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ON THE GENERALISED TATE CONJECTURE FOR PRODUCTS OF ELLIPTIC CURVES OVER FINITE FIELDS

BRUNO KAHN

Abstract. We prove the generalised Tate conjecture for $H^3$ of products of elliptic curves over finite fields, by slightly modifying the argument of M. Spiess concerning the Tate conjecture. We prove it in full if the elliptic curves run among at most 3 isogeny classes. We also show how things become more intricate from $H^4$ onwards, for more than 3 isogeny classes.

Let $\mathbb{F}_q$ be a finite field. It is known that the Tate conjecture for all smooth projective varieties over $\mathbb{F}_q$ implies the generalised Tate conjecture for all smooth projective varieties over $\mathbb{F}_q$ ([3, Rk. 10.3 2], [6, §1]); however, the proofs in these two references are non-effective. It is therefore of interest to ask if one can prove the generalised Tate conjecture for certain explicit classes of $\mathbb{F}_q$-varieties.

In [7], Michael Spiess proved the Tate conjecture for products of elliptic curves over a finite field: this provides a natural candidate for such a class. In this note, we show that a slight modification of his argument does yield the generalised Tate conjecture, in cohomological degree 3 or if the elliptic curves run over at most 3 distinct isogeny classes.

Contrary to [3] and [6], the proofs do not appeal to Honda’s existence theorem [1]. This theorem appears, however, when studying $H^4$ of a well-chosen product of 4 elliptic curves: this is directly related to the delicate combinatorics of Weil numbers; we illustrate the non-effectiveness of the arguments from [3] and [6] in this case.

Theorem 1. Let $X$ be a product of elliptic curves over $\mathbb{F}_q$. Then the generalised Tate conjecture holds for $H^3(\bar{X}, \mathbb{Q}_l)$: the subspace of Tate coniveau 1 coincides with the first step of the coniveau filtration.

Let $q = p^r$ for $p$ be a prime number and $r \geq 1$. As in [7], we write $[\rho]$ for the ideal generated by an algebraic integer $\rho$. As in [6] Def. 1

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The corresponding computation seems in contradiction with the one from [4, Claim p. 130].
we also say that a Weil $q$-number $\alpha$ is elliptic if it arises from the Frobenius endomorphism of an elliptic curve over $F_q$. There are two kinds of elliptic Weil $q$-numbers: the supersingular ones, of the form $\pm p^{r/2}$ and the ordinary ones, which generate a quadratic extension of $\mathbb{Q}$ in which $p$ is totally decomposed. In the latter case, if $[p] = p_1p_2$, then

$$[\alpha] = p_1^r \text{ or } p_2^r$$

(compare [4], Lemma 2.)

The main lemma is:

**Lemma 2.** Let $\alpha_1, \alpha_2, \alpha_3$ be 3 elliptic Weil $q$-numbers, generating a multiquadratic number field $K/\mathbb{Q}$. Suppose that

$$[\alpha_1\alpha_2\alpha_3] = [q\beta]$$

with $\beta$ an algebraic integer. Then there exist $i \neq j$ such that

$$[\alpha_i\alpha_j] = [q].$$

**Proof.** If two of the $\alpha_i$ are supersingular the assertion is obvious. Thus we may assume that at least two of the $\alpha_i$ are ordinary.

**Case 1:** one of the $\alpha_i$, say $\alpha_3$, is supersingular. If $[\alpha_1\alpha_2] \neq [q]$, one sees that $[\alpha_1\alpha_2]$ is not divisible by $[p]$. (Using (1) as in [7], proof of Lemma 3], either $\alpha_1$ and $\alpha_2$ generate the same quadratic field and then $[\alpha_1] = [\alpha_2]$, or $\alpha_1$ and $\alpha_2$ generate a biquadratic extension $K/\mathbb{Q}$ in which $[p] = q_1q_2q_3q_4$ and then without loss of generality, $[\alpha_1] = (q_1q_2)^r$ and $[\alpha_2] = (q_1q_3)^r$.) If $r > 1$, we get a contradiction. If $r = 1$, we have the equation $[\alpha_1\alpha_2] = [\sqrt{p}\beta]$ in $K(\sqrt{p})$. Since $p$ is totally ramified in $Q(\sqrt{p})$, the prime divisors of $[p]$ in $K$ are totally ramified in $K(\sqrt{p})$ and we get a new contradiction.

**Case 2:** all the $\alpha_i$ are ordinary. We assume again that the conclusion of the lemma is violated, and show that $[\alpha_1\alpha_2\alpha_3]$ is then not divisible by $[p]$.

If (say) $\alpha_1$ and $\alpha_2$ generate the same quadratic field, then as seen in Case 1, $[\alpha_1] = [\alpha_2]$ and $[\alpha_1\alpha_2\alpha_3]$ is not divisible by $[p]$. Suppose now that the $\alpha_i$ generate three distinct imaginary quadratic fields. In particular, $[K : \mathbb{Q}] \geq 4$. If $[K : \mathbb{Q}] = 4$, then $K = \mathbb{Q}(\alpha_1, \alpha_2)$ (say) and $\alpha_1, \alpha_2$ generate two distinct quadratic subextensions of $K$. Then $\alpha_3$ must generate the third quadratic subextension: but this is impossible because the latter is real. Thus $[K : \mathbb{Q}] = 8$.

We now set up some notation. Let $G = Gal(K/\mathbb{Q}) \simeq (\mathbb{Z}/2)^3$, and let $X(G)$ be the character group of $G$. The quadratic subextensions generated by $\alpha_1, \alpha_2, \alpha_3$ correspond to characters $\chi_1, \chi_2, \chi_3$ forming a basis of $X(G)$. Let $(\sigma_1, \sigma_2, \sigma_3)$ be the dual basis of $G$. Finally, let
$c \in G$ be the complex conjugation: since $\chi_i(c) = 1$ for all $i$, we find that $c = \sigma_1 \sigma_2 \sigma_3$. Note that, since the $\alpha_i$ are Weil $q$-numbers, we have $\alpha_i \alpha_i^c = q$.

Since $p$ is totally decomposed in all $\mathbb{Q}(\alpha_i)$, it is totally decomposed in $K$. Pick a prime divisor $\mathfrak{p}$ of $[p]$. We then have

$$[p] = \mathfrak{p}^{\sum_{\sigma \in G} \sigma}.$$ 

Since $\alpha_1$ is invariant under $\sigma_2$ and $\sigma_3$, we find from (1), up to changing $\alpha_1$ to $\alpha_1^c$:

$$[\alpha_1] = \mathfrak{p}^{r(1+\sigma_2)(1+\sigma_3)}$$

and similarly:

$$[\alpha_2] = \mathfrak{p}^{r(1+\sigma_1)(1+\sigma_3)}, \quad [\alpha_3] = \mathfrak{p}^{r(1+\sigma_1)(1+\sigma_2)}.$$ 

We now compute: $[\alpha_1 \alpha_2 \alpha_3] = \mathfrak{p}^{rm}$, with

$$m = (1 + \sigma_2)(1 + \sigma_3) + (1 + \sigma_1)(1 + \sigma_3) + (1 + \sigma_1)(1 + \sigma_2)$$

$$= 3 + 2(\sigma_1 + \sigma_2 + \sigma_3) + \sigma_2 \sigma_3 + \sigma_1 \sigma_3 + \sigma_1 \sigma_2.$$ 

This shows that $\mathfrak{p}^{rm}$ is not divisible by $[p]$ (the summand $\sigma_1 \sigma_2 \sigma_3$ is missing). Similarly, $[\alpha_1 \alpha_2 \alpha_3^c] = \mathfrak{p}^{rm'}$ with

$$m' = (1 + \sigma_2)(1 + \sigma_3) + (1 + \sigma_1)(1 + \sigma_3) + c(1 + \sigma_1)(1 + \sigma_2)$$

$$= 2 + \sigma_1 + \sigma_2 + 3\sigma_3 + 2\sigma_1 \sigma_3 + 2\sigma_2 \sigma_3 + \sigma_1 \sigma_2 \sigma_3$$

and $\mathfrak{p}^{rm'}$ is not divisible by $[p]$ (the summand $\sigma_1 \sigma_2$ is missing). The other possible products reduce to those by permutation of the $\alpha_i$ and conjugation by $c$: the proof is complete.

Proof of Theorem 3. It is sufficient to prove the equality after tensoring with a large enough number field $K$, Galois over $\mathbb{Q}$. We first observe that the Frobenius action on $H^*(X) := H^*(\bar{X}, \mathbb{Q}_l) \otimes K$ is semi-simple since $X$ is an abelian variety (compare [2] Lemma 1.9]). Let $v$ be an eigenvector of Frobenius, with eigenvalue $\rho$. Since $H^3(\bar{X}) = \Lambda^3 H^1(\bar{X})$ and $X$ is a product of elliptic curves, $v$ is a sum of vectors of the form $v_1 \wedge v_2 \wedge v_3$ where $v_i \in H^1(\bar{X})$ is an eigenvector with Frobenius eigenvalue $\alpha_i$ with $\alpha_1 \alpha_2 \alpha_3 = \rho$, $\alpha_i$ corresponds to an elliptic curve $E_i$ and $v_i$ comes from $H^1(E_i) \hookrightarrow H^1(X)$.

Suppose $\rho$ is divisible by $q$. Without loss of generality, we may assume that $v$ is a single vector $v_1 \wedge v_2 \wedge v_3$. By Lemma [2], up to renumbering we have $[\alpha_1 \alpha_2] = [q]$. As in [2] Corollary p. 288, there is an integer $N \geq 1$ such that $(\alpha_1 \alpha_2)^N = q^N$.

By the Tate conjecture in codimension 1 for $E_1 \times E_2$ (Deuring, cf. Tate [2]), $v_1 \wedge v_2 \otimes \mathbb{Q}_l(1) \in H^2(E_1 \times E_2)(1)$ is of the form $\text{cl}(\gamma)$ where
\[ \gamma \text{ is a cycle of codimension 1 on } E_1 \times E_2 \text{ and } \text{cl is the cycle class map.} \]

Hence \( v \otimes \mathbb{Q}_l(1) = \text{cl}(\pi^* \gamma) \cdot v_3 \), with \( \pi : X \rightarrow E_1 \times E_2 \) the projection. \( \square \)

**Theorem 3.** Let \( X \) be a product of elliptic curves, belonging to at most 3 distinct isogeny classes. Then the generalised Tate conjecture holds for \( X \) in all degrees and all coniveau.

The proof is a variant of the one above: in the proof of Lemma 2, Case 2, the computation showing that \([\alpha_1 \alpha_2 \alpha_3]\) and \([\alpha_1 \alpha_2 \alpha_3^c]\) are not divisible by \([p]\) extends to show that \([\alpha_1^{n_1} \alpha_2^{n_2} \alpha_3^{n_3}]\) and \([\alpha_1^{n_1} \alpha_2^{n_2} (\alpha_3^c)^{n_3}]\) are not divisible by \([p]\) for any nonnegative integers \(n_1, n_2, n_3\). This generalises Lemma 2 to any product of Weil \(q\)-numbers involved in the cohomology of \(X\). \( \square \)

Finally, we show what problems arise when one tries to replace 3 by 4 in Theorem 1 or 3. Start again with three non isogenous ordinary elliptic curves \(E_1, E_2, E_3\), with Weil numbers \(\alpha_1, \alpha_2, \alpha_3\). We retain the notation from Case 2 in the proof of Lemma 2. Apart from \(\chi_1, \chi_2\) and \(\chi_3\),

\[ \chi_1 \chi_2 \chi_3 \]

is the unique character which does not vanish on \(c\). In the corresponding quadratic subfield of \(K\), there is the possibility of a new Weil \(q\)-number \(\alpha_4\) with

\[ [\alpha_4] = p^{r(1+\sigma_1\sigma_2)(1+\sigma_1\sigma_3)}. \]

This can actually be achieved provided \(r\) is large enough. Since the class group \(\text{Cl}(O_K)\) is finite, we may choose \(r\) such that \(p^r = [\lambda]\). Then \(N_{K/\mathbb{Q}}(\lambda) = q\) (since \(K\) is totally imaginary) and we choose \(\alpha_4 = \lambda^{(1+\sigma_1\sigma_2)(1+\sigma_1\sigma_3)}.\)

Up to increasing \(r\), we may assume that the similar formulas hold for \(\alpha_1, \alpha_2\) and \(\alpha_3\).

By Honda’s theorem \([4]\), \(\alpha_4\) corresponds to a 4th (isogeny class of) elliptic curve \(E_4\). Now \(\alpha_1 \alpha_2 \alpha_3 \alpha_4^c = \lambda^{m''}\) with

\[ m'' = m + c(1 + \sigma_1\sigma_2)(1 + \sigma_1\sigma_3) \]

\[ = 3 + 2(\sigma_1 + \sigma_2 + \sigma_3) + \sigma_2\sigma_3 + \sigma_1\sigma_3 + \sigma_1\sigma_2 \]

\[ + \sigma_1\sigma_2\sigma_3(1 + \sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3) \]

\[ = N + 2(1 + \sigma_1 + \sigma_2 + \sigma_3) \]

with \(N = \sum_{\sigma \in G} \sigma\). Thus \(\alpha_1 \alpha_2 \alpha_3 \alpha_4^c = q\beta^2\), with

\[ \beta = \lambda^{(1+\sigma_1+\sigma_2+\sigma_3)}. \]

This \(\beta\) is a new Weil \(q\)-number; it generates \(K\) since the isotropy group of \([\beta]\) in \(G\) is trivial. By the Honda-Tate theorem, it corresponds
to the isogeny class of a simple $\mathbb{F}_q$-abelian variety $A$ of dimension 4 (see [3, p. 142 formula (7)]).

Let us say that a Weil $q$-number $\gamma$ is ordinary if $\gcd(\gamma, \gamma^c) = 1$. This is equivalent to requiring that $\gcd(p, \gamma + \gamma^c) = 1$, hence, by [9, Prop. 7.1], that the corresponding abelian variety be ordinary. Let $\gamma \in K$ be an ordinary Weil $q$-number. Since $\gamma \gamma^c = q$, the divisor of $\gamma$ is of the form $p^m \gamma$, where $m_\gamma \in \mathbb{Z}[G]$ is the sum of elements in a section of the projection $G \to G/\langle c \rangle$. These sections form a torsor under the group of maps from $G/\langle c \rangle$ to $\langle c \rangle$, so there are 16 of them. Up to conjugation by $c$, we get 8. Among these 8, 4 are given by the kernels of the characters $\chi_1, \chi_2, \chi_3$ and $\chi_1 \chi_2 \chi_3$, recovering $\alpha_1, \alpha_2, \alpha_3$ and $\alpha_4$. Among the 4 remaining ones, there is the one defining $\beta$; since the isotropy group of $[\beta]$ is trivial, the other ones are conjugate to it. We have exhausted the ordinary Weil $q$-numbers contained in $K$.

Let $X = \prod_{i=1}^4 E_i$. If we run the technique of proof of [3] or [8] to try and prove the generalised Tate conjecture for $N^1H^4(\bar{X})$, we end up with a Tate cycle in $H^6(\bar{X} \times \bar{A})(3)$. This Tate cycle is exotic in the sense that it is not a linear combination of products of Tate cycles of degree 2 (cf. [5, p. 136]), because the relation

$$\alpha_1 \alpha_2 \alpha_3 \alpha_4 \gamma^c (\beta^2)^c = q^3$$

cannot be reduced to relations of degree 2. I have no idea if the Tate conjecture can be proven for $X \times A$. Can the methods of [5] be used to answer this question?

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References


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