of  $S$  and of

the apparatus.

How can we now describe the situation with two identical systems ? Let us suppose for example that we want to describe the system  $S$  composed by two particles of spin  $\frac{1}{2}$  noted  $S_1$  and  $S_2$  . Let us suppose also that each of them is represented by a state vector  $|\psi_1\rangle$  and  $|\psi_2\rangle$ . The global Hilbert Space that describes the spin-state of the composed system  $S$  is  $\mathbb{C}^2 \otimes \mathbb{C}^2$ . If the two particles are initially independant from each other, the state vector in  $\mathbb{C}^2 \otimes \mathbb{C}^2$  will be  $|\psi_1\rangle \otimes |\psi_2\rangle$ . But we have to describe also the space-state of the particles. The Hilbert Space will thus be:  $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)$ . If  $f_1(X)$  and  $f_2(X)$  are two functions that describe the space-state of (respectively)  $S_1$  and  $S_2$ , the state vector of  $S$  will be:

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes f_1(X) \otimes f_2(X)$$

To work in a situation a little bit more general, we'll suppose now that the spin-state of  $S$  can be in an entangled state. For reason of simplicity we'll also suppose that the two space-functions  $f_1$  and  $f_2$  are not entangled. The state vector of  $S$  will thus be:

$$|\psi\rangle = |u\rangle \otimes f_1(X) \otimes f_2(X)$$

Let us suppose that we want to measure the spin of the particles with a Stern-Gerlach device. In the Hilbert Space  $H$  the corresponding operator will be:

$$\Sigma = \sigma \otimes Id_{\mathbb{C}^2} \otimes \chi(X) \otimes Id_{\mathbb{R}^3} + Id_{\mathbb{C}^2} \otimes \sigma \otimes Id_{\mathbb{R}^3} \otimes \chi(X)$$

where

- $\sigma$  is the operator to the Stern-Gerlach device in the Hilbert Space  $\mathbb{C}^2$ .
- $\chi(X)$  is the function that represent the space-state of the Stern-Gerlach device in  $L^2(\mathbb{R}^3)$

If  $(f_1.\chi)(X) \neq 0$  and  $(f_2.\chi)(X) \neq 0$ , the Stern-Gerlach device will have an effect on the two particles and will measure their spin. More precisely the effect of the operator  $\Sigma$  on the state vector  $|\Psi\rangle$  is :

$$\Sigma|\Psi\rangle = [(\sigma \otimes Id_{\mathbb{C}^2})|u\rangle] \otimes [\chi.f_1(X)] \otimes [I_{\mathbb{R}^3}.f_2(X)] + [(Id_{\mathbb{C}^2} \otimes \sigma)|u\rangle] \otimes [I_{\mathbb{R}^3}.f_1(X)] \otimes [\chi.f_2(X)]$$

Is it possible to measure only one particle? The only possibility consists in having the second term equal to zero. In order to do that, we have to suppose that the support of the two spatial functions  $f_2(X)$  and  $\chi(X)$  are disjoint, and by consequence that  $f_2(X).\chi(X) = 0$ . If we suppose also that the spatial function  $\chi(X)$  is equal to 1 everywhere the function  $f_1(X)$  is not equal to zero, and by consequence that  $f_1(X).\chi(X) = f_1(X)$ , we have :

- $[(Id_{\mathbb{C}^2} \otimes \sigma)|u\rangle] \otimes [I_{\mathbb{R}^3}.f_1(X)] \otimes [\chi.f_2(X)] = 0$

$$\bullet [(\sigma \otimes I_{\mathbb{C}^2})|u\rangle] \otimes [\chi \cdot f_1(X)] \otimes [I_{\mathbb{R}^3} \cdot f_2(X)] = [(\sigma \otimes I_{\mathbb{C}^2})|u\rangle] \otimes f_1(X) \otimes f_2(X)$$

And finally :

$$\Sigma|\Psi\rangle = [(\sigma \otimes I_{\mathbb{C}^2})|u\rangle] \otimes f_1(X) \otimes f_2(X)$$

If we want to measure the spin of a particle that is entangled with an other, this is the only way to do it.

### 3. ENTANGLED SYSTEMS ARE THEY EMERGENT SYSTEMS?

A lot of authors have argued that entangled systems are emergent system and their argumentations are always based on the same ideas. We can call this kind of argumentation 'the traditional argumentation' because this argumentation is now becomed very common and broadly accepted. This traditional argumentation compares the situation of a factorized vector and an entangled vector. Let us suppose that we have two distinct particles independent from each other and that they can be represented by two state vectors  $|\phi\rangle$  and  $|\Phi\rangle$  which belong to two Hilbert Spaces (resp.)  $H_1$  et  $H_2$ . According to Quantum Mechanics the global system S lives in a Hilbert Space  $H = H_1 \otimes H_2$  and its state vector is  $|\Sigma\rangle = |\phi\rangle \otimes |\Phi\rangle$ . This vector is a tensor product and so it has a factorized form. According to the traditional argumentation we have here a simple example of resultant system: the state of the global system S is reducible to the state of its two parts because its state vector can be written as a product of two state vectors.

What happens now if system S is in an entangled state? For example let us suppose that the two particles have come closer and have interacted in such a way that the state is now entangled. Its state vector  $|\Sigma\rangle$  cannot be factorized anymore and it is now a superposition of factorized vectors. For example it can be of this form:  $|\Sigma\rangle = \frac{1}{\sqrt{2}}(|+\rangle \otimes |-\rangle + |-\rangle \otimes |+\rangle)$ . Here it is impossible to give any state to the (supposed) parts of system S, that is to say: it is impossible to give any state to the particles that produced system S. So it is also impossible to reduce the state of the whole to the states of its (supposed) parts. According to the traditional argumentation this is a case of emergence. For example Andreas Hüttemann explains in [3] that:

Quantum entanglement is a counterexample to synchronic microexplanation [...]. The compound [system consisting of two nonidentical particles] is in a determinate state, but this cannot be explained in terms of the determinate states of its constituents. This is because there are states [...] that do not



allow the attribution of pure states to the parts of the compound. What we see is that synchronic microexplanation systematically fails. Thus, we have an example of emergence [...]. This is a case of emergence because it is, in principle, impossible to explain the behaviour of the compound (in this case: the state) in terms of the behaviour (states) of the parts.

A lot of authors has support this argumentation. See for example [7], [8], [9], [10],[11], [12].

#### 4. A NOTION OF DECOMPOSITION INTO SUBSYSTEMS INSIDE THE QUANTUM PARADIGM

##### 4.1. Decomposable states.

4.1.1. *Position of the problem.* As we said emergence and reduction are notions that concern composed systems: they are supposed to tell us what kind of relations exist between a whole and its parts. Thus there is no sense to ask whether a system is emergent or not if we cannot identify any parts inside this system, that is to say if this system is simple and not composed. A system that is not composed is neither emergent nor resultant: it just can not enter this discussion.

But what is less obvious is to determine if the entanglement systems are really composed systems or if they are actually simple systems. The traditional argumentation assumes that entangled systems are composed since it concludes that they are emergent but usually it isn't said explicitly why we have to consider them as composed systems.

So we need a criterion to clearly distinguish simple systems and composed systems. But we would like a criterion that would be independent (or as independent as it is possible) from metaphysical assumptions or ontological choices. For example with a corpuscular ontology a unique corpuscle is a simple system, and a composed system contains at least two corpuscles. The ontology fixes the criterion of composed system. But what we search here is a criterion that could potentially be accepted by everyone and that escapes metaphysical choices (or too heavy metaphysical choices).

One condition to reach this aim is to formalize the criterion inside the quantum formalism. That means that we cannot use any concept that doesn't belong to the standard theory.

4.1.2. *Criterion of composed system.* What can be the criterion of composed system ? Our answer is that this criterion should grasp what would be the motivation of a physicist to consider a system as a composed system and

to analyse it in many parts. It seems to us that a physicist would divide a system in many parts if it's possible to distinguish different parts that can be studied independently. This condition signifies first that it is possible to measure one part and not the others, that it to say it is possible to make partial measurements. This can be the first condition. But the possibility to make independent measurements implies also that the measurements have independent results. Thus the fact that we did a measurement in one part must have no consequence at all on the results that we could make on other part of the composed system. This can be the second condition.

Before giving the mathematical formulation of our criterion, we need to say that if  $\sigma$  is an operator defined in a Hilbert Space  $H$  then the physical measurement that corresponds to  $\sigma$  is called the ' $\sigma$ -measurement'.

Now we can give the formulation of our criterion:

*The system  $S$  that is described by a state vector in a Hilbert Space  $H$  is a system composed by the sub-systems  $S_1, S_2, \dots, S_n$  which are supposed to live respectively in the Hilbert Spaces  $H_1, H_2, \dots, H_n$  such that  $H = H_1 \otimes H_2 \otimes \dots \otimes H_n$  if and only if:*

- **Condition 1:** *if  $\sigma_1, \sigma_2, \dots, \sigma_n$  are  $n$  hermitian operators that are defined respectively in the Hilbert Spaces  $H_1, H_2, \dots, H_n$ , it is physically possible, for  $i = 1 \dots n$ , to realize a measurement corresponding to the operator  $\Sigma_i = I_1 \otimes I_2 \otimes \dots \otimes \sigma_i \otimes \dots \otimes I_n$ .*
- **Condition 2:** *if we note  $P(\Sigma_i)$  the probabilities of results for a  $\Sigma_i$ -measurement if we make only this measurement, and  $P(\Sigma_i/\Sigma_j)$  (with  $j \neq i$ ) the probabilities of results for a  $\Sigma_i$ -measurement if a  $\Sigma_j$ -measurement has been done before, we must have:  $P(\Sigma_i) = P_{\Sigma_j}(\Sigma_i)$  with  $j \neq i$ .*

Let us now give an explanation of the technical aspects of this criterion. The Condition 1 concerns the physical possibility of partial measurements. What we said is that a composed system can be studied part by part, that is to say that it is possible to realize physically a **separated measurement**, a measurement that concerns only  $S_i (i = 1 \dots n)$  and not the other parts  $S_j (j \neq i)$  of  $S$ . The problem here is that the state vector of the system  $S$  belongs to  $H$  : any measurement that we could make must be described in the same Hilbert Space. The solution is to use an operator defined in  $H$  but that will have effect only on the sub-space  $H_i$ . The general form of this kind of operator is:  $\Sigma_i = I_1 \otimes I_2 \otimes \dots \otimes \sigma_i \otimes \dots \otimes I_n$  (where  $\sigma_i$  is a hermitian operator in  $H_i$ ). This kind of operator concerns only  $H_i$  and has no effect on any other sub-system  $H_j$  (for  $j \neq i$  they correspond to the identity-operator).

The object of Condition 2 is the results of the separated measurements. As we say those results must be independent from each other. In Quantum Mechanics the prediction can only be probabilistic. So the independence of the results must concern the probabilities of results for any separated measurements: the function of probabilities of results for any partial measurement must be the same whether or not an other partial measurement has been done before. The independence of results of the measurements is so guaranteed by the independence of the probabilities of the measurements. That's why Condition 2 asserts that the probabilities of results for any measurement  $\Sigma_i$  must be the same whether or not a  $\Sigma_j$ -measurement ( $j \neq i$ ) has been done.

**4.2. Link with Entanglement.** Now the question is: are entangled systems composed systems? To answer this question we need only to see if entangled systems verify the two conditions.

4.2.1. *Condition 1.* See Appendix A below.

4.2.2. *Condition 2.* The second mathematical result is :

$|\Psi\rangle$  satisfies Condition 2  $\Leftrightarrow |\Psi\rangle$  is a tensor product of the form  $|\Psi\rangle = |u\rangle \otimes |v\rangle$  with  $|u\rangle$  and  $|v\rangle$  two vectors of (resp.)  $H_1$  et  $H_2$ .

It is obvious that all the tensor product of two vectors of the Hilbert Spaces  $H_1$  et  $H_2$  verifies Condition 2. But the reverse relation is less obvious. The demonstration of this result is given in appendix B.

This relation means that a system S that is in an entangled state cannot satisfy condition 2. Thus according to our criterion such a system is not a composed system.

## 5. LINK WITH "CLASSICAL" PARTITION

In this section we will emphasize some important differences between the situation in Quantum Mechanics according to our criterion and the situation in classical physics or in daily life.

The criterion we propose here is based on the idea that a system is composed if and only if it can be studied part by part, by partial measurements that must be independant. This idea corresponds to the intuition that a clock, by example, is a composed system because we can observe and study the behaviour of differents parts: a spring, the hour hand, the minute hand, a wheel, etc. The behaviour of the clock can be divided in many parts that can be studied independantly. We don't want to say that the behaviour itself

of each part is independent (which would be wrong). We just want to say that we can study part by part and begin with any part of the clock without any consequence of the measurements we'll make on the other parts. So the definition of our criterion seems correspond to our common intuition about what is a composed system.

But its consequences are rather different. First space doesn't play its classical role anymore. In classical Physics, physical matter is conceived in a strong link with space in the sense that two different pieces of matter are supposed to occupy two different regions of space. This is because of a property called 'antitypy'. This property has a long story in history of philosophy: Aristotle, the stoicians, the epicurians, but also Galileo and Newton claimed that 'antitypy' is one the properties of matter. According to all of them, two systems must be in two regions of space. But the property can lead us to think that, on the contrary, if a system occupies two separated regions of space, then it must be considered as a system composed by two different systems. In other word, a simple system is either here or there, but not at the two places at the same time. Space can then play an important role to count the number of systems. Each separated region of spaces can be associated with one system. Then the number of such regions of space is also the number of parts that the system contains. According to us, this is the first reason why entangled systems are usually considered as composed system. But this reason is not good because the role that space plays in Quantum Mechanics is very different that the one it plays in classical Physics. In Quantum Mechanics, space is treated as the other degrees of freedom, that is to say as spin or as polarization. That's why the space function of a system can be entangled with the spin or can be distributed in two separated continuous functions. In Quantum Mechanics it is wrong to think that a system can be associated with only one region of space and that one region of space can be associated with only one system. Space doesn't have the specific status it has in classical Physics.

The second reason why entangled system are usually considered as composed system is that it can be produced by two systems that are originally independent. For example we can take two different and independent electrons and then create entanglement by making them interact. Then we are tempted to considered that after interaction there is still two systems because they were two systems before. This is right for usual objects: the engine and the wheels of a car can interact, they are still different systems.

But it is no more the case in quantum mechanics. It is a consequence of our criterion that two systems can become only one system. Any interaction

that leads to entanglement is like a fusion of the two systems that don't exist anymore during entanglement. This situation correspond to what can happen in biology: for example two cells can fusion and leads to a new cell. In sociology also two systems (for example two social groups) can fusion into one single system (one new social group) and disappear. One other aspect must be emphasized in quantum mechanics: a measurement on a entanglement system leads necessary to a state vector that can be factorised. In other words, because of the measurement the parts that composed initially the system and that disappeared can now exist again. This is also the case in sociology because two groups can disappear and then be formed again.

Let us call the 'composability' of a system the fact it is or not a composed system. Now we can compare the composability in classical domain and in Quantum Mechanics. One feature of the classical composability is that it doesn't depend on time. If a system is a composed system, it will remain a composed system. The classical composability is then a property of the nature of the system (which cannot change) and not of this state (which depends on time). In Quantum Mechanics the composability can change during time: for example a system can be initially in a factorised state and then become entangled. The quantum composability is then a characteristic of the state and not of the nature of the system like for classical system.

But the link between quantum composability and classical composability is not like the link between quantum quantity and classical quantity (for example between the position in Quantum Mechanics and the position in classical Physics). In classical physics the state of a system is described by physical quantity as its mass, its volume, etc. These physical quantities are the properties of the system. Each property corresponds to one value of a physical quantity. So the state of a classical system is determinated by the values of all its properties. To be a simple system or a composed system is also a property: a system is either a simple system or a composed system. Let us call this property the 'classical composability'.

In Quantum Mechanics the physical quantity are observables. In general if we measure one observable, we can obtain many results beause the system can be in a state that corresponds to a superposition of the different values we can obtain. In other words, a 'quantum property' is characterised by the plurality of values at the same time and by the principle of superposition. Is quantum composability a quantum quantity? Like the spin and other quantum quantities, the quantum composability can take two values for the same system because its state can change during time. But the two values

that can take the quantum composability cannot be superposed. The answer is then no: quantum composability is not a quantum quantity like the spin or the polarization.

## 6. BACK TO EMERGENCE AND REDUCTION

The conclusion of this analyse is that an entangled system is not an emergent system. Any entangled system doesn't verify the first hypothesis of any model of emergence, that is to say that the system is a composed system (these hypothesis corresponds to point (a) of our model of emergentism). Thus emergentism cannot argue that entangled systems are examples of emergent systems, or that Quantum Mechanics provides a demonstration that emergent systems exist.

But we don't want to say that the behaviour of emergent systems could be explained by the behaviour of their parts and thus could be reduced to the latest: we say that entangled systems don't have any parts. In other terms this means that the debate between emergentism and reductionism doesn't concern entangled systems. So our conclusion cannot be used in favor of reductionism and against emergentism: it cannot be used by any of the two doctrines.

We would like to insist on that point: this study doesn't take any position against emergence in general. The fact that entangled systems are not emergent systems doesn't mean that there is no emergent system in nature. Our point is just that Quantum Mechanics doesn't fit the conceptual frame of this debate.

We would like to make two remarks to conclude.

First our criterion leads to the result that a system that is produced by two initially independant subsystems can become a non-composed system. As we said in the previous section, it is like these two subsystems have fusionned and disappeared as systems: they don't exist anymore. This result could remind us the model proposed by Paul Humphreys in [9]. In this model, emergence is conceived as a process of fusion. At the beginnig they are some instances of property of a first level (called the  $i$ -level). But at the end, they fusion and give birth to instances of properties of a higher level (the  $i+1$ -level). The instances of properties of the  $i$ -level don't exist anymore: the instances of properties of the  $i+1$ -level are the only ones to exist at that time. For that reason it is becomed impossible to reduce the instancies of the properties of  $i+1$ -level to the instances of properties of  $i$ -level. As he says,

when emergence occurs, the lower level property instances go out of existence by producing the higher level emergent instances

In that sense this operation of fusion can be viewed as a case of emergence. Then the author gives the example of entangled systems. For him, this kind of systems corresponds to the model of fusion he described: entangled systems are emergent systems according to his model of emergence by fusion. As he says:

The composite system can be in a pure state when the component systems are not, and the state of one component cannot be completely specified without reference to the state of the other component. Furthermore, the state of the compound system determines the states of the constituents, but not vice versa [...]. I believe that the interactions which give rise to these entangled states lend themselves to the fusion treatment described in the earlier part of this paper, because the essentially relational interactions between the 'constituents' (which no longer can be separately individuated within the entangled pair) have exactly the features required for fusion.

We disagree with this conclusion. In the model of fusion proposed by Humphreys, the instances of properties fusion but not the entities that support these properties. The entities of  $i$ -level still exist even if the properties of  $i$ -level don't exist anymore. This aspect of his model is not surprising because there is no sense to talk about emergence if they are not parts of the global system or if this system doesn't have parts anymore. Humphreys says explicitly that it is possible that an object of  $i+1$ -level occurs in the fusion operation but it is not necessary. But even in this case he doesn't say that the  $i$ -level entities disappear. That's why in the preceding citation he talks about the component of an entangled systems and supposes then that they still exist. He just remarks that the component of an entangled pair cannot be separately individuated. But this remark doesn't lead him to the conclusion that they don't exist anymore (and then that there is no emergence at all). So the difference between Humphreys and us is that he supposes that the parts still exist whereas we think that they don't exist anymore. That's why we claim that entangled systems are not emergent systems.

#### APPENDIX A. ON THE CONDITION 1

How is it possible in Quantum Mechanics to physically realize the operator  $\Sigma_i = I_1 \otimes I_2 \otimes \dots \otimes \sigma_i \otimes \dots \otimes I_n$ ? For example let's suppose that the system  $S$  is produced by two particles of spin  $\frac{1}{2}$ . The Hilbert Space that describes the spin-spaces is  $\mathbb{C}^2 \otimes \mathbb{C}^2$ . Let's suppose now that we want to measure the spin of only one particle with a Stern-Gerlach device. The problem is that in

this situation there is no reason why the Stern-Gerlach device measures only one particle and not the other. More precisely this kind of measurement will be formalized by this operator:

$$\Sigma = \sigma_1 \otimes I_2 + I_1 \otimes \sigma_2$$

Here the operator will act on the two particles.

How is it possible to transform  $\Sigma = \sigma_1 \otimes I_2 + I_1 \otimes \sigma_2$  in  $\Sigma = \sigma_1 \otimes Id_2$  which is the kind of operator we want to realize? The way to do this consists in using spatial functions: we'll describe not only the spin state of the particles but also their space-state. In order to do that we must introduce the Hilbert sub-space  $L^2(\mathbb{R}^3)$  which is the space of the square integrable functions. Suppose we can write  $H = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)$ . It is then possible to realise the kind of operator we want to do. Let's write  $\chi(X)$  the function describing in  $L^2(\mathbb{R}^3)$  the space-state of the measuring device (for example the space-state of the Stern-Gerlach device). If we've worked with only one particle of spin  $\frac{1}{2}$  the Hilbert Space would be  $H = \mathbb{C}^2 \otimes L^2(\mathbb{R}^3)$  and the hermitian operator of the Stern-Gerlach device would be  $\alpha = \sigma \otimes \chi$ .

But we work in the global Hilbert Space  $H = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)$ . So the hermitian operator will be :

$$\alpha = \sigma \otimes I_{\mathbb{C}^2} \otimes \chi \otimes I_{\mathbb{R}^3} + I_{\mathbb{C}^2} \otimes \sigma \otimes I_{\mathbb{R}^3} \otimes \chi$$

Suppose now that the state vector of the system S is :

$$|\Psi\rangle = |u\rangle \otimes f_1(X) \otimes f_2(X),$$

where  $|\Psi\rangle$  corresponds to the spin-state of S ( $|\Psi\rangle$  belongs to the Hilbert Space  $\mathbb{C}^2 \otimes \mathbb{C}^2$ ) and  $f_1(X) \otimes f_2(X)$  correspond to the space-states of the particles ( $f_1(X) \otimes f_2(X)$  belongs to the Hilbert Space  $L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)$ )

The effect of the operator  $\alpha$  on the state vector  $|\Psi\rangle$  is :

$$\alpha|\Psi\rangle = [(\sigma \otimes I_{\mathbb{C}^2})|u\rangle] \otimes [\chi \cdot f_1(X)] \otimes [I_{\mathbb{R}^3} \cdot f_2(X)] + [(I_{\mathbb{C}^2} \otimes \sigma)|u\rangle] \otimes [I_{\mathbb{R}^3} \cdot f_1(X)] \otimes [\chi \cdot f_2(X)]$$

The problem will be solved if the second term is equal to zero. The way to do that consists in supposing that

- the support of the two spatial functions  $f_2(X)$  and  $\chi(X)$  are disjoint, and by consequence that  $f_2(X) \cdot \chi(X) = 0$ ;
- the spatial function  $\chi(X)$  is equal to 1 everywhere the function  $f_1(X)$  is not equal to zero, and by consequence that  $f_1(X) \cdot \chi(X) = f_1(X)$ .

With this assumption we have :

- $[(I_{\mathbb{C}^2} \otimes \sigma)|u\rangle] \otimes [I_{\mathbb{R}^3} \cdot f_1(X)] \otimes [\chi \cdot f_2(X)] = 0$
- $[(\sigma \otimes I_{\mathbb{C}^2})|u\rangle] \otimes [\chi \cdot f_1(X)] \otimes [I_{\mathbb{R}^3} \cdot f_2(X)] = [(\sigma \otimes I_{\mathbb{C}^2})|u\rangle] \otimes f_1(X) \otimes f_2(X)$



And finally :

$$\alpha|\Psi\rangle = [(\sigma \otimes I_{\mathbb{C}^2})|u\rangle] \otimes f_1(X) \otimes f_2(X)$$

Now it is very simple to find the eigenvectors of the operator  $\alpha$  . In general, if  $\Sigma = \sigma \otimes \rho$  then  $Spect(\Sigma) = Spect(\sigma).Spect(\rho)$  and the eigenvectors of  $\Sigma$  are all the tensor products of a eigenvector of  $\sigma$  with an eigenvector of  $\rho$ .

So here we have  $Spect(\sigma.I_{\mathbb{C}^2}) = Spect(\sigma)$  and the eigenvectors of  $\sigma.I_{\mathbb{C}^2}$  are all the vectors of the form:  $|u_\lambda\rangle \otimes |v\rangle$  (it is easy to see that : if  $|u_\lambda\rangle$  is an eigenvector of  $\sigma$  associated with the eigen value  $\lambda$ , and if  $|v\rangle$  is a vector of the Hilbert Space  $\mathbb{C}^2$  , then:  $[\sigma \otimes I_{\mathbb{C}^2}][|u_\lambda\rangle \otimes |v\rangle] = [\sigma \otimes |u_\lambda\rangle] \otimes |v\rangle = \lambda|u_\lambda\rangle \otimes |v\rangle$ ).

If now we note  $|U_\lambda\rangle_{f_1 f_2}$  the vectors of this form:

$$|U_\lambda\rangle_{f_1 f_2} = |u_\lambda\rangle \otimes |v\rangle \otimes f_1(X) \otimes f_2(X),$$

we have :

$$\alpha|U_\lambda\rangle_{f_1 f_2} = [\sigma \otimes I_{\mathbb{C}^2} \otimes \chi \otimes I_{\mathbb{R}^3}]. [|u_\lambda\rangle \otimes |v\rangle \otimes f_1(X) \otimes f_2(X)] = [\sigma \otimes |u_\lambda\rangle] \otimes f_1(X) \otimes f_2(X),$$

And finally:

$$\alpha|U_\lambda\rangle_{f_1 f_2} = \lambda|U_\lambda\rangle_{f_1 f_2},$$

which means that all the vectors  $|U_\lambda\rangle_{f_1 f_2}$  are eigenvectors of  $\alpha$ .

## APPENDIX B. PROOFS

In order to simplify this part, we'll work in a 4-dimension Hilbert Space  $H$  and we'll try to generalize the result at the end. Let us take a system  $S$  that is described by a state vector  $|\Psi\rangle$  and suppose that we want to analyse it in two sub-systems  $S_1$  et  $S_2$  that live in two 2-dimension Hilbert Spaces  $H_1$  and  $H_2$ . We won't suppose the system  $S$  is in an entanglement state. On the contrary we'll show that :

*The state vector  $|\Psi\rangle$  of a system  $S$  verifies Condition 2  $\Rightarrow |\Psi\rangle$  is a tensor product of two vectors of the Hilbert Spaces  $H_1$  and  $H_2$ .*

Let's suppose that  $\sigma_1$  and  $\sigma_2$  are two hermitian operators belonging (resp.) to  $H_1$  and  $H_2$ . We'll note  $|\pm\rangle_1$  and  $|\pm\rangle_2$  their eigenvectors. Each pair of eigenvectors is a orthonormal basis. So we can write :

$$(1) \quad |\Psi\rangle = c_{++}|++\rangle + c_{+-}|+-\rangle + c_{-+}|-+\rangle + c_{--}|--\rangle$$

The condition for  $|\Psi\rangle$  to be a tensor product is:

$$(2) \quad c_{++}c_{--} = c_{+-}c_{-+}$$

We suppose that  $|\Psi\rangle$  verifies Condition 2. The general relation  $P(\Sigma_i) = P_{\Sigma_j}(\Sigma_i)$  with  $(j \neq i)$  must be right for any operator  $\Sigma_i$  and  $\Sigma_j$ . So it must be right in particular for the two operators  $\sigma_i$  and  $\sigma_j$ , and we can write  $P(\Sigma_2) = P_{\Sigma_1}(\Sigma_2)$ , with  $\Sigma_1 = \sigma_1 \otimes I_2$  and  $\Sigma_2 = I_1 \otimes \sigma_2$

From the expression 1 we have

$$\begin{aligned} P(\Sigma_2 = +) &= |c_{++}|^2 + |c_{-+}|^2 \\ P(\Sigma_2 = -) &= |c_{+-}|^2 + |c_{--}|^2 \end{aligned}$$

What are the probabilities of a  $\Sigma_2$ -measurement if we made a  $\Sigma_1$ -measurement before it? The result of the  $\Sigma_1$ -measurement can be either "+" or "-".

- If the result is "+", the state vector is:

$$|\Psi\rangle = \frac{1}{\sqrt{|c_{++}|^2 + |c_{+-}|^2}} |+\rangle \otimes [c_{++}|+\rangle + c_{+-}|-\rangle]$$

- Otherwise the state vector is:

$$|\Psi\rangle = \frac{1}{\sqrt{|c_{-+}|^2 + |c_{--}|^2}} |-\rangle \otimes [c_{-+}|+\rangle + c_{--}|-\rangle]$$

Then we make the  $\Sigma_2$ -measurement. We have:

$$\begin{aligned} P_+(+) &= \frac{|c_{++}|^2}{|c_{++}|^2 + |c_{+-}|^2} \\ P_-(+) &= \frac{|c_{-+}|^2}{|c_{-+}|^2 + |c_{--}|^2} \end{aligned}$$

According to the Condition 2, we must have:

$$\frac{|c_{++}|^2}{|c_{++}|^2 + |c_{+-}|^2} = \frac{|c_{-+}|^2}{|c_{-+}|^2 + |c_{--}|^2} = |c_{++}|^2 + |c_{-+}|^2$$

and

$$\frac{|c_{+-}|^2}{|c_{++}|^2 + |c_{+-}|^2} = \frac{|c_{--}|^2}{|c_{-+}|^2 + |c_{--}|^2} = |c_{+-}|^2 + |c_{--}|^2$$

From these relations we have

$$(3) \quad |c_{++}|^2 \cdot |c_{--}|^2 = |c_{+-}|^2 \cdot |c_{-+}|^2$$

But the the Condition 2 must be true with every hermitian operators  $\sigma_1$  and  $\sigma_2$  (and  $\Sigma_1 = \sigma_1 \otimes I_2$  and  $\Sigma_2 = I_1 \otimes \sigma_2$ ). Now we'll change the operator

$\sigma_1$  and take the operator  $\sigma_1(\theta)$  that is obtained by a rotation of  $\sigma_1$  by an angle equal to  $\theta$ .

The eigenvectors  $|\pm_\theta\rangle$  of  $\sigma_1(\theta)$  are:

$$\begin{pmatrix} |+\rangle_\theta \\ |-\rangle_\theta \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix}$$

So we have:

$$|\Psi\rangle = c_{++}(\theta)|+\theta+\rangle + c_{+-}(\theta)|+\theta-\rangle + c_{-+}(\theta)|-\theta+\rangle + c_{--}(\theta)|-\theta-\rangle$$

Now the relation 3 is transformed into the following relation :

$$(4) \quad |c_{++}(\theta)|^2 \cdot |c_{--}(\theta)|^2 = |c_{+-}(\theta)|^2 \cdot |c_{-+}(\theta)|^2$$

We also have

$$\begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} |+\rangle_\theta \\ |-\rangle_\theta \end{pmatrix}$$

Then :

$$\begin{aligned} |+\rangle &= \cos \theta |+\rangle_\theta - \sin \theta |-\rangle_\theta \\ |-\rangle &= \sin \theta |+\rangle_\theta + \cos \theta |-\rangle_\theta \end{aligned}$$

Because of 1, we have:

$$\begin{aligned} |\Psi\rangle &= (\cos \theta \cdot c_{++} + \sin \theta \cdot c_{-+}) |+\theta+\rangle + (\cos \theta \cdot c_{+-} + \sin \theta \cdot c_{--}) |+\theta-\rangle + \\ &\quad (\cos \theta \cdot c_{-+} - \sin \theta \cdot c_{++}) |-\theta+\rangle + (\cos \theta \cdot c_{--} - \sin \theta \cdot c_{+-}) |-\theta-\rangle \end{aligned}$$

(5)

So:

$$\begin{aligned} c_{++}(\theta) &= \cos \theta \cdot c_{++} + \sin \theta \cdot c_{-+} \\ c_{+-}(\theta) &= \cos \theta \cdot c_{+-} + \sin \theta \cdot c_{--} \\ c_{-+}(\theta) &= \cos \theta \cdot c_{-+} - \sin \theta \cdot c_{++} \\ c_{--}(\theta) &= \cos \theta \cdot c_{--} - \sin \theta \cdot c_{+-} \end{aligned}$$

We can thus write :

$$(6) \quad |\cos \theta \cdot c_{++} + \sin \theta \cdot c_{-+}|^2 \cdot |\cos \theta \cdot c_{--} - \sin \theta \cdot c_{+-}|^2 = |\cos \theta \cdot c_{+-} + \sin \theta \cdot c_{--}|^2 \cdot |\cos \theta \cdot c_{-+} - \sin \theta \cdot c_{++}|^2$$

All the number  $c_{ij}$  are complex. Let us write:

$$c_{\mu} = \rho_{\mu} e^{i\varphi_{\mu}}$$

with  $\mu = (++) , (+-), (-+), (--)$

We want now to rewrite 6 with this new parameters. We have:

$$|\cos \theta.c_{++} + \sin \theta.c_{-+}|^2 = (\cos \theta \rho_{++} e^{i\varphi_{++}} + \sin \theta \rho_{-+} e^{i\varphi_{-+}}) \cdot (\cos \theta \rho_{++} e^{-i\varphi_{++}} + \sin \theta \rho_{-+} e^{-i\varphi_{-+}})$$

So

$$|\cos \theta.c_{++} + \sin \theta.c_{-+}|^2 = \cos^2 \theta \rho_{++}^2 + \sin^2 \theta \rho_{-+}^2 + 2 \cos \theta \sin \theta \rho_{++} \rho_{-+} \cos(\varphi_{++} - \varphi_{-+})$$

We have other equalities of the same kind:

$$|\cos \theta.c_{+-} + \sin \theta.c_{--}|^2 = \cos^2 \theta \rho_{+-}^2 + \sin^2 \theta \rho_{--}^2 + 2 \cos \theta \sin \theta \rho_{+-} \rho_{--} \cos(\varphi_{+-} - \varphi_{--})$$

$$|\cos \theta.c_{-+} - \sin \theta.c_{++}|^2 = \cos^2 \theta \rho_{-+}^2 + \sin^2 \theta \rho_{++}^2 - 2 \cos \theta \sin \theta \rho_{-+} \rho_{++} \cos(\varphi_{-+} - \varphi_{++})$$

$$|\cos \theta.c_{--} - \sin \theta.c_{+-}|^2 = \cos^2 \theta \rho_{--}^2 + \sin^2 \theta \rho_{+-}^2 - 2 \cos \theta \sin \theta \rho_{--} \rho_{+-} \cos(\varphi_{--} - \varphi_{+-})$$

Now we can write

$$\begin{aligned} & [\cos^2 \theta \rho_{++}^2 + \sin^2 \theta \rho_{-+}^2 + 2 \cos \theta \sin \theta \rho_{++} \rho_{-+} \cos(\varphi_{++} - \varphi_{-+})] \cdot \\ & [\cos^2 \theta \rho_{--}^2 + \sin^2 \theta \rho_{+-}^2 - 2 \cos \theta \sin \theta \rho_{--} \rho_{+-} \cos(\varphi_{--} - \varphi_{+-})] = \\ & [\cos^2 \theta \rho_{+-}^2 + \sin^2 \theta \rho_{--}^2 + 2 \cos \theta \sin \theta \rho_{+-} \rho_{--} \cos(\varphi_{+-} - \varphi_{--})] \cdot \\ & [\cos^2 \theta \rho_{-+}^2 + \sin^2 \theta \rho_{++}^2 - 2 \cos \theta \sin \theta \rho_{-+} \rho_{++} \cos(\varphi_{-+} - \varphi_{++})] \end{aligned}$$

The relation 3 gives us:  $\rho_{++} \rho_{--} = \rho_{+-} \rho_{-+}$ . In order to simplify the formulation, we'll define the parameter  $k$ :

$$k = \frac{\rho_{++}}{\rho_{-+}} = \frac{\rho_{+-}}{\rho_{--}}$$

Then we have:

$$\begin{aligned} & [\cos^2 \theta k^2 + \sin^2 \theta + 2 \cos \theta \sin \theta k \cos(\varphi_{++} - \varphi_{-+})] \cdot \\ & [\cos^2 \theta + \sin^2 \theta k^2 - 2 \cos \theta \sin \theta k \cos(\varphi_{--} - \varphi_{+-})] = \\ & [\cos^2 \theta k^2 + \sin^2 \theta + 2 \cos \theta \sin \theta k \cos(\varphi_{+-} - \varphi_{--})] \cdot \\ & [\cos^2 \theta + \sin^2 \theta k^2 - 2 \cos \theta \sin \theta k \cos(\varphi_{-+} - \varphi_{++})] \end{aligned}$$

After developing and simplifying we have:

$$(k^2 + 1)(\cos \theta \sin \theta) \cos(\varphi_{--} - \varphi_{+-}) = (k^2 + 1)(\cos \theta \sin \theta) \cos(\varphi_{-+} - \varphi_{++})$$

And so:

- Either  $\theta = 0[\frac{\pi}{2}]$
- Either  $\cos(\varphi_{--} - \varphi_{+-}) = \cos(\varphi_{-+} - \varphi_{++})$

The result is:

- Either

$$(7) \quad (\varphi_{++} - \varphi_{-+}) = (\varphi_{+-} - \varphi_{--})[2\pi]$$

- Either

$$(8) \quad (\varphi_{++} - \varphi_{-+}) = -(\varphi_{+-} - \varphi_{--})[2\pi]$$

At this point we have  $\rho_{++}\rho_{--} = \rho_{+-}\rho_{-+}$  and the relations 7 and 8. The relation 7 corresponds to the relation on phases when the state is a tensor product (that is to say:7 is a consequence of 2). We'll show now that the relation 8 is impossible.

In order to do that we use  $\sigma(\omega)$  which is equal to  $N\sigma_2N^{-1}$  with N equal to :

$$\begin{pmatrix} \cos \omega & i \sin \omega \\ i \sin \omega & \cos \omega \end{pmatrix}$$

The eigenvectors  $|\pm_\omega\rangle$  of  $\sigma_2(\omega)$  are:

$$\begin{pmatrix} |+\rangle_\omega \\ |-\rangle_\omega \end{pmatrix} = \begin{pmatrix} \cos \omega & i \sin \omega \\ i \sin \omega & \cos \omega \end{pmatrix} \begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix}$$

We have two relations similar to 1 and 3:

$$(9) \quad |\Psi\rangle = c_{++}(\omega)|+\omega+\rangle + c_{+-}(\omega)|+\omega-\rangle + c_{-+}(\omega)|-\omega+\rangle + c_{--}(\omega)|-\omega-\rangle$$

And:

$$(10) \quad |c_{++}(\omega)|^2 \cdot |c_{--}(\omega)|^2 = |c_{+-}(\omega)|^2 \cdot |c_{-+}(\omega)|^2$$

We also have:

$$\begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix} = \begin{pmatrix} \cos \omega & -i \sin \omega \\ -i \sin \omega & \cos \omega \end{pmatrix} \begin{pmatrix} |+\rangle_\omega \\ |-\rangle_\omega \end{pmatrix}$$

Then:

$$\begin{aligned} |+\rangle &= \cos \omega |+\rangle_\omega - i \sin \omega |-\rangle_\omega \\ |-\rangle &= -i \sin \omega |+\rangle_\omega + \cos \omega |-\rangle_\omega \end{aligned}$$

From 1, we can write:

$$\begin{aligned} |\Psi\rangle &= (\cos \omega c_{++} - i \sin \omega c_{-+}) |+\omega +\rangle + (\cos \omega c_{+-} - i \sin \omega c_{--}) |+\omega -\rangle + \\ &\quad (\cos \omega c_{-+} - i \sin \omega c_{++}) |-\omega +\rangle + (\cos \omega c_{--} - i \sin \omega c_{+-}) |-\omega -\rangle \end{aligned}$$

So:

$$\begin{aligned} c_{++}(\omega) &= \cos \omega c_{++} - i \sin \omega c_{-+} \\ c_{+-}(\omega) &= \cos \omega c_{+-} - i \sin \omega c_{--} \\ c_{-+}(\omega) &= \cos \omega c_{-+} - i \sin \omega c_{++} \\ c_{--}(\omega) &= \cos \omega c_{--} - i \sin \omega c_{+-} \end{aligned}$$

From these we can write:

$$(11) \quad |\cos \omega c_{++} - i \sin \omega c_{-+}|^2 \cdot |\cos \omega c_{--} - i \sin \omega c_{+-}|^2 = |\cos \omega c_{+-} - i \sin \omega c_{--}|^2 \cdot |\cos \omega c_{-+} - i \sin \omega c_{++}|^2$$

We also have:

$$\begin{aligned} |\cos \omega c_{++} - i \sin \omega c_{-+}|^2 &= \\ &= \cos^2 \omega \rho_{++}^2 + \sin^2 \omega \rho_{-+}^2 + 2 \cos \omega \sin \omega \rho_{++} \rho_{-+} \cos(\varphi_{++} - (\varphi_{-+} - \frac{\pi}{2})) \\ &= \cos^2 \omega \rho_{++}^2 + \sin^2 \omega \rho_{-+}^2 + 2 \cos \omega \sin \omega \rho_{++} \rho_{-+} \cos(\varphi_{++} - \varphi_{-+} + \frac{\pi}{2}) \\ &= \cos^2 \omega \rho_{++}^2 + \sin^2 \omega \rho_{-+}^2 - 2 \cos \omega \sin \omega \rho_{++} \rho_{-+} \sin(\varphi_{++} - \varphi_{-+}) \end{aligned} \quad (12)$$

We can also write :

$$\begin{aligned} |\cos \omega c_{--} - i \sin \omega c_{+-}|^2 &= \cos^2 \omega \rho_{--}^2 + \sin^2 \omega \rho_{+-}^2 - 2 \cos \omega \sin \omega \rho_{--} \rho_{+-} \sin(\varphi_{--} - \varphi_{+-}) \\ |\cos \omega c_{+-} - i \sin \omega c_{--}|^2 &= \cos^2 \omega \rho_{+-}^2 + \sin^2 \omega \rho_{--}^2 - 2 \cos \omega \sin \omega \rho_{+-} \rho_{--} \sin(\varphi_{+-} - \varphi_{--}) \end{aligned}$$

$$|\cos \omega c_{-+} - i \sin \omega c_{++}|^2 = \cos^2 \omega \rho_{-+}^2 + \sin^2 \omega \rho_{++}^2 - 2 \cos \omega \sin \omega \rho_{++} \rho_{-+} \sin(\varphi_{-+} - \varphi_{++})$$

From these relations we can write

$$\begin{aligned} & [\cos^2 \omega \rho_{++}^2 + \sin^2 \omega \rho_{-+}^2 - 2 \cos \omega \sin \omega \rho_{++} \rho_{-+} \sin(\varphi_{++} - \varphi_{-+})]. \\ & \quad [\cos^2 \omega \rho_{--}^2 + \sin^2 \omega \rho_{+-}^2 - 2 \cos \omega \sin \omega \rho_{--} \rho_{+-} \sin(\varphi_{--} - \varphi_{+-})] = \\ & \quad [\cos^2 \omega \rho_{+-}^2 + \sin^2 \omega \rho_{--}^2 - 2 \cos \omega \sin \omega \rho_{+-} \rho_{--} \sin(\varphi_{+-} - \varphi_{--})]. \\ & \quad [\cos^2 \omega \rho_{-+}^2 + \sin^2 \omega \rho_{++}^2 - 2 \cos \omega \sin \omega \rho_{-+} \rho_{++} \sin(\varphi_{-+} - \varphi_{++})] \end{aligned}$$

With the parameters  $k$ , we obtain:

$$\begin{aligned} & [\cos^2 \omega k^2 + \sin^2 \omega - 2 \cos \omega \sin \omega k \sin(\varphi_{++} - \varphi_{-+})]. \\ & \quad [\cos^2 \omega + \sin^2 \omega k^2 - 2 \cos \omega \sin \omega k \sin(\varphi_{--} - \varphi_{+-})] = \\ & \quad [\cos^2 \omega + \sin^2 \omega k^2 - 2 \cos \omega \sin \omega k \sin(\varphi_{+-} - \varphi_{--})]. \\ & \quad [\cos^2 \omega k^2 + \sin^2 \omega - 2 \cos \omega \sin \omega k \sin(\varphi_{-+} - \varphi_{++})] \end{aligned}$$

After developing and simplifying we have:

$$(k^2 + 1)(\cos \omega \sin \omega) \sin(\varphi_{++} - \varphi_{-+}) = (k^2 + 1)(\cos \omega \sin \omega) \sin(\varphi_{+-} - \varphi_{--})$$

which is equivalent to

- Either  $\omega = 0[\frac{\pi}{2}]$
- Either  $\sin(\varphi_{++} - \varphi_{-+}) = \sin(\varphi_{+-} - \varphi_{--})$

From this we get these two relations:

- Either

$$(13) \quad (\varphi_{++} - \varphi_{-+}) = (\varphi_{+-} - \varphi_{--})[2\pi]$$

- Either

$$(14) \quad (\varphi_{++} - \varphi_{-+}) = \pi - (\varphi_{+-} - \varphi_{--})[2\pi]$$

It's easy now to see that

- 7 is equal to 13
- the relation 8 is not compatible neither with 13 nor 14. So the relation 8 is impossible.

Finally we get this conclusion : all the numbers  $C_{ij}$  that satisfy the Condition 2 satisfy also the factorising condition 2. We can conclude that all the state vector that satisfy the Condition 2 are in factorised state, that is to say that they are of the form:

$$|\Psi\rangle = |u\rangle \otimes |v\rangle$$

with  $|u\rangle$  and  $|v\rangle$  two vectors of (resp.)  $H_1$  and  $H_2$ .

It is obvious that all the tensor product of two vectors of the Hilbert Spaces  $H_1$  and  $H_2$  verifies B.

Now we can conclude :  $|\Psi\rangle$  satisfies Condition 2  $\Leftrightarrow |\Psi\rangle$  is a tensor product of the form  $|\Psi\rangle = |u\rangle \otimes |v\rangle$  with  $|u\rangle$  and  $|v\rangle$  two vectors of (resp.)  $H_1$  et  $H_2$ .

Although we will not give the proof here let us mention that the construction works as well in higher dimensions and for more than two particles.

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