



**HAL**  
open science

# On the generalised Tate conjecture for products of elliptic curves over finite fields

Bruno Kahn

► **To cite this version:**

Bruno Kahn. On the generalised Tate conjecture for products of elliptic curves over finite fields. 2010. hal-00544321v1

**HAL Id: hal-00544321**

**<https://hal.science/hal-00544321v1>**

Preprint submitted on 7 Dec 2010 (v1), last revised 9 Jan 2011 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# ON THE GENERALISED TATE CONJECTURE FOR PRODUCTS OF ELLIPTIC CURVES OVER FINITE FIELDS

BRUNO KAHN

ABSTRACT. We prove the generalised Tate conjecture for  $H^3$  of products of elliptic curves over finite fields, by slightly modifying the argument of M. Spiess [5] concerning the Tate conjecture. We prove it fully if the elliptic curves run among at most 3 isogeny classes. We also show how things become more intricate from  $H^4$  onwards, for more than 3 isogeny classes.

Let  $\mathbf{F}_q$  be a finite field. It is known that the Tate conjecture for all smooth projective varieties over  $\mathbf{F}_q$  implies the generalised Tate conjecture for all smooth projective varieties over  $\mathbf{F}_q$  ([3, Rk. 10.3 2], [4, §1]); however, the proofs in these two references are non-effective. It is therefore of interest to ask if one can prove the generalised Tate conjecture for certain explicit classes of  $\mathbf{F}_q$ -varieties.

In [5], Michael Spiess proved the Tate conjecture for products of elliptic curves over a finite field: this provides a natural candidate for such a class. In this note, we show that a slight modification of his argument does yield the generalised Tate conjecture, in cohomological degree 3 or if the elliptic curves run over at most 3 distinct isogeny classes.

We also illustrate the non-effectiveness of the arguments from [3] and [4] for  $H^4$  of a well-chosen product of 4 elliptic curves : it is directly related to the delicate combinatorics of Weil numbers.

**Theorem 1.** *Let  $X$  be a product of elliptic curves over  $\mathbf{F}_q$ . Then the generalised Tate conjecture holds for  $H^3(\bar{X}, \mathbf{Q}_l)$ : the subspace of Tate coniveau 1 coincides with the first step of the coniveau filtration.*

Let  $q = p^r$  for  $p$  be a prime number and  $r \geq 1$ . As in [5, Def. 1], we say that a Weil  $q$ -number  $\alpha$  is *elliptic* if it arises from the Frobenius endomorphism of an elliptic curve over  $\mathbf{F}_q$ . There are two kinds of elliptic Weil  $q$ -numbers: the supersingular ones, of the form  $\pm p^{r/2}$  ( $r$  even) and the ordinary ones, which generate a quadratic extension of  $\mathbf{Q}$  in which  $p$  is totally decomposed. In the latter case, if  $[p] = \mathfrak{p}_1 \mathfrak{p}_2$ ,

---

*Date:* Dec. 2, 2010.

then

$$(1) \quad [\alpha] = \mathfrak{p}_1^r \text{ or } \mathfrak{p}_2^r$$

(compare [5, Lemma 2].)

As in [5], we write  $[\rho]$  for the ideal generated by an algebraic integer  $\rho$ . The main lemma is:

**Lemma 2.** *Let  $\alpha_1, \alpha_2, \alpha_3$  be 3 elliptic Weil  $q$ -numbers, generating a multiquadratic number field  $K/\mathbf{Q}$ . Suppose that*

$$[\alpha_1\alpha_2\alpha_3] = [q\beta]$$

*with  $\beta$  an algebraic integer. Then there exist  $i \neq j$  such that*

$$[\alpha_i\alpha_j] = [q].$$

*Proof.* If two of the  $\alpha_i$  are supersingular the assertion is obvious. Thus we may assume that at least two of the  $\alpha_i$  are ordinary.

**Case 1:** one of the  $\alpha_i$ , say  $\alpha_3$ , is supersingular. If  $[\alpha_1\alpha_2] \neq [q]$ , one sees that  $[\alpha_1\alpha_2]$  is not divisible by  $[p]$ . (Using (1) as in [5, proof of Lemma 3], either  $\alpha_1$  and  $\alpha_2$  generate the same quadratic field and then  $[\alpha_1] = [\alpha_2]$ , or  $\alpha_1$  and  $\alpha_2$  generate a biquadratic extension  $K/\mathbf{Q}$  in which  $[p] = \mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3\mathfrak{q}_4$  and then without loss of generality,  $[\alpha_1] = (\mathfrak{q}_1\mathfrak{q}_2)^r$  and  $[\alpha_2] = (\mathfrak{q}_1\mathfrak{q}_3)^r$ .) Since  $r$  is even, hence  $> 1$ , we get a contradiction.

**Case 2:** all the  $\alpha_i$  are ordinary. We assume again that the conclusion of the lemma is violated, and show that  $[\alpha_1\alpha_2\alpha_3]$  is then not divisible by  $[p]$ .

If (say)  $\alpha_1$  and  $\alpha_2$  generate the same quadratic field, then as seen in Case 1,  $[\alpha_1] = [\alpha_2]$  and  $[\alpha_1\alpha_2\alpha_3]$  is not divisible by  $[p]$ . Suppose now that the  $\alpha_i$  generate three distinct imaginary quadratic fields. In particular,  $[K : \mathbf{Q}] \geq 4$ . If  $[K : \mathbf{Q}] = 4$ , then  $K = \mathbf{Q}(\alpha_1, \alpha_2)$  (say) and  $\alpha_1, \alpha_2$  generate two distinct biquadratic subextensions of  $K$ . Then  $\alpha_3$  must generate the third biquadratic subextension: but this is impossible because the latter is real. Thus  $[K : \mathbf{Q}] = 8$ .

We now set up some notation. Let  $G = \text{Gal}(K/\mathbf{Q}) \simeq (\mathbf{Z}/2)^3$ , and let  $X(G)$  be the character group of  $G$ . The quadratic subextensions generated by  $\alpha_1, \alpha_2, \alpha_3$  correspond to characters  $\chi_1, \chi_2, \chi_3$  forming a basis of  $X(G)$ . Let  $(\sigma_1, \sigma_2, \sigma_3)$  be the dual basis of  $G$ . Finally, let  $c \in G$  be the complex conjugation: since  $\chi_i(c) = 1$  for all  $i$ , we find that  $c = \sigma_1\sigma_2\sigma_3$ . Note that, since the  $\alpha_i$  are Weil  $q$ -numbers, we have  $\alpha_i\alpha_i^c = q$ .

Since  $p$  is totally decomposed in all  $\mathbf{Q}(\alpha_i)$ , it is totally decomposed in  $K$ . Pick a prime divisor  $\mathfrak{p}$  of  $[p]$ . We then have

$$[p] = \mathfrak{p}^{\sum_{\sigma \in G} \sigma}.$$

Since  $\alpha_1$  is invariant under  $\sigma_2$  and  $\sigma_3$ , we find from (1), up to changing  $\alpha_1$  to  $\alpha_1^c$ :

$$[\alpha_1] = \mathfrak{p}^{r(1+\sigma_2)(1+\sigma_3)}$$

and similarly:

$$[\alpha_2] = \mathfrak{p}^{r(1+\sigma_1)(1+\sigma_3)}, \quad [\alpha_3] = \mathfrak{p}^{r(1+\sigma_1)(1+\sigma_2)}.$$

We now compute:  $[\alpha_1\alpha_2\alpha_3] = \mathfrak{p}^{rm}$ , with

$$\begin{aligned} m &= (1 + \sigma_2)(1 + \sigma_3) + (1 + \sigma_1)(1 + \sigma_3) + (1 + \sigma_1)(1 + \sigma_2) \\ &= 3 + 2(\sigma_1 + \sigma_2 + \sigma_3) + \sigma_2\sigma_3 + \sigma_1\sigma_3 + \sigma_1\sigma_2. \end{aligned}$$

This shows that  $\mathfrak{p}^{rm}$  is not divisible by  $[p]$  (the summand  $\sigma_1\sigma_2\sigma_3$  is missing). Similarly,  $[\alpha_1\alpha_2\alpha_3^c] = \mathfrak{p}^{rm'}$  with

$$\begin{aligned} m' &= (1 + \sigma_2)(1 + \sigma_3) + (1 + \sigma_1)(1 + \sigma_3) + c(1 + \sigma_1)(1 + \sigma_2) \\ &= 2 + \sigma_1 + \sigma_2 + 3\sigma_3 + 2\sigma_1\sigma_3 + 2\sigma_2\sigma_3 + \sigma_1\sigma_2\sigma_3 \end{aligned}$$

and  $\mathfrak{p}^{rm'}$  is not divisible by  $[p]$  (the summand  $\sigma_1\sigma_2$  is missing). The other possible products reduce to those by permutation of the  $\alpha_i$  and conjugation by  $c$ : the proof is complete.  $\square$

*Proof of Theorem 1.* It is sufficient to prove the equality after tensoring with a large enough number field  $K$ , Galois over  $\mathbf{Q}$ . We first observe that the Frobenius action on  $H^*(\bar{X}) := H^*(\bar{X}, \mathbf{Q}_l) \otimes K$  is semi-simple since  $X$  is an abelian variety (compare [2, Lemma 1.9]). Let  $v$  be an eigenvector of Frobenius, with eigenvalue  $\rho$ . Since  $H^3(\bar{X}) = \Lambda^3 H^1(\bar{X})$ ,  $v$  is of the form  $v_1 \wedge v_2 \wedge v_3$  where  $v_i \in H^1(\bar{X})$  is an eigenvector with Frobenius eigenvalue  $\alpha_i$ . Since  $X$  is a product of elliptic curves, each  $\alpha_i$  corresponds to an elliptic curve  $E_i$  with each  $v_i$  coming from  $E_i$ .

Suppose  $\rho$  is divisible by  $q$ . By Lemma 2, up to renumbering we have  $[\alpha_1\alpha_2] = [q]$ . As in [5, Corollary p. 288], there is an integer  $N \geq 1$  such that  $(\alpha_1\alpha_2)^N = q^N$ .

By the Tate conjecture in codimension 1 for  $E_1 \times E_2$  (Deuring, cf. Tate [6]),  $v_1 \wedge v_2 \otimes \mathbf{Q}_l(1) \in H^2(\bar{E}_1 \times \bar{E}_2)(1)$  is of the form  $\text{cl}(\gamma)$  where  $\gamma$  is a cycle of codimension 1 on  $\bar{E}_1 \times \bar{E}_2$  and  $\text{cl}$  is the cycle class map. Hence  $v \otimes \mathbf{Q}_l(1) = \text{cl}(\pi^*\gamma) \cdot v_3$ , with  $\pi : X \rightarrow E_1 \times E_2$  the projection.  $\square$

**Theorem 3.** *Let  $X$  be a product of elliptic curves, belonging to at most 3 distinct isogeny classes. Then the generalised Tate conjecture holds for  $X$  in all degrees.*

The proof is a variant of the one above: in the proof of Lemma 2, Case 2, the computation showing that  $[\alpha_1\alpha_2\alpha_3]$  and  $[\alpha_1\alpha_2\alpha_3^c]$  are not divisible by  $[p]$  extends to show that  $[\alpha_1^{n_1}\alpha_2^{n_2}\alpha_3^{n_3}]$  and  $[\alpha_1^{n_1}\alpha_2^{n_2}(\alpha_3^c)^{n_3}]$  are not divisible by  $[p]$  for any nonnegative integers  $n_1, n_2, n_3$ . This

generalises Lemma 2 to any product of Weil  $q$ -numbers involved in the cohomology of  $X$ .  $\square$

Finally, we show what problems arise when one tries to replace 3 by 4 in Theorem 1 or 3. Start again with three non isogenous ordinary elliptic curves  $E_1, E_2, E_3$ , with Weil numbers  $\alpha_1, \alpha_2, \alpha_3$ . We retain the notation from Case 2 in the proof of Lemma 2. Apart from  $\chi_1, \chi_2$  and  $\chi_3$ ,

$$\chi_1\chi_2\chi_3$$

is the unique character which does not vanish on  $c$ . In the corresponding quadratic subfield of  $K$ , there is the possibility of a new Weil  $q$ -number  $\alpha_4$  with

$$[\alpha_4] = \mathfrak{p}^{r(1+\sigma_1\sigma_2)(1+\sigma_1\sigma_3)}.$$

This can actually be achieved provided  $r$  is large enough. Since the class group  $Cl(O_K)$  is finite, we may choose  $r$  such that  $\mathfrak{p}^r$  is principal, say  $\mathfrak{p}^r = [\lambda]$ . Then  $N_{K/\mathbf{Q}}(\lambda) = q$  (since  $K$  is totally imaginary) and we choose  $\alpha_4 = \lambda^{(1+\sigma_1\sigma_2)(1+\sigma_1\sigma_3)}$ .

Up to increasing  $r$ , we may assume that the similar formulas hold for  $\alpha_1, \alpha_2$  and  $\alpha_3$ .

By Honda's theorem [1],  $\alpha_4$  corresponds to a 4th (isogeny class of) elliptic curve  $E_4$ . Now  $\alpha_1\alpha_2\alpha_3\alpha_4^c = \lambda^{m''}$  with

$$\begin{aligned} m'' &= m + c(1 + \sigma_1\sigma_2)(1 + \sigma_1\sigma_3) \\ &= 3 + 2(\sigma_1 + \sigma_2 + \sigma_3) + \sigma_2\sigma_3 + \sigma_1\sigma_3 + \sigma_1\sigma_2 \\ &\quad + \sigma_1\sigma_2\sigma_3(1 + \sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3) \\ &= N + 2(1 + \sigma_1 + \sigma_2 + \sigma_3) \end{aligned}$$

with  $N = \sum_{\sigma \in G} \sigma$ . Thus  $\alpha_1\alpha_2\alpha_3\alpha_4^c = q\beta^2$ , with

$$\beta = \lambda^{(1+\sigma_1+\sigma_2+\sigma_3)}.$$

This  $\beta$  is a new Weil  $q$  number; it generates  $K$  since the isotropy group of  $[\beta]$  in  $G$  is trivial. By the Honda-Tate theorem, it corresponds to the isogeny class of a simple  $\mathbf{F}_q$ -abelian variety  $A$  of dimension 4 (see [7, p. 142 formula (7)]).

Let  $\gamma$  be a Weil  $q$ -number contained in  $K$ . Since  $\gamma\gamma^c = q$ , the divisor of  $\gamma$  is of the form  $\mathfrak{p}^{m_\gamma}$ , where  $m_\gamma \in \mathbf{Z}[G]$  is the sum of elements in a section of the projection  $G \rightarrow G/\langle c \rangle$ . These sections form a torsor under the group of maps from  $G/\langle c \rangle$  to  $\langle c \rangle$ , so there are 16 of them. Up to conjugation by  $c$ , we get 8. Among these 8, 4 are given by the kernels of the characters  $\chi_1, \chi_2, \chi_3$  and  $\chi_1\chi_2\chi_3$ , recovering  $\alpha_1, \alpha_2, \alpha_3$  and  $\alpha_4$ . Among the 4 remaining ones, there is the one defining  $\beta$ ; since

the isotropy group of  $[\beta]$  is trivial, the other ones are conjugate to it. We have exhausted the Weil  $q$ -numbers contained in  $K$ .

Let  $X = \prod_{i=1}^4 E_i$ . If we run the technique of proof of [3] or [4] to try and prove the generalised Tate conjecture for  $F^1 H^4(\bar{X})$ , we end up with a Tate cycle in  $H^8(\bar{X} \times \bar{A})(4)$ . This Tate cycle is exotic in the sense that it is not a linear combination of products of Tate cycles of degree 2 (cf. [7, p. 82]), because the relation

$$\alpha_1 \alpha_2 \alpha_3 \alpha_4^c (\beta^2)^c = q^2$$

cannot be reduced to relations of degree 2. I have no idea if the Tate conjecture can be proven for  $X \times A$ .

**Acknowledgements.** Most of this work was done during a stay at IMPA (Rio de Janeiro) in November 2010, in the framework of the France-Brazil cooperation. I thank the first for its hospitality and the second for its support.

#### REFERENCES

- [1] T. Honda *Isogeny classes of abelian varieties over finite fields*, J. Math. Soc. Japan **20** (1968) 83–95.
- [2] B. Kahn *Équivalences rationnelle et numérique sur certaines variétés de type abélien sur un corps fini*, Ann. Sci. Éc. Norm. Sup. **36** (2003), 977–1002.
- [3] B. Kahn *Zeta functions and motives*, Pure Appl. Math. Quarterly **5** (2009), Special Issue: In honor of Jean-Pierre Serre, 507–570 [2008].
- [4] J.S. Milne, N. Ramachandran *Motivic complexes over finite fields and the ring of correspondences at the generic point*, Pure Appl. Math. Q. **5** (2009), Special Issue: In honor of John Tate, 1219–1252.
- [5] M. Spiess *Proof of the Tate conjecture for products of elliptic curves over finite fields*, Math. Ann. **314** (1999), 285–290.
- [6] J.T. Tate *Endomorphisms of abelian varieties over finite fields*, Invent. Math. **2** (1965), 134–144.
- [7] J.T. Tate *Conjectures on algebraic cycles in  $\ell$ -adic cohomology*, Proceedings of Symposia in Pure mathematics **55** (I) (1994), 71–83.

INSTITUT DE MATHÉMATIQUES DE JUSSIEU, UPMC - UFR 929, MATHÉMATIQUES, 4 PLACE JUSSIEU, 75005 PARIS, FRANCE

*E-mail address:* kahn@math.jussieu.fr